The Structure of C^*-algebras Associated with Hyperbolic Dynamical Systems

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Dedicated to Marc Rieffel on the occasion of his sixtieth birthday.

Abstract. We consider the stable, unstable C^*-algebras and the Ruelle algebras associated to a mixing Smale space. In the case of a shift of finite type, these are the AF-algebras studied by W. Krieger and the (stabilized) Cuntz-Krieger algebras. In the general case, we show that the stable and unstable algebras are simple and amenable. We also show the Ruelle algebras are simple, amenable and purely infinite.

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1. Introduction and Statement of the Main Results

Our interest in this paper is in Smale spaces and their associated C^*-algebras. For more detailed information, we refer the reader to [7,8,11]. We will also give explicit descriptions of the basic ideas in the next section, but for the moment, a Smale space is a homeomorphism, ϕ, of a compact metric space, (X, d), having specific properties. Roughly, these mean that X has local canonical coordinates of contracting and expanding directions for ϕ. We will also assume throughout that ϕ is mixing [7,8].

These systems include Anosov diffeomorphisms (the smooth case), shifts of finite type (the zero dimensional case) and other interesting examples from the theory of self-similar tilings [3].

We consider the notions of stable and unstable equivalence; x and y are stably (unstably) equivalent if $d(\phi^n(x), \phi^n(y))$ tends to zero as n tends to plus (minus, respectively) infinity. We let G_s and G_u denote these equivalence relations, *i.e.* principal groupoids. They may be topologized and given Haar systems so we may consider their C^*-algebras:

$$S = C^*(G_s), \qquad U = C^*(G_u).$$

In the case of a shift of finite type, these are the AF-algebras considered by Krieger [4].

In general, ϕ induces automorphisms of G_s and G_u and we may form groupoids $G_s \rtimes \mathbb{Z}$ and $G_u \rtimes \mathbb{Z}$, whose C^*-algebras are $*$-isomorphic to the crossed products

$$C^*(G_s) \rtimes \mathbb{Z}, \qquad C^*(G_u) \rtimes \mathbb{Z}.$$

These are denoted R_s and R_u, respectively and we refer to them as the Ruelle algebras. Again for a shift of finite type, these are the (stabilized) Cuntz-Krieger algebras.

Here, we are interested in analyzing the structure of the C^*-algebras: S, U, R_s, R_u. We state the main results as follows.

Theorem 1.1. *The groupoids G_s and G_u are amenable in the sense of Renault. Hence, we have*

$$S = C^*(G_s) \cong C^*_{\mathrm{red}}(G_s),$$

$$U = C^*(G_u) \cong C^*_{\mathrm{red}}(G_u),$$

and these are amenable C^*-algebras.

Theorem 1.2. *The groupoids $G_s \rtimes \mathbb{Z}$ and $G_u \rtimes \mathbb{Z}$ are amenable in the sense of Renault. Hence, we have*

$$R_s = C^*(G_x \rtimes \mathbb{Z}) \cong C^*_{\text{red}}(G_s \rtimes \mathbb{Z})$$

$$R_u = C^*(G_u \rtimes \mathbb{Z}) \cong C^*_{\text{red}}(G_u \rtimes \mathbb{Z})$$

and these are amenable C^*-algebras.

Theorem 1.3. *The C^*-algebras S and U are simple.*

Theorem 1.4. *The C^*-algebras R_s and R_u are simple.*

Theorem 1.5. *The C^*-algebras R_s and R_u are purely infinite.*

The main technique is to use ideas and results of Muhly, Renault and Williams (building on earlier work of Rieffel) regarding equivalence of groupoids and strong Morita equivalence of their C^*-algebras. A common set-up is to have a groupoid G with an abstract transversal $T \subseteq G^\circ$ (the unit space of G). In our situation, in considering G_s, we show how we can use any single unstable equivalence class as a transversal. The subtlety here lies in the fact that such a set is dense in X and its relative topology is rather unwholesome. It does, however, possess a nice topology in a very natural way. We show how the Muhly-Renault-Williams machine may be adapted to such a situation. Reducing G_s on such a transversal yields an r-discrete groupoid, because of the transverse nature of the local stable and unstable co-ordinates. It is then much simpler to analyze these groupoids and translate the results back to the original algebras using the strong Morita equivalence.

2. Smale Spaces

Here, we give the basic definitions of a Smale space along with the constructions of the groupoids associated with them. This is taken more or less directly from [10], but we present it for completeness.

Let (X, d) be a compact metric space and let ϕ be a homeomorphism of X. We will assume throughout (X, ϕ) is topologically mixing [7]. (This is not part of the usual definition of Smale space.) We assume that we have constants

$$\epsilon_0 > 0, \quad 0 < \lambda_0 < 1$$

and a continuous map

$$(x, y) \in \{(x, y) \in X \times X \mid d(x, y) \leq 2\epsilon_0\} \to [x, y] \in X$$

satisfying axioms as in [7,8,10,11].

For $0 < \epsilon \leq \epsilon_0$, we define

$$V^s(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon, \quad [x, y] = y\}$$

$$V^u(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon, \quad [y, x] = y\}$$

so we have (as an axiom)

$$d\left(\phi(y),\, \phi(z)\right) \leq \lambda_0\, d(y, z), \quad y, z \in V^s(x, \epsilon)$$

$$d\left(\phi^{-1}(y),\, \phi^{-1}(z)\right) \leq \lambda_0\, d(y, z), \quad y, z \in V^u(x, \epsilon).$$

That is, ϕ contracts on $V^s(x, \epsilon)$ while ϕ^{-1} contracts on $V^u(x, \epsilon)$. The axioms imply that the map sending (y, z) in $V^u(x, \epsilon) \times V^s(x, \epsilon)$ to $[y, z]$ is a homeomorphism onto a neighbourhood of x. Such a neighbourhood is called a rectangle.

Next, we define, for any x in X,

$$V^s(x) = \bigcup_{n \geq 0} \phi^{-n}\left(V^s\left(\phi^n(x), \epsilon\right)\right)$$

$$V^u(x) = \bigcup_{n \geq 0} \phi^{n}\left(V^u\left(\phi^{-n}(x), \epsilon\right)\right),$$

both being independent of $\epsilon > 0$. Each set $\phi^{-n}\left(V^s\left(\phi^n(x), \epsilon\right)\right)$ is given the relative topology of X, while $V^s(x)$ is given the inductive limit topology. In this topology it is a locally compact, non-compact Hausdorff space. On the other hand, if we assume that (X, ϕ) is mixing, then $V^s(x)$ is dense in X [10]. We treat $V^u(x)$ in an analogous way.

We recall from [7,8],

$$G_s^1 = \{(x,y) \in X \times X \mid y \in V^s(x,\epsilon_0)\}$$

$$G_u^1 = \{(x,y) \in X \times X \mid y \in V^u(x,\epsilon_0)\}$$

$$G_s^n = (\phi \times \phi)^{-n+1}\ (G_s^1), \quad n \geq 2$$

$$G_u^n = (\phi \times \phi)^{n-1}\ (G_u^1), \quad n \geq 2$$

$$G_s = \bigcup_{n \geq 1} G_s^n$$

$$G_u = \bigcup_{n \geq 1} G_u^n.$$

Then G_s and G_u are equivalence relations on X, called stable and unstable equivalence. Each G_s^n, G_u^n are given the relative topologies of $X \times X$ and G_s, G_u are given the inductive limit topologies. Notice that the G_s-equivalence class of x in X is simply $V^s(x)$. Finally, we let

$$G_a^n = G_s^n \cap G_u^n, \quad n \geq 1$$

$$G_a = \bigcup_{n \geq 1} G_a^n.$$

Again, each G_a^n is given the relative topology of X while G_a is given the inductive limit topology. G_a is also an equivalence relation on X. For each x in X, we denote its G_a-equivalence class by $V^a(x)$; it is countable and dense in X if (X,ϕ) is mixing [10].

We regard G_s, G_u, G_a as principal groupoids. With their topologies they are locally compact and Hausdorff. Moreover, G_a is r-discrete and counting measure is a Haar system. Haar systems

$$\{\mu_s^x \mid x \in X\}\,, \quad \{\mu_u^x \mid x \in X\}$$

for G_s and G_u, respectively, are described in [7,8]. We let $S(X,\phi)$, $U(X,\phi)$ and $A(X,\phi)$ denote the C^*-algebras of G_s, G_u and G_a, respectively.

The map $\phi \times \phi$ acts as automorphisms of G_s, G_u and G_a (scaling the Haar systems in the first two). We form the semi-direct products as follows:

$$G_s \rtimes \mathbb{Z} = \{(x,n,y) \mid n \in \mathbb{Z},\ (\phi^n(x),y) \in G_s\}$$

$$G_u \rtimes \mathbb{Z} = \{(x,n,y) \mid n \in \mathbb{Z},\ (\phi^n(x),y) \in G_u\}$$

$$G_a \rtimes \mathbb{Z} = \{(x,n,y) \mid n \in \mathbb{Z},\ (\phi^n(x),y) \in G_a\}$$

with groupoid operations

$$(x,n,y)\cdot(x',n',y') = (x,\,n+n',\,y') \text{ if } y = x'$$

$$(x,n,y)^{-1} = (y,\,-n,\,x).$$

Observe that $G_s \subseteq G_s \rtimes \mathbb{Z}$, $G_u \subseteq G_u \rtimes \mathbb{Z}$, $G_a \subseteq G_a \rtimes \mathbb{Z}$ by identifying (x,y) in G_s with $(x,0,y)$ in $G_s \rtimes \mathbb{Z}$, for example.

Notice that $G_s^o = (G_s \rtimes \mathbb{Z})^o$, $G_u^o = (G_u \rtimes \mathbb{Z})^o$, $G_a^o = (G_a \rtimes \mathbb{Z})^o$, with the identifications above.

Finally, the map, η, sending (x,y,n) in $G_s \times \mathbb{Z}$ to $(x,\,n,\,\phi^n(y))$ in $G_s \rtimes \mathbb{Z}$ is bijective, and we transfer the product topology from $G_s \times \mathbb{Z}$ over via this map.

For any x in X, $(x,0,x)$ is in the unit space $(G_s \rtimes \mathbb{Z})^o$ and

$$\begin{aligned} r^{-1}\{(x,0,x)\} &= \{(x,n,y) \in G_s \rtimes \mathbb{Z}\} \\ &= \bigcup_{n\in\mathbb{Z}} \eta\left(\{(x,y) \in G_s\} \times \{n\}\right). \end{aligned}$$

Using this decomposition, we define a Haar system λ_s^x on $G_s \rtimes \mathbb{Z}$ by setting

$$\lambda_s^x \mid \{(x,y) \mid y \in V^s(x)\} \times \{n\} = \mu_s^x \circ \eta^{-1}.$$

We treat $G_u \rtimes \mathbb{Z}$ and $G_a \rtimes \mathbb{Z}$ similarly.

The C^*-algebras $C^*(G_s \rtimes \mathbb{Z})$, $C^*(G_u \rtimes \mathbb{Z})$ and $C^*(G_a \rtimes \mathbb{Z})$ are denoted R_s, R_u and R_a and are called the Ruelle algebras.

Another description of these algebras is to consider the automorphisms α_s, α_u and α_a of S, U and A, respectively, which are induced by the automorphisms $\phi \times \phi$ of G_s, G_u

and G_a and take the C^*-crossed products by $\mathbb{Z}$. That is,

$$R_s \cong S \rtimes \mathbb{Z}$$

$$R_u \cong U \rtimes \mathbb{Z}$$

$$R_a \cong A \rtimes \mathbb{Z}.$$

3. Generalized Transversals

In this section, we present a general result on groupoids. The idea is to show how the techniques of Muhly, Renault and Williams [6] on equivalence of groupoids may be applied to certain situations involving "generalized transversals". Let us begin by giving a simple example to motivate our result.

Let θ be a fixed irrational number between 0 and 1. Let G be the groupoid of the Kronecker flow on the two-torus, $\mathbb{T}^2$, determined by θ. That is,

$$G = \mathbb{T}^2 \times \mathbb{R}$$

$$(w_1, w_2, s) \cdot (z_1, z_2, t) = (w_1, w_2,\ s + t)$$

$$\text{if } z_1 = e^{2\pi i s}\, w_1 \quad \text{and} \quad z_2 = e^{2\pi i s\theta}\, w_2.$$

An example of an "abstract transversal" in this situation is

$$T = \mathbb{T} \times \{1\} \times \{0\} \subseteq G^0.$$

The reduction of G on T is:

$$G_T^T = \{g \in G \mid r(g),\ s(g) \in T\}$$

and can, in this case, be identified with

$$\mathbb{T} \times \mathbb{Z}$$

$$(w, k) \cdot (z, \ell) = (w,\ k + \ell) \ \text{ if } \ z = e^{2\pi i k\theta}\, w,$$

in a straightforward way.

Our point is that there is another, less obvious, choice. Pick any irrational α, between 0 and 1 and unequal to θ. Let

$$T = \left\{ \left(e^{2\pi i t},\ e^{2\pi i \alpha t},\ 0 \right) \in G^0 \mid t \in \mathbb{R} \right\}$$

which is a line, winding densely in $\mathbb{T}^2 \cong G^0$ and transverse to each G-orbit. This T can also be used as a transversal to G; of course, its relative topology in G is rather horrid. Instead we want to use its natural topology as a line. In this case

$$G_T^T \cong \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}$$

$$(x, k, \ell) \cdot (y, m, n) = (x,\ k+m,\ \ell+n)$$

$$\text{if}\ \ y = x + k + \ell\theta.$$

The difficulty lies in showing that this "re-topologizing" of the transversal can be incorporated into the equivalence of Muhly, *et al.*

¿From now on, we assume that G is a second countable locally compact, Hausdorff groupoid with Haar system. Let T be a locally compact, second countable, Hausdorff space and let $f : T \to G^0$ be a continuous, injective map.

We say that open sets $U \subseteq G$ and $V^r \subseteq T$ satisfy (Ar) if, for all x in U, there is a unique y in U with $s(x) = s(y)$ and $r(y) \in f(V^r)$.

We note that if (U, V^r) satisfy (Ar) then so does the pair $\left(U,\ f^{-1}\, r(U) \cap V^r\right)$. We will say that open sets $U \subseteq G$, $V^s \subseteq T$ satisfy (As) if (U^{-1}, V^s) satisfies (Ar).

Finally, we say that open sets $U \subseteq G$, V^r, $V^s \subseteq T$ satisfy (A) if

(i) for all t in V^r, there is a unique y in U with $r(y) = f(t)$ and $s(y)$ in $f(V^s)$

and

(ii) for all t in V^s, there is a unique y in U with $s(y) = f(t)$ and $r(y)$ in $f(V^r)$.

On the groupoid G, space T and continuous injective map $f : T \to G^0$, we consider the following conditions.

T1. For any x in G with $r(x)$ in $f(T)$, and open sets

$$x \in U_0 \subseteq G, \quad f^{-1} r(x) \in V_0 \subseteq T,$$

we may find open sets $U \subseteq G$, $V \subseteq T$ such that $x \in U \subseteq U_0$, $f^{-1}r(x) \in V \subseteq V_0$ and (U, V) satisfy (Ar).

T1′. For any x in G with $s(x)$ in $f(T)$, and open sets

$$x \in U_0 \subseteq G, \quad f^{-1}s(x) \in V_0 \subseteq T,$$

we may find open sets $U \subseteq G$, $V \subseteq T$ such that $x \in U \subseteq U_0$, $f^{-1}s(x) \in V \subseteq V_0$ and (U, V) satisfy (As).

T2. For any x in G with $r(x)$ and $s(x)$ in $f(T)$ and open sets

$$x \in U_0 \subseteq G$$

$$f^{-1}r(x) \in V_0^r \subseteq T$$

$$f^{-1}s(x) \in V_0^s \subseteq T,$$

there are open sets

$$x \in U \subseteq U_0$$

$$f^{-1}r(x) \in V^r \subseteq V_0^r$$

$$f^{-1}s(x) \in V^s \subseteq V_0^s$$

such that (U, V^r, V^s) satisfy (A).

T3. For any x in G, there is y in G with $r(y) = r(x)$, $s(y) \in f(T)$.

We let

$$G_T = \{x \in G \mid s(x) \in f(T)\}$$

$$G_T^T = \{x \in G \mid r(x),\, s(x) \in f(T)\}\,.$$

(Note that these are $G_{f(T)}$ and $G_{f(T)}^{f(T)}$, in the notation of [6].)

Lemma 3.1. *Suppose* G, T, f *satisfy* T1. *Then*

(i) G, T, f *satisfy* T1′.

(ii) G, T, f *satisfy* T2.

(iii) *the collection of sets*

$$U \cap s^{-1} f(V^s) \cap r^{-1} f(V^r),$$

where $U \subseteq G$, V^s, $V^r \subseteq T$ are open, forms a base for a topology on G_T^T.

(iv) *the collection of sets*

$$U \cap s^{-1} f(V^s),$$

where $U \subseteq G$, $V^s \subseteq T$ are open, forms a base for a topology on G_T.

Proof. The proof of (i) is straightforward and we omit it. As for (ii), consider x, U_0, V_0^r, V_0^s as in condition T2. We apply T1 and T1$'$ to obtain open sets $U_1, U_2 \subseteq U_0$ with $x \in U_1,\ x \in U_2$, and open sets V_1^r, V_1^s in T with $f^{-1}r(x) \in V_1^r \subseteq V_0^r$, $f^{-1}s(x) \in V_1^s \subseteq V_0^s$, such that (U_1, V_1^r) satisfy (Ar) while (U_2, V_1^s) satisfy (As). Writing $x = r(x)\,x$, we may find open sets U_3 and U_4 in G, $r(x) \in U_3$, $x \in U_4$ and $U_3U_4 \subseteq U_1 \cap U_2$. We once again apply T1 and T1$'$ to $r(x) \in U_3,\ f^{-1}r(x) \in V_1^r$ and $x \in U_4,\ f^{-1}s(x) \in V_1^s$ to obtain open sets U_5, U_6 in G with $r(x) \in U_5 \subseteq U_3$, $x \in U_6 \subseteq U_4$, and open sets V_2^r, V_2^s in T with $f^{-1}r(x) \in V_2^r \subseteq V_1^r$, $f^{-1}s(x) \in V_2^s \subseteq V_1^s$, such that (U_5, V_2^r) satisfy (Ar) and (U_6, V_2^s) satisfy (As). We let $U = U_5U_6$, $V^r = f^{-1}\,r(U) \cap V_2^r$ and $V^s = f^{-1}\,s(U) \cap V_2^s$. Let us prove (i) holds in (A). Suppose t is in V^r. Then $f(t) = r(yz)$, for some y in U_5, z in U_6. By (As), there is z' in U_6 with $r(z') = r(z)$ and $s(z')$ in $f(V_2^s)$. Then yz' is in U, $r(yz') = f(t)$ and $s(yz') = s(z') \in f(V^s)$. As for the uniqueness, suppose z_1 and z_2 are both in U with $r(z_1) = r(z_2) = f(t)$ and $s(z_1)$, $s(z_2)$ are both in $f(V^s)$. Then $z_1, z_2 \in U = U_5U_6 \subseteq U_3U_4 \subseteq U_1 \cap U_2 \subseteq U_2$, $r(z_1) = r(z_2)$, and $s(z_1)$, $s(z_2)$ are in $f(V^s) \subseteq f(V_1^s)$. By the uniqueness part of condition (As), we have $z_1 = z_2$. The proof of (ii) of (A) is similar. We omit the details.

Parts (iii) and (iv) are routine and we omit the details. ∎

Definition 3.2. We let H and Ω denote G_T^T and G_T, respectively, with the topologies given in the Lemma.

It is immediate that H and Ω are second countable. Also observe that, a sequence $\{x_n\}$ converges to x in H if and only if

$$\lim x_n = x \ \text{ in } \ G$$

$$\lim f^{-1}r(x_n) = f^{-1}r(x)$$

$$\text{and } \lim f^{-1}s(x_n) = f^{-1}s(x), \text{ in } T.$$

Similarly, $\{x_n\}$ converges to x in Ω if and only if

$$\lim x_n = x \text{ in } G$$

$$\lim f^{-1}s(x_n) = f^{-1}s(x) \text{ in } T.$$

The following is an immediate consequence of the definitions.

Lemma 3.3. *Suppose G, T, f satisfy* T1. *The collection of sets*

$$U \cap r^{-1}f(V^r) \cap s^{-1}f(V^s)$$

where U, V^r, V^s are open and satisfy (A) forms a base for the topology of H.

The collection of sets

$$U \cap s^{-1} f(V^s)$$

where $U \subseteq G$, $V^s \subseteq T$ are open and satisfy (As), *forms a base for the topology of Ω.*

Our aim is to show that conditions T1 and T3 imply that H is a locally compact, Hausdorff r-discrete groupoid with counting measure as a Haar system and that Ω is a G-H equivalence bimodule in the sense of [6].

The following is an immediate consequence of the definitions and we omit the proof.

Lemma 3.4. *Suppose (U, V^r, V^s) are open and satisfy* (A) *Let*

$$N = U \cap r^{-1}f(V^r) \cap s^{-1}f(V^s).$$

Then

$$r : N \longrightarrow f(V^r)$$

$$s : N \longrightarrow f(V^s)$$

are bijective.

Lemma 3.5. *Let t be in T and $x = f(t)$. Suppose (U, V^r, V^s) are open, satisfy* (A) *and $x \in U$, $f^{-1}r(x) = f^{-1}s(x) = t \in V^r \cap V^s$. Then there is $V \subseteq V^r \cap V^s$, $t \in V$ and V open such that*

$$U \cap r^{-1}f(V) \cap s^{-1}f(V) \subseteq G^0.$$

Proof. By definition, $U \cap G^0$ is open in G^0. As f is continuous, we may find V, $t \in V \subseteq V^r \cap V^s$ with $f(V) \subseteq U \cap G^0$. Now suppose y is in $U \cap r^{-1}f(V) \cap s^{-1}f(V)$. So $s(y)$ is in $f(V)$ and $f(V) \subseteq U$, so $s(y)$ is also in U. We have:

$$y \in U,\ r(y) \in f(V) \subseteq f(V^r)$$

$$s(y) \in f(V) \subseteq f(V^s)$$

$$s(y) \in U,\ r\left(s(y)\right) = s(y) \in f(V) \subseteq f(V^r)$$

$$s\left(s(y)\right) = s(y) \in f(V) \subseteq f(V^s)$$

$$\text{and } s(y) = s\left(s(y)\right).$$

Hence by (A) (ii), $y = s(y)$ by uniqueness. This implies y is in G^0. ∎

Theorem 3.6. *H is a second countable, locally compact, Hausdorff, r-discrete groupoid, with counting measure as Haar system.*

Proof. That $H = G_T^T$ is a groupoid, in the purely algebraic sense, is immediate. It is also immediate from the lemma and the facts that both T and G are second countable, that H is also. It is straightforward to check that the groupoid operations on H are continuous and we omit the details.

Lemma 3.5 shows that H^0 is open in H. The map $f : T \longrightarrow H^0$ is clearly bijective and it is easy to check that it is a homeomorphism.

We will now prove that $r : H \longrightarrow H^0$ is a local homeomorphism; the remaining conclusions follow from this.

First, by 3.3, we have a base for the topology of H consisting of sets

$$N = U \cap r^{-1}f(V^r) \cap s^{-1}f(V^s)$$

where U, V^r, V^s are open and satisfy (A). For such a set $r(N) = f(V^r)$, which is open in H^0, as f is a homeomorphism from T to H^0. Therefore, $r : H \longrightarrow H^0$ is open. By 3.4, r is bijective, and is continuous because the groupoid operations are. Therefore, r is a homeomorphism from N to $f(V^r)$. This completes the proof. ∎

Let us now bring in $\Omega = G_T$, with the topology as given earlier.

Theorem 3.7. *Suppose G, T, f satisfy* T1 *and* T3. *Then Ω is a G-H equivalence bimodule in the sense of* [6].

Proof. We must show:

(i) Ω is a left principal G-space: *i.e.* the left action of G is free and the map sending (x, y) in $G * \Omega$ to $(xy,\, y)$ in $\Omega \times \Omega$ is proper.

(ii) Ω is a right principal H-space.

(iii) the G and H-actions commute.

(iv) $r : \Omega/H \longrightarrow G^0$ is a homeomorphism.

(v) $s : G\backslash\Omega \longrightarrow H^0$ is a homeomorphism.

Notice that condition (iii) and the freeness conditions of (i) and (ii) do not involve any topology. Their proofs are exactly as in [6].

We will make use of the following characterization of proper maps, which is a relative exercise in topology. Let X and Y be second countable Hausdorff spaces and let $\pi : X \to Y$ be a continuous map. Then π is proper if and only if, for every sequence $\{x_n\}_1^\infty$ in X such that $\{\pi(x_n)\}_{n=1}^\infty$ is convergent in Y, $\{x_n\}_1^\infty$ has a convergent subsequence in X.

Suppose then that $\{(x_n, y_n)\}_1^\infty$ is a sequence in $G * \Omega$ (*i.e.* $s(x_n) = r(y_n)$, for all n) such that $\{(x_n y_n, y_n)\}_1^\infty$ has limit (z, y) in $\Omega \times \Omega$. Thus,

$$\lim x_n y_n = z, \quad \text{in } G,$$

$$\lim y_n = y, \quad \text{in } G$$

and

$$\lim\, f^{-1}s(x_n y_n) = f^{-1}s(z)$$

$$\lim f^{-1}s(y_n) = f^{-1}s(y)$$

in T. Also $s(x_n y_n) = s(y_n)$ and hence $s(z) = s(y)$. Immediately, $\{y_n\}$ converges to y in Ω and

$$\lim\ x_n = \lim\ x_n y_n\, y_n^{-1}$$

$$= z \cdot y^{-1}, \quad \text{in } G.$$

We conclude that

$$\lim(x_n, y_n) = (zy^{-1},\, y)$$

in $G * \Omega$.

We move on to the map

$$(x, y) \in \Omega * H \longrightarrow (x,\, xy) \in \Omega \times \Omega.$$

Suppose $\{(x_n, y_n)\}^\infty$ is in $\Omega * H$ (*i.e.* $s(x_n) = r(y_n) \in f(T)$, $s(y_n) \in f(T)$) and $(x_n,\, x_n y_n)$ converges to (x, z) in $\Omega \times \Omega$. This means that

$$\lim\, x_n = x \text{ in } G,$$

$$\lim\, x_n y_n = z \text{ in } G,$$

$$\lim\, f^{-1}s(x_n) = f^{-1}s(x) \text{ in } T,$$

$$\lim\, f^{-1}s(x_n y_n) = f^{-1}s(z) \text{ in } T.$$

Then we have

$$\begin{aligned} \lim\, y_n &= \lim\, x_n^{-1}\, x_n y_n \\ &= x^{-1}z \text{ in } G \end{aligned}$$

and

$$\begin{aligned} \lim\, f^{-1}s(y_n) &= \lim\, f^{-1}s(x_n y_n) \\ &= f^{-1}s(z) \\ &= f^{-1}s(x^{-1}z) \text{ in } T, \\ \lim\, f^{-1}r(y_n) &= \lim\, f^{-1}s(x_n) \\ &= f^{-1}s(x) \\ &= f^{-1}r(x^{-1}z) \text{ in } T. \end{aligned}$$

Hence y_n converges to $x^{-1}z$ in H and (x_n, y_n) converges to $(x,\, x^{-1}z)$ in $\Omega * H$. Thus, the map is proper.

To verify (iv) and (v), it suffices to show that $r : \Omega \to G^0$ and $s : \Omega \to H^0$ are continuous and open. In fact, since $f : T \to H^0$ is a homeomorphism, we will discuss

$f^{-1} \circ s : \Omega \to T$, rather than s. Suppose $\{x_n\}_1^\infty$ is a sequence converging to x in Ω. Then, we have

$$\lim x_n = x \quad \text{in} \quad G$$

$$\lim f^{-1} s(x_n) = f^{-1}\, s(x) \quad \text{in} \quad T.$$

It follows at once that $f^{-1}s$ is continuous and r is continuous on G and so from Ω to G^0. As for openness, it suffices to consider a set $U \cap s^{-1}\, f(V^s)$, $U \subseteq G$ open, $V^s \subseteq T$ open and (U, V^s) satisfy (As). It follows from (As) that $r\left(U \cap s^{-1}\, f(V^s)\right) = r(U)$, which is open since $r : G \to G^0$ is open [9]. Also, we have

$$s\left(U \cap s^{-1}\, f(V^s)\right) = s(U) \cap f(V^s)$$

and

$$f^{-1}\, s\left(U \cap s^{-1}\, f(V^s)\right) = f^{-1}\, s(U) \cap V^s$$

which is open in T since $s : G \to G^0$ is open and $f : T \to G^0$ is continuous. This completes the proof. ∎

4. Reduction of Stable and Unstable Equivalence

The aim of this section is to show that the results of Section 3 may be applied to the groupoids of Section 2. Specifically, we consider $G = G_s$ and $G = G_s \rtimes \mathbb{Z}$ as in Section 2 and, for any x_0 in X, the transversal $T = V^u(x_0)$. The map f is just the inclusion of T in X, regarded as the unit space of G. More accurately, in the case $G = G_s$

$$f(x) = (x, x), \qquad x \in V^u(x_0)$$

and in the case $G = G_s \rtimes \mathbb{Z}$,

$$f(x) = (x, 0, x), \qquad x \in V^u(x_0).$$

Let us also note here that the results immediately apply to $G = G_u$, $G = G_u \rtimes \mathbb{Z}$ and $T = V^s(x_0)$, by simply considering the Smale space (X, d, ϕ^{-1}) and noting, for example, $G_s(X, \phi^{-1}) = G_u(X, \phi)$.

It is worth stressing that the topology on $V^u(x_0)$ is that given in Section 2 and not the relative topology of X.

Lemma 4.1.

(a) *Let x_0 be in X. Define $f : V^u(x_0) \to G^0$ by $f(x) = (x, x)$, $x \in V^u(x_0)$. Then f is continuous and injective.*

(b) *Let x be in X, $V \subseteq V^u(x, \epsilon_0)$ $W_1, W_2 \subseteq V^s(x, \epsilon_0)$ open in the relative topologies of $V^u(x)$ and $V^s(x)$, respectively, and x in W_1. Let*

$$U = \{(x', y') \mid [x', x] = [y', x] \in V,\ [x, x'] \in W_1,\ [x, y'] \in W_2\}.$$

Then U is an open subset of G_s and (U, V) satisfy (Ar).

Proof. The proof of (a) is clear. For (b), it is easy to check that U is in G_s and is open. We must check (Ar). Suppose (x', y') is in U. Then it is easy to verify that $([x', x], y')$ is in U, $r([x', x], y') = f([x', x])$ is in $f(V)$ and $s([x', x], y') = s(x', y') = (y', y')$. As for uniqueness, suppose (x'', y'') is in U, $r(x'', y'')$ is in $f(V)$ and $s(x'', y'') = s(x', y')$. Then we see at once that $y'' = y'$. As (x'', y'') is in U, and $x'' \in V$,

$$x'' = [x'', x] = [y'', x] = [y', x] = [x', x].$$

This completes the proof. ∎

Theorem 4.2. *Let (X, d, ϕ) be a mixing Smale space and let x_0 be in X. Then $G = G_s$, $T = V^u(x_0)$ and f as above satisfy* T1 *and* T3.

Proof. Let us first suppose that (x, y) is in G_s^0, with x in $V^u(x_0)$. Suppose also that we have open sets $(x, y) \in U_0 \subseteq G_s$, $x \in V_0 \subseteq V^u(x_0)$. First, we may find an open set $x \in V_1 \subseteq V^u(x, \epsilon_0)$ with $V_1 \subseteq V_0$. Next, since the rectangles in X form a base for its topology, we may find open sets $x \in V_2 \subseteq V^u(x, \epsilon_0)$, $x \in W_2 \subseteq V^s(x, \epsilon_0)$, $x \in V_3 \subseteq V^u(x, \epsilon_0)$, $y \in W_3 \subseteq V^s(x, \epsilon_0)$ such that $[V_2, W_2] \times [V_3, W_3]$ contains (x, y) and is contained in U_0. Let $V = V_1 \cap V_2 \cap V_3$ and

$$U = \{(x', y') \mid [x', x] = [y', x] \in V, \quad [x, x'] \in W_2, \quad [x, y'] \in W_3\}.$$

Then $(x, y) \in U \subseteq U_0$, $x \in V \subseteq V_0$ and (U, V) satisfies (Ar) by 4.1.

For a general (x, y) in G_s, we have (x, y) is in G_s^n, for some n. We may apply the above arguments to $(\phi^n(x), \phi^n(y))$, $(\phi \times \phi)^n(U_0)$ and $\phi^n(V_0)$ to obtain the result. We omit the details.

It remains to verify T3. Let x be any point in X. As $V^a(x)$ is dense in X, we may find y in $V^a(x)$ with $d(x_0, y) < \epsilon_0$. Then $(x, [y, x_0])$ is in G_s, $r(x, [y, x_0]) = (x, x)$ and $s(x, [y, x_0] = [y, x_0])$ is in $V^u(x_0)$. Condition T3 follows. ∎

Theorem 4.3. *Let (X, d, ϕ) be a mixing Smale space and let x_0 be any point of X. Then $G = G_s \rtimes \mathbb{Z}$, $T = V^u(x_0)$, f as before satisfy the conditions* T1 *and* T3.

Proof. Property T3 follows easily from the fact that it holds for G_s and $G_s \subseteq G_s \rtimes \mathbb{Z}$ with $G_s^0 = G_s \rtimes \mathbb{Z}^0$.

As for T1, suppose (x, n, y) is in $G_s \rtimes \mathbb{Z}$ and U_0, V_0 are as in T1. Without loss of generality we may assume that

$$U_0 \subseteq \{(x', n, y') \mid (\phi^n(x'), y') \in G_s\}.$$

We may apply T1 for G_s from 4.2 to $(\phi^n(x), y) \in G_s$,

$$\tilde{U}_0 = \{(\phi^n(x'), y') \mid (x', n, y') \in U_0\}$$

$$\tilde{V}_0 = \phi^n(V_0)$$

to obtain $\tilde{U}$, $\tilde{V}$ satisfying (Ar). Now let

$$U = \left\{(x', n, y') \mid (\phi^n(x'), y') \in \tilde{U}\right\}$$

$$V = \phi^{-n}(\tilde{V}).$$

It is easy to check (U, V) satisfies (Ar). ∎

Definition 4.4. For x_0 in X, we let $G_s(x_0)$ denote the groupoid H of 4.2 in the case $G = G_s$, $T = V^u(x_0)$. We let $G_s(x_0) \rtimes \mathbb{Z}$ denote the groupoid H in the case

$$G = G_s \rtimes \mathbb{Z}, \qquad T = V^u(x_0).$$

Similarly, we define $G_u(x_0)$ is H in the case $G = G_u$ and $T = V^s(x_0)$ and $G_u(x_0) \rtimes \mathbb{Z}$ is H in the case $G = G_u \rtimes \mathbb{Z}$ and $T = V^s(x_0)$.

It is worth noting that all of these groupoids are r-discrete; $G_s(x_0)$, $G_u(x_0)$ are also principal. We may identify unit spaces: $G_s(x_0)^0$, $G_s(x_0) \rtimes \mathbb{Z}^0$ with $V^u(x_0)$. Note that the $G_s(x_0)$-equivalence class of x in $V^u(x_0)$ is $V^a(x)$.

The notation $G_s(x_0) \rtimes \mathbb{Z}$ may be somewhat misleading: this is the semi-direct product groupoid *only* in the case $V^u(x_0)$ contains a fixed-point of ϕ and hence is ϕ-invariant.

5. Proofs of the Main Results

We begin with Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We show that G_s is amenable in the sense of Renault. The case for G_u is analogous.

We construct a sequence $\{f_n\}$ in $C_c(G_s)$ such that $f_n f_n^*$ converges to 1 uniformly on compact subsets of G_s. (Note that the other condition of II.3.6 of [9] follows since the unit space of G_s is compact.) Let Δ denote the unit space of G_s. Let g in $C_c(G_s)$ be chosen so that g is non-negative and strictly positive on Δ. Then

$$
\begin{aligned}
gg^*(x,x) &= \int_{y \in V^s(x)} |g(x,\, y)|^2 \; d\mu_s^x(y) \\
&> 0, \quad \text{for all } x \text{ in } X.
\end{aligned}
$$

Let

$$f(x,y) = gg^*(x,x)^{-\frac{1}{2}} \; g(x,y), \qquad (x,y) \in G_s.$$

Then f is in $C_c(G_s)$ and

$$ff^*(x,x) = 1, \quad \text{for all } x \text{ in } X.$$

Define

$$
\begin{aligned}
f_n &= \lambda^{-\frac{n}{2}} \, f \circ (\phi \times \phi)^n \\
&= \lambda^{\frac{n}{2}} \; \alpha_s^{-n}(f)
\end{aligned}
$$

(with $\log(\lambda)$ equal to the entropy of ϕ). Then

$$\begin{aligned} f_n f_n^* &= \lambda^{\frac{n}{2}}\ \alpha_s^{-n}(f)\ \lambda^{\frac{n}{2}}\ \alpha_s^{-n}(f^*) \\ &= \lambda^n\ \alpha_s^{-n}(ff^*) \\ &= ff^* \circ (\phi \times \phi)^n. \end{aligned}$$

Now for any compact set $K \subseteq G_s$ and $\epsilon > 0$, there is a $\delta > 0$ so that $\|ff^* - 1\| < \epsilon$ on the set

$$\Delta_\delta = \{(x, y) \in G_s \mid d(x, y) < \delta\},$$

because ff^* is continuous and $ff^* \mid \Delta = 1$. Choose N sufficiently large so that

$$(\phi \times \phi)^n(K) \subseteq \Delta_\delta, \qquad \text{for all } n \geq N.$$

Then $n \geq N$ implies $\|f_n f_n^* - 1\| < \epsilon$ on K. We have now shown the existence of $\{f_n\}$ as desired.

The remainder of the proof follows from [2,9]. ▮

Proof of 1.2. The amenability of $G_s \rtimes \mathbb{Z}$ results from 1.1 and the amenability of $\mathbb{Z}$ as follows. Define

$$c : G_s \rtimes \mathbb{Z} \longrightarrow \mathbb{Z}$$

by $c(x, n, y) = n$, $(x, n, y) \in G_s \rtimes \mathbb{Z}$. The kernel of c is G_s and this situation satisfies the hypotheses of Theorem 5.2.13 [2]. It follows that $G_s \rtimes \mathbb{Z}$ is amenable. Again, the rest follows from [2,9]. ▮

Before beginning the proof of 1.3, we need a dynamical result and a result regarding equivalence of amenable groupoids.

Lemma 5.1. *Let x_0 be in X and x be in $V^u(x_0)$. Then $V^a(x)$ is dense in $V^u(x_0)$, in the new topology introduced in Section* 2.

Proof. It suffices to show that $V^a(x) \cap V^u(x_0, \epsilon_0)$ is dense in $V^u(x_0, \epsilon_0)$. Let U be an open set in $V^u(x_0, \epsilon_0)$. Then $[U, V^s(x_0, \epsilon_0)]$ is open in X. Since $V^a(x)$ is dense, we may find y in $V^a(x)$ in this set. Let $z = [y, x_0]$, which is in U. Also y and z are stably equivalent, so z is stably equivalent to x. Also, z and x are both in $V^u(x_0)$, hence they are unstably equivalent. Thus z is in $V^a(x)$, as desired. ▮

Proof of 1.3. First of all G_s is amenable. Let x_0 be in X. By [6], $G_s(x_0)$ is equivalent to G_s. Hence $G_s(x_0)$ is also amenable by Theorem 2.2.13 of [2].

Therefore the C^*-algebras

$$C^*(G_s) \cong C^*_{\text{red}}(G_s)$$

$$C^*\left(G_s(x_0)\right) \cong C^*_{\text{red}}\left(G_s(x_0)\right)$$

have the same ideal structure. Now $G_s(x_0)$ is an r-discrete groupoid and so its ideals are described completely by II.4.6 of [9]. In particular, in view of the last lemma, $C^*_{\text{red}}\left(G_s(x_0)\right)$ is simple. The conclusion follows. ▮

Proof of 1.4. The argument begins in the same way as 1.3. We use the fact that $G_s \rtimes \mathbb{Z}$ is equivalent to $G_s(x_0) \rtimes \mathbb{Z}$, which is an r-discrete groupoid. To apply II.4.6 of [9] to show $C^*_{\text{red}}\left(G_s(x_0) \rtimes \mathbb{Z}\right)$ is simple, we must again see two things: $G_s(x_0) \rtimes \mathbb{Z}$ is minimal and essentially principal. Minimality is the same as for $G_s(x_0)$. In both cases, the unit space is $V^u(x_0)$ and, for a given x in $V^u(x_0)$, its $G_s(x_0) \rtimes \mathbb{Z}$-equivalence class contains its $G_s(x_0)$-equivalence class which is $V^a(x)$, which is already dense in $V^u(x_0)$. It remains to show that $G_s(x_0) \rtimes \mathbb{Z}$ is essentially principal; that is, the isotropy

$$r^{-1}\left\{(x,0,x)\right\} \cap s^{-1}\left\{(x,0,x)\right\} = \left\{(x,n,x) \in G_s(x_0) \rtimes \mathbb{Z} \mid n \in \mathbb{Z}\right\}$$

is trivial (equals $\{(x,0,x)\}$) for a dense set of x in $V^u(x_0)$. We will, in fact, show that the set of x for which this is non-trivial is countable. As $V^u(x_0)$ is locally compact and has no isolated points, the conclusion follows.

This will be divided into three Lemmas.

Lemma 5.2. *In a mixing Smale space (X, ϕ), the set of periodic points of period n,*

$$\mathrm{Per}_n = \{x \mid \phi^n(x) = x\}$$

is finite, for any positive integer n.

Proof. As noted in [8,10,11], ϕ is expansive. That is, there is an $\epsilon_1 > 0$ so that for any x, y in X, if $d\left(\phi^k(x), \phi^k(y)\right) < \epsilon_1$, for all k in $\mathbb{Z}$, then $x = y$. We may then choose ϵ_n sufficiently small so that $d(x, y) < \epsilon_n$ implies $d\left(\phi^i(x), \phi^i(y)\right) < \epsilon_1$, for $0 \leq i < n$. Then it is easy to check that ϕ^n is also expansive, with constant ϵ_n. From this, it follows that the distance between any two fixed-points of ϕ^n is at least ϵ_n. The result follows since X is compact. ∎

Lemma 5.3. *Suppose $\phi^n(x)$ is in $V^s(x)$, for some x in X, $n \geq 1$. Then*

$$\lim_{k \to +\infty} \phi^{nk}(x)$$

exists and is in Per_n.

Proof. Suppose z is a limit point of $\left\{\phi^{nk}(x) \mid k \geq 1\right\}$. Then

$$\begin{aligned} \phi^n(z) &= \phi^n\left(\lim_i \phi^{nk_i}(x)\right) \\ &= \lim_i \phi^{nk_i}\left(\phi^n(x)\right) \\ &= \lim_i \phi^{nk_i}\ (x) \\ &= z \end{aligned}$$

since x and $\phi^n(x)$ are stably equivalent.

Thus, the limit points of $\left\{\phi^{nk}(x) \mid k \geq 1\right\}$ — which exist as X is compact — are contained in Per_n. We must show there is at most one such point.

Let $\mathrm{Per}_n = \{x_1, \cdots, x_m\}$ (by 5.2) and choose open neighbourhoods U_i of x_i such that $\phi^n(U_i) \cap U_j = \emptyset$ for $i \neq j$. If there are infinitely $k \geq 1$ such that $\phi^{nk}(x)$ is not in the union of the U_i, then this sequence has a limit point in $X - U_1 - U_2 - \cdots - U_m$, by compactness. This limit point is in Per_n, but Per_n is contained in $U_1 \cup U_2 \cup \cdots \cup U_m$, a contradiction.

Thus, for some $k_0 \geq 0$, $\phi^{nk}(x)$ is in $U_1 \cup \cdots \cup U_m$, for all $k \geq k_0$. But as $\phi^n(U_i) \cap U_j = \emptyset$, for all $i \neq j$, $\phi^{nk}(x)$ must all be in the same U_i, for $k \geq k_0$. It follows then that

$$\lim_h \phi^{nk}(x) = x_i$$

as x_i is the only point of Per_n in $\overline{U_i}$. This completes the proof. ∎

Lemma 5.4. *The set of x in $V^u(x_0)$ such that $\phi^n(x)$ is in $V^s(x)$, for some $n \neq 0$, is countable.*

Proof. It is clearly sufficient to prove this for a fixed $n \neq 0$.

Suppose x is such that $\phi^n(x)$ is in $V^s(x)$, x in $V^u(x_0)$. Then by Lemma 5.3,

$$\lim_{k \to +\infty} \phi^{nk}(x) = y,$$

for some y in Per_n. It is then easy to see that x is in $V^s(y)$. So the set of x under consideration is contained in

$$\bigcup_{y \in \mathrm{Per}_n} V^s(y) \cap V^u(x_0).$$

Now, Per_n is finite and we noted earlier that for any x_0, y

$$V^s(y) \cap V^u(x_0)$$

is countable. This completes the proof. ∎

We can now complete the proof of 1.4 outlined earlier. The groupoid $G_s(x_0) \rtimes \mathbb{Z}$ is minimal as described above. For a fixed unit $(x, 0, x)$ with non-trivial isotropy, (x, n, x) is in $G_s(x_0) \rtimes \mathbb{Z}$ for some $n \neq 0$. This means $\phi^n(x)$ is in $V^s(x)$. The set of such x is countable. Hence the points of non-trivial isotropy are countable and their compliment is dense. That is, $G_s(x_0) \rtimes \mathbb{Z}$ is essentially principal. The conclusion follows. ∎

Toward the proof of 1.5, we begin with the following.

Proposition 5.5. *Let A and B be simple separable C^*-algebras which are strongly Morita equivalent. If A is purely infinite then so is B.*

Proof. Let K denote the C^*-algebra of compact operators on the Hilbert space $\ell^2(\mathbb{N})$. For each i, j in $\mathbb{N}$, e_{ij} denotes the operator

$$(e_{ij}\ \xi)(k) = \begin{cases} \xi(j) & \text{if } i = k \\ 0 & \text{otherwise,} \end{cases}$$

for ξ in $\ell^2(\mathbb{N})$, k in $\mathbb{N}$.

As A and B are separable and strongly Morita equivalent, we have

$$A \otimes K \cong B \otimes K.$$

First, we show $A \otimes K$ is purely infinite. Let C be any hereditary subalgebra of $A \otimes K$. Choose $1 \geq x \geq 0$, $x \neq 0$, in C. For some i in $\mathbb{N}$,

$$(1 \otimes e_{ii})\, x(1 \otimes e_{ii}) = x_{ii} \otimes e_{ii}$$

is non-zero, where $0 \leq x_{ii} \leq 1$ is in A. As A is purely infinite, there is an infinite projection p in $x_{ii}Ax_{ii}$. Then $p \otimes e_{ii}$ is an infinite projection in C.

Next, as $A \otimes K \cong B \otimes K$, the latter is purely infinite. Finally B is isomorphic to $B \otimes e_{11}$ which is a hereditary subalgebra of $B \otimes K$, and hence purely infinite. ▮

Definition 5.6 ([1]). A topological groupoid G is called *locally contracting* if, for every non-empty open set $U \subseteq G^0$, there is an open G-set Δ such that

$$r(\bar{\Delta}) \subsetneq s(\Delta) \subseteq U.$$

Also, compare this definition with that of a "local boundary" contained in [5].

Proposition 5.7. *For any x_0 in X, $G_s(x_0) \rtimes \mathbb{Z}$ is locally contracting.*

Proof. Suppose $U \subseteq V^u(x_0)$ is non-empty and open. Then for some $n \geq 1$,

$$\phi^{-n}(U) \cap V^u\left(\phi^{-n}(x_0),\ \epsilon_0\right)$$

is non-empty and open in $V^u\left(\phi^{-n}(x_0),\ \epsilon_0\right)$. Consider

$$\left[\phi^{-n}(U) \cap V^u\left(\phi^{-n}(x_0),\ \epsilon_0\right), \quad V^s\left(\phi^{-n}(x_0),\ \epsilon_0\right)\right]$$

which is open in X. Thus, it contains a periodic point for ϕ, say x_1, with $\phi^N(x_1) = x_1$, for some $N \geq 1$.

Find $\epsilon_1 > 0$ such that

$$\left[V^u(x_1, \epsilon_1),\, \phi^{-n}(x_0)\right] \subseteq \phi^{-n}(U) \cap V^u\left(\phi^{-n}(x_0),\ \epsilon_0\right).$$

As $V^u(x_1, \epsilon_1)$ is not discrete, we may find $m \geq 1$ such that

$$\begin{aligned}\phi^{-mN}\left(V^u(x_1, \epsilon_1)\right) &\subseteq V^u\left(x_1,\, \epsilon_1 \lambda^{-mN}\right) \\ &\subsetneqq V^u(x_1,\, \epsilon_1).\end{aligned}$$

For each y in $V^u(x_1, \epsilon_1)$, $\phi^{-mN}(y)$ is also in the same set and

$$\begin{aligned}\left[y,\ \phi^{-n}(x_0)\right] &\in V^s(y) \\ \left[y,\ \phi^{-n}(x_0)\right] &\in V^u\left(\phi^{-n}(x_0),\, \epsilon_0\right) \\ \left[\phi^{-mN}(y),\ \phi^{-n}(x_0)\right] &\in V^s\left(\phi^{-mN}(y)\right) \\ \left[\phi^{-mN}(y),\ \phi^{-n}(x_0)\right] &\in V^u\left(\phi^{-n}(x_0),\, \epsilon_0\right).\end{aligned}$$

Let

$$\Delta = \left\{ \left(\phi^n \left[\phi^{-mN}(y),\ \phi^{-n}(x_0)\right],\ mN,\ \phi^n \left[y,\, \phi^{-n}(x_0)\right]\right) \,\Big|\, y \in V^u(x_1, \epsilon_1) \right\}.$$

It is easy to check that $\Delta \subseteq G_s(x_0) \rtimes \mathbb{Z}$ and is a $G_s(x_0) \rtimes \mathbb{Z}$-set. Moreover,

$$\begin{aligned}r(\bar{\Delta}) &= \phi^n \left[\phi^{-mN}\left(V^u(x_1, \epsilon_1)\right), \quad \phi^{-n}(x_0)\right] \\ s(\Delta) &= \phi^n \left[V^u(x_1, \epsilon_1), \quad \phi^{-n}(x_0)\right]\end{aligned}$$

and the desired conclusion follows. ∎

The proof of 1.5 follows immediately from 5.5, 5.7, Proposition 2.4 of [1] and the fact that $G_s(x_0) \rtimes \mathbb{Z}$ is essentially principal, as shown in the proof of 1.4.

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