Binary factors of shifts of finite type

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Abstract

We construct two new classes of topological dynamical systems; one is a factor of a one-sided shift of finite type while the second is a factor of the two-sided shift. The data is a finite graph which presents the shift of finite type, a second finite directed graph and a pair of embeddings of it into the first, satisfying certain conditions. The factor is then obtained from a simple idea based on binary expansion of real numbers. In both cases, we construct natural metrics on the factors and, in the second case, this makes the system a Smale space, in the sense of Ruelle. We compute various algebraic invariants for these systems, including the homology for Smale space developed by the author and the K-theory of various C^*-algebras associated to them, in terms of the pair of original graphs.

1 Introduction

In the subject of topological dynamical systems, a crucial part has been played by systems which display some type of hyperbolicity; that is, systems which have some type of local expanding or contracting/expanding behaviour. This began in the smooth category with the study of Anosov diffeomorphisms [15]. Smale's seminal paper [32] showed that, even for smooth

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systems, the hyperbolic behaviour may be limited to the non-wandering set which may be far from being a submanifold. Motivated by this, David Ruelle gave a purely topological definition of hyperbolicity which he called a Smale space [28]. This broader framework includes shifts of finite type, which are highly combinatorial in nature (see section 2 for the definition). Restricted to irreducible systems, these are precisely the Smale spaces whose underlying space is totally disconnected.

One of the fundamental ideas for these systems is that they may be coded by a Markov partition [15]. Ignoring some minor technical issues, the existence of a Markov is equivalent to the condition that there is an almost-one-to-one factor map from a shift of finite type onto the given system. As noted by Adler in [1], the simplest example of this is binary expansion of real numbers: the space of one-sided $0, 1$-sequences maps onto the unit circle via the familiar formula sending a sequence $x_n, n \geq 1$, to $\exp(2\pi i \sum_n x_n 2^{-n})$. Moreover, this map intertwines the dynamics of the left shift map on the sequences with the squaring map on the circle. ('Binary' may be changed to 'decimal' by replacing certain 2's with 10's.) One tends to regard decimal expansion as a bijection, but the two spaces here are very different topologically.

Returning to the general situation, we recall the following seminal result. In the form stated, it is due to Rufus Bowen [4], but it builds on the work of many others, including Sinai [31], Adler and Weiss [2] and others. Let (X, d) be a compact metric space and φ be a homeomorphism of X such that (X, φ, d) is a irreducible Smale space (see section 4 for the definition). Alternately, let (X, φ) be the restriction of an Axiom A system to a basic set. Then there is a irreducible shift of finite type (Σ, σ) and a continuous surjection $\pi : \Sigma \to X$ such that $\pi \circ \varphi = \sigma \circ \pi$.

There are several interesting invariants of a Smale space. First, one can construct a number of different C^*-algebras from a single Smale space. For any Smale space, (X, d, φ) and choice of a finite φ-invariant set $P \subseteq X$, there are C^*-algebras $S(X, \varphi, P)$ and $U(X, \varphi, P)$ based on stable and unstable equivalence, respectively. Each has a canonical isomorphism induced by φ and we let $R^s(X, \varphi, P) = S(X, \varphi, P) \rtimes_\varphi \mathbb{Z}$ and $R^u(X, \varphi, P) = U(X, \varphi, P) \rtimes_\varphi \mathbb{Z}$ be the associated crossed product C^*-algebras (see section 6).

This was done initially for shifts of finite type by Cuntz and Krieger [7] and later by Ruelle [29] for general Smale spaces. The K-theory groups of these C^*-algebras have been investigated quite thoroughly and provide interesting data. For shifts of finite type, these include Krieger's dimension group invariant as well as the Bowen-Franks groups [6]. There are, in fact, two dimension groups which are associated with right and left tail-equivalence, respectively. These are described in section 5 but we note that for a shift of finite type associated with a finite directed graph G, these have simple combinatorial descriptions in terms of G and we denote them by $D^s(G)$ and $D^u(G)$. We let A_G denote the adjacency matrix associated with G. The matrices A_G^T and A_G induce automorphisms of $D^s(G)$ and $D^u(G)$, respectively.

In another direction, the author constructed a homology theory for Smale spaces [24] which generalizes Krieger's dimension group for shifts of finite type. Indeed, the dimension group is a key part of the construction. The existence of this theory, which provides a Lefschetz formula, was conjectured by Bowen [5]. For any Smale space (X, d, φ), there are groups $H_N^s(X, \varphi)$, $H_N^u(X, \varphi), N \in \mathbb{Z}$, and each has a canonical automorphism induced by φ.

It is worth noting that recent work of Proietti and Yamashita [20, 21] shows that there are very close relations between the K-theory of the C^*-algebras and the Smale space homology, as well the cohomology of certain groupoids.

The main goal of this paper is a kind of reverse-engineering of Bowen's result, where the Smale space is the end product of the construction, rather than the initial object of interest. We begin with a some combinatorial data: a pair of shifts of finite type associated with finite directed graphs G and H with a pair of embeddings of the latter into the former $\xi^0, \xi^1 : H \to G$. We let A_G and A_H denote the adjacency matrices for the two graphs. We defer the details of the construction as well as certain hypotheses on our data to section 3 and focus on the properties of the resulting systems. In particular, the *standing hypotheses* are described in Definition 3.1.

As described in detail in section 2, we let (X_G^+, σ) and (X_G, σ) denote the one and two-sided shifts, respectively, associated with the graph G. Adding a basic idea modeled on binary expansion, we construct topological dynamical systems (X_ξ^+, σ_ξ) and (X_ξ, σ_ξ) along with factor maps $\pi_\xi : (X_G^+, \sigma) \to (X_\xi^+, \sigma_\xi)$ and $\pi_\xi : (X_G, \sigma) \to (X^+, \sigma_\xi)$. Both systems are continuous and surjective and the second is actually a homeomorphism. We also construct specific metrics d_ξ on our two spaces having nice properties. Indeed, most of the effort lies in producing the metrics.

Let us summarize the main results on these systems. The first is a local expansiveness property for the first system.

Theorem 3.17. *If x, y in X_ξ^+ satisfy $d_\xi(x, y) \leq 2^{-2}$, then*

$$2d_\xi(x, y) \leq d_\xi(\sigma_\xi(x), \sigma_\xi(y)) \leq 8d_\xi(x, y).$$

In fact, the map is actually a local homeomorphism.

Theorem 3.19. *If x, y are in X_ξ^+ with $d_\xi(x, \sigma_\xi(y)) \leq 2^{-1}$, then there is z in X_ξ^+ with $d_\xi(z, y) \leq 2^{-1} d_\xi(x, \sigma_\xi(y))$ and $\sigma_\xi(z) = x$. In particular, we have*

$$X_\xi^+(\sigma_\xi(y), \epsilon) \subseteq \sigma_\xi(X_\xi^+(y, 2^{-1}\epsilon)),$$

for any $\epsilon < 2^{-1}$ and the map σ_ξ is open.

The actual geometric structure of X_ξ^+ is rather curious. We offer an intriguing picture of a single example in section 7,but we also note the following.

Corollary 7.7. *The connected subsets of X_ξ^+ are either points or circles and both occur.*

The property analogous to locally expanding for the second system is that it possess local coordinates of contracting and expanding directions. In short, it is a Smale space. (We review the definition in section 4.)

Theorem 4.2. *$(X_\xi, d_\xi, \sigma_\xi)$ is a Smale space.*

We also note the following (abridged version). Here, $X^s(x)$ denotes the global stable set of the point x in a Smale space X.

Theorem 4.3. *The map $\pi_\xi : (X_G, \sigma) \to (X_\xi, \sigma_\xi)$ is s-bijective; that is, for every x in X_G, $\pi_\xi | X_G^s(x)$ is a bijection from $X_G^s(x)$ to $X_\xi^s(\pi_\xi(x))$.*

This means, in particular, that the local stable sets of X_ξ are Cantor sets.

The following summarizes our computation of the homology theory.

Theorem 5.1. *Under the standing hypotheses, we have*

$$\begin{array}{rclcrcl}
H_0^s(X_\xi, \sigma_\xi) & \cong & D^s(G), & (\sigma_\xi)_*^{-1} & \cong & A_G, \\
H_1^s(X_\xi, \sigma_\xi) & \cong & D^s(H), & (\sigma_\xi)_*^{-1} & \cong & A_H, \\
H_0^u(X_\xi, \sigma_\xi) & \cong & D^u(G), & (\sigma_\xi)_* & \cong & A_G^T, \\
H_1^u(X_\xi, \sigma_\xi) & \cong & D^u(H), & (\sigma_\xi)_* & \cong & A_H^T, \\
H_k^s(X_\xi, \sigma_\xi) & \cong & 0, & k & \neq & 0, 1, \\
H_k^u(X_\xi, \sigma_\xi) & \cong & 0, & k & \neq & 0, 1.
\end{array}$$

In the first two lines, we regard σ_ξ as an s-bijective factor map from (X_ξ, σ_ξ) to itself and our description of the induced map on homology is interpreted via the isomorphism which precedes it. In the next two lines, we regard σ_ξ as a u-bijective factor map.

The next two results summarize our computations for the K-theory of the C^*-algebras.

Theorem 6.1. *Under the standing hypotheses, we have*

1. $K_0(S(X_\xi, \sigma_\xi, P_\xi)) \cong D^u(G)$ *as ordered abelian groups and, under this isomorphism, the automorphism induced by* σ_ξ *is* A_G^T.

2. $K_1(S(X_\xi, \sigma_\xi, P_\xi)) \cong D^u(H)$ *and, under this isomorphism, the automorphism induced by* σ_ξ *is* A_H^T.

3. $K_0(R^s(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}^{G^0}/(I - A_G^T)\mathbb{Z}^{G^0} \oplus \ker(I - A_H^T : \mathbb{Z}^{H^0} \to \mathbb{Z}^{H^0})$.

4. $K_1(R^s(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}^{H^0}/(I - A_H^T)\mathbb{Z}^{H^0} \oplus \ker(I - A_G^T : \mathbb{Z}^{G^0} \to \mathbb{Z}^{G^0})$.

Theorem 6.2. *Under the standing hypotheses, we have*

1. $K_0(U(X_\xi, \sigma_\xi, P_\xi)) \cong D^s(G)$ *as ordered abelian groups and, under this isomorphism, the automorphism induced by* σ_ξ *is* A_G^{-1}.

2. $K_1(U(X_\xi, \sigma_\xi, P_\xi)) \cong D^s(H)$ *and, under this isomorphism, the automorphism induced by* σ_ξ *is* A_H^{-1}.

3. $K_0(R^u(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}^{G^0}/(I - A_G)\mathbb{Z}^{G^0} \oplus \ker(I - A_H : \mathbb{Z}^{H^0} \to \mathbb{Z}^{H^0})$.

4. $K_1(R^u(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}^{H^0}/(I - A_H)\mathbb{Z}^{H^0} \oplus \ker(I - A_G : \mathbb{Z}^{G^0} \to \mathbb{Z}^{G^0})$.

A curious consequence of these computations and the Phillips-Kirchberg classification theorem for C^*-algebras is the following.

Theorem 6.5. *Let* (X, φ, d) *be a mixing Smale space and* Q *be a finite* φ*-invariant subset of* X*. There exist finite directed graphs,* G, H *and embeddings* $\xi^0, \xi^1 : H \to G$ *satisfying the standing hypotheses such that* $R^s(X_\xi, \sigma_\xi, P_\xi) \cong R^s(X, \varphi, Q)$*, for any* P*, a finite* σ*-invariant subset of* X_G.

The second section contains basic background information on shifts of finite type, or at least edge shifts for finite directed graphs. The third section is the construction of our factor system. At this stage, we begin only with one-sided shifts of finite type and construct factors which are continuous topological dynamical systems, but not invertible. Much of the effort here is focused on the construction of a specific metric with nice properties. In the fourth section, we continue the construction to produce invertible systems.

This is a standard technique by inverse limits, but again, we construct a specific metric which makes the result a Smale space. We also review the definitions of a Smale space. The fifth section is devoted to the computation of the homology theory for our Smale spaces. The sixth deals with the various C^*-algebras associated with our examples, including computations of their K-theories. In the seventh section, we return to the factors of the one-sided shifts of finite type. Geometrically, shifts of finite type are not particularly interesting; at least they are no more interesting the Cantor ternary set. This not is true of our factors and we spend some time giving a description of their geometry. In particular, we show that the spaces can be embedded in $\mathbb{R}^3$. We also give a couple of simple examples, which can actually be embedded in the plane.

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2 Preliminaries

In this section, we set out some well-known preliminaries on shift spaces. An excellent reference is the book of Lind and Marcus [17]. As discussed in the introduction, we are considering shifts of finite type. However, we will deal exclusively with edge shifts of finite directed graphs. These are dynamically equivalent, as a consequence of Example 1.5.10 and Theorem 2.3.2 of [17].

By a *finite directed graph*, G, we mean two finite sets G^0 (the vertex set) and G^1 (the edge set) along with maps $i, t : G^1 \to G^0$. A path of length n in G is a finite sequence $p = (p_1, p_2, \ldots, p_n)$ in G^1 such that $t(p_i) = i(p_{i+1})$, for all $1 \leq i < n$. We define $i(p) = i(p_1)$ and $t(p) = t(p_n)$.

The *adjacency matrix* for the graph is denoted A_G and we regard it as the function on $G^0 \times G^0$ whose value at (v, w) is the number of edges e with $i(e) = w$ and $t(e) = v$. We also regard it an endomorphism of $\mathbb{Z}^{G^0}$ by matrix multiplication.

We say G is *irreducible* if, for every ordered pair of vertices v, w, there is a path p with $i(p) = v, t(p) = w$. Equivalently, G is irreducible, if for every (v, w), there is a positive integer k with $A_G^k(w, v)$ positive. We say G is *primitive* if there is a positive integer k such that $A_G^k(w, v)$ is positive, for all v, w.

We let X_G^+ be the one-sided infinite path space: an element is a sequence $(x_1, x_2, \ldots)$ in G^1 with $t(x_n) = i(x_{n+1})$ for all $n \geq 1$. The bi-infinite path space X_G is defined analogously with sequences indexed by the integers. We let χ_G^+ denote the obvious map from X_G to X_G^+, which simply restricts the domain of the sequence.

The spaces X_G^+ and X_G both have canonical metrics. For x, y in X_G^+, we define

$$d_G(x,y) = \inf\{2^{-n} \mid n \geq 0, x_i = y_i, 1 \leq i \leq n\}.$$

and for x, y in X_G, we define

$$d_G(x,y) = \inf\{2^{-n} \mid n \geq 0, x_i = y_i, 1-n \leq i \leq n\}.$$

Using the same notation should cause no confusion. Observe that χ_G^+ is a contraction.

The spaces X_G^+ and X_G both carry dynamical systems. We define σ by $\sigma(x)_n = x_{n+1}$. Here, either x is in X_G^+ and $n \geq 1$ or x is in X_G and n is an integer, making σ a self-map of either space. Both are continuous and surjective and the latter is a homeomorphism. We make the easy observation that $d_G(\sigma^n(x), \sigma^n(y)) \leq 2^n d_G(x,y)$, for x, y in X_G^+.

Let G, H be two finite directed graphs. By a *graph homomorphism* from H to G we mean a function, $\xi : H^0 \cup H^1 \to G^0 \cup G^1$, such that $\xi(H^0) \subseteq G^0, \xi(H^1) \subseteq G^1$ and satisfying $t_G \circ \xi|_{H^0} = \xi \circ t_H|_{H^0}, i_G \circ \xi|_{H^0} = \xi \circ i_H|_{H^0}$. It will usually not cause any confusion to drop the subscripts on i, t. We usually write such a function as $\xi : H \to G$. A *graph embedding* is a graph homomorphism which is injective.

If (X, d) is a metric space, x is in X and ϵ is positive, then we let $X(x, \epsilon)$ denote the ball centred at x of radius ϵ. If A is any subset of X, we let $Cl(A)$ denote its closure. If (X_1, d_1) and (X_2, d_2) are metric spaces, we let

$$d_1 \times d_2((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2),$$

for all x_1, y_1 in X_1 and x_2, y_2 in X_2, which is a metric for the product space $X_1 \times X_2$.

3 Construction

In this section, we describe our basic construction.

Definition 3.1. *For finite directed graphs, G, H, a pair of graph embeddings $\xi^0, \xi^1 : H \to G$ are said to satisfy*

1. *(H0) if $\xi^0|_{H^0} = \xi^1|_{H^0}$,*

2. *(H1) if $\xi^0(H^1) \cap \xi^1(H^1)$ is empty,*

3. *(H2) if, for every y in H^1, there is x in G^1 with $t(x) = t(\xi^0(y)), i(x) = i(\xi^0(y))$ and $x \notin \xi^0(H^1) \cup \xi^1(H^1)$.*

We say that $G, H, \xi = (\xi^0, \xi^1)$ satisfies the standing hypotheses *if G is primitive and ξ satisfies (H0), (H1) and (H2).*

The first two conditions will be essential for our construction. The third is a convenience and is not always needed. It also seems likely that the hypothesis that G is primitive can be weakened.

We observe a simple consequence of (H0) and (H1): if y is any edge in H^1, then $\xi^0(y)$ and $\xi^1(y)$ are distinct, but have the same initial and terminal vertices in G^0.

Definition 3.2. *Let $\xi^0, \xi^1 : H \to G$ satisfy (H0) and (H1) of 3.1.*

1. *We let $\xi(H^1) = \xi^0(H^1) \cup \xi^1(H^1)$.*

2. *For x in $\xi(H^1)$, we let $\varepsilon(x) = 0, 1$ be such that x is in $\xi^{\varepsilon(x)}(H^1)$.*

3. *We let G_ξ be the graph with $G^0_\xi = G^0$ and G^1_ξ obtained by identifying $\xi^0(y)$ with $\xi^1(y)$, for every y in H^1. We also let τ_ξ denote the obvious quotient map from G^1 to G^1_ξ as well as the associated map from X^+_G to $X^+_{G_\xi}$ and the map from X_G to X_{G_ξ}.*

Definition 3.3. *Let $\xi^0, \xi^1 : H \to G$ satisfy (H0) and (H1) of 3.1.*

1. *For x in X^+_G, let $\kappa(x)$ be the number of positive integers n such that x_n is* not *in $\xi(H^1)$, allowing the values 0 and ∞.*

2. *For $0 \leq k \leq \infty$, let $X^+_k = \kappa^{-1}\{k\}$.*

3. *For x in X^+_G with $\kappa(x) > 0$, we define $n(x)$ to be the least positive integer with $x_{n(x)}$* not *in $\xi(H^1)$.*

The following result is worth noting but its proof is trivial and we omit it.

Proposition 3.4. *1. The sets $X_k^+, 0 \leq k \leq \infty$, are pairwise disjoint and, for any $0 \leq k < \infty$, $\cup_{j=0}^k X_j^+$ is closed in X_G^+.*

2. If x, y in X_G^+ satisfy $\tau_\xi(x) = \tau_\xi(y)$, then $\kappa(x) = \kappa(y)$ and $n(x) = n(y)$ if $\kappa(x) > 0$.

3. If x is in X_G^+ with $0 < \kappa(x) < \infty$, then $\kappa(\sigma^{n(x)}(x)) = \kappa(x) - 1$.

The following is a summary of some convenient topological properties of the sets $X_k^+, k \geq 0$. Essentially, these are consequences of our property (H2) and G being primitive.

Proposition 3.5. *Suppose that G, H, ξ satisfy the standing hypotheses.*

1. If x is in X_j^+ and $j < k < \infty$, then there is a sequence $x^l, l \geq 1$, in X_k^+ which converges to x in X_G^+.

2. The closure of X_k^+ in X_G^+ is $\cup_{j=0}^k X_j^+$.

3. If x is in X_G^+, then there is a sequence $x^l, l \geq 1$, converging to x in X_G^+ with x^l in X_l^+, for all $l \geq 1$.

4. The closure of $\cup_{k=0}^\infty X_k^+$ in X_G^+ is X_G^+.

Proof. We first observe that it follows from hypothesis (H2) and G being primitive that $G^1 - \xi(H^1)$ is connected.

For x in X_j^+, there are only finitely many n with x_n not in $\xi(H^1)$. Choose n_0 such that x_n is in $\xi(H^1)$, for all $n \geq n_0$. For $l \geq 1$, we define x^l in three segments, as follows. Define $x_n^l = x_n$, for $1 \leq n \leq n_0 + l$. Then choose $x_n^l, n_0 + l < n \leq n_0 + l + k - j$ to be any path in $G^1 - \xi(H^1)$ going through the same vertices as x_n, by property (H2). Finally, set $x_n^l = x_n$ for $n > n_0 + l + k - j$. It is clear that this sequence has the desired properties. The first part shows that the closure of X_k^+ contains $\cup_{j=0}^k X_j^+$. The reverse containment follows from the fact that the latter is closed, as we noted in 3.4.

As G is primitive, we may find $l_0 \geq 1$ such that, for any ordered pair of vertices in G^0 and any $l \geq l_0$, there is a path of length l between these two vertices. From property (H2), we may assume such a path lies in $G^1 - \xi(H^1)$. For $l > l_0$, we again define x^l in three segments as follows. For the first segment, we set $x_n^l = x_n$, for $1 \leq n \leq l - l_0$. Let j be the number of integers, n, between 1 and $l - l_0$ with x_n not in $\xi(H^1)$. Obviously, $j \leq l - l_0$. For the second segment, $x^l_n, l-l_0 < n \leq 2l-l_0-j$, we choose any path in $G^1-\xi(H^1)$ from $t(x_{l-l_0})$ to any vertex of $\xi^0(H^0)$ having length $l-j \geq l_0$. Finally, we define the third segment $x^l_n, 2l-l_0-j < n < \infty$, to be any path in $\xi(H^1)$. The points $x^l, 1 \leq l \leq l_0$, may be chosen arbitrarily from X^+_l. Again, it is clear that this sequence has the desired properties. The last statement follows at once. □

We now give our two main definitions. The influence of binary expansion should be clear in the first.

Definition 3.6. *Let x be in X^+_G.*

1. *If*
$$x = (x_1, x_2, \cdots, x_{n-1}, \xi^{1-i}(y_n), \xi^i(y_{n+1}), \xi^i(y_{n+2}), \ldots),$$
for some $n \geq 1$, $y_n, y_{n+1}, \ldots$ in H^1 and $i = 0, 1$, then we define
$$x' = (x_1, x_2, \cdots, x_{n-1}, \xi^i(y_n), \xi^{1-i}(y_{n+1}), \xi^{1-i}(y_{n+2}), \ldots).$$

2. *If*
$$x = (x_1, x_2, \cdots, x_n, \xi^i(y_{n+1}), \xi^i(y_{n+2}), \ldots),$$
for some $n \geq 1$ and $i = 0, 1$, $y_{n+1}, y_{n+2}, \ldots$ in H^1, x_n is not in $\xi(H^1)$, and $i = 0, 1$, then we define
$$x' = (x_1, x_2, \cdots, x_n, \xi^{1-i}(y_{n+1}), \xi^{1-i}(y_{n+2}), \xi^{1-i}(y_{n+3}), \ldots).$$

3. *We define $x \sim_\xi x'$, whenever x, x' are as above. Also, we define $x \sim_\xi x$, for every x in X^+_G.*

It is worth noting that the n in parts 1 and 2 is unique (if it exists). The proofs of the following easy facts are left to the interested reader.

Proposition 3.7.

1. *$\sim_\xi$ is an equivalence relation and each equivalence class has either one or two elements.*

2. *If $x \sim_\xi x'$, then $\tau_\xi(x) = \tau_\xi(x')$.*

3. *For each $k \geq 0$, the set X^+_k is $\sim_\xi$-invariant. That is, if x is in X^+_k and $x \sim_\xi x'$, then x' is in X^+_k.*

4. *If $x \sim_\xi x'$, then $\sigma(x) \sim_\xi \sigma(x')$.*

5. *Applying σ to the $\sim_\xi$-equivalence class of x yields the $\sim_\xi$-equivalence class of $\sigma(x)$.*

We can now define our main item of interest.

Definition 3.8. *Let G, H be finite directed graphs and $\xi^0, \xi^1 : H \to G$ be two graph embeddings satisfying (H0) and (H1) of Definition 3.1. We define X_ξ^+ to be the quotient space of X_G^+ by the equivalence relation $\sim_\xi$, endowed with the quotient topology. We let $\pi_\xi : X_G^+ \to X_\xi^+$ be the quotient map. It follows from part 2 of Proposition 3.7 that there is a function we denote $\rho_\xi : X_\xi^+ \to X_{G_\xi}$ satisfying $\rho_\xi \circ \pi_\xi = \tau_\xi$.*

We also define $\sigma_\xi : X_\xi^+ \to X_\xi^+$ by $\sigma_\xi(\pi_\xi(x)) = \pi_\xi(\sigma(x))$, for all x in X_G^+, which is well-defined by Proposition 3.7. We let $X_{\xi,k}^+ = \pi_\xi(X_k^+)$, for each $k \geq 0$.

As indicated above, the space X_ξ^+ is given the quotient topology. It is not hard to use general topological results to show that X_ξ^+ is metrizable. That is not sufficient for our goals as we will need a metric with specific properties. The construction is rather subtle.

We begin the construction of a pseudo-metric, d, on X_G^+. We will show that $d(x,y) \leq 3d_G(x,y)$, for all x, y in X_G^+. In addition, we will prove $d(x,y) = 0$ if and only if $x \sim_\xi y$, for any x, y in X_G^+. The immediate consequence is that d induces a metric on the quotient space X_ξ^+, denoted by d_ξ, whose topology is the the quotient topology. This means that an alternate definition of our space X_ξ^+ is as the metric space naturally induced by the pseudo-metric d on X_G^+. The definition given in 3.6 and 3.8 seems more intuitive.

For x in X_G^+, if $\kappa(x) = 0$, we define

$$\theta(x) = \exp\left(2\pi i \sum_{j=1}^{\infty} \varepsilon(x_j) 2^{-j}\right).$$

If $\kappa(x) > 0$, we define

$$\theta(x) = \exp\left(2\pi i \sum_{j=1}^{n(x)-1} \varepsilon(x_j) 2^{-j}\right).$$

Obviously, $\theta(x)$ lies on the unit circle, $\mathbb{T}$, in either case and is a $2^{n(x)-1}$-th root of unity in the latter.

For $k \geq 0$, we define $\lambda_k : X_k^+ \times X_k^+ \to [0, \infty)$ inductively as follows. First, we set

$$\begin{aligned} \lambda_0(x, y) &= d_{\mathbb{T}}(\theta(x), \theta(y)) \\ &= \inf\{|t| \mid t \in \mathbb{R}, e^{2\pi i t}\theta(x) = \theta(y)\}, \end{aligned}$$

for x, y in X_0^+.

For x, y in X_k^+ with $k > 0$, we define

$$\begin{aligned} \lambda_k(x, y) &= |2^{-n(x)} - 2^{-n(y)}| + d_{\mathbb{T}}(\theta(x), \theta(y)) \\ &\quad + \delta((n(x), \theta(x)), (n(y), \theta(y))) 2^{-2-n(x)} \lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y)), \end{aligned}$$

where we use $\delta(\cdot, \cdot)$ for the Kronecker delta function.

Lemma 3.9. *For $k \geq 0$, we have*

$$d_{\mathbb{T}}(\theta(x), \theta(y)) \leq d_G(x, y),$$

and

$$\lambda_k(x, y) \leq 2 d_G(x, y),$$

for all x, y in X_k^+.

Proof. We proceed by induction on k. For $k = 0$, suppose $d_G(x, y) = 2^{-m}$, for some $m \geq 0$, which implies $x_j = y_j$ for $1 \leq j \leq m$. We have

$$\begin{aligned} d_{\mathbb{T}}(\theta(x), \theta(y)) &\leq \sum_{j=1}^{\infty} |\varepsilon(x_j) - \varepsilon(y_j)| 2^{-j} \\ &\leq \sum_{j=m+1}^{\infty} |\varepsilon(x_j) - \varepsilon(y_j)| 2^{-j} \\ &\leq 2^{-m} \\ &= d_G(x, y). \end{aligned}$$

For $k = 0$, the second statement follows from the first as $\lambda_0(x, y) = d_{\mathbb{T}}(\theta(x), \theta(y))$.

Now assume that $k > 0$ and again $d_G(x, y) = 2^{-m}$, so $x_j = y_j$ for $1 \leq j \leq m$. If either $n(x) \leq m$ or $n(y) \leq m$, then $n(x) = n(y)$ and $|2^{-n(x)} - 2^{-n(y)}| = 0$. In addition, we have $\theta(x) = \theta(y)$ so $d_{\mathbb{T}}(\theta(x), \theta(y)) = 0$.

Otherwise, we have $n(x), n(y) > m$ and

$$\begin{aligned} d_{\mathbb{T}}(\theta(x), \theta(y)) &\leq |\sum_{j=m+1}^{n(x)} \varepsilon(x_j) 2^{-j} - \sum_{j=m+1}^{n(y)} \varepsilon(y_j) 2^{-j}| \\ &\leq \sum_{j=m+1}^{\infty} 2^{-j} \\ &= 2^{-m} \\ &= d_G(x, y). \end{aligned}$$

This establishes the first inequality. In addition, we note that

$$|2^{-n(x)} - 2^{-n(y)}| \leq 2^{-m-1} < d_G(x, y).$$

Now, we consider $\lambda_k(x, y)$. If either $n(x) \neq n(y)$ or $\theta(x) \neq \theta(y)$, then we have

$$\lambda_k(x, y) = |2^{-n(x)} - 2^{-n(y)}| + d_{\mathbb{T}}(\theta(x), \theta(y)) \leq d_G(x, y) + d_G(x, y).$$

In the case $n(x) = n(y) = n$ and $\theta(x) = \theta(y)$, from the definition and the induction hypothesis, we have

$$\begin{aligned} \lambda_k(x, y) &= 2^{-2-n} \lambda_{k-1}(\sigma^n(x), \sigma^n(y)) \\ &\leq 2^{-2-n} \cdot 2 d_G(\sigma^n(x), \sigma^n(y)) \\ &\leq 2^{-2-n} \cdot 2 \cdot 2^n d_G(x, y) \\ &< d_G(x, y). \end{aligned}$$

□

We can now begin our definition of our pseudo-metric on X_G^+. Initially, we treat the spaces X_k^+ separately.

Proposition 3.10. *1. For each $k \geq 0$,*

$$d_k(x, y) = d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + \lambda_k(x, y),$$

for x, y in X_k^+, is a pseudo-metric on X_k^+.

2. For x, y in X_k^+, we have

$$d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) \leq d_k(x, y) \leq 3 d_G(x, y).$$

3. For x, y in X_k^+, $d_k(x,y) = 0$ if and only if $x \sim_\xi y$.

Proof. For the first part, the d_{G_ξ} term is clearly a pseudometric, so it suffices to show that λ_k is also, which we do by induction on k. This is clear for $k = 0$. For $k > 0$, a simple induction argument shows that λ_k is reflexive and symmetric. Now suppose that x, y, z are in X_k^+. We first consider the case $(n(x), \theta(x)) \neq (n(y), \theta(y))$, where we have

$$\begin{array}{rcl} \lambda_k(x,y) & = & |2^{-n(x)} - 2^{-n(y)}| + d_{\mathbb{T}}(\theta(x), \theta(y)) \\ & \leq & |2^{-n(x)} - 2^{-n(z)}| + d_{\mathbb{T}}(\theta(x), \theta(z)) \\ & & |2^{-n(z)} - 2^{-n(y)}| + d_{\mathbb{T}}(\theta(z), \theta(y)). \end{array}$$

If the first two terms are both zero, then their sum is certainly less than or equal to $\lambda_k(x,z)$. On the other hand, if either is non-zero, then their sum equals $\lambda_k(x,z)$, by definition. The third and fourth terms are done similarly.

Now suppose that $(n(x), \theta(x)) = (n(y), \theta(y))$. If $(n(z), \theta(z)) = (n(x), \theta(x))$ then,

$$\begin{array}{rcl} \lambda_k(x,y) & = & 2^{-2-n(x)}\lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y)) \\ & \leq & 2^{-2-n(x)}\lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(z)}(z)) \\ & & +2^{-2-n(x)}\lambda_{k-1}(\sigma^{n(z)}(z), \sigma^{n(y)}(y)) \\ & = & \lambda_k(x,z) + \lambda_k(z,y) \end{array}$$

follows from the induction hypothesis.

We next consider the case $n(z) \neq n(x)$ and so $n(z) \neq n(y)$ also. For any positive integers $m \neq n$, we have

$$|2^{-m} - 2^{-n}| = 2^{-\min\{m,n\}}|1 - 2^{-|m-n|}| \geq 2^{-\min\{m,n\}-1} \geq 2^{-n-1}$$

and so

$$\lambda_k(x,z) \geq |2^{-n(x)} - 2^{n(z)}| \geq 2^{-n(x)+1}.$$

A similar estimate holds for $\lambda_k(z,y)$.

On the other hand, we also have

$$\begin{array}{rcl} 2^{-2-n(x)}\lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y)) & \leq & 2^{-2-n(x)}2d_G(\sigma^{n(x)}(x), \sigma^{n(y)}(y)) \\ & \leq & 2^{-1-n(x)}. \end{array}$$

Together this yields

$$\begin{aligned}
\lambda_k(x,y) &= 2^{-2-n(x)}\lambda_{k-1}(\sigma^{n(x)}(x),\sigma^{n(y)}(y)) \\
&\leq 2^{-1-n(x)} \\
&= 2^{-2-n(x)} + 2^{-2-n(y)} \\
&\leq \lambda_k(x,z) + \lambda_k(z,y).
\end{aligned}$$

The final case to consider is $n(x) = n(y) = n(z)$ and $\theta(z) \neq \theta(x)$, so $\theta(z) \neq \theta(y)$ also. Then, as $\theta(x), \theta(y), \theta(z)$ are all $2^{n(x)-1}$-th roots of unity, we have

$$d_{\mathbb{T}}(\theta(x),\theta(z)), d_{\mathbb{T}}(\theta(z),\theta(y)) \geq 2^{-n(x)}.$$

It follows that

$$\begin{aligned}
\lambda_k(x,y) &= 2^{-2-n(x)}\lambda_{k-1}(\sigma^{n(x)}(x),\sigma^{n(y)}(y)) \\
&\leq 2^{-1-n(x)} \\
&= 2^{-2-n(x)} + 2^{-2-n(y)} \\
&\leq d_{\mathbb{T}}(\theta(x),\theta(z)) + d_{\mathbb{T}}(\theta(z),\theta(y)) \\
&= \lambda_k(x,z) + \lambda_k(z,y).
\end{aligned}$$

The second part is immediate from the second statement of Lemma 3.9.

For the third part, we consider the 'if' direction first. We proceed by induction on k, beginning with $k = 0$. Suppose $x \sim_\xi y$, where x, y are in X_0^+. From part 2 of Proposition 3.4, we know $\tau_\xi(x) = \tau_\xi(y)$. In addition, $\varepsilon(x_j), j \geq 1$ and $\varepsilon(y_j), j \geq 1$ are two $0,1$-sequences and the definition of $\sim_\xi$ implies that these sequences are dyadic representations of the same real number, modulo the integers. This implies $\theta(x) = \theta(y)$ and so

$$d_0(x,y) = d_{\mathbb{T}}(\theta(x),\theta(y)) + d_{G_\xi}(\tau_\xi(x),\tau_\xi(y)) = 0 + 0 = 0.$$

Now assume the result is true for some k and let $x \sim_\xi y$ be in X_{k+1}^+. From the Proposition 3.4, we have $n(x) = n(y)$ and $x_i = y_i$, for all $1 \leq i \leq n(x)$. It follows that $\theta(x) = \theta(y)$ and

$$d_{k+1}(x,y) = 2^{-2-n(x)}\lambda_k(\sigma^k(x),\sigma^k(y)) \leq 2^{-2-n(x)}d_k(\sigma^k(x),\sigma^k(y)).$$

It follows from Proposition 3.4 $\sigma^k(x) \sim_\xi \sigma^k(y)$, so the conclusion follows from the induction hypothesis.

Now we consider the 'only if' direction. Suppose $d_0(x,y)=0$, where x,y are in X_0^+. This means that x_i, y_i are in $\xi(H^1)$, for all $i \geq 1$. The hypothesis obviously implies that $d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) = 0$ so $\tau_\xi(x_i) = \tau_\xi(y_i)$, for all $i \geq 1$. In other words, there is a path z in X_H^+ such that $\{x_i, y_i\} \subseteq \{\xi^0(z_i), \xi^1(z_i)\}$ for all i. The fact that $\lambda_0(x,y) = 0$ implies that $\theta(x) = \theta(y)$ and this implies that the sequences $\varepsilon(x_i), \varepsilon(y_i), i \geq 1$ are the binary expansions of the same real number, modulo the integers. This implies $x \sim_\xi y$.

We now consider $k > 0$ and $d_k(x,y) = 0$, where x,y are in X_k^+. The first consequence is that $\tau_\xi(x) = \tau_\xi(y)$. This implies that $n(x) = n(y) = n$ and since $x_{n(x)}$ and $y_{n(y)}$ are not in $\xi^0(H) \cup \xi^1(H)$, we have $x_{n(x)} = y_{n(y)}$. In addition, there is a path $h_1, \ldots, h_{n-1}$ such that x_i, y_i are in $\{\xi^0(h_i), \xi^1(h_i)\}$, for all $1 \leq i < n$. The fact that $\lambda_k(x,y) = 0$ implies that $\theta(x) = \theta(y)$. The map which takes a $0,1$ sequence $\varepsilon_1, \ldots, \varepsilon_{n-1}$ to $exp(2\pi \sum_{j=1}^{n-1} \varepsilon_j 2^{-j})$ is injective and so we conclude that $\varepsilon(x_i) = \varepsilon(y_i)$, for all $1 \leq i < n$. It follows that $x_i = y_i$, for $1 \leq i \leq n$. The fact $\tau_\xi(x) = \tau_\xi(y)$ implies that $\tau_\xi(\sigma^{n(x)}(x)) = \tau_\xi(\sigma^{n(y)}(y))$ also and so we have

$$d_{k-1}(\sigma^n(x), \sigma^n(y)) = \lambda_{k-1}(\sigma^n(x), \sigma^n(y)) = 2^{2+n}\lambda_k(x,y) = 0.$$

By our induction hypothesis, $\sigma^n(x) \sim_\xi \sigma^n(y)$. Together with the fact that $x_i = y_i$ for $1 \leq i \leq n$ implies $x \sim_\xi y$. □

It will be useful to have a characterization of Cauchy sequences in our pseudo-metric.

Proposition 3.11. *1. A sequence $x^l, l \geq 1$, in X_0^+ is Cauchy in d_0 if and only if $\tau_\xi(x^l), l \geq 1$, is convergent in $X_{G_\xi}^+$ and $\theta(x^l), l \geq 1$ is convergent in $\mathbb{C}$.*

2. For $k > 0$, a sequence $x^l, l \geq 1$, in X_k^+ is Cauchy in d_k if and only only $\tau_\xi(x^l), l \geq 1$, is convergent in $X_{G_\xi}^+$ and either

(a) $\lim_l n(x^l) = \infty$ and $\theta(x^l)$, $l \geq 1$, is convergent in $\mathbb{C}$ or

(b) the sequences $n(x^l)$ and $\theta(x^l)$ are eventually constant and $\sigma^{n(x^l)}(x^l), l \geq 1$, is Cauchy in d_{k-1}.

Proof. The first part is clear from the definition of d_0

For the second part we begin with the 'only if' direction. If $x^l, l \geq 1$, is Cauchy in d_k, then $\tau_\xi(x^l), l \geq 1$ is clearly Cauchy and hence convergent in X_{G_ξ}. In addition, both sequences $2^{-n(x^l)}$ and $\theta(x^l)$ are Cauchy. There are clearly two cases: either $n(x^l), l \geq 1$, tends to infinity or it is eventually constant. In the first case, we have case (a). In the second case, if $n(x^l) = n$ for all l sufficiently large, then for such l, $\theta(x^l)$ lies in a finite set, so to be Cauchy it must be eventually constant also, say with value θ. Then for sufficiently large values of l, m, we have

$$\begin{aligned} d_k(x^l, x^m) &= d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(x^m)) + 2^{-2-n}\lambda_{k-1}(\sigma^n(x^l), \sigma^n(x^m)) \\ &\geq 2^{-n} d_{G_\xi}(\tau_\xi(\sigma^n(x^l)), \tau_\xi(\sigma^n(x^m))) \\ &\quad +2^{-2-n}\lambda_{k-1}(\sigma^n(x^l), \sigma^n(x^m)) \\ &\geq 2^{-2-n} d_{k-1}(\sigma^n(x^l), \sigma^n(x^m)). \end{aligned}$$

It follows that $\sigma^n(x^l), l \geq 1$ is Cauchy in d_{k-1}.

Conversely, if $x^l, l \geq 1$ is any sequence in X_k^+, then, for any l, m, $d_k(x^l, x^m)$ is either equal to either

$$d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(x^m)) + |2^{-n(x^l)} - 2^{-n(x^m)}| + d_{\mathbb{T}}(\theta(x^l), \theta(x^m)),$$

or

$$d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(x^m)) + 2^{-2-n(x^l)}\lambda_{k-1}(\sigma^{n(x^l)}(x^l), \sigma^{n(x^l)}(x^m)).$$

It is then immediate that if $\tau_\xi(x^l), l \geq 1$ being convergent and either of the two conditions imply the sequence is Cauchy in d_k. □

At this point, we have defined a pseudo-metric on X_k^+, for each value of k. Assuming condition (H2) and that G is primitive, we also know that X_k^+ will have limit points in X_j^+, for every $0 \leq j < k$. We now need to show these different pseudo-metrics are compatible, in an obvious sense. The following concerns a special case ($j = 0$) but will be used in the proof of the general statement, as well.

Lemma 3.12. *Let $k \geq 1$ and suppose that $x^l, l \geq 1$, is a sequence in X_k^+ which converges to x in X_G^+ and such that $n(x^l)$ tends to infinity as l tends to infinity. Then x is in X_0^+ and* $\lim_{l\to\infty} \theta(x^l) = \theta(x)$.

Proof. For any $n \geq 1$, we have $x_n = \lim_l x_n^l$ and x_n^l is in $\xi(H^1)$, provided $n(x^l) > n$. It follows that x_n is in $\xi(H^1)$, for every n, so x is in X_0^+.

If we fix $m \geq 1$, we may choose l_0 sufficiently large so that $n(x^l) > m$ and $x_i = x_i^l$, for all $1 \leq i \leq m$ and $l \geq l_0$. For such l, we have

$$\begin{aligned} |\theta(x) - \theta(x^l)| &= |\exp(2\pi i \sum_{j=1}^{\infty} \varepsilon(x_j) 2^{-j}) - \exp(2\pi i \sum_{j=1}^{n(x^l)-1} \varepsilon(x_j^l) 2^{-j})| \\ &= |\exp(2\pi i \sum_{j=m+1}^{\infty} \varepsilon(x_j) 2^{-j}) - \exp(2\pi i \sum_{j=m+1}^{n(x^l)-1} \varepsilon(x_j^l) 2^{-j})| \\ &\leq 2\pi |\sum_{j=m+1}^{\infty} \varepsilon(x_j) 2^{-j} - \sum_{j=m+1}^{n(x^l)-1} \varepsilon(x_j^l) 2^{-j}| \\ &\leq 2\pi 2^{-m}. \end{aligned}$$

As m was arbitrary, this completes the proof. □

The following summarizes the compatibility of the different pseudo-metrics.

Proposition 3.13. *Suppose $0 \leq j < k$, the sequences $x^l, l \geq 1$ and $y^l, l \geq 1$ are in X_k^+, x and y are in X_j^+, $x^l, l \geq 1$, converges to x and $y^l, l \geq 1$, converges to y in X_G^+, then*

$$d_j(x, y) = \lim_{l \to \infty} d_k(x^l, y^l).$$

Proof. We first consider the case $n(x^l), l \geq 1$, tends to infinity. The fact that $x^l, l \geq 1$, converges in X_G^+, along with part 2 of Proposition 3.10 shows that $x^l, l \geq 1$, is Cauchy in d_k. Proposition 3.11 shows that there are two cases to consider. We begin by assuming that (a) holds so $n(x^l)$ tends to infinity while $\theta(x^l), l \geq 1$, converges.

Lemma 3.12 implies that x is in X_0^+. Hence, we have $j = 0$. We now claim that the sequence $n(y^l)$ must also tend to infinity. By the same argument as above, $y^l, l \geq 1$ is Cauchy in d_k. If $n(y^l), l \geq 1$ is eventually constant, say n, then $y_n = \lim_l y_n^l$ is not in $\xi(H^1)$ which means that y is not in X_0^+, a contradiction.

Now, we consider the case $\theta(x) \neq \theta(y)$. From Lemma 3.12, we see that $\theta(x^l) \neq \theta(y^l)$ for l sufficiently large. In this case, we know that

$$d_k(x^l, y^l) = d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(y^l)) + |2^{-n(x^l)} - 2^{-n(y^l)}| + d_{\mathbb{T}}(\theta(x^l), \theta(y^l)).$$

The first term converges to $d_{G_\xi}(\tau_\xi(x), \tau_\xi(y))$, the second to zero and the third to $d_{\mathbb{T}}(\theta(x), \theta(y))$. The desired equality follows from the definition of $d_0(x, y)$.

We next turn to the case $\theta(x) = \theta(y)$. Here, we have either

$$d_k(x^l, y^l) = d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(y^l)) + |2^{-n(x^l)} - 2^{-n(y^l)}| + d_{\mathbb{T}}(\theta(x^l), \theta(y^l)).$$

or else

$$d_k(x^l, y^l) = d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(y^l)) + 2^{-2-n(x^l)}\lambda_0(\sigma^{n(x^l)}(x^l), \sigma^{n(y^l)}(y^l)).$$

In either case, the limit is

$$d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) = d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + d_{\mathbb{T}}(\theta(x), \theta(y)) = d_0(x, y).$$

We are left to consider the case that $n(x^l)$ is eventually constant. It follows that $\theta(x^l)$ is also. Suppose these values are n and θ, respectively. It follows that $n(x) = n, \theta(x) = \theta$ and $j > 0$. If the sequence $n(y^l)$ is unbounded, then, arguing as before for $n(x^l)$, we have $j = 0$, which is a contradiction. Hence the sequences $n(y^l)$ and $\theta(y^l)$ are also eventually constant, say with values n' and θ', respectively.

As x_i^l (y_i^l) converges to x_i (y_i, respectively), for $1 \leq i \leq n$ ($1 \leq i \leq n'$, respectively, we know then that $n(x) = n$, $\theta(x) = \theta$, $n(y) = n'$ and $\theta(y) = \theta'$. Let us first assume that $(n, \theta) \neq (n', \theta')$. In this case, we have, for l sufficiently large,

$$d_k(x^l, y^l) = d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(y^l)) + |2^{-n} - 2^{-n'}| + d_{\mathbb{T}}(\theta, \theta')$$

which clearly converges to

$$d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + |2^{-n} - 2^{-n'}| + d_{\mathbb{T}}(\theta, \theta') = d_j(x, y).$$

The second case to consider is $(n, \theta) = (n', \theta')$. In this case, we have, for l sufficiently large,

$$\begin{aligned} d_k(x^l, y^l) &= d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(y^l)) \\ &\quad +2^{-2-n}\lambda_{k-1}(\sigma^n(x^l), \sigma^n(y^l)) \\ &= d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(y^l)) \\ &\quad -2^{-2-n}d_{G_\xi}(\tau_\xi(\sigma^n(x^l)), \tau_\xi(\sigma^n(y^l))) \\ &\quad +2^{-2-n}d_{k-1}(\sigma^n(x^l), \sigma^n(y^l)). \end{aligned}$$

It is clear that the $\sigma^n(x^l), \sigma^n(y^l), l \geq 1$ and points $\sigma^n(x), \sigma^n(y)$ satisfy the hypotheses for integer for $0 \leq j-1 < k-1$, so by the induction hypothesis, the sequence above converges to

$$\begin{array}{rc} & d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) \\ & -2^{-2-n} d_{G_\xi}(\tau_\xi(\sigma^n(x)), \tau_\xi(\sigma^n(y))) \\ & +2^{-2-n} d_{j-1}(\sigma^n(x), \sigma^n(y)) \\ = & d_j(x, y). \end{array}$$

□

We are now ready to unify our metrics d_k into a single pseudo-metric.

Let us take a moment to recall (one approach to) taking the completion of a metric space (X, d) (see page 196 of Kelley [16]). A key step is showing that, if $x^l, y^l, l \geq 1$ are two Cauchy sequences in X with respect to d, then the sequence of real numbers, $d(x^l, y^l), l \geq 1$, is also Cauchy and hence convergent. This proof only needs symmetry and the triangle inequality, and works equally well if d is a pseudo-metric.

Let us describe how we will use this. For x, y in X_k^+, we have the inequality $d_k(x, y) \leq 3d_G(x, y)$. If we choose any sequences $x^l, y^l, l \geq 1$, in X_k^+, converging to x, y in X_G^+, respectively, then both are Cauchy in d_G and hence also in d_k. The argument outlined above then implies that $\lim_{l \to \infty} d_k(x^l, y^l)$ exists and it can be shown to be independent of the choice of sequences. If x, y are actually in X_k^+, then this limit agrees with $d_k(x, y)$. We refer to this as an *extension* of d_k. We use this repeatedly in the following.

Theorem 3.14. *Assume that G, H, ξ satisfies the standing hypotheses. There exists a pseudo-metric, d, on X_G^+ satisfying the following.*

1. $$d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) \leq d(x, y) \leq 3d_G(x, y),$$
   for all x, y in X_G^+,

2. $d(x, y) = 0$ *if and only if $x \sim_\xi y$, for all x, y in X_G^+,*

3. $d(x, y) = d_k(x, y)$*, for all x, y in X_k^+.*

Proof. For $k \geq 0$, the space X_k^+ carries two pseudo-metrics, d_G and d_k. The former dominates the latter (part 2 of Proposition 3.10), at least up to a factor of 3. In addition, it is clear that $d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) \leq d_k(x, y)$, for all x, y in X_k^+. We also recall from Proposition 3.5 that the closure of X_k^+ in X_G^+ is $Cl(X_k^+) = \cup_{j=0}^k X_j^+$. It follows that d_k extends to a pseudo-metric on $Cl(X_k^+)$, which we denote $\bar{d}_k$, as described above which is also dominated by $3d_G$ and bounded below by $d_{G_\xi}(\pi_\xi(\cdot), \pi_\xi(\cdot))$. In addition, it follows from Proposition 3.13 that the restriction of $\bar{d}_k$ to $X_j^+ \times X_j^+$ equals d_j, for any $0 \leq j \leq k$. It also follows that $\bar{d}_k$ equals $\bar{d}_j$ after extending the latter to $\cup_{i=0}^j X_i^+$.

We may then define a pseudo-metric, d_∞, on $\cup_{j=0}^\infty X_j^+$ which is $\bar{d}_k$ on $\cup_{j=0}^k X_j^+$. This is also bounded by $3d_G$ and so extends continuously to a pseudo-metric d on the closure of $\cup_{j=0}^\infty X_j^+$ which is X_G^+ (Proposition 3.4) and is bounded below by $d_\xi(\pi_\xi(\cdot), \pi_\xi(\cdot))$ and above by $3d_G$.

The first and third properties of d in the conclusion are immediate. For the second, we first suppose that $x \sim_\xi y$. It follows from the definition of $\sim_\xi$ that x, y lie in X_k^+, for some $0 \leq k < \infty$ and hence $d(x, y) = d_k(x, y) = 0$, by Proposition 3.10.

Now assume that x, y are in X_G^+ and $d(x, y) = 0$. It follows that $\tau_\xi(x) = \tau_\xi(y)$ which, in turn implies that $\kappa(x) = \kappa(y)$. If this is finite, say k, then x, y are in X_k^+ and the conclusion follows from Proposition 3.10. We now assume $\kappa(x) = \infty$ and $\tau_\xi(x) = \tau_\xi(y)$ also implies that $n(x) = n(y)$. From Proposition 3.5, we may find sequences, x^l, y^l in X_l^+ converging to x, y, respectively. This means that

$$0 = d(x, y) = \lim_l d(x^l, y^l) = \lim_l d_l(x^l, y^l).$$

There exists $l_0 \geq 1$ such that, for all $l \geq l_0$, $x_n = x_n^l, y_n = y_n^l$, for $1 \leq n \leq n(x)$. From this, it follows that $n(x^l) = n(x) = n(y) = n(y^l)$ and $\theta(x) = \theta(x^l), \theta(y) = \theta(y^l)$, for $l \geq l_0$. If $\theta(x^l)$ and $\theta(y^l)$ are distinct for all l, then

$$d_l(x^l, y^l) \geq d_{\mathbb{T}}(\theta(x^l), \theta(y^l)) \geq 2^{-n(x)}$$

which is a contradiction. It follows that, for some l, we have $\theta(x) = \theta(x^l) = \theta(y^l) = \theta(y)$. Together with that the fact that $\tau_\xi(x_n) = \tau_\xi(y_n)$, this implies that $x_n = y_n$, for $1 \leq n \leq n(x)$.

We note that for $l \geq l_0$, $\sigma^{n(x)}(x^l), \sigma^{n(x)}(y^l)$ are in X^+_{l-1} and we have

$$\begin{aligned} d_{l-1}(\sigma^{n(x)}(x^l), \sigma^{n(x)}(y^l)) &= d_{G/\xi}(\sigma^{n(x)}(x^l), \sigma^{n(x)}(y^l)) \\ &\quad +\lambda_{l-1}(\sigma^{n(x)}(x^l), \sigma^{n(x)}(y^l))) \\ &= 2^{n(x)} d_{G/\xi}(x^l, y^l) + \lambda_{l-1}(\sigma^{n(x)}(x^l), \sigma^{n(x)}(y^l))) \\ &\leq 2^{n(x)} d_l(x^l, y^l). \end{aligned}$$

After re-indexing the sequence, we can apply the same argument to $\sigma^{n(x)}(x), \sigma^{n(x)}(x), \sigma^{n(x)}(x^{l-1}), \sigma^{n(x)}(y^{l-1})$ which shows that $x_n = y_n$, for $1 \leq n \leq n(x) + n(\sigma^{n(x)})$. Continuing in this way, we see that $x = y$, as desired. □

Corollary 3.15. *The space topological X^+_ξ has a metric, d_ξ, inducing the topology such that*

1. $d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) \leq d_\xi(\pi_\xi(x), \pi_\xi(y)) \leq 3d_G(x, y)$, *for all x, y in X^+_G,*
2. $d_\xi(\pi_\xi(x), \pi_\xi(y)) = d_k(x, y)$, *for all x, y in X^+_k.*

Theorem 3.16. 1. *The map $\rho_\xi : X^+_\xi \to X^+_{G_\xi}$ satisfies*

$$d_{G_\xi}(\rho_\xi(x), \rho_\xi(y)) \leq d_\xi(x, y),$$

for all x, y in X^+_ξ.

2. *The map $\theta : X^+_G \to \mathbb{T}$ is constant on $\sim_\xi$-equivalence classes and so there is a map (also) denoted by $\theta : X^+_\xi \to \mathbb{T}$ which satisfies*

$$d_{\mathbb{T}}(\theta(x), \theta(y)) \leq d_\xi(x, y),$$

for all x, y in X^+_ξ.

Proof. It is a trivial consequence of the definition of d_k that the inequality holds with d_k on the right hand side for x, y in X^+_k, for any $k \geq 1$. The conclusion holds since d_ξ is the common extension of all d_k.

For any $k \geq 1$ and x, y in X^+_k, we have

$$d_{\mathbb{T}}(\theta(x), \theta(y)) \leq \lambda_k(x, y) \leq d_k(x, y).$$

Again, as d_ξ is the common extension of all d_k, this also holds for d_ξ on X_ξ. □

As we mentioned earlier, it is an easy matter to see that X_ξ^+ is metrizable. We have gone to considerable trouble to actually produce a metric. The reason is that it satisfies some nice properties and the following is the most important. We remark that the same property holds for the map σ on (X_G^+, d_G).

Theorem 3.17. *If x, y in X_ξ^+ satisfy $d_\xi(x,y) \leq 2^{-2}$, then*

$$2d_\xi(x,y) \leq d_\xi(\sigma_\xi(x), \sigma_\xi(y)) \leq 8d_\xi(x,y).$$

Proof. We will prove the analogous statement for d_k, σ and x, y in X_k^+, for some fixed $k \geq 0$.

We first consider $k = 0$ and note that $\sigma(X_0^+) = X_0^+$. For x, y in X_0^+, we have

$$d_0(x,y) = d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + d_{\mathbb{T}}(\theta(x), \theta(y)).$$

Our hypothesis implies both terms on the right are at most 2^{-2}. This implies that $\tau_\xi(x_1) = \tau_\xi(y_1)$ and hence

$$d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) = d_{G_\xi}(\sigma \circ \tau_\xi(x), \sigma \circ \tau_\xi(y)) = 2d_{G_\xi}(\tau_\xi(x), \tau_\xi(y))$$

and $d_{\mathbb{T}}(\theta(x), \theta(y)) \leq 2^{-2}$ and hence

$$d_{\mathbb{T}}(\theta(\sigma(x)), \theta(\sigma(y))) = d_{\mathbb{T}}(\theta(x)^2, \theta(y)^2) = 2d_{\mathbb{T}}(\theta(x), \theta(y)).$$

It follows that $d_0(\sigma(x), \sigma(y)) = 2d_0(x,y)$ and we are done.

Now assume $k \geq 1$. First, we note that $d_k(x,y) \leq 2^{-2}$ implies that $d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) \leq 2^{-2}$ so $\tau_\xi(x_i) = \tau_\xi(y_i)$, for $i = 1, 2$. It follows that $n(x) = n(y)$ or both are at least 3.

Our first case is $n(x) = 1 = n(y)$. It follows that $\sigma(x), \sigma(y)$ are both in X_{k-1}^+. We have $n(x) = 1 = n(y)$ and $\theta(x) = \theta(y) = 1$ so

$$\begin{aligned} d_\xi(x,y) &= d_k(x,y) \\ &= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + \lambda_k(x,y) \\ &= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + 2^{-3}\lambda_{k-1}(\sigma(x), \sigma(y)) \\ &= 2^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + 2^{-3}\lambda_{k-1}(\sigma(x), \sigma(y)). \end{aligned}$$

The right hand side is clearly bounded above by

$$2^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + 2^{-1}\lambda_{k-1}(\sigma(x), \sigma(y)) = 2^{-1}d_{k-1}(\sigma(x), \sigma(y))$$

and below by

$$8^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)),\tau_\xi(\sigma(y)))+8^{-1}\lambda_{k-1}(\sigma(x),\sigma(y))=8^{-1}d_{k-1}(\sigma(x),\sigma(y)).$$

Next, we consider $n(x)=n(y)=2$. Also, $\theta(x),\theta(y)$ are both ± 1. If they are distinct, we have

$$d_\xi(x,y)\geq \lambda_k(x,y)\geq d_{\mathbb{T}}(\theta(x),\theta(y))\geq 2^{-1},$$

which is a contradiction. Hence, we have $\theta(x)=\theta(y)$. So we have x,y are still in X_k^+, $n(\sigma(x))=n(x)-1=n(\sigma(y))$ and $\theta(\sigma(x))=\theta(x)^2=\theta(y)^2=\theta(\sigma(y))$ and

$$\begin{aligned}
d_\xi(x,y) &= d_k(x,y)\\
&= d_{G_\xi}(\tau_\xi(x),\tau_\xi(y))+\lambda_k(x,y)\\
&= 2^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)),\tau_\xi(\sigma(y)))+2^{-n(x)-2}\lambda_{k-1}(\sigma^{n(x)}(x),\sigma^{n(y)}(y))\\
&= 2^{-1}\big(d_{G_\xi}(\tau_\xi(\sigma^{n(x)}(x)),\tau_\xi(\sigma^{n(x)}(y)))\\
&\quad +2^{-1}2^{-n(x)-1}\lambda_{k-1}(\sigma(\sigma(x)),\sigma(\sigma(y)))\big)\\
&= 2^{-1}\big(d_{G_\xi}(\tau_\xi(\sigma^{n(x)}(x)),\tau_\xi(\sigma^{n(x)}(y)))\\
&\quad +2^{-1}\lambda_k(\sigma(x),\sigma(y))\big)\\
&= 2^{-1}d_k(\sigma(x),\sigma(y)).
\end{aligned}$$

Finally, we consider the case $n(x),n(y)>2$. Here again, we have $n(\sigma(x))=n(x)-1$, $n(\sigma(y))=n(y)-1$, $\theta(\sigma(x))=\theta(x)^2$ and $\theta(\sigma(y))=\theta(y)^2$. Let us first assume that $(n(x),\theta(x))\neq(n(y),\theta(y))$. We claim that $(n(\sigma(x)),\theta(\sigma(x)))\neq(n(\sigma(y)),\theta(\sigma(y)))$. If they are equal then $n(x)-1=n(y)-1$ so $n(x)=n(y)$. It follows that $\theta(x)\neq\theta(y)$, while $\theta(x)^2=\theta(y)^2$ which implies $\theta(x)=-\theta(y)$. This in turn implies that $d_k(x,y)\geq d_{\mathbb{T}}(\theta(x),\theta(y))=2^{-1}$ which is a contradiction. From the fact that $2^{-2}\geq d_k(x,y)\geq d_{\mathbb{T}}(\theta(x),\theta(y))$, we see that $d_{\mathbb{T}}(\theta(x)^2,\theta(y)^2)=2d_{\mathbb{T}}(\theta(x),\theta(y))$.

Now we have

$$\begin{aligned}
d_\xi(x,y) &= d_k(x,y)\\
&= d_{G_\xi}(\tau_\xi(x),\tau_\xi(y))+\lambda_k(x,y)\\
&= d_{G_\xi}(\tau_\xi(x),\tau_\xi(y))+|2^{-n(x)}-2^{-n(y)}|+d_{\mathbb{T}}(\theta(x),\theta(y))\\
&= 2^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)),\tau_\xi(\sigma(y)))+2^{-1}|2^{1-n(x)}-2^{1-n(y)}|\\
&\quad +2^{-1}d_{\mathbb{T}}(\theta(x)^2,\theta(y)^2)\\
&= 2^{-1}d_k(\sigma(x),\sigma(y))\\
&= 2^{-1}d_\xi(\sigma(x),\sigma(y)).
\end{aligned}$$

The final case is $(n(x), \theta(x)) = (n(y), \theta(y))$. Here, we have $(n(\sigma(x)), \sigma(\theta(x)) = (n(x) - 1, \theta(x)^2) = (n(\sigma(y)), \sigma(y))$ and

$$
\begin{aligned}
d_\xi(x, y) &= d_k(x, y) \\
&= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + \lambda_k(x, y) \\
&= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + \lambda_k(x, y) \\
&= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + 2^{-n(x)-2}\lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y)) \\
&= 2^{-1}\big(d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) \\
&\quad + 2^{1-n(x)-2}\lambda_{k-1}(\sigma^{n(x)-1}(\sigma(x)), \sigma^{n(y)-1}(\sigma(y)))\big) \\
&= 2^{-1}\big(d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + \lambda_k(\sigma(x), \sigma(y))\big) \\
&= 2^{-1}d_k(\sigma(x), \sigma(y)) \\
&= 2^{-1}d_\xi(\sigma(x), \sigma(y)).
\end{aligned}
$$

□

We finish this section with an alternate version of the local expanding property which will be useful in the later sections.

Lemma 3.18. *If x, y are in X_G^+ satisfy $x_1 = y_1$ and $d_\xi(\sigma(x), \sigma(y)) \leq 2^{-1}$, then $d_\xi(x, y) \leq 2^{-1}d_\xi(\sigma(x), \sigma(y))$.*

Proof. By Lemma 3.5, we may find sequences $x^k, k \geq 0$, and $y^k, k \geq 0$, converging to x and y respectively, and x^k, y^k are in X_k^+. So it suffices to prove the statement for x, y in X_k^+, for $k \geq 2$.

We consider three cases separately. The first is when $x_1 = y_1$ is not in $\xi(H^1)$. This implies that $n(x) = n(y) = 1$, $\theta(x) = \theta(y) = 1$ and $n(\sigma(x)) = n(\sigma(y)) = n(x) - 1$. It also means that $\sigma(x), \sigma(y)$ are in X_{k-1}^+. It follows

$$
\begin{aligned}
d_\xi(\sigma(x), \sigma(y)) &= d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + \lambda_{k-1}(\sigma(x), \sigma(y)) \\
&= 2d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + 2^3\lambda_k(x, y) \\
&\geq 2d_\xi(x, y).
\end{aligned}
$$

The second case is $x_1 = y_1$ is in $\xi(H^1)$, which implies that $n(x), n(y) > 1$, and $(n(x), \theta(x)) \neq (n(y), \theta(y))$. We have $n(\sigma(x)) = n(x) - 1$ and $n(\sigma(y)) = n(y) - 1$, $\theta(\sigma(x)) = \theta(x)^2, \theta(\sigma(y)) = \theta(y)^2$. It also means that $\sigma(x), \sigma(y)$ are in X_k^+. We have

$$\begin{aligned} d_\xi(\sigma(x),\sigma(y)) &= d_{G_\xi}(\tau_\xi(\sigma(x)),\tau_\xi(\sigma(y))) + \lambda_k(\sigma(x),\sigma(y)) \\ &= 2d_{G_\xi}(\tau_\xi(x),\tau_\xi(y)) + |2^{-n(x)+1} - 2^{-n(y)+1}| \\ &\quad + d_{\mathbb{T}}(\theta(x)^2,\theta(y)^2) \\ &= 2d_{G_\xi}(\tau_\xi(x),\tau_\xi(y)) + 2|2^{-n(x)} - 2^{-n(y)}| \\ &\quad + d_{\mathbb{T}}(\theta(x)^2,\theta(y)^2). \end{aligned}$$

The hypothesis that $d_\xi(\sigma(x),\sigma(y)) \leq 2^{-1}$ implies that $d_{\mathbb{T}}(\theta(x)^2,\theta(y)^2) \leq 2^{-1}$. The fact that $x_1 = y_1$ implies $\varepsilon(x_1) = \varepsilon(y_1)$. Together these imply that $d_{\mathbb{T}}(\theta(x)^2,\theta(y)^2) = 2d_{\mathbb{T}}(\theta(x),\theta(y))$. Putting this in the last line above yields $d_\xi(\sigma(x),\sigma(y)) = 2d_\xi(x,y)$.

The final case to consider is that $x_1 = y_1$ is in $\xi(H^1)$, which implies that $n(x), n(y) > 1$, and $(n(x),\theta(x)) = (n(y),\theta(y))$. Again, we know $\sigma(x),\sigma(y)$ are in X_k^+. Here we have $n(\sigma(x)) = n(x) - 1 = n(\sigma(y)) = n(y) - 1$ and $\theta(\sigma(x)) = \theta(x)^2 = \theta(\sigma(y)) = \theta(y)^2$ We have

$$\begin{aligned} d_\xi(\sigma(x),\sigma(y)) &= d_{G_\xi}(\tau_\xi(\sigma(x)),\tau_\xi(\sigma(y))) + \lambda_k(\sigma(x),\sigma(y)) \\ &= 2d_{G_\xi}(\tau_\xi(x),\tau_\xi(y)) + 2^{-2-(n(x)-1)} \\ &\quad \lambda_{k-1}(\sigma^{n(x)-1}(\sigma(x)),\sigma^{n(y)-1}(\sigma(y)) \\ &= 2\left(d_{G_\xi}(\tau_\xi(x),\tau_\xi(y)) + 2^{-2-n(x)}\lambda_k(\sigma^{n(x)}(x),\sigma^{n(y)}(y))\right) \\ &= 2d_\xi(x,y). \end{aligned}$$

□

Theorem 3.19. *If x, y are in X_ξ^+ with $d_\xi(x,\sigma_\xi(y)) \leq 2^{-1}$, then there is z in X_ξ^+ with $d_\xi(z,y) \leq 2^{-1}d_\xi(x,\sigma_\xi(y))$ and $\sigma_\xi(z) = x$. In particular, we have*

$$X_\xi^+(\sigma_\xi(x),\epsilon) \subseteq \sigma_\xi(X_\xi^+(x,2^{-1}\epsilon))$$

and the map σ_ξ is open.

Proof. Write $x = \pi_\xi(x')$, for some x' in X_G^+. Let y' be in X_G^+ with $\pi_\xi(y') = y$. As $\epsilon \leq 2^{-3}$, we know that $d_{G_\xi}(\tau_\xi(x'),\tau_\xi(\sigma(y'))) \leq 2^{-3}$ also. This implies that $\tau_\xi(x'_1) = \tau_\xi(y'_2)$, which in turn implies that $x'_1 = y'_2$. It follows that $z' = (y'_1, x'_1, x'_2, \ldots)$ is in X_G^+. Applying Lemma 3.17 to z', y' gives

$$d_\xi(y,z) = d_\xi(y',z') \leq 2^{-1}d_\xi(\sigma(y'),\sigma(z')) = 2^{-1}d_\xi(\sigma(y),x).$$

□

4 Smale spaces

The results of section 3 showed a construction of a compact metric space (X_ξ^+, d_ξ) along with a map $\sigma_\xi : X_\xi^+ \to X_\xi^+$ which is continuous, surjective and open. It also satisfies a local expansiveness condition Theorem 3.17. Our goal in this section is to replace this system with another where the dynamics is actually a homeomorphism. This is a standard construction via inverse limits. Without giving the precise definition for the moment, this introduces a new component to the space where the dynamics is contracting. In short, we are going to build a Smale space. Two references for Smale spaces are Ruelle [28] and Putnam [24].

Let us begin with recalling the definition of a Smale space. First of all, (X, d) is a compact metric space and φ is a homeomorphism of X. We assume that there are constants $\epsilon_X > 0, 0 < \lambda < 1$ and a continuous map

$$[\cdot,\cdot] : \{(x, y) \mid x, y \in X, d(x, y) \leq \epsilon_X\} \to X,$$

which satisfies the following:

B1 $[x, x] = x$,

B2 $[x, [y, z]] = [x, z]$,

B3 $[[x, y], z] = [x, z]$,

B4 $[\varphi(x), \varphi(y)] = \varphi[x, y]$,

for all x, y, z in X, whenever both sides of an equation are defined, and finally

C1 if $[x, y] = y$, then $d(\varphi(x), \varphi(y)) \leq \lambda d(x, y)$,

C2 if $[x, y] = x$, then $d(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \lambda d(x, y)$,

for x, y with $d(x, y) \leq \epsilon_X$.

We say that (X, d, φ) is a Smale space. We note that it is not necessary to specify the bracket map as part of the structure, only its existence: if it exists, it is essentially unique.

We now consider our specific case of interest. We will usually work under the standing hypotheses. It seems likely that it is sufficient for G to be irreducible. That is only used in an essential way in Proposition 3.5, but it seems possible that some other hypothesis on ξ may be needed for the construction of the last section to work.

Definition 4.1. *Let G, H, ξ satisfy the standing hypotheses. We define X_ξ to be the inverse limit of the system*

$$X_\xi^+ \xleftarrow{\sigma_\xi} X_\xi^+ \xleftarrow{\sigma_\xi} \qquad \cdots$$

That is,

$$X_\xi = \{(x^0, x^1, \ldots) \mid x^n \in X_\xi^+, x^n = \sigma_\xi(x^{n+1}), n \geq 0\}.$$

We also define $\sigma_\xi : X_\xi \to X_\xi$ by $\sigma_\xi(x)^n = \sigma_\xi(x^n)$, for $n \geq 0$ and $x^n, n \geq 1$ in X_ξ. It is a simple matter to check that the inverse is given by

$$\sigma_\xi^{-1}(x)^n = x^{n+1}, n \geq 0,$$

for any x in X_ξ.

Finally, we define a metric, also denoted d_ξ, on X_ξ by

$$d_\xi(x, y) = \sup\{2^{-n} d_\xi(x^n, y^n) \mid n \geq 0\},$$

for $x = (x^n)_{n\geq 0}, y = (y^n)_{n\geq 0}$ in X_ξ.

We remark that if we replace X_ξ^+, σ_ξ by X_G^+, σ_G the result is X_G, σ_G in a canonical way by identifying x in X_G with the sequence $x^n = \chi_G^+(\sigma^{-n}(x)) = (x_{1-n}, x_{2-n}, \ldots)$, for $n \geq 0$. Pursuing this remark a little further, it is a fairly simple matter to check that the map which sends x in X_G to the sequence $\pi_\xi(\chi_G^+(\sigma^{-n}(x))), n \geq 0$, in X_ξ is well-defined. We denote the map by π_ξ. It is immediate that $\pi_\xi \circ \sigma = \sigma_\xi \circ \pi_\xi$.

The system (X_G, d_G, σ) is a Smale space with constants $\epsilon_{X_G} = \lambda = 2^{-1}$ and bracket defined as $[x, y] = (\ldots y_{-1}, y_0, x_1, \ldots)$, for $d_G(x, y) \leq 2^{-1}$, which ensures $x_0 = y_0, x_1 = y_1$.

To see that $(X_\xi, d_\xi, \sigma_\xi)$ is a Smale space, we need to define the bracket map. Let $x = (x^n)_{n\geq 0}, y = (y^n)_{n\geq 0}$ be in X_ξ and assume that $d_\xi(x, y) \leq 2^{-n-1}$.

We inductively define $z^n, n = 0, 1, 2, 3, \ldots$ in X_ξ^+ satisfying

1. $d_\xi(z^n, y^n) \leq 2^{-n-1}$,
2. $\sigma_\xi(z^{n+1}) = z^n$.

Begin with $z^0 = x^0$. Observe the first condition holds for $n = 0$ and

$$d_\xi(x^0, y^0) \leq d_\xi(x, y) \leq 2^{-1}.$$

To define z^{n+1}, we have $d_\xi(z^n, \sigma_\xi(y^{n+1})) = d_\xi(z^n, y^n) \leq 2^{-n-1}$. By Lemma 3.18, we may find z^{n+1} in X_ξ^+ with $\sigma_\xi(z^{n+1}) = z^n$ and $d_\xi(z^{n+1}, y^{n+1}) \leq 2^{-1} d_\xi(z^n, y^n) \leq 2^{-n-2}$. So $z = (z^0, z^1, \ldots)$ lies in X_ξ.

We use the constants $\epsilon_{X_\xi} = \lambda = 2^{-1}$. Notice that $d_\xi(x, y) \leq 2^{-1}$ implies that $d_\xi(x^0, y^0) \leq 2^{-1}$, which ensures that $[x, y]$ is defined.

We will not verify the axioms B1-B4. As for C1, suppose that $z = [x, y] = y$. It follows that $y^0 = x^0$ and so

$$\begin{aligned}
d(\sigma_\xi(x), \sigma_\xi(y)) &= \sup\{2^{-n} d_\xi(\sigma_\xi(x^n), \sigma_\xi(y^n)) \mid n \geq 0\} \\
&= \sup\{2^{-n} d_\xi(\sigma_\xi(x^n), \sigma_\xi(y^n)) \mid n \geq 1\} \\
&= \sup\{2^{-n} d_\xi(x^{n-1}, y^{n-1}) \mid n \geq 1\} \\
&= 2^{-1} \sup\{2^{-m} d_\xi(x^m, y^m) \mid m \geq 0\} \\
&= 2^{-1} d_\xi(x, y).
\end{aligned}$$

We also verify C2: if $z = [x, y] = x$, then we have

$$\begin{aligned}
d(\sigma_\xi^{-1}(x), \sigma_\xi^{-1}(y)) &= \sup\{2^{-n} d_\xi(x^{n+1}, y^{n+1}) \mid n \geq 0\} \\
&= \sup\{2^{-n} 2^{-1} d_\xi(x^n, y^n) \mid n \geq 0\} \\
&= 2^{-1} d_\xi(x, y).
\end{aligned}$$

Theorem 4.2. *With constants $\epsilon_{X_\xi} = \lambda = 2^{-1}$ and bracket map as defined above, $(X_\xi, d_\xi, \sigma_\xi)$ is a Smale space.*

Theorem 4.3.

1. *The map $\pi_\xi : (X_G, d_G, \sigma) \to (X_\xi, d_\xi, \sigma_\xi)$ is a factor map.*
2. *For each x, y in X_G such that $d_G(x, y) \leq 2^{-3}$, both $[x, y]$ and $[\pi_\xi(x), \pi_\xi(y)]$ are defined and $\pi_\xi[x, y] = [\pi_\xi(x), \pi_\xi(y)]$.*
3. *Points x, y in X_G satisfy $\pi_\xi(x) = \pi_\xi(y)$ if and only if exactly one of the following hold:*
    (a) *$x = y$,*
    (b) *there exists $i = 0, 1$ and z in X_H such that $\xi^i(z_n) = x_n, \xi^{1-i}(z_n) = y_n$, for all integers n,*

(c) *there is an integer* m*,* $i = 0, 1$ *and* z *in* X_H *such that*

   i. $x_n = y_n$*, for all* $n < m$*,*

   ii. $x_m = y_m$ *is not in* $\xi(H)$ *or* $\xi^{1-i}(z_m) = x_m, \xi^i(z_m) = y_m$ *and*

   iii. $\xi^i(z_n) = x_n, \xi^{1-i}(z_n) = y_n$*, for all integers* $n > m$*.*

4. *The map* π_ξ *is s-bijective; that is, for every* x *in* X_G*,* $\pi_\xi | X^s_G(x)$ *is a bijection from* $X^s_G(x)$ *to* $X^s_\xi(\pi_\xi(x))$*.*

Proof. The first part is easy, as we have already noted.

For the second part, if $d_G(x, y) \leq 2^{-3}$, then $[x, y]$ is defined. In addition, by 3.15, $d_\xi(\pi_\xi(x), \pi_\xi(y)) \leq 3 \cdot 2^{-3} < 2^{-1}$, so $[\pi_\xi(x), \pi_\xi(y)]$ is also defined.

We define, for $n \geq 0$, $z^n = \pi_\xi(y_{-n}, \cdots, y_{-1}, x_0, x_1, \ldots)$ in X^+_ξ. It is clear that $z^0 = \pi_\xi(\chi^+_G(x))$ and $\sigma_\xi(z^n) = z^{n-1}$, for all $n \geq 1$. We claim that $d_\xi(z^n, y^n) \leq 2^{-n-1}$, which we show by induction. The case $n = 0$ is simply the fact that $d_\xi(\pi_\xi(x), \pi_\xi(y)) \leq 2^{-1}$, which we have already established. For any $n \geq 1$, Lemma 3.18 and the induction hypothesis show that

$$d_\xi(z^n, y^n) \leq 2^{-1} d_\xi(z^{n-1}, y^{n-1}) \leq 2^{-n-1}.$$

It follows from the definition of the bracket on X_ξ that $z = [\pi_\xi(x), \pi_\xi(y)]$. On the other hand, we have

$$(\pi_\xi[x, y])^n = \pi_\xi([x, y]^n) = \pi_\xi(\chi^+_G(\sigma^{-n}[x, y])) = z^n,$$

by definition.

For the third part, it is clear that if x, y satisfy conditions (b) or (c), then $\chi^+_G(\sigma^{-n}(x)) \sim_\xi \chi^+_G(\sigma(y))$, for all integers n and it follows that $\pi_\xi(x)^n = \pi_\xi(y)^n$, for all integers n.

Conversely, if $\pi_\xi(x) = \pi_\xi(y)$, then $\chi^+_G(\sigma^{-n}(x)) \sim_\xi \chi^+_G(\sigma^{-n}(y))$, for all integers n. It follows that there is an integer n such that, there is a path z^n in X^+_H and $i_n = 0, 1$ such that

$$\chi^+_G(\sigma^{1-n}(x) = \xi^{i_n}(\chi^+_G(\sigma^{1-n}(z^n)), \chi^+_G(\sigma^{1-n}(y) = \xi^{1-i_n}(\chi^+_G(z^n)).$$

If n satisfies this condition, then so does $n+1$ and $i_{n+1} = i_n$, $\chi^+_G(\sigma^n(z^{n+1}) = \chi^+_G(\sigma^{1-n}(z^n)$. There are then two cases to consider. In the first, the condition is satisfied by all integers n. In this case, (b) holds with $z = \lim_{n \to -\infty} z^n$ (and possibly) reversing x and y if $i_m = 1$. Otherwise, let m be the greatest integer not satisfying the condition. It is a simple matter to check that (c) holds in this case.

For the last part, if $\pi_\xi(x) = \pi_\xi(y)$, then by part 3, x and y are not stably equivalent (i.e. right tail-equivalent) unless $x = y$. This shows that $\pi_\xi | X_G^s(x)$ is injective. The surjectivity follows from the fact that G primitive implies (X_G, σ) is non-wandering, (Proposition 2.2.14 of [17]), and Theorem 2.5.8 of [24]. $\square$

5 Homology

In [24], a homology theory for (non-wandering) Smale spaces is described. Specifically, given a non-wandering Smale space (X, d, φ) and integer n, there are two countable abelian groups, $H_n^s(X, \varphi)$ and $H_n^u(X, \varphi)$. The former invariant is covariant for s-bijective factor maps and contravariant for u-bijective factor maps while the latter is covariant for u-bijective factor maps and contravariant for s-bijective factor maps. As such the map φ induces a pair of automorphisms of each. The aim of this section is to compute these invariants for the systems $(X_\xi, d_\xi, \sigma_\xi)$ we have constructed.

Let us begin with some preliminary notions. If G is any finite directed graph, we may consider the free abelian group on the vertex set G^0, denoted $\mathbb{Z}G^0$. This comes with two canonical endomorphisms

$$\gamma_G^s(v) = \sum_{i(e)=v} t(e), \qquad \gamma_G^u(v) = \sum_{t(e)=v} i(e),$$

for any v in G^0. This is needed at various places in the theory and in our proofs below. However, for the statements of the results, we can use the more familiar group $\mathbb{Z}^{G^0}$, which is isomorphic to $\mathbb{Z}G^0$ in an obvious way. Under this isomorphism, the map γ_G^s becomes multiplication by the matrix A_G while γ_G^u is multiplication by its transpose. Then we define $D^s(G)$ as the inductive limit of the sequence

$$\mathbb{Z}^{G^0} \stackrel{A_G}{\to} \mathbb{Z}^{G^0} \stackrel{A_G}{\to} \mathbb{Z}^{G^0} \stackrel{A_G}{\to} \cdots$$

Similarly, the invariant $D^u(G)$ is computed in a similar way, but with the transpose, A_G^T, replacing A_G.

With this notation, we can summarize our results with the following theorem.

Theorem 5.1. *Let G, H, ξ satisfy the standing hypotheses. We have*

$$
\begin{array}{rclccl}
H^s_0(X_\xi,\sigma_\xi) & \cong & D^s(G), & (\sigma_\xi)_*^{-1} & \cong & A_G, \\
H^s_1(X_\xi,\sigma_\xi) & \cong & D^s(H), & (\sigma_\xi)_*^{-1} & \cong & A_H, \\
H^u_0(X_\xi,\sigma_\xi) & \cong & D^u(G), & (\sigma_\xi)_* & \cong & A_G^T, \\
H^u_1(X_\xi,\sigma_\xi) & \cong & D^u(H), & (\sigma_\xi)_* & \cong & A_H^T, \\
H^s_k(X_\xi,\sigma_\xi) & \cong & 0, & k & \neq & 0,1, \\
H^u_k(X_\xi,\sigma_\xi) & \cong & 0, & k & \neq & 0,1.
\end{array}
$$

In the first two lines, we regard σ_ξ as an s-bijective factor map from (X_ξ, σ_ξ) to itself and our description of the induced map on homology is interpreted via the description which precedes it. In the next two lines, we regard σ_ξ as a u-bijective factor map

In fact, we will prove only the first, second and fifth parts as the other three are quite similar.

The starting point for computation of the homology theory for a general Smale space, (X, φ) is to find an s/u-bijective pair: Smale spaces (Y, ψ) and (Z, ζ) such that $Y^u(y)$ and $Z^s(z)$ are totally disconnected, for all y in Y and z in Z, along with an s-bijective factor map $\pi_s : (Y, \psi) \to (X, \varphi)$ and a u-bijective factor map $\pi_u : (Z, \zeta) \to (X, \varphi)$. Usually, the collection $(Y, \psi, \pi_s, Z, \zeta, \pi_u)$ is written as π.

Given such an s/u-bijective pair, one proceeds to define dynamical systems as fibred products: for $L, M \geq 0$,

$$
\begin{aligned}
\Sigma_{L,M}(\pi) \ = \ & \{(y^0, \ldots, y^L, z^0, \ldots, z^M) \in Y^{L+1} \times Z^{M+1} \\
& \mid \pi_s(y^l) = \pi_u(z^m), \text{ all } l, m\}.
\end{aligned}
$$

Each of these systems is a shift of finite type and $\Sigma_{L,M}(\pi)$ carries an obvious action of the group $S_{L+1} \times S_{M+1}$.

Theorem 5.2. *Let G, H, ξ satisfy the standing hypotheses. Then $(X_G, \sigma, \pi_\xi, X_\xi, \sigma_\xi, id_{X_\xi})$ is an s/u-bijective pair for (X_ξ, σ_ξ), which we denote by π_ξ. Moreover, we have $\Sigma_{0,0}(\pi_\xi) = X_G$ and $C^s_{\mathcal{Q},\mathcal{A}}(\pi_\xi)_{0,0} \cong D^s(G)$.*

The next step is to find a graph which gives a symbolic presentation for $\Sigma_{0,0}(\pi_\xi)$. This requires that, for any x, y in X_g satisfying $t(x_0) = t(y_0)$, then $\pi_\xi[x, y] = [\pi_\xi(x), \pi_\xi(y)]$, meaning that both sides are defined. Unfortunately, G will not suffice but this is not a major obstacle.

We review the standard notion of the higher block coding of G (see section 2.3 [17]). We let G^K be the set of all paths of length K in G, for any $K \geq 2$, and define $i, t : G^K \to G^{K-1}$ by

$$i(x_1, \ldots, x_K) = (x_1, \ldots, x_{K-1}), \qquad i(x_1, \ldots, x_K) = (x_2, \ldots, x_K),$$

for $(x_1, \ldots, x_K)$ in G^K. In this way, we interpret G^K, G^{K-1}, i, t as a graph. For any $1 \leq k \leq K$, the map sending $(x_1, \ldots, x_K)$ in G^K to x_k induces a homeomorphism between X_{G^K} and X_G which intertwines the shift maps. It will not be necessary for us to write this map explicitly: we simply regard X_{G^K} and X_G as equal, in an obvious way.

We use $K = 7$ and $k = 4$. The point is that, if x, y are in X_G and $t(x_0) = t(y_0)$, then regarding them as elements in X_{G^7}, means $x_{[-2,3]} = y_{[-2,3]}$, which implies $d(x, y) \leq 2^{-3}$ and so part 2 of Theorem 4.3 applies and G^7 satisfies the conditions of Definition 2.6.8 of [24].

We remark that the map $t : G^6 \to G^0$ induces an isomorphism between $D^s(G^7)$ and $D^s(G)$.

Our two maps are $\pi_s = \pi_\xi$ and π_u being the identity on X_ξ. Due to part 3 of Theorem 4.3, we see that $\#\pi_s\{x\} \leq 2 = L_0$ and obviously $\#id^{-1}\{x\} \leq 1 = M_0$, for all x in X_ξ, so by Theorem 5.1.10 of [24], our complex consists of only two non-zero groups, $C^s_{\mathcal{Q},\mathcal{A}}(\pi_\xi)_{0,0}$ and $C^s_{\mathcal{Q},\mathcal{A}}(\pi_\xi)_{1,0}$, and a single homomorphism from the latter to the former. So what are these groups, how do we compute them and what is the homomorphism between them? The first is given already in 5.2.

We now come to the second group, $C^s_{\mathcal{Q},\mathcal{A}}(\pi_\xi)_{1,0}$, where it will be necessary to use our higher block presentation. The system $\Sigma_{(1,0)}(\pi_\xi)$ of [24] consists of pairs

$$\Sigma_{(1,0)}(\pi_\xi) = \{(x^0, x^1) \mid x^0, x^1 \in X_G, \pi_\xi(x^0) = \pi_\xi(x^1)\}.$$

Fortunately, Theorem 4.3 gives a nice description of these. The fact that our map is regular means that this is the shift of finite type associated with an obvious subgraph of $G^7_1 \subseteq G^7 \times G^7$, whose vertex set is $G^6_1 \subseteq G^6 \times G^6$, with obvious maps i, t. (This suppresses the fact that an infinite sequence of pairs can also be seen as a pair of infinite sequences.) This graph is obtained by simply taking pairs in $\Sigma_{(1,0)}(\pi_\xi)$ and finding all words of length 6 and 7.

We may partition our vertex set $G^6_1 = V_0 \cup V_1 \cup \cdots \cup V_6$ as follows. First, V_0 consists of all pairs (x, x), where x is in G^6. Also, V_6 consists of all pairs

$(\xi^0(y), \xi^1(y)), (\xi^1(y), \xi^0(y))$, where y is in H^6. For $1 \leq k < 6$, we let V_k be all pairs of the following types

$$(x\xi^0(y), x\xi^1(y)), (x\xi^1(y), x\xi^0(y)),$$

where, x is in G^{n-k}, y is in H^k and $t(g) = \xi^0(i(y))$, along with

$$\begin{array}{ll} (x\xi^1(y_1)\xi^0(y_2), \ldots, \xi^0(y_{k+1})), & x\xi^0(y_1)\xi^1(y_2), \ldots, \xi^1(y_{k+1]}))), \\ (x\xi^0(y_1)\xi^1(y_2), \ldots, \xi^1(y_{k+1})), & x\xi^1(y_1)\xi^0(y_2), \ldots, \xi^0(y_{k+1})), \end{array}$$

where x is in G^{n-k-1}, y is in H^{k+1} and $t(g) = \xi^0(i(y))$.

There is an analogous partition $G_1^7 = E_0 \cup \cdots \cup E_7$. Notice that $i(E_0) \subseteq V_0$, $i(E_i) \subseteq V_{i-1}$, if $1 \leq i \leq 7$, $t(E_i) \subseteq V_i$,if $0 \leq i \leq 6$, and $t(E_7) \subseteq V_6$. That is, V_0 and V_6 are two components of G_1^7 and the remaining edges move from V_0 to V_6.

Observe that the permutation group on 2-symbols acts in an obvious manner on all these objects.

We consider the free abelian group on G_1^6 with an inductive limit given by G_1^7. Before doing so, we must take a quotient, moding out by the subgroup generated by vertices v with $v = v \cdot \alpha$ and all elements of the form $v - v \cdot (1, 2)$, where α interchanges entries 0 and 1. The first means that we are removing V_0 from consideration. Among the remaining vertices, only the ones in V_6 are the terminus of a path of any length greater than 6 and it follows that the inductive limit only using that part. On the other hand, when we quotient by $v - v \cdot \alpha$, the result is clearly isomorphic to the free abelian group on H^6, with generating set $(\xi^0(y), \xi^1(y))$, y in H^6. We conclude that $C^s_{\mathcal{Q},\mathcal{A}}(\pi_\xi)_{1,0}$ is isomorphic to to the inductive limit of this group under the map induced by H^7. We have proved the following.

Lemma 5.3. *We have $C^s_{\mathcal{Q},\mathcal{A}}(\pi_\xi)_{1,0} \cong D^s(H)$.*

We are now left to consider the boundary map between the two groups and a key technical point in its computation is the following. Our notation is slightly different from [24], so we give a self-contained statement here.

Lemma 5.4. *For the regular s-bijective factor map $\pi_\xi : (X_G, \sigma) \to (X_\xi, \sigma_\xi)$, the constant $K_\xi = 0$ satisfies the conditions of Lemma 2.7.2 of [24]. That is, if x^1, x^2, y^1, y^2 are all in X_G with $\pi_\xi(x^1) = \pi_\xi(x^2), \pi_\xi(y^1) = \pi_\xi(y^2)$, $x^1_k = y^1_k$, for $k \geq k_0$, and x^2, y^2 stably equivalent, then $x^1_k = y^1_k$, for $k \geq k_0$ also.*

Proof. The statement is trivial if either $x^1 = x^2$ or $y^1 = y^2$. It remains to consider when both pairs are distinct. This situation is described explicitly in Theorem 4.3. The rest of the proof is done by checking the different cases, which we leave to the reader. $\square$

Lemma 5.5. *The boundary map*

$$d^s_{\mathcal{Q}} : C^s_{\mathcal{Q},\mathcal{A}}(\pi_\xi)_{1,0} \to C^s_{\mathcal{Q},\mathcal{A}}(\pi_\xi)_{0,0}$$

is the zero map.

Proof. Via the isomorphism of Lemma 5.3, this group is generated by classes of the form $(\xi^0(y), \xi^1(y))$, y in H^6. Using the formula given in Definition 4.2.1, we have

$$d^s_{\mathcal{Q}}(\xi^0(y), \xi^1(y)) = \xi^0(y) - \xi^1(y),$$

for each y in H^6, where the term on the right is interpreted as an element of $\mathbb{Z}G^6$. We know that the map t induces an isomorphism between $D^s(G^7)$ and $D^s(G)$ and we have

$$t(\xi^0(y) - \xi^1(y)) = t(\xi^0(y)) - t(\xi^1(y)) = 0,$$

by condition (H0). $\square$

The computations of the homology groups, $H^s_*(X_\xi, \sigma_\xi)$, as summarized in Theorem 5.1, follow from Theorem 5.2 and Lemmas 5.3 and 5.5.

Let us mention an alternate proof of these results. This is based on two sets of results. The first are those of Proietti and Yamashita [20] and [21], who show that the Smale space homology here agrees with the groupoid homology as studied by Matui. Further, Matui has recently the adapted results which are used in the next section for the K-theory of the C^*-algebras to the case of groupoid homology: see [18]. These require the unit space to be totally disconnected and so only apply to parts 1, 2 and 5 of Theorem 5.1.

6 Groupoids, C^*-algebras and K-theory

The goal of this section is to describe the stable and unstable equivalence relations (or groupoids) of our Smale space, $(X_\xi, d_\xi, \sigma_\xi)$, their associated C^*-algebras and their K-theory.

We begin with a short review of what is involved for general Smale space. For any Smale space, (X, d, φ), two points x, y in X are *stably* equivalent if

$$\lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0.$$

Unstable equivalence is defined by simply replacing both occurrences of φ by φ^{-1}. We define

$$\begin{aligned} G^s(X, \varphi) &= \{(x, y) \in X \times X \mid \lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0\}, \\ G^u(X, \varphi) &= \{(x, y) \in X \times X \mid \lim_{n \to +\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0\}. \end{aligned}$$

Each is an equivalence relation and has a natural topology. It will not be necessary to describe this here in detail, but we refer the reader to [22]. In addition, if (X, φ) is non-wandering, then, as groupoids, each has a Haar system. Again, we do not need a detailed description. For any x in X, we let $X^s(x)$ and $X^u(x)$ be the equivalence classes of x in the two equivalence relations. Each carries a natural topology which is finer than the relative topology of X but makes each locally compact Hausdorff.

Generally speaking, étale groupoids [27] are much easier to deal with and, thanks to the work of Muhly, Renault and Williams [19], our groupoids are equivalent to étale groupoids by reducing to an abstract transversal. With the slightly stronger hypothesis that (X, φ) is irreducible, we select a finite set P with the property that $\varphi(P) = P$ (i.e. consisting of periodic points), we let

$$X^u(P) = \cup_{p \in P} X^u(p), \qquad X^s(P) = \cup_{p \in P} X^s(p)$$

and these function as natural transversals to $G^s(X, \varphi), G^u(X, \varphi)$, respectively. Note that these spaces are given locally compact topologies which are finer than the relative topology of X. We refer the reader to [23]. We denote the reduction to these transversals by

$$\begin{aligned} G^s(X, \varphi, P) &= G^s(X, \varphi) \cap (X^u(P) \times X^u(P)), \\ G^u(X, \varphi, P) &= G^u(X, \varphi) \cap (X^s(P) \times X^s(P)) \end{aligned}$$

Each is an étale equivalence relation. Different choices of P yield equivalent groupoids. In [23], it is shown that these groupoids are amenable, so we do not need to make a distinction between the full and reduced C^*-algebras. We define

$$\begin{aligned} S(X, \varphi, P) &= C^*(G^s(X, \varphi, P)), \\ U(X, \varphi, P) &= C^*(G^u(X, \varphi, P)). \end{aligned}$$

The map which sends a continuous function of compact support, f, on either of the two groupoids, to $f \circ (\varphi \times \varphi)^{-1}$ extends to an automorphism of the respective C^*-algebra, which is denoted by φ. The Ruelle C^*-algebras are defined as the associated crossed-products:

$$\begin{aligned} R^s(X, \varphi, P) &= S(X, \varphi, P) \rtimes_\varphi \mathbb{Z}, \\ R^u(X, \varphi, P) &= U(X, \varphi, P) \rtimes_\varphi \mathbb{Z}. \end{aligned}$$

Let us review some basic properties of these C^*-algebras. If (X, φ) is mixing then $S(X, \varphi, P)$ and $U(X, \varphi, P)$ are separable, simple, finite, stable, satisfy the Universal Coefficient Theorem (UCT) [23] and have finite nuclear dimension [9]. If (X, φ) is irreducible, then $R^s(X, \varphi, P)$ and $R^u(X, \varphi, P)$ are separable, simple, purely infinite, stable, satisfy the Universal Coefficient Theorem (UCT) [23] and have finite nuclear dimension [9]. All of these C^*-algebras come under the Elliott classification scheme. (We include a small remark: the results of [9] need the hypothesis that the C^*-algebras contain a projection. This was later shown to hold in general [10], but, in our case, we will explicitly provide clopen subsets of the unit spaces of both $G^s(X_\xi, \sigma_\xi, P_\xi)$ and $G^s(X_\xi, \sigma_\xi, P_\xi)$ which shows this holds.)

In our particular case for $(X_\xi, d_\xi, \sigma_\xi)$, we can actually go a little further and the first step is to make a particular choice for P, as we describe.

Under the standing hypotheses, we select a periodic point as follows. We have already observed that $G^1 - \xi(H^1)$ is also primitive. We select a cycle $C^1 \subseteq G^1 - \xi(H^1)$ of minimal length. This implies that $t|_{C^1}$ is injective. We let $C^0 = t(C^1)$. We let $P \subseteq X_G$ be the finite set of infinite paths which simply repeat the cycle C^1. We note that an element p of P is uniquely determined in P by $i(p_1)$ in C^0. Also notice that π_ξ is a bijection from P to $P_\xi = \pi_\xi(P)$ from part 3 of Theorem 4.3.

The s-bijective map $\pi_\xi : X_G \to X_\xi$ induces maps at the level of groupoids and also C^*-algebras

$$\begin{aligned} (\pi_\xi)^* : \quad & S(X_\xi, \sigma_\xi, P_\xi) \to S(X_G, \sigma, P), \\ (\pi_\xi)_* : \quad & U(X_G, \sigma, P) \to U(X_\xi, \sigma_\xi, P_\xi). \end{aligned}$$

The C^*-algebras $S(X_G, \sigma, P)$ and $U(X_G, \sigma, P)$ are both AF-algebras, their K-zero groups are $D^u(G)$ and $D^s(G)$, respectively, while their K-one groups are trivial.

We now state the two main results of this section. Of course, they could easily be assembled into a single result, but we separate them as the proofs are rather long and also rather different and they will be divided into two subsections.

Theorem 6.1. *If G, H, ξ satisfy the standing hypotheses, we have*

1. *$K_0(S(X_\xi, \sigma_\xi, P_\xi)) \cong D^u(G)$ as ordered abelian groups and, under this isomorphism, the automorphism induced by σ_ξ is A_G^T.*

2. *$K_1(S(X_\xi, \sigma_\xi, P_\xi)) \cong D^u(H)$ and, under this isomorphism, the automorphism induced by σ_ξ is A_H^T.*

3. *$K_0(R^s(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}^{G^0}/(I - A_G^T)\mathbb{Z}^{G^0} \oplus \ker(I - A_H^T : \mathbb{Z}^{H^0} \to \mathbb{Z}^{H^0})$.*

4. *$K_1(R^s(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}^{H^0}/(I - A_H^T)\mathbb{Z}^{H^0} \oplus \ker(I - A_G^T : \mathbb{Z}^{G^0} \to \mathbb{Z}^{G^0})$.*

Theorem 6.2. *If G, H, ξ satisfy the standing hypotheses, we have*

1. *$K_0(U(X_\xi, \sigma_\xi, P_\xi)) \cong D^s(G)$ as ordered abelian groups and, under this isomorphism, the automorphism induced by σ_ξ is A_G^{-1}.*

2. *$K_1(U(X_\xi, \sigma_\xi, P_\xi)) \cong D^s(H)$ and, under this isomorphism, the automorphism induced by σ_ξ is A_H^{-1}.*

3. *$K_0(R^u(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}^{G^0}/(I - A_G)\mathbb{Z}^{G^0} \oplus \ker(I - A_H : \mathbb{Z}^{H^0} \to \mathbb{Z}^{H^0})$.*

4. *$K_1(R^u(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}^{H^0}/(I - A_H)\mathbb{Z}^{H^0} \oplus \ker(I - A_G : \mathbb{Z}^{G^0} \to \mathbb{Z}^{G^0})$.*

Remark 6.3. *Both results are undoubtedly true if the hypothesis that G is primitive is weakened to (X_G, σ_G) being non-wandering. However, our proofs rely heavily on results of two papers [25] and [12] which require the respective equivalence relations $G^s(X_G, \sigma, P)$ and $G^u(X_G, \sigma, P)$ to have all equivalence classes dense and this is equivalent to G being primitive. It would not be difficult to adjust these results to the more general situation, but it would complicate matters.*

Remark 6.4. *The alert reader may be somewhat perplexed by the switching of 's' and 'u', between the two sides of the first two parts of both results. This is due to an historical anomaly: the superscript on G^s was chosen for 'stable' equivalence and it seems logical to use S for its C^*-algebra. However, the elements of the K_0-group of this C^*-algebra are realized by characteristic functions of clopen subsets of its unit space, which is $X^u(P)$. So the notation for $D^u(G)$ was chosen to coincide with this interpretation.*

Let us mention one other groupoid, without going into great detail. It follows from Theorems 3.17 and 3.19 that the map σ_ξ of X_ξ^+ is locally expanding and also a local homeomorphism. We can associate to it its Deaconu-Renault groupoid, as follows. For each x, y in X_ξ^+ and positive integers m, n satisfying $\sigma_\xi^m(x) = \sigma_\xi^n(y)$, we consider the triple $(x, m-n, y)$. The set of all such triples is a groupoid with the product $(x, k, y) \cdot (x', k', y')$ defined when $y = x'$ and result $(x, k+k', y')$. It can be given a topology it which it is étale [8]. The associated groupoid C^*-algebra has the nice feature of being unital, and it is also equivalent to $R^u(X_\xi, \sigma_\xi, P_\xi)$.

The result of our K-theory computations has an interesting consequence that our construction exhausts all possible Ruelle algebras from mixing Smale spaces, as we describe below.

We begin with some simple observations. Let (X, φ, d) be a mixing Smale space and Q be a finite φ-invariant subset of X. The C^*-algebras $R^s(X, \varphi, Q)$ and $R^u(X, \varphi, Q)$ are Spanier-Whitehead duals of each other (see Definition 4.1 and Theorem 1.1 of [14]). As noted in section 4 of [14], this implies that their K-groups are finitely generated. Proietti and Yamashita have recently shown that $K_*(S(X, \varphi))$ and $K_*(U(X, \varphi))$ are finite rank (Theorem 5.1 of [21]) and, as explained in section 4 of [14], this implies that $K_0(R^s(X, \varphi, Q))$ and $K_1(R^s(X, \varphi, Q))$ have the same rank. Putting all of this together, we conclude that there is an integer $k \geq 0$ and finite abelian groups G_0 and G_1 such that

$$\begin{aligned} K_0(R^s(X, \varphi, Q)) &\cong \mathbb{Z}^k \oplus G_0, \\ K_1(R^s(X, \varphi, Q)) &\cong \mathbb{Z}^k \oplus G_1. \end{aligned}$$

We add that another consequence of Theorem 5.1 of [21], as described in section 4 of [14] is that

$$R^s(X, \varphi, Q) \cong R^u(X, \varphi, Q).$$

Corollary 6.5. *Let (X, φ, d) be a mixing Smale space and Q be a finite φ-invariant subset of X. There exist finite directed primitive graphs, G, H and embeddings $\xi^0, \xi^1 : H \to G$ satisfying (H0), (H1) and (H2) such that $R^s(X_\xi, \sigma_\xi, P_\xi) \cong R^s(X, \varphi, Q)$, for any P, a finite σ-invariant subset of X_G.*

We begin the proof with a simple algebraic result.

Lemma 6.6. *Let G be a finitely generated abelian group and d_0, M_0 be positive integers. There exist $d \geq d_0$ and a $d \times d$ integer matrix A such that*

$$\mathbb{Z}^d / (I - A)\, \mathbb{Z}^d \cong G$$

and each entry of A is greater than or equal to M_0.

Proof. From the classification theorem for finitely generated abelian groups (Theorem 2.2, page 76 of Hungerford [13]), we know that there are integers $k \geq 0$, $j_1, \ldots, j_l \geq 2$, $l \geq 0$ such that

$$G \cong \mathbb{Z}^k \oplus (\mathbb{Z}/j_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/j_l\mathbb{Z}) .$$

Let D be the diagonal matrix whose diagonal entries are 0, k times, $j_1, \ldots, j_l$ and then a collection of 1's so that there is at least one and $d_0 - l - k$, so the the size of D, which we call d, is at least d_0.

It is immediate that $\mathbb{Z}^d/D\mathbb{Z}^d \cong G$. Observe that this isomorphism still holds if we multiply D on either side by an integer matrix of determinant one. In particular, it remains true if we add a multiple of a row or column to another. Using column operations, the 1 in the d, d-entry of D can then be used to ensure row d has all positive integer entries. Then using row operations, we can ensure every row, other than row d has entries at least M_0. Finally, we add row 1 to row d and the resulting matrix matrix B has all entries at least M_0. Then $A = B + I$ satisfies the desired conclusion. $\square$

Corollary 6.5. We apply Lemma 6.6 to the group $G = K_1(R^s(X, \varphi, Q))$ with $d_0 = M_0 = 1$ to obtain a $d \times d$ matrix A. We apply Lemma 6.6 a second time to the group $Tor(K_0(R^s(X, \varphi, Q)))$, with $d_0 = d$ and $M_0 = \max\{2A(i, j)+1 \mid 1 \leq i, j \leq d\}$ to obtain a $d' \times d'$ matrix B.

We let H be the directed graph with d vertices and adjacency matrix A^T and G be the directed graph with d' vertices and adjacency matrix B^T. As $d' \geq d$, we may find an embedding of H^0 in G^0 and from the choice of B we can find embeddings ξ^0, ξ^1 of H^1 in G^1 which satisfy (H0), (H1) and (H2).

We note that as

$$\mathbb{Z}^{d'} / (I - B)\, \mathbb{Z}^{d'} \cong Tor(K_0(R^s(X, \varphi, Q))),$$

which is finite, $\ker(I - B)$ is free abelian and rank zero, so is trivial. If we let k be the rank of $K_1(R^s(X, \varphi, Q))$, then $\ker(I - A)$ also has rank k and is free abelian. We conclude from Theorem 6.1 that

$$\begin{array}{lll} K_0(R^s(X,\varphi,Q)) \cong & \mathbb{Z}^k \oplus Tor(K_0(R^s(X,\varphi,Q))) \cong & K_0(R^s(X_\xi,\sigma_\xi,P_\xi)) \\ K_1(R^s(X,\varphi,Q)) \cong & K_1(R^s(X_\xi,\sigma_\xi,P_\xi)) & \end{array}$$

As noted in [14] these C^*-algebras fall under the Kirchberg-Phillips classification Theorem, so $R^s(X,\varphi,Q) \cong R^s(X_\xi,\sigma_\xi,P_\xi)$. □

6.1 Stable equivalence

In this subsection, we focus on the C^*-algebras $S(X_\xi,\sigma_\xi,P_\xi)$ and $R^s(X_\xi,\sigma_\xi,P_\xi)$.

Lemma 6.7. *1. The set*

$$Y_G^u(P) = \{y \in X_G^u(P) \mid y_n = p_n, n \leq 0, \text{ some } p \in P\}$$

is a compact open subset of $X_G^u(P)$. Its relative topology from $X_G^u(P)$ agrees with the relative topology from X_G. If x is any point of X_G, then x is stably equivalent to some y in $Y_G^u(P)$. Finally, we have $\sigma^{-1}(Y_G^u(P)) \subseteq Y_G^u(P)$.

2. The set $Y_\xi^u(P) = \pi_\xi(Y_G^u(P))$ is a compact open subset of $X_\xi^u(P)$. Its relative topology from $X_\xi^u(P)$ agrees with the relative topology from X_ξ. If x is any point of $X_\xi^u(P)$, then x is stably equivalent to some y in $Y_\xi^u(P)$. Finally, we have $\sigma_\xi^{-1}(Y_\xi^u(P)) \subseteq Y_\xi^u(P)$.

Proof. For the first part, the statements about the topology are standard. As G is primitive, we may $l > 0$ such that there is a path of length l between any two vertices of G. If x in in X_G, let p be any point of P. Define y by $y_n = p_n, n \leq 0$, $y_1, \ldots, y_l$ is any path from $t(p_0)$ to $t(x_l)$, and $y_n = x_n, n > l$. So y is stably equivalent to x and lies in $Y_G^u(P)$.

For the second part, $Y_\xi^u(P)$ is compact since $Y_G^u(P)$ is. Moreover, $Y_G^u(P)$ is clearly invariant under $\sim_\xi$ and π_ξ is continuous, so $Y_G^u(P) = \pi_\xi^{-1}(Y_\xi^u(P))$ which implies that $Y_\xi^u(P)$ is open also. The remaining statements are clear. □

The following is an immediate consequence.

Proposition 6.8. *Let us denote by $G^s(\xi, P)$ the reduction of the groupoid $G^s(X_\xi, \sigma_\xi)$ to $Y^u_\xi(P)$. It is étale, equivalent to $G^s(X_\xi, \sigma_\xi)$ and also $G^s(X_\xi, \sigma_\xi, P_\xi)$, in the sense of Muhly, Renault and Williams [19]. In particular, we have containments and a commutative diagram*

$$\begin{array}{ccc} C^*(G^s(\xi, P)) & \subseteq & C^*(G^s(X_\xi, \sigma_\xi, P_\xi)) \\ \downarrow \sigma_\xi^{-1} & & \downarrow \sigma_\xi^{-1} \\ C^*(G^s(\xi, P)) & \subseteq & C^*(G^s(X_\xi, \sigma_\xi, P_{|xi})). \end{array}$$

The proof of Theorem 6.1 involves an application of the results of [12]. Let us provide a brief discussion of the set-up there. It will be fairly similar to the one we consider here, but there are a couple of differences which we need to address.

We begin with two Bratteli diagrams, $(V, E), (W, F)$ and two embeddings of the latter into the former satisfying conditions analogous to (H0) and (H1). Section 3 of [12] describes the construction of a quotient of the path space of the Bratteli diagram X_E, denoted X_ξ. The equivalence relation of tail equivalence on X_E, denoted, R_E then descends to an étale equivalence relation, denoted R_ξ. It is then shown (Theorem 1.1) that the $K_0(C^*(R))$ is isomorphic to the dimension group of the Bratteli diagram (V, E), while $K_1(C^*(R))$ is isomorphic to that of (W, F).

The first point to note is that in our current situation will we be using (V, E) and (F, W) stationary diagrams given by the matrices G and H, respectively. That is, at least approximately, $V_n = G^0, E_n = G^1, W_n = H^0, F_n = H^1$, for all n.

There is a second minor problem with the convention of [25] that the Bratteli diagram begins with V_0 as a single vertex. This is also solved easily: we let V_0 be a single vertex, $V_1 = t(C)$, our selected minimal cycle, and E_1 to have one edge from V_0 to each vertex, v, of V_1 which is simply the unique edge e of C with $t(e) = v$. Then, for $n \geq 2$, we inductively $E_n = i^{-1}(V_{n-1})$ and $V_n = t(E_n)$. As we assume that G is primitive, there will be some $n_0 \geq 1$ such that $V_n = G^0$ and $E_n = G^1$, for all $n \geq n_0$. With these definitions, it is immediate that the path space X_E of [12] coincides with $Y^u_G(P)$, the space X_ξ of [12] coincides with $Y^u_\xi(P)$, the groupoid R_E coincides with the reduction of $G^s(X_G, \sigma, P)$ to $Y^u_G(P)$ and the groupoid R_ξ coincides with $G^s(\xi, P)$.

There is a somewhat different solution to this problem for (W, F). We simply drop the assumption on the start and say W_n is only defined for

$n \geq n_0$, the two embeddings ξ^0, ξ^1 are the given ones. The results of [12] still hold verbatim.

For the third and fourth parts of Theorem 6.1, we use the Pimsner-Voiculescu exact sequence [30]. For notational purposes, we use S instead of $S(X_\xi, \sigma_\xi, P_\xi)$ and $R^s = S \rtimes_{\sigma_\xi} \mathbb{Z}$:

$$\begin{array}{ccccc} K_0(S) & \xrightarrow{id-(\sigma_\xi)_*} & K_0(S) & \longrightarrow & K_0(R^s) \\ \uparrow & & & & \downarrow \\ K_1(R^s) & \longleftarrow & K_1(S) & \xleftarrow[id-(\sigma_\xi)_*]{} & K_1(S) \end{array}$$

This immediately yields two short exact sequences

$$0 \to K_*(S)/\left(id - (\sigma_\xi)_*\right) K_*(S) \to K_*(R^s) \to \ker\left(id - (\sigma_\xi)_*\right) \to 0.$$

We know that $K_0(S) \cong D^u(G)$, which is an inductive limit of groups $\mathbb{Z}^{G^0}$. This means that there is a natural map we denote $i_1 : \mathbb{Z}^{G^0} \to D^u(G)$ as the first group in the inductive limit. This intertwines multiplication by A_G^T with the automorphism A_G^T and it is an easy exercise in algebra to show that this induces isomorphisms

$$\begin{aligned} \ker\left(I - A_G^T : \mathbb{Z}^{G^0} \to \mathbb{Z}^{G^0}\right) &\cong \ker\left(id - (\sigma_\xi)_*\right) \\ \mathbb{Z}^{G^0}/\left(I - A_G^T\right)\mathbb{Z}^{G^0} &\cong D^u(G)/\left(id - A_G^T\right) D^u(G). \end{aligned}$$

The quotients in the two exact sequences above are now seen to be subgroups of $\mathbb{Z}^{G^0}$ and hence are finitely generated and free. It follows that the two sequences both split. This completes the proof of Theorem 6.1.

6.2 Unstable equivalence

In this subsection, we focus on the C^*-algebras $U(X_\xi, \sigma_\xi, P_\xi)$ and $R^u(X_\xi, \sigma_\xi, P_\xi)$.

Lemma 6.9. *1. The set*

$$Y_G^s(P) = \{y \in X_G^s(P) \mid y_n = p_n, n \geq -1, \text{ some } p \in P\}$$

is a compact open subset of $X_G^s(P)$. Its relative topology from $X_G^s(P)$ agrees with the relative topology from X_G. If x is any point of X_G, then x is unstably equivalent to some y in $Y^s_G(P)$. Finally, we have $\sigma(Y^s_G(P)) \subseteq Y^s_G(P)$.

2. *The set $Y^s_\xi(P) = \pi_\xi(Y^s_G(P))$ is a compact open subset of $X^s_\xi(P_\xi)$. Its relative topology from $X^s_\xi(P_\xi)$ agrees with the relative topology from X_ξ. If x is any point of $X^s_\xi(P_\xi)$, then x is unstably equivalent to some y in $Y^s_\xi(P)$. Finally, we have $\sigma_\xi(Y^s_\xi(P)) \subseteq Y^s_\xi(P)$.*

Proof. The first part is all easy topological facts while the second follows as π_ξ is a homeomorphism when restricted to $X^s_G(P)$. □

The next result follows immediately from the last.

Proposition 6.10. *Let us denote by $G^u(\xi, P)$ the reduction of the groupoid $G^u(X_\xi, \sigma_\xi)$ to $Y^s_\xi(P)$. It is étale equivalent to $G^u(X_\xi, \sigma_\xi)$ and also to $G^u(X_\xi, \sigma_\xi, P_\xi)$, in the sense of Muhly, Renault and Williams [19]. In particular, we have containments and a commutative diagram*

$$\begin{array}{ccc} C^*(G^u(\xi, P)) & \subseteq & C^*(G^u(X_\xi, \sigma_\xi, P_\xi)) \\ \downarrow \sigma_\xi & & \downarrow \sigma_\xi \\ C^*(G^u(\xi, P)) & \subseteq & C^*(G^u(X_\xi, \sigma_\xi, P_\xi)). \end{array}$$

We begin our analysis with a technical result on the structure of $G^u(\xi, P)$.

Lemma 6.11. *Let x, y be in $Y^s_G(P)$ and assume that $\pi_\xi(x)$ and $\pi_\xi(y)$ are unstably equivalent in (X_ξ, σ_ξ). Then either*

1. *there exists $N < 0$ such that $x_n = y_n$, for all $n \leq N$, or*
2. *there exists $N < 0, i = 0, 1$ and z in X_H such that $x_n = \xi^i(z_n), y_n = \xi^{1-i}(z_n)$, for all $n \leq N$.*

Proof. We recall that in a general Smale space (X, φ, d), if points x, y are unstably equivalent, then there exists $n_0 < 0$ such that $\varphi^{n_0}(x)$ and $\varphi^{n_0}(y)$ are in the same local unstable set, meaning that $d(\varphi^{n_0}(x), \varphi^{n_0}(y)) < \epsilon_X$ and $[\varphi^{n_0}(x), \varphi^{n_0}(y)] = \varphi^{n_0}(x)$ (see Proposition 2.1.11 of [24]). It follows by induction that for all $n \leq n_0$, we have $d(\varphi^n(x), \varphi^n(y)) < \epsilon_X \lambda^{n-n_0}$. In our case, this means that

$$d_\xi(\sigma^n_\xi(\pi_\xi(x)), \sigma^n_\xi(\pi_\xi(y))) < 2^{-3+n-n_0},$$

for all $n \leq n_0$. From the definition of the metric d_ξ on the inverse limit space X_ξ, we have $d_\xi(\sigma_\xi^n(\pi_\xi(x))^0, \sigma_\xi^n(\pi_\xi(y)^0) < 2^{-3+n-n_0}$. Recalling the definition of π_ξ on X_G, we have $d_\xi(\pi_\xi(\chi_G^+(\sigma^n(x))), \pi_\xi(\chi_G^+(\sigma^n(y)))) < 2^{-3+n-n_0}$.

It follows from Theorem 3.9 that we have

$$\begin{aligned} d_{G_\xi}(\tau_\xi(\chi_G^+(\sigma^n(x))), \tau_\xi(\chi_G^+(\sigma^n(y)))) &< 2^{-3-n_0+n} \quad (1) \\ d_{\mathbb{T}}(\theta(\chi_G^+(\sigma^n(x)))\theta(\chi_G^+(\sigma^n(y)))) &< 2^{-3-n_0+n}, \quad (2) \end{aligned}$$

for $n \leq n_0$. The first of these inequalities immediately implies that $\tau_\xi(x_n) = \tau_\xi(y_n)$, for all $n < n_0$. If x_n is not in $\xi(H^1)$, it follows that $x_n = y_n$.

Any x in $Y_G^s(P)$ is right-tail equivalent to a point in P so, for any integer n, there is a least $m \geq n$ such that x_m is not in $\xi(H^1)$. We note then that

$$\theta(\chi_G^+(\sigma^n(x))) = \exp\left(2\pi i \sum_{j=1}^{m-n-1} \varepsilon(x_{n+j})2^{-j}\right).$$

As a consequence of the sum being finite, if the quantity above equals one, then $\varepsilon(x_k) = 0$, for $n + 1 \leq k < m$.

For $0 \leq t < 1$, we let $\exp(2\pi i t)^{1/2} = \exp(\pi i t)$. Observe that

$$\begin{aligned} \theta(\chi_G^+(\sigma^{n-1}(x))) &= \exp(\varepsilon(x_n)\pi i)\theta(\chi_G^+(\sigma^n(x)))^{1/2} \\ &= (-1)^{\varepsilon(x_n)}\theta(\chi_G^+(\sigma^n(x)))^{1/2}. \end{aligned}$$

Suppose for some $n < n_0 - 1$, we have $x_{n+1} = y_{n+1}$. We will show $x_n = y_n$ also. There are several cases to consider. First, suppose that $x_{n+1} = y_{n+1}$ is not in $\xi(H^1)$. If x_n is also not in $\xi(H^1)$, then $x_n = y_n$. If x_n is in $\xi(H^1)$, then $\theta(\chi_G^+(\sigma^{n-1}(x))) = \exp(2\pi i\varepsilon(x_n))$ and $\theta(\chi_G^+(\sigma^{n-1}(y))) = \exp(2\pi i\varepsilon(y_n))$. The second inequality above then implies that $\varepsilon(x_n) = \varepsilon(y_n)$ and hence $x_n = y_n$ also.

Now, suppose $x_{n+1} = y_{n+1}$ is in $\xi(H^1)$. If $\varepsilon(x_{n+1}) = 0$, then both $\chi_G^+(\sigma^n(x))$ and $\chi_G^+(\sigma^n(y))$ lie in $\{\exp(2\pi i t) \mid 0 \leq t < 2^{-1}\}$. If $\varepsilon(x_{n+1}) = 1$, then both $\chi_G^+(\sigma^n(x))$ and $\chi_G^+(\sigma^n(y))$ lie in $\{\exp(2\pi i t) \mid 2^{-1} \leq t < 1\}$. In either case, we have

$$d_{\mathbb{T}}(\theta(\chi_G^+(\sigma^n(x)))^{1/2}, \theta(\chi_G^+(\sigma^n(y)))^{1/2}) = 2^{-1}d_{\mathbb{T}}(\theta(\chi_G^+(\sigma^n(x))), \theta(\chi_G^+(\sigma^n(y))))$$

and hence

$$\begin{aligned} 2^{-2-n_0+n} &> d_{\mathbb{T}}(\theta(\chi_G^+(\sigma^{n-1}(x))), \theta(\chi_G^+(\sigma^{n-1}(y)))) \\ &= d_{\mathbb{T}}(\theta(\chi_G^+(\sigma^n(x)))^{1/2}(-1)^{\varepsilon(x_n)}, \theta(\chi_G^+(\sigma^n(y)))^{1/2}(-1)^{\varepsilon(y_n)}) \\ &\geq d_{\mathbb{T}}(\theta(\chi_G^+(\sigma^n(x)))^{1/2}(-1)^{\varepsilon(x_n)}), \theta(\chi_G^+(\sigma^n(x)))^{1/2}(-1)^{\varepsilon(y_n)}) \\ &\quad -d_{\mathbb{T}}(\theta(\chi_G^+(\sigma^n(x)))^{1/2}(-1)^{\varepsilon(y_n)}, \theta(\chi_G^+(\sigma^n(y)))^{1/2}(-1)^{\varepsilon(y_n)}) \\ &= d_{\mathbb{T}}((-1)^{\varepsilon(x_n)}, (-1)^{\varepsilon(y_n)}) \\ &\quad -2^{-1} d_{\mathbb{T}}(\theta(\chi_G^+(\sigma^n(x))), \theta(\chi_G^+(\sigma^n(y)))) \\ &\geq d_{\mathbb{T}}((-1)^{\varepsilon(x_n)}, (-1)^{\varepsilon(y_n)}) - 2^{-4-n_0+n}. \end{aligned}$$

From which it follows that $\varepsilon(x_n) = \varepsilon(y_n)$ and so $x_n = y_n$.

We have shown that if $x_{n+1} = y_{n+1}$, for some $n < n_0 - 1$, then $x_n = y_n$ also. It follows by induction that we are in case 1.

It remains to consider the case $x_n \neq y_n$, which implies x_n is in $\xi(H^1)$ and $\varepsilon(x_n) \neq \varepsilon(y_n)$, for all $n < n_0$. Without loss of generality, assume $\varepsilon(x_n) = 0, \varepsilon(y_n) = 1$, for some $n < n_0$. We will show that if $\varepsilon(x_{n-1}) = 1, \varepsilon(y_{n-1}) = 0$, then $x_{n-2} = y_{n-2}$ contradicting our hypothesis. The only remaining possibility is case 2, so this will complete the proof.

Under the conditions $\varepsilon(x_n) = 0, \varepsilon(y_n) = 1, \varepsilon(x_{n-1}) = 1, \varepsilon(y_{n-1}) = 0$, we have $\theta(\chi_G^+(\sigma^{n-2}(x))$ lies in $\{\exp(2\pi i t) \mid 2^{-1} \leq t < 1 - 2^{-2}\}$ while $\theta(\chi_G^+(\sigma^{n-2}(y))$ lies in $\{\exp(2\pi i t) \mid 2^{-2} \leq t < 2^{-1}\}$. It follows that

$$d_{\mathbb{T}}(\theta(\chi_G^+(\sigma^n(x)))^{1/2}, \theta(\chi_G^+(\sigma^n(y)))^{1/2}) = 2^{-1} d_{\mathbb{T}}(\theta(\chi_G^+(\sigma^n(x))), \theta(\chi_G^+(\sigma^n(y))))$$

and the same calculation as before shows that $\varepsilon(x_{n-2}) = \varepsilon(y_{n-2})$ and hence $x_{n-2} = y_{n-2}$. □

The proof of Theorem 6.2 involves an application of the results of [25]. Let us provide a brief discussion of the set-up there. It will be fairly similar to the one we consider here, but there are a couple of differences which we need to address.

We begin with two Bratteli diagrams, $(V, E), (W, F)$ and two embeddings of the latter into the former satisfying conditions analogous to (H0) and (H1). Section 2 of [25] describes the construction of an étale equivalence relation, R, on the path space of the diagram (V, E), X_E, which contains tail equivalence, R_E, as an open subequivalence relation. It is then shown that the $K_0(C^*(R))$ is isomorphic to the dimension group of the Bratteli diagram (V, E), while $K_1(C^*(R))$ is isomorphic to that of (W, F).

The first point to note is that in our current situation will we be using (V, E) and (W, F) stationary diagrams given by the matrices G and H, respectively. That is, at least approximately, $V_n = G^0, E_n = G^1, W_n = H^0, F_n = H^1$, for all n.

The first minor annoyance is that, because we are studying unstable or left-tail equivalence, our graph G is going in the wrong direction. (As we were also looking at right-tail equivalence in the last subsection, this was kind of inevitable.) This can be repaired easily by simply looking at the opposite graphs of G and H; that is, simply reverse the maps i, t. This means that the dimension groups of (V, E) and (W, F) will be isomorphic to $D^s(G)$ and $D^s(H)$, respectively.

There is a second minor problem with the convention of [25] that the Bratteli diagram begins with V_0 and W_0 as a single vertex. This is also solved easily exactly as we did for the stable case.

The directional reversal does pose some notational problems. The simplest solution is the following. We observe the following: if y is any path in $Y^s_G(P)$, then letting $\tilde{y}_n = y_{-n}$, for $n \geq 1$, defines an infinite path in X_E. In fact, this association is a homeomorphism between $Y^s_G(P)$ and X_E which induces an isomorphism between $G^u(X_G, \sigma, Y^s_G(P))$ and R_E. Furthermore, bearing in mind that π_ξ is an homeomorphism between $Y^s_G(P)$ and its image in X_ξ, Lemma 6.11 shows that π_ξ induces a bijection between R, as described in [25], and $G^u(\xi, P)$. We will suppress this map and simply write our sequences as indexed by $n \leq 1$.

One technical issue remains: as a consequence of Lemma 6.11, we know that, under this identification, $G^u(\xi, P)$ and R agree as sets, but the former is given a topology based on the Smale space structure, while the latter was given a rather ad-hoc topology in [25]. We must check these coincide. This is the content of the following.

Lemma 6.12. *Let (x, y) be in R, as above. There exists a positive integer n and a compact, open neighbourhood U of (x, y) in R such that*

$$\pi_\xi \times \pi_\xi(U) = \{(z, \sigma^n[\sigma^{-n}(y), \sigma^{-n}(z)]) \mid z \in \pi_\xi(r(U))\}$$

is a compact, open subset of $G^u(\xi, P)$ and $\pi_\xi(r(U))$ is a compact, open subset of $Y^s_\xi(P)$.

Proof. We first consider the case that (x, y) is in R_E, meaning that x, y are themselves unstably equivalent in X_G. We can then choose $n \geq 3$ such that

$x_i = y_i$, for all $i \leq -n+3$. We define U to be the set of all pairs (z, z') such that $z_i = z'_i$, for $i \leq -n$, $z_i = x_i, z'_i = y_i$, for all $i \geq -n$. Notice that $d_G(\sigma^{-n}(y), \sigma^{-n}(z)) \leq 2^{-3}$ and $[\sigma^{-n}(y), \sigma^{-n}(z)] = \sigma^{-n}(z')$, for all such z, z'. The first desired property of U follow at once. The last statements follow from the fact that π_ξ is a homeomorphism on $Y^s_G(P)$.

The second case, is that (x, y) is not in R_E. By Lemma 6.11, we may find $n \geq 1, j = 0, 1$ and a path w in X_H such that such that $x_i = \xi^j(w_i), y_i = \xi^{1-j}(w_i)$, for all $i \leq -n+3$. As before, let U be the collection of all pairs z, z' such that $z_i = x_i, z'_i = y_i$, for all $i \geq -n$. If we let $p = x_{-n+1}, \ldots, x_0, q = y_{-n+1}, \ldots, y_0$, then the set $\delta^{j,1-j}_{p,q}$ as defined following Lemma 2.4 in [25], is precisely U and hence is a compact, open subset of R. If z, z' is in U, let z'' be the unique element of X_G with $z'' \neq z, z'' \cong_\xi z$. It follows that $d_G(\sigma^{-n}(y), \sigma^{-n}(z)) \leq 2^{-3}$ and we have

$$\begin{aligned} \pi_\xi[\sigma^{-n}(y), \sigma^{-n}(z'')] &= [\pi_\xi\sigma^{-n}(y), \pi_\xi\sigma^{-n}(z'')] \\ &= [\sigma^{-n}\pi_\xi(y), \sigma^{-n}\pi_\xi(z'')] \\ &= [\sigma^{-n}\pi_\xi(y), \sigma^{-n}\pi_\xi(z)]. \end{aligned}$$

The first part of the conclusion follows. The last properties are as before. □

The computation of the groups $K_*(U(X_\xi, \sigma_\xi, P_\xi))$ as stated in Theorem 6.2 is an immediate consequence of Theorem 1.1 of [25] and the construction given there of R.

We next turn to the claim of the maps on these groups induced by σ_ξ. First, because of the commutative diagram of Proposition 6.10, it suffices to compute the map induced by σ_ξ, which is an endomorphism of $C^*(G^u(\xi, P))$. For the K_0 group, the results of [25] actually show that the inclusion of $C^*(R_E)$ in $C^*(R)$ induces an isomorphism on K_0. The former is an AF-algebra with stationary Bratteli diagram, so the induced map is given by the matrix $(A^T_G)^{-1}$, as claimed.

For K_1, let x be an element of $Y^s_G(P)$ such that, for some $n \leq -1$, $i(x_n)$ is in $\xi^0(H^0)$. Letting $p = (x_n, \ldots x_0)$, Remark 3.6 of [25] gives an explicit description of a partial isometry, v_p, in $C^*(R)$ so that $v_p + (1 - v^*_p v_p)$ is a unitary. This gives an explicit group isomorphism between $D^s(G)$ and $K_1(C^*(R))$.

Moreover, it is an easy exercise to check that if we let $q = (x_{n-1}, \ldots, x_{-1})$, then $v_q = \sigma_\xi(v_p)$ and it follows that the isomorphism intertwines the automorphism $(A^T_G)^{-1}$ and $(\sigma_\xi)_*$.

The proofs of the last two parts of Theorem 6.2 are completely analogous to those of Theorem 6.1 and we omit the details.

7 Realizations

Our goal in this section is to provide a more concrete description of the space X_ξ^+. More specifically, we give an explicit embedding of it into $\mathbb{R}^3$. We also examine a couple of simple examples more closely.

Definition 7.1. *For $k \geq 0$, define $\zeta_k : X_k^+ \to \mathbb{C}$ inductively by setting $\zeta_0 = \theta$ on X_0^+ and*

$$\zeta_k(x) = \left(1 - 2^{1-n(x)}\right)\theta(x) + 2^{-3-n(x)}\zeta_{k-1}(\sigma^{n(x)}(x)),$$

for x in X_k^+ and $k \geq 1$.

A remark is probably in order on the factor $1-2^{1-n(x)}$: this is normalized so that, if $n(x) = 1$, then the first term is zero. Ultimately, when we extend the definition of ζ_k to a map ζ on all of X_ξ^+, this will have the effect that $\zeta(x) = 0$ exactly when x_n is not in $\xi(H^1)$, for any $n \geq 1$.

Lemma 7.2. 1. *For x in $X_k^+, k \geq 0$, we have $|\zeta_k(x)| \leq 1$.*

2. *For any x, y in $X_k^+, k \geq 0$, we have $|\zeta_k(x) - \zeta_k(y)| \leq 8d_k(x, y)$,*

3. *There is a unique continuous map $\zeta_\xi : X_G^+ \to \mathbb{C}$ such that*

   (a) *for x in $X_k^+, k \geq 0$, $\zeta_\xi(x) = \zeta_k(x)$,*

   (b) *for x, y in X_G^+,*

   $$|\zeta_\xi(x) - \zeta_\xi(y)| \leq 8d_\xi(x, y).$$

   (c) *for x in X_G^+, $|\zeta_\xi(x)| \leq 1$.*

Proof. We prove the first statement by induction. It obviously holds for $k = 0$. Let $k \geq 1$, assume the statement is true for $k - 1$ and x be in X_k^+. We have

$$\begin{aligned} |\zeta_k(x)| &\leq 1 - 2^{1-n(x)} + 2^{-3-n(x)}|\zeta_{k-1}(\sigma^{n(x)}(x))| \\ &\leq 1 - 2^{1-n(x)} + 2^{-3-n(x)} \\ &< 1. \end{aligned}$$

We prove the second part by induction on k. For $k=0$, we have

$$2\pi d_0(x,y) = 2\pi d_{\mathbb{T}}(\theta(x),\theta(y)) \geq |\theta(x)-\theta(y)| = |\zeta_0(x)-\zeta_0(y)|,$$

for all x, y in X_0^+.

We now assume the result holds for $k-1$, with $k \geq 1$. For x, y in X_k^+, we consider two cases separately. The first is that $(n(x),\theta(x)) \neq (n(y),\theta(y))$. In this case, we have have

$$d_k(x,y) = d_{G_\xi}(\tau_\xi(x),\tau_\xi(y)) + |2^{-n(x)} - 2^{-n(y)}| + d_{\mathbb{T}}(\theta(x),\theta(y)).$$

We claim that this is bounded below by 2^{-1-n}, where $n = \min\{n(x), n(y)\}$. Without loss of generality, we assume $n(x) \leq n(y)$. First consider the case $n(x) < n(y)$, where we have

$$d_k(x,y) \geq |2^{-n(x)} - 2^{-n(y)}| \geq 2^{-n} - 2^{-n-1} = 2^{-1-n}.$$

We are left to consider $n(x) = n(y)$ and $\theta(x) \neq \theta(y)$. Here, we have

$$d_k(x,y) \geq d_{\mathbb{T}}(\theta(x),\theta(y)) \geq 2^{-n},$$

since $\theta(x)$ and $\theta(y)$ are distinct 2^{n-1}-th roots of unity.

On the other hand, we have

$$\begin{aligned} |\zeta_k(x)-\zeta_k(y)| &\leq |(1-2^{1-n(x)})\theta(x) - (1-2^{1-n(y)})\theta(y)| \\ &\quad + |2^{-2-n(x)}\zeta_{k-1}(\sigma^{n(x)}) - 2^{-2-n(x)}\zeta_{k-1}(\sigma^{n(y)})|. \end{aligned}$$

For the first term, we use the triangle inequality as follows

$$\begin{aligned} & |(1-2^{1-n(x)})\theta(x) - (1-2^{1-n(y)})\theta(y)| \\ \leq\ & |(1-2^{1-n(x)})\theta(x) - (1-2^{1-n(y)})\theta(x)| \\ & + |(1-2^{1-n(y)})\theta(x) - (1-2^{1-n(y)})\theta(y)| \\ \leq\ & |2^{1-n(x)} - 2^{1-n(y)}| \\ & + (1-2^{1-n(y)})|\theta(x)-\theta(y)| \\ \leq\ & 2|2^{-n(x)} - 2^{-n(y)}| + |\theta(x)-\theta(y)| \\ \leq\ & 2|2^{-n(x)} - 2^{-n(y)}| + 2\pi d_{\mathbb{T}}(\theta(x),\theta(y)) \\ \leq\ & 2\pi d_k(x,y). \end{aligned}$$

For the second term, as ζ_{k-1} is in the unit disc, this is bounded by

$$2^{-2-n(x)} + 2^{-2-n(y)} \leq 2^{-1-n} \leq d_k(x,y)$$

from our claim above. The conclusion follows as $2\pi + 1 \leq 8$.

It remains for us to consider the case $(n(x), \theta(x)) = (n(y), \theta(y))$. Here, we make use of the induction hypothesis in estimating

$$\begin{aligned}
|\zeta_k(x) - \zeta_k(y)| &= |2^{-3-n(x)}\zeta_{k-1}(\sigma^{n(x)}(x)) - 2^{-3-n(y)}\zeta_{k-1}(\sigma^{n(y)}(x))| \\
&= 2^{-3-n(x)}|\zeta_{k-1}(\sigma^{n(x)}(x)) - \zeta_{k-1}(\sigma^{n(y)}(x))| \\
&\leq 2^{-3-n(x)} 8 d_{k-1}(\sigma^{n(x)}(x)), \sigma^{n(y)}(y))) \\
&= 2^{-n(x)} \left[d_{G_\xi}(\sigma^{n(x)}(x)), \sigma^{n(y)}(y)) \right. \\
&\quad \left. + \lambda_{k-1}(\sigma^{n(x)}(x)), \sigma^{n(y)}(y)) \right] \\
&= \cdot 2^{-n(x)} d_{G_\xi}(\sigma^{n(x)}(x)), \sigma^{n(y)}(y)) + 2^2 \lambda_k(x, y) \\
&\leq \cdot d_{G_\xi}(x, y) + 2^2 \lambda_k(x, y) \\
&\leq 8 d_k(x, y).
\end{aligned}$$

The third part is an immediate consequence of the first two and the definition of d_ξ. ☐

Lemma 7.3. *Let x, y be in X_G^+ with $\kappa(x) > 0$.*

1. *We have $(1 - 2^{1-n(x)})\theta(x) - \zeta_\xi(x) = 2^{-3-n(x)}\zeta_\xi(\sigma^{n(x)}(x))$.*
2. *If $\zeta_\xi(x) = \zeta_\xi(y)$ and $\tau_\xi(x_m) = \tau_\xi(y_m)$, for all $1 \leq m \leq n(x)$, then $x_m = y_m$, for all $1 \leq m \leq n(x)$.*

Proof. Choose sequences x^l, y^l in X_l^+, $l \geq 1$ converging to x and y respectively. For l sufficiently large, we have $x_m^l = x_m, 1 \leq m \leq n(x)$ and $n(x^l) = n(x)$, $\theta(x^l) = \theta(x)$, $\sigma^{n(x^l)}(x^l)$ is in X_{l-1}^+ and converges to $\sigma^{n(x)}(x)$. It follows that

$$\begin{aligned}
\zeta_\xi(\sigma^{n(x)}(x)) &= \lim_{l\to\infty} \zeta_{l-1}(\sigma^{n(x^l)}(x^l)) \\
&= \lim_{l\to\infty} 2^{3+n(x^l)} \left((1 - 2^{1-n(x^l)})\theta(x^l) - \zeta_l(x^l) \right) \\
&= 2^{3+n(x^l)} \left((1 - 2^{1-n(x)})\theta(x) - \zeta_\xi(x) \right)
\end{aligned}$$

which proves the first part.

For the second part, the second hypothesis implies that $n(y) = n(x) = n$. Moreover, if $n = 1$, then x_1, y_1 are not in $\xi(H^1)$ and we are done. It remains to consider the case $n > 1$.

We will first show that $\sin(\pi 2^{-n}) - 2^{-2-n} > 0$ for $n \geq 1$, by induction. For $n = 1$, we have

$$\sin(\pi 2^{-n}) - 2^{-2-n} = 1 - 2^{-2} > 0.$$

Now assume this is positive for some $n \geq 1$. Making use of the formula $\sin(t) = 2\sin(t/2)\cos(t/2)$, we have

$$\begin{aligned}
\sin(\pi 2^{-(n+1)}) - 2^{-2-(n+1)} &= \sin(\pi 2^{-n-1}) - 2^{-3-n} \\
&= 2^{-1}\left(\cos(\pi 2^{-n-1})\right)^{-1}\sin(\pi 2^{-n}) - 2^{-3-n} \\
&\geq 2^{-1}\left(\sin(\pi 2^{-n}) - 2^{-2-n}\right) \\
&> 0
\end{aligned}$$

by the induction hypothesis.

We may choose l_0 sufficiently large so that, for $l \geq l_0$, we have

$$\begin{aligned}
|\zeta_\xi(x) - \zeta_l(x^l)| &< 2^{-1}\left(\sin(\pi 2^{-n}) - 2^{-2-n}\right), \\
|\zeta_\xi(y) - \zeta_l(y^l)| &< 2^{-1}\left(\sin(\pi 2^{-n}) - 2^{-2-n}\right), \\
x^l_m &= x_m, \\
y^l_m &= y_m,
\end{aligned}$$

for all $1 \leq m \leq n$. It follows that, for such l,

$$\begin{aligned}
(1 - 2^{1-n})|\theta(x) - \theta(y)| &= (1 - 2^{1-n})|\theta(x^l) - \theta(y^l)| \\
&\leq |\zeta_l(x^l) - \zeta_l(y^l)| + 2^{-2-n} \\
&\leq |\zeta_l(x^l) - \zeta_\xi(x)| \\
&\quad + |\zeta_\xi(y) - \zeta_l(y^l)| + 2^{-2-n} \\
&< \sin(\pi 2^{-n}).
\end{aligned}$$

So we have

$$|\theta(x) - \theta(y)| < (1 - 2^{1-n})^{-1}\sin(\pi 2^{-n}) \leq 2\sin(\pi 2^{-n}).$$

Both $\theta(x)$ and $\theta(y)$ are among the 2^{n-1}-th roots of unity and it is an elementary exercise that the minimum distance between any two is $2\sin(\pi 2^{-n})$. Hence, we see $\theta(x) = \theta(y)$, which implies that $\varepsilon(x_m) = \varepsilon(y_m)$, for all $1 \leq m < n$. Together with the fact that $\tau_\xi(x_m) = \tau_\xi(y_m)$, for $1 \leq m \leq n$ implies $x_m = y_m$, for $1 \leq m \leq n$, as claimed. □

Theorem 7.4. *Under the standing hypotheses, the map from X_ξ^+ to $X_{G_\xi}^+ \times \mathbb{C}$ sending $\pi_\xi(x), x \in X_G^+$, to the pair $(\tau_\xi(x), \zeta_\xi(x))$ is well-defined, continuous and injective.*

Proof. The facts that the map is well-defined and continuous follows immediately from Theorem 3.7 and part 2 of Lemma 7.2. It remains for us to prove that it is injective. It suffices to show that, for any x, y in X_G^+, if $\tau_\xi(x) = \tau_\xi(y)$ and $\zeta_\xi(x) = \zeta_\xi(y)$, then $x \sim_\xi y$. The first hypothesis implies that $\kappa(x) = \kappa(y)$.

If $\kappa(x) = \kappa(y) = 0$, then x, y are in X_0^+, so by definition, $\zeta_\xi(x) = \theta(x), \zeta_\xi(y) = \theta(y)$ and hence $d_0(x, y) = 0$ so $x \sim_\xi y$.

We now assume $\kappa(x) = \kappa(y) > 0$. It is an immediate consequence of part 2 of Lemma 7.3 that $x_m = y_m, 1 \leq m \leq n(x) = n(y)$. It is then obvious that $\tau_\xi(\sigma^{n(x)}(x)) = \sigma^{n(x)}(\tau_\xi(x)) = \sigma^{n(x)}(\tau_\xi(y)) = \tau_\xi(\sigma^{n(x)}(y))$. We also note that $\kappa(\sigma^{n(x)}(x)) = \kappa(x) - 1 = \kappa(y) - 1 = \kappa(\sigma^{n(y)}(y))$. In addition, $x_m = y_m, 1 \leq m \leq n(x) = n(y)$ implies that $\theta(x) = \theta(y)$ and hence, from part 1 of Lemma 7.3 that $\zeta_\xi(\sigma^{n(x)}(x)) = \zeta_\xi(\sigma^{n(y)}(y))$. If $\kappa(x)$ is finite, we may repeat this argument to find a finite n such that $x_m = y_m$ for $1 \leq m \leq n$ and $\kappa(\sigma^n(x)) = \kappa(\sigma^n(y)) = 0$. It follows that $x \sim_\xi y$. If $\kappa(x)$ is infinite, we may repeat this argument to show that that $x_m = y_m$ for all $1 \leq m$, so $x = y$. □

Under our hypotheses, X_{G_ξ} is homeomorphic to the Cantor ternary set, so we conclude the following.

Corollary 7.5. *Under the standing hypotheses, the space X_ξ^+ can be embedded in $\mathbb{R} \times \mathbb{C}$.*

We also observe the following, which already appeared in [12]. We provide a different proof here which employs our metric.

Corollary 7.6. *Recall that we have factor maps*

$$(X_G^+, \sigma_G) \xrightarrow{\pi_\xi} (X_\xi^+, \sigma_\xi) \xrightarrow{\rho_\xi} (X_{G_\xi}^+, \sigma_{G_\xi})$$

and $\rho_\xi \circ \pi_\xi = \tau_\xi$.

Under the standing hypotheses, for each x in X_G^+, $\rho_\xi^{-1}\{\tau_\xi(x)\}$ is

1. *a finite collection of pairwise disjoint circles if $\kappa(x)$ is finite,*

2. *2^m points if $m = \#\{n \mid x_n \in \xi(H^1)\}$ is finite,*

3. *totally disconnected if $\kappa(x)$ is infinite.*

Proof. If we apply the map of Theorem 7.4 from X_ξ^+ to $X_{G_\xi} \times \mathbb{C}$ and restrict it to $\rho_\xi^{-1}\{\tau_\xi(x)\}$, it is a homeomorphism to its image, which is $\{\tau_\xi(x)\} \times \zeta_\xi(\tau_\xi^{-1}\{\tau_\xi(x)\})$. Hence, it suffices for us to prove $\zeta_\xi(\tau_\xi^{-1}\{\tau_\xi(x)\})$ is as described, in each case.

For x in X_G^+, define $I(x)$ to be the set of positive integers such that x_n is not in $\xi(H^1)$ while $t(x_n)$ is in $\xi^0(H^0)$. For any n in $I(x)$, we define $P(x,n)$ to be the set of all paths $p = (p_1, \ldots, p_n)$ in G such that $\tau_\xi(p_m) = \tau_\xi(x_m)$, for all $1 \leq m \leq n$, which is obviously a finite set. For such a path p, let $C(p,x)$ denote the set of y in X_G^+ such that $y_m = p_m$, for $1 \leq m \leq n$ and $\tau_\xi(y_m) = \tau_\xi(x_m)$, for $m > n$.

If $\kappa(x) = 0$, then it is a simple matter to check that

$$\zeta_0(\tau_\xi^{-1}(\{\tau_\xi(x)\}) = \mathbb{T}.$$

If $0 < \kappa(x) < \infty$, then $I(x)$ is finite. Let n be its maximum element. It is clear that $\tau_\xi^{-1}\{\tau_\xi(x)\} = \cup_{p \in P(x,n)} C(p,x)$. Again, an easy computation shows that, for each p, $\zeta_\xi(C(p))$ is a circle (of radius 2^{-3-n}) and several applications of part 2 of Lemma 7.3 proves that they are pairwise disjoint, for different values of p. This proves the first statement.

In the second case, we have $\tau_\xi^{-1}\{\tau_\xi(x)\}$ is finite, so its image under ζ_ξ is also.

For the third, it suffices to consider the case when x_n is in $\xi(H^1)$, for infinitely many n and not in $\xi(H^1)$ for infinitely many n. In this case, $I(x)$ is infinite.

Suppose $\tau_\xi(y) = \tau_\xi(z) = \tau_\xi(x)$ and $\zeta_\xi(y) \neq \zeta_\xi(z)$. This implies that $y \neq z$. It follows that we may find n in $I(x)$ and $1 \leq m \leq n$ such that $y_m \neq z_m$. As before, the collection $\zeta_\xi(C(p,x)), p \in P(x,n)$, is a finite number of pairwise disjoint circles. Moreover, from part 1 of Lemma 7.3, $\zeta_\xi(\tau_\xi^{-1}\{\tau_\xi(x)\})$ is contained in these circles together with their interior discs. These form a partition of $\zeta_\xi(\tau_\xi^{-1}\{\tau_\xi(x)\})$ into pairwise disjoint closed, and hence also open, sets. Moreover, from our choice on n, $\zeta_\xi(y)$ and $\zeta_\xi(z)$ lie in distinct elements. This completes the proof. □

Corollary 7.7. *Under the standing hypotheses, the connected subsets of X_ξ^+ are either points or circles and both occur.*

Proof. If C is a connected subset of X_ξ then $\rho_\xi(C)$ is a connected subset of X_{G_ξ} and hence is a single point, say x. So C is a subset of $\rho_\xi^{-1}\{x\}$ and the conclusion follows from Corollary 7.6. The fact that both circles and single points occur follows from the fact easy fact that both cases 1 and 2 occur in 7.6. □

We finish this section by looking at a couple of specific examples. The first is instructive even if it does not satisfy hypothesis (H2).

Example 7.8. *Suppose $G^0 = H^0$ contains a single vertex, H^1 contains a single edge and G^1 contains exactly two edges. There is essentially only one choice for ξ. The space X_G^+ may be identified with $\{0,1\}^{\mathbb{N}}$ in an obvious way. Then X_ξ is the unit circle and π_ξ is binary expansion.*

Rather more generally (as noted in [12]), if $G^0 = H^0$ and $G^1 = H^1 \times \{0,1\}$ with ξ^i being the identity map on H^0 and $\xi^i(x) = (x,i)$, for x in H^1, then X_ξ^+ is homeomorphic to $X_H^+ \times \mathbb{T}$. This can be seen from the first part of Theorem 3.5 and the fact that X_k^+ is empty, for $k \geq 1$.

Example 7.9. *Suppose $G^0 = H^0$ contains a single vertex, H^1 contains a single edge and G^1 contains exactly three edges. There is essentially only one choice for ξ. The space X_G^+ may be identified with $\{0,1,2\}^{\mathbb{N}}$ in an obvious way. Consider the set $A = \{x \in X_1^+ \mid n(x) \leq 4\}$. The following is a picture of $\zeta_0(X_0^+) \cup \zeta_1(A)$:*

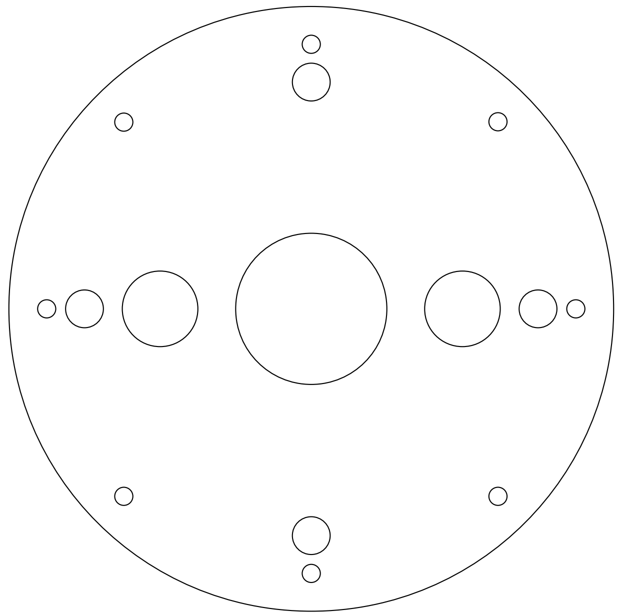

Figure 1

Hopefully, the reader can see how to sketch all of $\zeta_1(X_1^+)$. To get an idea of $\zeta_2(X_2^+)$, we suggest the reader verifies the following easy result:

$$\bigcup_{j=0}^{k} \zeta_\xi(X_j^+) = \bigcup \quad c + r\zeta_\xi(X_0^+ \cup X_1^+),$$

where the union is over all c in $\mathbb{C}$ and positive real numbers r such that $c + r\mathbb{T} \subseteq \cup_{j=0}^{k-1}\zeta_\xi(X_j^+)$. That is, for every k, the space $\cup_{j=0}^{k}\zeta_\xi(X_k^+)$ is a union of circles. To get the next one, one replaces each circle, $c + r\mathbb{T}$ in the current one, by $c + r\zeta_\xi(X_0^+ \cup X_1^+)$.

Finally, $\zeta_\xi(X_\xi^+)$ is the closure of the union of $\zeta_\xi(X_k^+)$, over all $k \geq 0$.

We leave it as an exercise to check that, in this case, ζ_ξ alone is injective so X_ξ^+ can be embedded in the plane.

It appears that this set is an example of a fractal with condensation set (see pages 91-94 of [3]). Probably a caution is in order: the embedding to the plane we have given depends on several parameters which were chosen rather arbitrarily. The underlying iterated function system would be affected by these choices and perhaps in a rather bad way.

Question 7.10. *In which cases can X_ξ^+ be embedded in the plane?*

References

[1] Adler, Roy, *Symbolic Dynamics and Markov Partitions*, Bulletin of the Amer. Math. Soc. **35**, No.1, 1998, 1–56.

[2] Adler, Roy and Weiss, Benjamin, *Similarity of Automorphisms of the Torus*, Memoirs Amer. Math. Soc. **98**, 1970.

[3] Michael F. Barnsly, *Fractals Everywhere*, Academic Press, 2nd ed., 1993.

[4] Bowen, Rufus, *Markov partitions for Axiom A diffeomorphisms* Amer. J. Math. **3**, 1970, 725–747.

[5] Bowen, Rufus, *On Axiom A diffeomorphisms*, CBMS Reg. Conf. **71**, Providence, 1978.

[6] Bowen, Rufus and Franks, John, *Homology for zero-dimensional non-wandering sets*, Ann. of Math. **106** No. 1, 1977, 73-92.,

[7] Cuntz, Joachim and Krieger, Wolfgang, *A class of C^*-algebras and topological Markov chains* Invent. Math. **56**, 1980, 251-268.

[8] Deaconu, Valentin *Groupoids associated with endomorphisms*, Trans. Amer. Math. Soc. **347** 1995, 1779–1786.

[9] Deeley, Robin and and Strung, Karen R., *Nuclear dimension and classification of C^*-algebras associated to Smale spaces*, Trans. Amer. Math. Soc. **370** No. 5, 2018, 3467–3485.

[10] Deeley, Robin J, Goffeng, Magnus and Yashinski, Alan, *Smale space C^*-algebras have nonzero projections*, Proceedings Amer. Math. Soc. **148** No. 4, 2020, 1625–1639.

[11] Haslehurst, Mitch, *Relative K-theory for C^*-algebras*, Rocky Mountain J. Math., to appear, https://arxiv.org/pdf/2106.02620.pdf

[12] Haslehurst, Mitch, *Some examples of factor groupoids*, arXiv:2204.13888v2.

[13] Hungerford, Thomas W., *Algebra*, Graduate Texts in Mathematics **73**, Springer, New York, 1974.

[14] Kaminker, Jerome and Putnam, Ian F. and Whittaker, Michael F., *K-theoretic duality for hyperbolic dynamical systems*, J. fur die Reine und Ange. Math. **730**, 2017, 257-288.

[15] Katok, Anatole and Hasselblatt, Boris, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995.

[16] Kelley, John L., *General Topology*, Graduate Texts in Mathematics **27**, Springer, Berlin Heidelberg New York, 1955.

[17] Lind, Douglas and Marcus, Brian H., *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press. 1995.

[18] Matui, Hiroki, *Long exact sequences of homology groups of étale groupoids*, Discrete and Continuous Dynamical Systems, http://dx.doi.org/10.3934/dcds.2022095

[19] Muhly, Paul S. and Renault, Jean N. and Williams, Dana P., *Equivalence and isomorphism for groupoid C^*-algebras* J. Operator Theory **17**, 1987, 3-22.

[20] Proietti, Valerio and Yamashita, Makoto *Homology and K-theory of dynamical systems. II. Smale spaces with totally disconnected transversal*, arXiv:2104.10938

[21] Proietti, Valerio and Yamashita, Makoto *Homology and K-theory of dynamical systems. III. Beyond totally disconnected case*, arXiv:2207.03118

[22] Putnam, Ian F., *C^*-algebras from Smale spaces*, Canad. J. Math. **48**, 1996, 175-195.

[23] Putnam, Ian F. and Spielberg, Jack S., *The structure of C^*-algebras associated with hyperbolic dynamical systems* J. Funct. Anal. **163**, 1999, 279-299.

[24] Putnam, Ian F., *A homology theory for Smale spaces*, Memoirs Amer. Math. Soc. **232**, 2014.

[25] Putnam, Ian F., *Some classifiable groupoid C^*-algebras with prescribed K-theory* Mathematische Annalen **370** No. 3-4, 2018, 1361-1387.

[26] Putnam, Ian F., *An excision theorem for the K-theory of C^*-algebras, with applications to groupoid C^*-algebras*, Münster Journal of Mathematics **14**, 2021, 349-402.

[27] Renault, Jean *A groupoid approach to C^*-algebras* Lecture Notes in Mathematics **793**, Springer, Berlin, 1980.

[28] Ruelle, David, *Thermodynamic formalism*, Cambridge University Press, Cambridge, 2004.

[29] Ruelle, David, *Noncommutative Algebras for Hyperbolic Diffeomorphisms* Invent. Math. **93**, 1988, 1-13.

[30] Rørdam, M., Larsen, F., Laustsen, N.J., *An Introduction to K-theory for C^*-algebras*, London Mathematical Society Student Texts vol. **49**, Cambridge University Press, Cambridge-New York-Melbourne, 2000.

[31] Sinai, Yakov, *Markov partitions and C-diffeomorphisms*, Func. Anal. and its Appl. **2**, 1968,64-89.

[32] Smale, Stephen, *Differentiable dynamical systems*, Bulletin Amer. Math. Soc. **73**, 1967, 747-817.