K-theory and Asymptotic Index for Certain Almost One-To-One Factors

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§1. Introduction

Recently, there has been interest in using C^*-algebraic tools, especially K-theory, in the study of minimal topological dynamical systems. See, for example, [BH1, F, GPS, GW, HPS]. This paper is an attempt to apply some of these ideas to the question of factor maps between Cantor minimal systems.

Recall [HPS] that (X, ϕ) is a Cantor minimal system if (X, d) is a compact, totally disconnected metric space with no isolated points and ϕ is a homeomorphism of X such that every ϕ-orbit is dense in X or, equivalently, the only closed ϕ-invariant sets are X and the empty set.

For such an (X, ϕ), recall from [HPS] that we define an ordered abelian group with order unit, denoted $K^0(X, \phi)$, as follows. First, we let $C(X, \mathbb{Z})$ denote the continuous integer-valued functions on X. Regard $C(X, \mathbb{Z})$ as an abelian group with point-wise addition. For a set $E \subseteq X$, we let χ_E denote its characteristic function. If E is both closed and open – we use the term clopen – χ_E is continuous. In $C(X, \mathbb{Z})$, the set

$$\{f - f \circ \phi \mid f \in C(X, \mathbb{Z})\}$$

is a subgroup – called the coboundaries – and we let $K^0(X, \phi)$ be the corresponding quotient group. For f in $C(X, \mathbb{Z})$, we let $[f]$ denote its coset in $K^0(X, \phi)$. We also define a positive cone

$$K^0(X, \phi)^+ = \{[f] \mid f \geq 0, \quad f \in C(X, \mathbb{Z})\}$$

and order unit $[1]$.

In [HPS], it is shown that $K^0(X, \phi)$ is a simple dimension group and every such group arises in this way.

This group also possesses the following property, demonstrated in [GPS]. Given $0 < a < b < [1]$ in $K^0(X, \phi)$, then there are clopen set $\phi \neq E \underset{\neq}{\subset} F \underset{\neq}{\subset} X$ such that $a = [\chi_E]$, $b = [\chi_F]$.

Given a dimension group, G, with order unit u, a trace or state ω on G is a group homomorphism $\omega : G \to \mathbb{R}$ such that $\omega(G^+) \subseteq [0, \infty)$ and $\omega(u) = 1$. We let $S(G)$ denote the set of states on G which is a convex metric space [GH]. There is a natural positive group homomorphism from G into $\mathrm{Aff}\,(S(G))$, the continuous affine real-valued functions on $S(G)$ by $\hat{g}(\omega) = \omega(g)$. In the case $G = K^0(X,\phi)$, where (X,ϕ) is a minimal Cantor system, each ϕ-invariant probability measure, μ, on X defines a state $\tilde{\mu}$ on G by

$$\tilde{\mu}[f] = \int f\, d\mu, \qquad f \in C(X,\mathbb{Z}).$$

In [HPS], it is shown that this map $\mu \to \tilde{\mu}$ is an affine isomorphism between the space of ϕ-invariant probabilities measure on X and $S\left(K^0(X,\phi)\right)$.

Given two such systems (X,ϕ) and (Y,ψ), we say that a continuous map $\pi : X \to Y$ is a factor from (X,ϕ) to (Y,ψ) if $\pi \circ \phi = \psi \circ \pi$. We also say (X,ϕ) is an extension of (Y,ψ). We usually write $\pi : (X,\phi) \to (Y,\phi)$ for convenience.

If $\pi : (X,\phi) \to (Y,\psi)$ is a factor between Cantor minimal systems then $\pi^*[f] = [f \circ \pi]$, f in $C(Y,\mathbb{Z})$ defines a positive, order-unit preserving group homomorphism $\pi^* : K^0(Y,\psi) \to K^0(X,\phi)$. It is shown in [GW] that π^* is injective. In fact, π^* is an order embedding; *i.e.* $\pi^*[f] \geq 0$ if and only if $[f] \geq 0$.

This follows from the discussion above and the fact that, for any ψ-invariant probability measure μ on Y, there is a ϕ-invariant measure ν on X such that $\mu = \nu \circ \pi^{-1}$. (See 3.11 of [DGS].)

If $i : H \to G$ is an order embedding between simple dimension groups then i induces $i^* : S(G) \to S(H)$ by $i^*(\omega) = \omega \circ i$, for ω in $S(G)$. In this situation i^* is always surjective. We also say that $i(H)$ is order-dense in G if, given $g_1 < g_2$ in G, there is h in H so that $g_1 < i(h) < g_2$.

Proposition 1.1. *Let $i : H \to G$ be an order embedding of simple dimension groups. Then $i(H)$ is order dense in G if and only if $i^* : S(G) \to S(H)$ is one-to-one. Moreover, if $G/i(H)$ is cyclic, then both these conditions are satisfied.*

Proof. First, we suppose i^* is not injective; *i.e.* $i^*(\omega_1) = i^*(\omega_2)$ for some $\omega_1 \neq \omega_2$ in $S(G)$. Then $\omega_1 - \omega_2 : G \to \mathbb{R}$ is a group homomorphism and $\omega_1 - \omega_2 \mid i(H) = 0$. Since $\omega_1 \neq \omega_2$, there is a g in G such that $\omega_1(g) - \omega_2(g) \neq 0$. For each r in $\mathbb{R}$, $r\,\hat{g}$ is an affine function on $S(G)$. Then

$$\begin{aligned}\mathrm{Aff}\, S(G)(\omega_1 - \omega_2) &\supseteq \{r\hat{g}(\omega_1 - \omega_2) \mid r \in \mathbb{R}\} \\ &= \mathbb{R}.\end{aligned}$$

Now by 4.10 of [GH], the image of G under $\hat{}$ is dense in $\mathrm{Aff}(S(G))$ and it follows that $(\omega_1 - \omega_2)(G)$ is dense in $\mathbb{R}$.

It then follows that $G/i(H)$ cannot be cyclic. Also, to see that $i(H)$ is not dense in G, take g in G such that $(\omega_2 - \omega_1)(g) > 2$. If h in H is such that $g < i(h) < g + u$, where u is the order unit of G, then

$$\begin{aligned}\omega_1 \circ i(h) &\leq \omega_1(g+u)\\ &= \omega_1(g) + 1\\ &< \omega_2(g) - 2 + 1\\ &\leq \omega_2 \circ i(h) - 1,\end{aligned}$$

which would contradict $i^*(\omega_1) = i^*(\omega_2)$. We conclude that $i(H)$ is not order dense in G.

For the converse, if i^* is one-to-one then it is a homeomorphism between $S(H)$ and $S(G)$ and induces an isomorphism between $\mathrm{Aff}(S(H))$ and $\mathrm{Aff}(S(G))$. As $\hat{H}$ is dense in the former, $i(\hat{H})$ is dense in the latter and thus in $\hat{G}$ as well. The conclusion follows. ∎

In the present paper, we concentrate our attention on a special class of factor maps.

Definition 1.2. In the system (X, ϕ), two points x, x' are *asymptotic* if

$$\lim_{n\to\pm\infty} d(\phi^n(x),\ \phi^n(x')) = 0.$$

Definition 1.3. Let $\pi : (X, \phi) \to (Y, \psi)$ be a factor map. We say π is *asymptotic* if, for any x, x' in X with $\pi(x) = \pi(x')$, x and x' are asymptotic.

Definition 1.4. Let $\pi : (X, \phi) \to (Y, \psi)$ be a factor map.

(i) We say π satisfies A1 if π is one-to-one or two-to-one on X; *i.e.*

$$|\pi^{-1}\{y\}| \leq 2$$

for all y in Y.

(ii) We say π satisfies A2 if for all $\epsilon > 0$,

$$\{(x, x') \mid x, x' \in X, \quad d(x, x') \geq \epsilon, \quad \pi(x) = \pi(x')\}$$

is finite.

It is worthwhile to note that condition A2 (and for that matter, the notion of asymptotic points) can be described without using a metric. Condition A2 becomes: for every open set $U \subseteq X \times X$ containing $\Delta = \{(x, x) \mid x \in X\}$,

$$\{(x, x') \mid x, x' \in X, \quad (x, x') \notin U, \quad \pi(x) = \pi(x')\}$$

is finite. The equivalence of the two definitions is an easy consequence of the compactness of X. We omit the details.

It is easy to see that if π satisfies $A2$, then it is asymptotic. It also follows from A1 and A2 that

$$X_2 = \{x \in X \mid \pi(x) = \pi(x') \quad \text{for some } x' \neq x\}$$

is countable and ϕ-invariant.

For an example of a factor satisfying A1 and A2, consider two Denjoy homeomorphisms (as described in [PSS]) restricted to their minimal Cantor sets. If (Y, ψ) is constructed from an irrational rotation by "cutting the circle" at some set $Q \subseteq S^1$ and (X, ϕ) from the same irrational rotation "cutting the circle" at $Q' \supseteq Q$ then there is an obvious factor $\pi : (X, \phi) \to (Y, \psi)$ satisfying A1 and A2.

For a subset $E \subseteq X$, we let

$$D_E = \{(x, x') \in X \times X \mid \pi(x) = \pi(x'), \quad x \in E, \quad x' \notin E\}.$$

Lemma 1.5. *Let $\pi : (X, \phi) \to (Y, \psi)$ be a factor map satisfying* A1. *Suppose $\mathcal{F}$ is a collection of open sets in X which generate its topology. Suppose that for all F in $\mathcal{F}$, D_F is finite. Then π satisfies* A2.

Proof. It is easy to check that, for any sets E, F,

$$D_{E \cap F} \subseteq D_E \cup D_F$$

$$D_{E \cup F} \subseteq D_E \cup D_F .$$

Therefore, D_F is finite for all F in $\tilde{\mathcal{F}}$, the algebra generated by $\mathcal{F}$. These sets form a base for the topology. Let U be a neighbourhood of Δ in $X \times X$. By a standard compactness argument, we may find $F_1, F_2, \cdots, F_n$ in $\tilde{\mathcal{F}}$ such that

$$U \supseteq \bigcup_i F_i \times F_i \supseteq \Delta.$$

Then, we have

$$\{(x, x') \in X \times X \mid \pi(x) = \pi(x') \quad (x, x') \notin U\}$$

$$\subseteq \bigcap_i \{(x, x') \in X \times X \mid \pi(x) = \pi(x') \quad (x, x') \notin F_i \times F_i\}$$

We claim this set is finite. Suppose (x, x') is in this intersection. Since $\Delta \subseteq \cup F_i \times F_i$, x lies in some F_j. For that j, $(x, x') \notin F_j \times F_j$ then implies x' is not in F_j. Thus (x, x') is in D_{F_j}. We have just shown

$$\bigcap_i \{(x, x') \in X \times X \mid \pi(x) = \pi(x'), \quad (x, x') \notin F_i \times F_i\} \subseteq \bigcup_j D_{F_j}$$

which is finite. ∎

The aim of the paper is two-fold. First, we show that factors $\pi : (X, \phi) \to (Y, \psi)$ between Cantor minimal systems satisfying A1 and A2 have certain K-theoretic properties. Specifically, $\pi^* \left(K^0(Y, \psi)\right)$ is order-dense in $K^0(X, \phi)$ and the quotient of the latter by the former is free abelian. In fact, the quotient is $\mathbb{Z}^K$, where $2K$ is the number of distinct orbits in X_2. We henceforth adopt the notation, in the case $K = \infty$, that $\mathbb{Z}^\infty$ denotes the free abelian group on countably infinitely many generators. We also mean $k = 1, 2, 3, \cdots$ when we write $k = 1, \cdots, K$ in the case $K = \infty$.

Secondly we consider the situation of being given a Cantor minimal system (Y, ψ), simple dimension group G and order embedding

$$i : K^0(Y, \psi) \to G$$

with order-dense image such that $G/i\left(K^0(Y, \psi)\right)$ is torsion free. We show that such a situation is realized by an extension of (Y, ψ) satisfying A1.

The main tool is a notion of "asymptotic index". Given asymptotic points x_0, x_1 in (X, ϕ), we define a natural group homomorphism from $K^0(X, \phi)$ to $\mathbb{Z}$. This is done in Section 2. Section 3 and 4 are devoted to the two main results (3.1 and 4.1) mentioned above.

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§2. Asymptotic Index

Let (X, d) be any compact, totally disconnected metric space and let ϕ be any homeomorphism of X. We do not assume minimality here.

Lemma 2.1. *Let x_0, x_1 be asymptotic points in (X, ϕ) and suppose $E \subseteq X$ is clopen. Then*

$$\chi_E(\phi^n(x_0)) - \chi_E(\phi^n(x_1))$$

is zero for all but finitely many n in $\mathbb{Z}$.

Proof. Since E is compact and open, there is an $\epsilon > 0$ such that

$$B_\epsilon(E) = \{y \in X \mid d(y, x) < \epsilon, \quad \text{for some } x \text{ in } E\}$$

is exactly E. For all but finitely many n, $d(\phi^n(X_0), \phi^n(x_1)) < \epsilon$ and so, for such n, $\phi^n(x_0)$ and $\phi^n(x_1)$ are either both in E or in $X - E$. In this case, the number we are considering is zero. ▮

Lemma 2.2. *Let x_0, x_1 be asymptotic points in (X, ϕ) and let f be in $C(X, \mathbb{Z})$. Then,*

$$\partial(f) = \partial_{x_0, x_1}(f) = \sum_{n \in \mathbb{Z}} f(\phi^n(x_0)) - f(\phi^n(x_1))$$

is a well-defined integer.

Moreover, we have

(i) $\partial(f + g) = \partial(f) + \partial(g)$

(ii) $\partial(f - f \circ \phi) = 0.$

Proof. The first statement follows at once from Lemma 2.1. The last two parts are left as easy exercises for the reader. ▮

Definition 2.3. For asymptotic points x_0 and x_1 in X, we define the *asymptotic index*

$$\partial = \partial_{x_0, x_1} : K^0(X, \phi) \to \mathbb{Z}$$

by

$$\partial_{x_0,x_1}[f] = \partial_{x_0,x_1}(f)$$

as in 2.3.

Notice that condition (ii) of 2.2 insures that ∂ is well-defined. We quickly note the following.

(1) ∂ is <u>not</u> a positive map. Also, $\partial[1] = 0$.

(2) $\partial_{x_1,x_0} = -\partial_{x_0,x_1}$ and $\partial_{x_0,x_0} = 0$.

Remark 2.4. This index is a special case of a more general construction given [Put]. We will comment on this again in Section 3.

§3. K-Theory of Factors

Theorem 3.1. *Let $\pi : (X, \phi) \to (Y, \psi)$ be a factor map between two Cantor minimal systems satisfying* A1 *and* A2.

(i) *Then $\pi^*\left(K^0(Y, \psi)\right)$ is order dense in $K^0(X, \phi)$.*

(ii) *Suppose that for $k = 1, \cdots, K$, we have pairs of asymptotic points $\left(x_0^{(k)}, x_1^{(k)}\right)$ so that*

$$x_0^{(k)} \neq x_1^{(k)}$$

$$\pi\left(x_0^{(k)}\right) = \pi\left(x_1^{(k)}\right)$$

and $\left\{\pi\left(x_0^{(k)}\right) \mid 1 \le k \le K\right\}$ contains exactly one point from each ψ-orbit of $\pi(X_2)$. Define

$$\partial : K^0(X, \phi) \longrightarrow \bigoplus_1^K \mathbb{Z}$$

by

$$\partial = \bigoplus_{K} \partial_{x_0^{(k)}, x_1^{(k)}}$$

then the following sequence is exact.

$$0 \longrightarrow K^0(Y,\psi) \xrightarrow{\pi^*} K^0(X,\phi) \xrightarrow{\partial} \mathbb{Z}^K \longrightarrow 0.$$

Proof. **(i)**. As we noted in Section 1, conditions A1 and A2 imply that π is one-to-one, except on a countable set of points. Any ϕ-invariant probability measure on X cannot have atoms and must be zero on any countable set. Therefore the map induced by π from the ϕ-invariant probability measures on X to the ψ-invariant probability measures on Y is injective. As mentioned in Section 1, these can be identified with the state spaces of $K^0(X,\phi)$ and $K^0(Y,\psi)$, respectively. The conclusion then follows from 1.1.

(ii) Begin by observing that ∂ is well-defined. The only new item to observe is that in the case $K = \infty$, the range of ∂ is contained in the direct sum because of condition A2.

We show ∂ is onto. Fix k and let δ denote the element of $\mathbb{Z}^K$,

$$\delta_\ell = \begin{cases} 1 & \text{if } \ell = k \\ 0 & \text{otherwise.} \end{cases}$$

Choose a clopen set E containing $x_0^{(k)}$, not containing $x_1^{(k)}$. As $\pi\{x \mid (x,x') \in D_E\}$ is finite, choose a clopen neighbourhood , F, of $\pi(x_0^{(k)})$, disjoint from the other elements of this set. Let $G = E \cap \pi^{-1}(F)$, which is clopen in X. Clearly, $x_0^{(k)}$ is in G while $x_1^{(k)}$ is not. On the other hand, suppose x and x' are in X with $\pi(x) = \pi(x')$. We compare $\chi_G(x)$ and $\chi_G(x')$ as follows. First, we have

$$\chi_G = \chi_E \cdot \chi_{\pi^{-1}(F)}$$

$$= \chi_E \cdot \chi_F \circ \pi$$

and since $\pi(x) = \pi(x')$, $\chi_G(x)$ and $\chi_G(x')$ are unequal only if $\chi_E(x)$ and $\chi_E(x')$ are (hence (x',x) or (x,x') is in D_E) and $\chi_F \circ \pi(x) = 1$ (hence $\pi(x)$ is in F). In this case, we have $(x,x') = \left(x_0^{(k)}, x_1^{(k)}\right)$, or the other way around, from the choice of F. It easily follows that $\partial[\chi_G] = \delta$ as desired.

Next, we show that if f is in $C(X,\mathbb{Z})$ and $\partial[f]=0$, then $[f]$ is in $\pi^*\left(K^0(Y,\psi)\right)$. Define

$$D_f = \{(x,y) \mid \pi(x)=\pi(y), \quad f(x)\neq f(y)\}.$$

Since f is continuous and using A2, D_f is finite. If D_f is empty, then it follows that $f = g\circ\pi$ for some continuous $g : Y \to \mathbb{Z}$ and hence $[f]=\pi^*[g]$ as desired. We will show that $[f]=[f']$ for some f' in $C(X,\mathbb{Z})$ with $D_{f'}$ empty, and we will be done.

If D_f is non-empty, we claim that there is $h : X \to \mathbb{Z}$ which is continuous and $D_{f+h-h\circ\phi} \subsetneq D_f$. We may repeat this process, each time obtaining a new $f' = f+h-h\circ\phi$ such that $D_{f'} \subsetneq D_f$ and eventually, since the original D_f is finite, we will have $D_{f'}$ empty. Moreover, $[f']=[f]$ since they differ by a coboundary.

Choose (x,y) in D_f; *i.e.* $f(x)\neq f(y)$ and $\pi(x)=\pi(y)$. Since $\partial f=0$, $\partial_{x,y} f = 0$ ($\pi(x)$ is in the orbit of some $\pi(x_0^{(k)})$) and so, for some $n\neq 0$,

$$f\left(\phi^{-n}(x)\right)\neq f\left(\phi^{-n}(y)\right).$$

We assume $n>0$. We may argue exactly as in the proof that ∂ is onto, to find a clopen set $G\subseteq X$ such that $D_G=\{(x,y)\}$; *i.e.* $x\in G$, $y\notin G$. Define h by

$$h = (f(y)-f(x))\cdot\sum_{i=0}^{n-1}\chi_G\circ\phi^i.$$

Then we have

$$h-h\circ\phi = (f(y)-f(x))\left[\chi_G-\chi_G\circ\phi^n\right].$$

We see that

$$D_{h-h\circ\phi}=\left\{(x,y),\ \left(\phi^{-n}(x),\ \phi^{-n}(y)\right),\ (y,x),\ \left(\phi^{-n}(y),\ \phi^{-n}(x)\right)\right\}$$

and so

$$D_{f+h-h\circ\phi}\subseteq D_f\cup D_{h-h\circ\phi}\subseteq D_f.$$

Also, we have

$$\begin{aligned}
&(f+h-h\circ\phi)(x)-(f+h-h\circ\phi)(y)\\
&\quad= f(x)+(f(y)-f(x))\ \chi_G(x)-(f(y)-f(x))\ \chi_G\left(\phi^n(x)\right)\\
&\qquad - f(y)-(f(y)-f(x))\ \chi_G(y)\\
&\qquad + (f(y)-f(x))\ \chi_G\left(\phi^n(y)\right).
\end{aligned}$$

Now recall that, $\chi_G(x) = 1$, $\chi_G(y) = 0$ and $\chi_G(\phi^n(x)) = \chi_G(\phi^n(y))$. Thus, the above expression equals zero and so $(x, y) \notin D_{f+h-h\circ\phi}$. Therefore $D_{f+h-h\circ\phi} \subsetneq D_f$. This completes the proof that $\ker \partial \subseteq \mathrm{Im}(\pi^*)$. The reverse inclusion is straight-forward:

$$\begin{aligned}\partial_{x,y} \circ \pi^*[f] &= \sum_n f \circ \pi(\phi^n(x)) - f \circ \pi(\phi^n(y)) \\ &= 0\end{aligned}$$

since $\pi \circ \phi^n(x) = \pi \circ \phi^n(y)$, for all n.

The fact that π^* is injective was shown in [GW]. ∎

Remark 3.2. Let us observe another proof of (ii) of 3.1. One can duplicate the set-up of Example 2.2 of [Put], taking into account K-asymptotic pairs instead of just one. To be specific, let $\Gamma = \mathbb{Z}$, $H^0 = \{1, 2, \cdots, K\} \times \mathbb{Z}$, (or $\mathbb{N} \times \mathbb{Z}$ if $K = \infty$) $H = \{1, 2, \cdots, K\} \times \mathbb{Z} \times \mathbb{Z}$, with the co-trivial structure on the first and the trivial on the last two. Define

$$i_j(k, m, n) = \left(\phi^m(x_j^{(k)}),\ n - m\right)$$

for $j = 0, 1$, $1 \leq k \leq K$, $m, n \in \mathbb{Z}$. In this case

$$\begin{aligned}C_r^*(H) &\cong \bigoplus_1^K \mathcal{K} \\ K_0(C_r^*(H)) &\cong \mathbb{Z}^K \\ K_1(C_r^*(H)) &\cong 0.\end{aligned}$$

Moreover, since (X, ϕ) and (Y, ψ) are minimal Cantor systems

$$K_1(C_r^*(G)) \cong K_1(C_r^*(G')) \cong \mathbb{Z}$$

and, as both groups are generated by the classes of the canonical unitaries in the crossed product, the map α_* between them is an isomorphism. The reason we do not complete the proof in this fashion is because we would need to show that the map $[i_0, i_1]_*$ of 2.1 of [Put] agrees with our ∂. This is not so arduous, but is probably less "dynamically friendly".

Remark 3.3. It is important to note that the hypotheses that (X, ϕ) and (Y, ψ) are Cantor systems is important. Consider the example of (Y, ψ) being an irrational rotation by angle $2\pi\theta$ of the circle and (X, ϕ) is a Denjoy homeomorphism of the Cantor set made from (Y, ψ) by "cutting" a single orbit [PSS]. There is a natural factor map $\pi : (X, \phi) \to (Y, \psi)$ satisfying A1 and A2 (with $K = 1$). In this case, we have

$$K^0(X, \phi) \cong K_0\left(C(X) \times_\phi \mathbb{Z}\right) \cong \mathbb{Z} + \theta\mathbb{Z}$$

$$\cong K_0\left(C(Y) \times_\phi \mathbb{Z}\right)$$

and π^* is an order isomorphism. (Note that since $\dim(Y) > 0$, we do not have the description of $K_0\left(C(Y) \times \mathbb{Z}\right)$ given in Section 1 as $C(Y, \mathbb{Z})$ modulo the coboundaries.) It is interesting to note, however, that the set-up of Example 2.2 of [Put] still holds and there is a six-term exact sequence from Theorem 2.1 of [Put]. In this case, $\partial = [i_0, i_1]_* = 0$.

§4. Realization by Extensions

Our aim is to prove a realization theorem of the following kind. Begin with a Cantor minimal system (Y, ψ). (We will let H denote $K^0(Y, \psi)$ later for convenience.) Suppose G is a simple dimension group which is an extension of H by a torsion free group Q. We also assume the inclusion of H in G is a dense order embedding. Then this inclusion is realized by (X, ϕ) a Cantor minimal extension of (Y, ψ). More precisely, we will prove:

Theorem 4.1. *Let (Y, ψ) be a Cantor minimal system and suppose*

$$0 \longrightarrow K^0(Y, \psi) \xrightarrow{\ i\ } G \xrightarrow{\ q\ } Q \longrightarrow 0$$

is a short exact sequence of abelian groups, with Q countable and torsion free. Also suppose that G is a simple dimension group, that i is an order embedding and that $i\left(K^0(Y, \psi)\right)$ is order dense in G.

Then there is a Cantor minimal system (X, ϕ), an almost one-to-one factor map satisfying A1

$$\pi : (X, \phi) \longrightarrow (Y, \psi)$$

and an order isomorphism

$$\alpha : K^0(X, \phi) \longrightarrow G$$

such that $\alpha \circ \pi^* = i$.

Moreover, if Q is free abelian then the factor may be chosen to satisfy A2 *as well.*

The first step of the proof is to begin with a Bratteli-Vershik model, D, for (Y, ψ) as described in [HPS] and [GPS]. That is, D is an ordered Bratteli diagram whose path space we identify with Y. Although Q has no inherent order, we implicitly give it one so as to find another Bratteli diagram D_Q, whose associated dimension group is Q. This is for convenience, but it does require Q being torsion free.

We will need to have D_Q embedded inside D so that its image is fairly small. This is achieved by standard telescoping arguments. Here, "small" should be interpreted as follows. The path space of D_Q, denoted Z, will be a closed subset of Y. This set should have measure zero under all finite ψ-invariant measures on Y.

The final step is to construct, for certain g in G, $0 < g < 1$, an open subset of Y. These sets will not be closed, in fact their boundaries are contained in the orbit of Z. We will add these to the Boolean algebra of clopen sets in Y to obtain a new Boolean algebra whose spectrum will be our extension. The real difficulty lies in choosing these open sets in a "consistent" way for different g in G.

Let us set out some notation. If $D = (V(D), E(D), \geq)$ is our Bratteli-Vershik model for (Y, ψ), then each vertex v in $V(D)$ determines a positive element of $K^0(Y, \psi)$. We denote $K^0(Y, \psi)$ by H for convenience, and this element is denoted by h_v.(See [GPS].)

As Q is a countable torsion free group, it may be embedded in $\mathbb{R}$. The relative order from $\mathbb{R}$ then makes Q into a simple dimension group. We will construct a Bratteli diagram for Q, D_Q. As above for H, each vertex v in this diagram determines an element q_v in Q.

Let us recall some notation from [HPS]. If $(e_1, e_2, \cdots, e_n) = p$ is any finite path from v_0 in $V_0(D)$, then

$$U(p) = \{(f_1, f_2, \cdots) \in Y \mid f_i = e_i, \quad 1 \leq i \leq n\}$$

is a clopen subset of Y. Also, for any p which is not maximal, $\psi(U(p)) = U(p')$, where p' is the successor of p. Henceforth, we let $p + 1$ denote the successor of p.

Lemma 4.2. *There is a Bratteli-Vershik model, D, for (Y, ψ) having a sub-diagram, D_Q, which represents Q as above and so that*

(i) *$E(D_Q)$ contains no maximal or minimal edges of $E(D)$,*

(ii) *letting*

$$Z = \{(e_1, e_2, \cdots) \in Y \mid e_i \in E(D_Q), \text{ for all } i\},$$

$\mu(Z) = 0$, for all ψ-invariant probability measures, μ, on Y. In particular, $\bigcup_{k\in\mathbb{Z}} \psi^k(Z)$ is a set of first category.

We may also select a sequence $\epsilon_0 > \epsilon_1 > \cdots$ in H^+ and elements g_v in G, for each v in $V(D_Q)$, so that

(iii) *$q(g_v) = q_v$,*

(iv) *$0 < \epsilon_n < g_v < h_v - \epsilon_n$, for all v in $V_n(D_Q)$,*

(v) *$h_v\,|E_n(D_Q)| < \epsilon_{n-1}$, for all v in $V_n(D_Q)$*

and

(vi) *$2\,|E_n(D_Q)|\,\epsilon_n < \epsilon_{n-1}$.*

Proof. Begin by choosing any Bratteli diagram D_Q for Q and any Bratteli-Vershik model D' for (Y, ψ). Our diagram D will actually be a telescope of D'. That is, for each n, $V_n(D)$ will be $V_{m_n}(D')$, for some m_n.

First, by telescoping, replacing vertices with edges (as in 3.1 of [HPS]) and telescoping again, we may assume that the size of $V_n(D')$ and the minimum number of edges between vertices at levels n and $n+1$ are both increasing with n.

Our choices for the g_v, h_v, ϵ_n and m_n as above, will be made inductively on n. We begin with $V_0(D) = V_0(D')$ (*i.e.* $m_0 = 0$) and so $h_{v_0} = 1$, the order unit for H. As H is dense in G, so is the coset $q^{-1}(q_{v_0})$. So choose g_{v_0}such that $q(g_{v_0}) = q_{v_0}$ and

$$0 < g_{v_0} < 1.$$

Finally choose ϵ_0 in H such that

$$0 < \epsilon_0 < g_{v_0} < 1 - \epsilon_0.$$

For the induction step, suppose all the items have been chosen to level n. First, we select our next vertex level in D', $V_{n+1}(D) = V_m(D')$. The value m should be chosen sufficiently large so that all of the following hold:

(a) For any w in $V_m(D')$,

$$h_w \cdot |E_{n+1}(D_Q)| < \epsilon_n .$$

This is possible since the "maximum size" of the generators, h_w, tends to zero in a simple dimension group.

(b) $|V_m(D')| \geq |V_{n+1}(D_Q)|$,

(c) The minimum number of paths from v_0 to any vertex of $V_m(D')$ is greater than 2^{n+1} times the total number of paths from v_0 to $V_{n+1}(D_Q)$ in D_Q.

(d) The number of paths in D' from any vertex of $V_n(D) = V_{m_n}(D')$ to one of $V_m(D')$ exceeds $|E_{n+1}(D_Q)| + 2$.

Having selected such an m, we let $V_{n+1}(D) = V_m(D')$ (*i.e.* $m_{n+1} = m$) and by (b) and (c), we may embed $V_{n+1}(D_Q)$ and $E_{n+1}(D_Q)$ in $V_{n+1}(D)$ and $E_{n+1}(D)$, respectively. Moreover, because of the $+2$ in (d), the embedded $E_{n+1}(D_Q)$ contains no maximal or minimal edges of $E_{n+1}(D)$.

As noted before, as H is dense in G, so is each H-coset. So we may choose g_v, for each v in $V_{n+1}(D_Q)$, such that

$$q(g_v) = q_v \quad \text{and} \quad 0 < g_v < h_v .$$

Finally, select ϵ_{n+1} so that

$$0 < \epsilon_{n+1} < g_v, \quad h_v - g_v \,,$$

for all v in $V_{n+1}(D_Q)$,

$$\epsilon_{n+1} < h_v,$$

for all v in $V_{n+1}(D) - V_{n+1}(D_Q)$, and

$$2\,|E_{n+1}(D_Q)|\;\epsilon_{n+1} < \epsilon_n .$$

This completes the induction step. All the desired properties are clear except for (ii). To prove this, we proceed as follows.

For each $n = 1, 2, \cdots$, let

$$Z_n = \{(e_1, e_2, \cdots) \in Y \mid e_i \in E(D_Q), \quad 1 \leq i \leq n\} .$$

Then clearly, $Z = \bigcap_n Z_n$ and $Z_1 \supseteq Z_2 \supseteq \cdots$. Hence it suffices to prove

$$\lim_{n \longrightarrow \infty} \mu(Z_n) = 0,$$

for any ψ-invariant probability measure μ on Y. For each v in $V_n(D)$, choose a path p_v in D from v_0 to v. Let k_v denote the number of paths in D_Q from v_0 to v and ℓ_v denotes the number of paths in D from v_0 to v. From the ψ-invariance of μ,

$$\mu\left(U(p)\right) = \mu\left(U(p')\right)$$

for any two finite paths in D, provided $r(p) = r(p')$. So we have

$$\mu(Z_n) = \sum \mu\left(U(p)\right),$$

where the sum is over all paths p in D_Q from v_0 to $V_n(D_Q)$,

$$= \sum_{v \in V_n(D_q)} k_v \cdot \mu\left(U(p_v)\right)$$

$$\leq \sum_{v \in V_n(D_Q)} 2^{-n}\, \ell_v\, \mu\left(U(p_v)\right),$$

by (c),

$$\leq 2^{-n} \sum_{v \in V_n(D)} \ell_v\, \mu\left(U(p_v)\right)$$

$$\leq 2^{-n}\, \mu(Y)$$

$$= 2^{-n}.$$

The desired conclusion follows. It also follows from the ψ-invariance of μ that $\mu\left(\psi^k(Z)\right) = 0$, for all integers k. As the support of μ is Y, $\psi^k(Z)$ has empty interior and so $\bigcup_k \psi^k(Z)$ is of first category. ▮

Lemma 4.3. *Let v be in $V_n(D_Q)$.For any w in $V_{n+1}(D_Q)$, let R_{vw} denote the number of edges from v to w in $E(D_Q)$. That is,*

$$q_v = \sum_w R_{vw}\, q_w,$$

where the sum is over w in $V_{n+1}(D_Q)$. Then, we have

(i) $g_v - \sum_w R_{vw}\, g_w$ is in H,

(ii) $0 < g_v - \sum R_{vw}\, g_w < h_v - \sum R_{vw}\, h_w$.

Proof. First, we note that

$$0 < \sum_w R_{vw}\, h_w$$

$$< \sum_w R_{vw}\, |E_{n+1}(D_Q)|^{-1}\, \epsilon_n$$

by (v) of 4.2

$$< \epsilon_n,$$

since the number of edges from v to w in D_Q is R_{vw}. Now, using (iv) of 4.2,

$$\begin{aligned} g_v - \sum R_{vw}\, g_w &< g_v \\ &< h_v - \epsilon_n \\ &< h_v - \sum R_{vw}\, h_w \end{aligned}$$

and

$$\begin{aligned} g_v - \sum R_{vw}\, g_w &> g_v - \sum R_{vw}\, h_w \\ &> g_v - \epsilon_n \\ &> 0. \end{aligned}$$

As for part (i), we use (iii) of 4.2,

$$\begin{aligned} q\left(g_v - \sum R_{vw}\, g_w\right) &= q(g_v) - \sum R_{vw}\, q(g_w) \\ &= q_v - \sum R_{vw}\, q_w \\ &= 0 \end{aligned}$$

from the definition of R. The conclusion follows since $H = \ker q$. ▮

If p_1 and p_2 are finite paths in the Bratteli diagram and $r(p_1) = s(p_2)$, then we may form their concatenation, which we denote $p_1 p_2$.

Lemma 4.4. *Let P_n (P, respectively) be all finite paths, p, in D with $s(p) = v_0$ and $r(p) \in V_n(D_Q)$ ($V(D_Q)$, respectively). There is a collection of clopen sets, $\{C(p) \mid p \in P\}$, such that*

(i) $C(p+1) = \psi(C(p))$, *if p is not maximal,*

(ii) $\emptyset \neq C(p) \subsetneq U(p)$, *for all $p \in P$,*

(iii) $[\chi_{C(p)}] = g_v - \sum R_{vw}\, g_w$, *where $v = r(p)$, for all $p \in P$,*

(iv) *if p is in P and p_1 and p_2 are two paths in D_Q with*

$$s(p_1) = s(p_2) = r(p) \quad \textit{and} \quad p_1 \neq p_2,$$

then $C(pp_1)$ and $C(pp_2)$ are disjoint.

Proof. For each vertex v in $V_n(D_Q)$, let p_v denote the minimal path in D from v_0 to v. Define

$$D(p_v) = \left\{(e_1, e_2, \cdots) \in Y \;\middle|\; (e_1, \cdots, e_n) = p_v \text{ and } e_{n+1} \in E_{n+1}(D_Q)\right\}.$$

Notice that we may also write

$$D(p_v) = \dot{\bigcup}\; U(p_v f),$$

where the union is over f in $E_{n+1}(D_Q)$ with $s(f) = v$. (We use $\dot{\cup}$to denote disjoint union.) Thus it follows that

$$\begin{aligned}
\left[\chi_{D(p_v)}\right] &= \sum_f \left[\chi_{U(p_v f)}\right] \\
&= \sum_f h_{r(f)} \\
&= \sum_{w \in V_{n+1}(D_Q)} R_{vw}\, h_w
\end{aligned}$$

since there are exactly R_{vw} edges in $E_{n+1}(D_Q)$ from v to w. We may apply (ii) of 4.3 to see

$$\begin{aligned}
0 &< g_v - \sum R_{vw}\, g_w \\
&< h_v - \sum R_{vw} h_w \\
&= \left[\chi_{U(p_v)}\right] - \left[\chi_{D(p_v)}\right] \\
&= \left[\chi_{U(p_v) - D(p_v)}\right],
\end{aligned}$$

since $D(p_v)$ is contained in $U(p_v)$. Thus, there is a clopen set $C(p_v)$,

$$\emptyset \neq C(p_v) \subsetneq U(p_v) - D(p_v),$$

$$\left[\chi_{C(p_v)}\right] = g_v - \sum R_{vw}\, g_w.$$

Now, if p is any other path in P, then p is the kth-successor of some minimal p_v ($v = r(p) \in V_n(D_Q)$).

We then set

$$C(p) = \psi^k\left(C(p_v)\right).$$

Notice that $C(p) \subseteq U(p)$ and if $(e_1, e_2, \cdots)$ is in $C(p)$, then e_{n+1} is not in $E_{n+1}(D_Q)$.

Properties (i), (ii) and (iii) are immediate. It remains to check (iv). Since $C(p) \subseteq U(p)$, (iv) is also clear if pp_1 and pp_2 differ in some entry. The only remaining case to consider is when pp_2 is an extension of pp_1; that is, $p_2 = p_1 p_2'$. Suppose $r(p_1)$ is in $V_n(D_Q)$. Then as noted above, if $(e_1, e_2, \cdots)$ is in $C(pp_1)$, then e_{n+1} is not in $E_{n+1}(D_Q)$. But since p_2' is a path in D_Q, its first edge lies in $E_{n+1}(D_Q)$. This means $C(pp_1)$ and $C(pp_2)$ are disjoint. ∎

Lemma 4.5. *For each p in P, let*

$$G(p) = C(p) \cup \left[\bigcup C(pp')\right],$$

where the union is over all finite paths p' in D_Q with $s(p') = r(p)$. Then, we have

(i) $G(p+1) = \psi\left(G(p)\right)$, *if p is not maximal.*

(ii) $\emptyset \neq G(p) \subsetneq U(p)$,

(iii) $G(p) = C(p) \cup \left[\cup_e G(pe)\right]$, *where the union is over e in $E(D_Q)$ with $s(e) = r(p)$.*

(iv) *$G(p)$ is open, but not closed and $U(p) - G(p)$ has non-empty interior.*

(v) *The boundary of $G(p)$ is contained in $\psi^k(Z)$, for some k in $\mathbb{Z}$, where Z is as in* 4.2.

(vi) $\sum_{p'} \left[\chi_{C(pp')}\right] + \left[\chi_{C(p)}\right] = g_{r(p)}$, *where the series converges in the order topology of G.*

Proof. (i) and (ii) follow at once from (i) and (ii), respectively, of 4.4.

The collection of paths p' in D_Q with $s(p') = r(p)$, may be partitioned into a finite number of sets, according to their first edge. This immediately yields (iii).

It follows from (iv) of 4.4 that the sets $C(p)$ and $C(pp')$ in the definition of $G(p)$ are pairwise disjoint. The first part of (iv) is an easy consequence of this and the fact that these sets are clopen and non-empty.

As for the second part, we have

$$\begin{aligned} U(p) - G(p) &= U(p) - C(p) - \cup_e G(pe) \\ &\supseteq U(p) - C(p) - \cup_e U(pe) \\ &= U(p) - C(p) - D(p), \end{aligned}$$

in the notation from the proof of 4.4, and, as seen there, this set is non-empty and open. This proves the second part of (iv).

As for (v), since $r(p)$ is in $V_n(D_Q)$, there is some other path $\tilde{p}$ in D_Q from v_0 to $r(p)$. Now p is the kth successor of $\tilde{p}$, for some k in $\mathbb{Z}$. (Here k negative means p is the kth predecessor of $\tilde{p}$). Applying ψ^{-k} we can simply consider the case $p = \tilde{p}$; *i.e.* p lies in D_Q.

We have our description of $G(p)$ as a countable union of pairwise disjoint non-empty clopen sets. It is then easily seen that any boundary point, say x, of $G(p)$ is a limit of a sequence $\{x_1, x_2, \cdots\}$ where the x_i come from distinct clopen sets in the collection. That is, we may find paths $p_1, p_2, \cdots$ in D_Q so that x_n is in $C(pp_n)$. By passing to a subsequence, we may assume the lengths of the p_n are increasing. Write $x = (e_1, e_2, \cdots)$ as an edge list and $x_n = (e_1^{(n)}, e_2^{(n)}, \cdots)$. Since x_n converges to x, for any fixed k, $e_k^{(n)} = e_k$, for n sufficiently large. However, once the length of p_n plus the length of p exceeds k, $e_k^{(n)}$ is in D_Q. Thus e_k is in D_Q, for all k, and so x is in Z.

For part (vi), we start with 4.4(iii)

$$[\chi_{C(p)}] = g_v - \sum_w R_{vw}\, g_w,$$

where $v = r(p)$. From this, it is easy to prove inductively that if $m > n$,

$$g_v = \sum_{p'} [\chi_{C(pp')}] + \sum_w \tilde{R}_{vw}\, g_w,$$

where the first sum is over all p', a path in $D_Q$with $s(p') = v$ and length less than $m - n$, the second sum is over all w in $V_{m+1}(D_Q)$ and $\tilde{R}_{vw}$ denotes the number of paths from v

to w in $E(D_Q)$. Thus, it suffices to show that

$$\sum_w \tilde{R}_{vw}\, g_w$$

tends to zero in G as m goes to infinity. First, $0 < g_w < \epsilon_m / |E_{m+1}(D_Q)|$ by 4.2(iv) and (v) and

$$\sum_{v,w} \tilde{R}_{vw} \leq |E_{n+1}(D_Q)| \cdots |E_{m+1}(D_Q)| \, .$$

So, we have

$$0 < \sum \tilde{R}_{vw}\, g_w \leq |E_{n+1}(D_Q)| \cdots |E_m(D_Q)|\ \epsilon_m$$

$$\leq |E_{n+1}(D_Q)| \cdots |E_{m-1}(D_Q)|\ \epsilon_{m-1} 2^{-1}$$

by (vi) of 4.2. Continuing, we see that

$$\Sigma\, \tilde{R}_{vw}\, g_w \leq 2^{-(m-n)} \epsilon_n,$$

and the conclusion follows. ▮

Lemma 4.6. *Define*

$$\mathcal{A} = \operatorname{span}\left\{ \chi_{U(p)} \;\middle|\; p \textit{ a finite path in } D \right\},$$

$$\mathcal{B}_n = \mathcal{A} + \operatorname{span}\left\{ \chi_{G(p)} \;\middle|\; p \textit{ a path in } P_n \right\},$$

for $n = 0, 1, 2, \cdots$, $\mathcal{B} = \bigcup_n \mathcal{B}_n$. *Then we have*

(i) *for all* n, $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$,

(ii) $\mathcal{B}_n \cong \mathcal{A} \oplus \left[\bigoplus_{p \in P_n} \mathbb{C}\chi_{G(p)} \right]$, *as linear spaces,*

(iii) $\mathcal{B}$ *is a* $*$*-algebra,*

(iv) *for* f *in* $\mathcal{B}$, $f \circ \psi^{-1}$ *is in* $\mathcal{B}$.

Proof. (i) follows from 4.5(iii). For (ii), consider an element of $\mathcal{B}_n$

$$f + \sum \lambda_p\, \chi_{G(p)} = 0$$

where f is in $\mathcal{A}$. Pick any p_0 in P_n and multiply both sides by $\chi_{U(p_0)}$. Then we have

$$f \cdot \chi_{U(p_0)} + \lambda_{p_o}\, \chi_{G(p_0)} = 0.$$

The first function is continuous and the second is only if $\lambda_{p_0} = 0$, by 4.5(iv). Thus $\lambda_{p_0} = 0$ and as p_0 was arbitrary we may conclude all λ_p's are zero and hence $f = 0$. This proves (ii).

For (iii), we first observe that $\mathcal{A}$ is a $*$-algebra. Next we claim that if p is in P_n, then

$$\chi_{G(p)} \cdot \mathcal{A} \subseteq \mathcal{B}.$$

It is sufficient to consider a path q in D and

$$\chi_{G(p)} \cdot \chi_{U(q)}.$$

In the case q is length n, then this product is either $\chi_{G(p)}$ if $p = q$ and zero otherwise. If the length of q is less than n, then the product is again $\chi_{G(p)}$, if $p = qq'$, for some q', and zero otherwise. Finally, if the length of q is greater than n, we may use (iii) of 4.5 to replace $\chi_{G(p)}$ by a sum of elements of $\mathcal{A}$ and $\chi_{G(pp')}$ where pp' is the same length as q, and then appeal to the case above.

This same technique can be used to show $\chi_{G(p)}\chi_{G(q)}$ is in $\mathcal{B}$, whenever p and q are in P. We omit the details.

Part (iv) is clear for $\mathcal{A}$ and it remains to consider $f = \chi_{G(p)}$. Now if p is not maximal, $f \circ \psi^{-1}$ is in $\mathcal{B}$ by 4.5(i). If p is maximal, we use 4.5(iii) to write $\chi_{G(p)}$ as a sum of an element of $\mathcal{A}$ and some terms $\chi_{G(pe)}$, where e is in $E_{n+1}(D_Q)$. Since $E(D_Q)$ contains no maximal edges, none of the paths, pe, are maximal and the result follows as above. ∎

The C^*-algebra generated by $\mathcal{B}$, *i.e.* its completion, will be a unital commutative C^*-algebra whose spectrum we will define to be X. That is, X is the set of non-zero multiplicative linear functionals on the completion of $\mathcal{B}$.

Lemma 4.7. *Let α be a non-zero multiplicative linear functional on the completion of $\mathcal{B}$. The restriction of α to the completion of $\mathcal{A}$, which may be identified with $C(Y)$, is given by point evaluation. That is, there is a unique y in Y such that*

$$\alpha(f) = f(y),$$

for all f in $\mathcal{A}$. Let p be in P.

(i) *If* y *is in* $G(p)$*, then* $\alpha(\chi_{G(p)}) = 1$.

(ii) *If* y *is not in* $\overline{G(p)}$*, then* $\alpha(\chi_{G(p)}) = 0$.

(iii) *If* y *is in* $\partial G(p)$*, then* $\alpha(\chi_{G(p)}) = 0$ *or* 1.

Moreover, if y *is in* $\partial G(p)$*, for some* p *in* P*, then* α *is uniquely determined by* y *and* $\alpha(\chi_{G(p)})$.

Proof. We know that $G(p)$ is the countable union of clopen sets and if U is the one containing y then

$$1 = \alpha(1) \geq \alpha(\chi_{G(p)}) \geq \alpha(\chi_U) = \chi_U(y) = 1$$

and (i) follows.

If y is not in $\overline{G(p)}$, we may find a clopen set U containing y and disjoint from $G(p)$. Then we have

$$\begin{aligned}\alpha\left(\chi_{G(p)}\right) &= \alpha\left(\chi_{G(p)}\right) \cdot \chi_U(y) \\ &= \alpha\left(\chi_{G(p)} \cdot \chi_U\right) \\ &= \alpha(0) \\ &= 0.\end{aligned}$$

As α is multiplicative and $\chi_{G(p)}$ is idempotent, $\alpha(\chi_{G(p)})$ is idempotent so (iii) is true.

Suppose $r(p)$ is in $V_n(D_Q)$. For any other p' with $r(p')$ in $V_n(D_Q)$, $G(p)$ and $G(p')$ are contained in $U(p)$ and $U(p')$ respectively. This means that y is not in $\overline{G(p')}$, for any $p' \neq p$. So $\alpha\left(G(p')\right) = 0$ by (ii), for $p' \neq p$. Thus $\alpha \mid \mathcal{B}_n$ is determined by $\alpha\left(\chi_{G(p)}\right)$ and $\alpha \mid \mathcal{A}$.

The fact that the sets $U(p)$ are clopen and pairwise disjoint (as p runs over all paths to level $n+1$) and parts (ii) and (iii) of 4.5, together imply that y is in $\partial G(pe)$, for exactly one edge e in $E_{n+1}(D_Q)$, $s(e) = r(p)$. For this one particular e, $\alpha(\chi_{G(pe)})$ is uniquely determined by $\alpha(\chi_{G(p)})$, $\alpha \mid \mathcal{A}$ and

$$\chi_{G(p)} = \chi_{C(p)} + \sum_{e'} \chi_{G(pe')}$$

which follows from 4.5(iii). Thus, $\alpha \mid \mathcal{B}_{n+1}$ is uniquely determined by $\alpha(\chi_{G(p)})$ and $\alpha \mid \mathcal{A}$. Continuing inductively, the final statement follows. ∎

Theorem 4.8. *Let X be the spectrum of the commutative C^*-algebra $\overline{\mathcal{B}}$. Since $C(Y) \cong \overline{\mathcal{A}}$ is a unital C^*-subalgebra of $\overline{\mathcal{B}}$, we have a natural continuous surjection $\pi : X \to Y$. The map sending f in $\mathcal{B}$ to $f \circ \psi^{-1}$ extends to an automorphism of $\overline{\mathcal{B}}$ and we let ϕ denote the associated homeomorphism of X. Then*

(i) $\pi \circ \phi = \psi \circ \pi$

(ii) *for y not in $\bigcup_k \psi^k(Z)$, and, in particular, if y is the unique path of maximal (or minimal) edges, then $\pi^{-1}\{y\}$ is a single point.*

(iii) *for any y in Y, $\pi^{-1}\{y\}$ is at most two points.*

(iv) *each ψ-invariant probability measure lifts uniquely to a ϕ-invariant probability measure on X.*

Moreover, (X, ϕ) is a Cantor minimal system.

Proof. Part (i) follows at once from the definitions. Part (ii) follows from (v) of 4.5, (i) and (ii) of 4.7. Part (iii) follows from (iii) and the final statement of 4.7.

For part (iv), every ψ-invariant measure μ is zero on Z by 4.2(ii), and so by (ii), π is one-to-one on a set whose image is of full μ-measure in Y. The conclusion follows.

The fact that X is totally disconnected follows from 4.6(ii) and (iii). To see that (X, ϕ) is minimal, we proceed as follows. Suppose C is a closed, non-empty, ϕ-invariant subset of X. Then $\pi(C)$ is a closed ψ-invariant subset of Y. By the minimality of (Y, ψ), $\pi(C) = Y$. Let y be any point of Y where $\pi^{-1}\{y\}$ is a single point, say x in X. Then x is in C and so it suffices to show that the one-to-one set is dense in X.

By Lemma 4.2(ii), the ψ-orbit of Z is a set of first category and so every open set of Y contains a point y where $\pi^{-1}\{y\}$ is a single point.

Fix a vertex level n and consider the linear span of the characteristic functions of the following sets: $U(p)$, with $r(p)$ not in D_Q, and $G(p)$ and $U(p) - G(p)$, for p with $r(p)$ in D_Q. From the proof of 4.6, this is a finite dimensional $*$-subalgebra of $\mathcal{B}_n$. If we union over n, we get a dense subalgebra of $\overline{\mathcal{B}}$. To complete the proof, it suffices to find, given any multiplicative linear functional $\tilde{x}$ on $\overline{\mathcal{B}}$, one which is the same on our finite dimensional algebra and in our one-to-one set. Now, $\tilde{x}$ will have the value one on the characteristic function of one of the sets above, say A, and zero on the others. By Lemma 4.5(iv), A has a non-empty interior so there is a point y in A with $\pi^{=1}\{y\} = \{x\}$ in the one-to-one set. By Lemma 4.7, x and $\tilde{x}$ are equal on our finite dimensional algebra. This completes the proof. ▮

We now have constructed a Cantor minimal system (X, ϕ) and the factor map π. It remains to construct the group homomorphism α. To do this, we first let

$$C_n = C(Y, \mathbb{Z}) + \sum_p \mathbb{Z}\, \chi_{G(p)},$$

where the sum is over p in P_n, for $n = 1, 2, 3, \cdots$. It follows from 4.6 that

$$C_n \subseteq C_{n+1}$$

$$\bigcup_n C_n = C(X, \mathbb{Z}).$$

Define

$$\tilde{\alpha} : C_n \longrightarrow G$$

by

$$\tilde{\alpha}\left(f + \sum n_p\, \chi_{G(p)}\right) = i[f] + \sum_p n_p\, g_{r(p)}.$$

It follows from 4.6(ii) that $\tilde{\alpha}$ is well-defined and is consistently defined for different values of n by 4.5(iii) and 4.4(iii).

We will establish the following three assertions:

(i) If $g \geq 0$ in $C(X, \mathbb{Z})$, then $\tilde{\alpha}(g) \geq 0$ in G.

Every such g may be written as a positive combination of functions of the form:

$$\chi_{U(p)}, \quad r(p) \notin V(D_Q),$$

$$\chi_{G(p)},\ \chi_{U(p)-G(p)}, \quad r(p) \in V(D_Q).$$

For the first,

$$\tilde{\alpha}\left(\chi_{U(p)}\right) = i\left[\chi_{U(p)}\right]$$

which is positive since $[\chi_{U(p)}]$ is. For p in P,

$$\tilde{\alpha}\left(\chi_{G(p)}\right) = g_{r(p)} \geq 0$$

$$\begin{aligned}\tilde{\alpha}\left(\chi_{U(p)-G(p)}\right) &= \tilde{\alpha}\left(\chi_{U(p)} - \chi_{G(p)}\right)\\ &= i\left[\chi_{U(p)}\right] - g_{r(p)}\\ &= h_{r(p)} - g_{r(p)} \geq 0\end{aligned}$$

by 4.2. The claim follows.

(ii) $\tilde{\alpha}(g - g \circ \phi) = 0$, for all g in $C(X, \mathbb{Z})$.

Adding a constant to g does not change $g - g \circ \phi$, so replacing g by $g - g(x_0)$, where x_0 denotes the infinite path of minimal edges in Y, we may assume $g(x_0) = 0$. Next, choose n sufficiently large so g is in C_n and $g \mid U(p) = 0$, for every minimal path p from v_0 to $V_n(D)$. Then $g \circ \phi \mid U(p) = 0$, for every maximal path p from v_0 to $V_n(D)$. If we write

$$g = f + \sum n_p \, \chi_{G(p)},$$

then,

$$g - g \circ \phi = f - f \circ \psi + \sum (n_p - n_{p+1}) \, \chi_{G(p)}$$

where $p+1$ is the successor of p if p is not maximal, while $n_{p+1} = 0$ for p maximal. Then

$$\tilde{\alpha}(g - g \circ \phi) = i\,[f - f \circ \psi] + \sum (n_p - n_{p+1}) \, g_{r(p)}.$$

First, $[f - f \circ \psi] = 0$ in $K^0(Y, \psi)$ and in the sum, if we fix $r(p)$, the sum telescopes to give 0. This establishes the claim.

(iii) If $\tilde{\alpha}(g) = 0$, for some g in $C(X, \mathbb{Z})$, then $g = g'' - g'' \circ \phi$, for some g'' in $C(X, \mathbb{Z})$.

First, we write

$$g = f + \sum n_p \, \chi_{G(p)}.$$

consider $q\left(\tilde{\alpha}(g)\right) = q(0) = 0$, and

$$\begin{aligned}q\left(\tilde{\alpha}(g)\right) &= q\left(i[f] + \sum n_p \, g_{r(p)}\right)\\ &= 0 + \sum n_p \, q_{r(p)}\end{aligned}$$

which can be regarded as an element of Q_n. Since it is zero in Q, the inductive limit of the Q_n's, we may assume that n is chosen sufficiently large so that this element is zero in Q_n. That is, for fixed v in $V_n(D_Q)$,

$$\sum_{r(p)=v} n_p = 0.$$

Define

$$g' = \sum_{v \in V_n(D_Q)} \sum_{r(p)=v} \left(\sum_{p' < p} -n_{p'} \right) \chi_{G(p)},$$

which is in C_n. It is easy to check that

$$g' - g' \circ \phi = \sum n_p \, \chi_{G(p)}.$$

Now

$$f = g - (g' - g' \circ \phi)$$

and

$$i[f] = \tilde{\alpha}(g) - \tilde{\alpha}(g' - g' \circ \phi)$$

$$= 0 - 0 \quad \text{(by (ii))},$$

so $[f] = 0$ in $K^0(Y, \psi)$. We may find f' such that $f = f' - f' \circ \psi$. The function $g'' = f' + g'$ satisfies the desired conclusion. ∎

We are now ready to give a proof of 4.1. We have already constructed

$$\pi : (X, \phi) \longrightarrow (Y, \psi).$$

Since the map

$$\tilde{\alpha} : C(X, \mathbb{Z}) \longrightarrow G$$

has kernel equal to the ϕ-coboundaries it induces an injective map

$$\alpha : K^0(X, \phi) \longrightarrow G.$$

By property (i) above, α is positive. We also note here that $\alpha \circ \pi^* = i$ is clear from the definition of α. This implies that the range of α contains $i\left(K^0(Y,\psi)\right)$. Also, it is clear that

$$\alpha\left(\left[\chi_{G(p)}\right]\right) = g_{r(p)},$$

and since G is generated by $i\left(K^0(Y,\psi)\right)$ and the g_v's, it follows that α is onto.

All that remains is to show that if $\alpha([g]) \geq 0$ in G, then $[g] \geq 0$ in $K^0(X,\phi)$. To do this we proceed by analyzing the states as follows.

If β is a positive homomorphism between dimension groups, then it induces a map, $\hat{\beta}$, between their state spaces, $\hat{\beta}(\sigma) = \sigma \circ \beta$. Now the states on $K^0(X,\phi)$ and $K^0(Y,\psi)$ are obtained from the invariant measures [HPS] and by 4.8(iv), $\widehat{\pi^*}$ is a bijection. As the image of i is dense in G, $\hat{\imath}$ is also a bijection. It follows that $\hat{\alpha} = \widehat{\pi^*}^{-1} \circ \hat{\imath}$ is also a bijection. Since an element of a simple dimension group is positive if and only if it is zero or else has positive image under every state, we conclude that if $\alpha[g]$ is positive then so is $[g]$.

Let us give a sketch of the proof of the final assertion; if Q is free abelian, then the factor can be constructed to satisfy A2. If Q is free abelian, then $Q \cong \mathbb{Z}^K$, for some $K = 1,2,3,\cdots,\infty$. We select for D_Q the diagram having n vertices at level n when $1 \leq n \leq K$ and K vertices for all higher levels. Each vertex connects via an edge to exactly one vertex at the next level, except for one in V_n which connects to two in V_{n+1}, when $1 \leq n < K$.

We leave it to the reader to verify that the factor will satisfy A2. ∎

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