Functoriality of the C^*-algebras Associated with Hyperbolic Dynamical Systems

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Abstract

We consider a mixing Smale space, the relations of stable and unstable equivalence on such a space and the C^*-algebras which are constructed from them. In general, these associations are not functorial. However, we show that, if one restricts to the class of s-resolving, finite-to-one factor maps, then the construction of the stable C^*-algebra is contravariant, while that of the unstable C^*-algebra is covariant. We also discuss the constructions of these C^*-algebras for Smale spaces which are not mixing.

1 Introduction and notation

There are several papers [Rue, Put1, Put2, PS] discussing the construction of various C^*-algebras from certain hyperbolic dynamical systems. More specifically, one begins with a mixing Smale, (X, d, ϕ). This means that (X, d) is a compact metric space and ϕ is a homeomorphism of X possessing canonical coordinates of contracting and expanding directions. We refer to [Put1] for the definitions. These arise as the non-wandering set for Smale's Axiom A systems. We will note here that the key item is a map $[,]$, which

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is defined on a pair x, y in X such that $d(x,y) \leq \epsilon_X$, where ϵ_X is some fixed positive constant. The idea is that $[x,y]$ is where the local stable set (i.e. contracting direction) of x meets the local unstable set (i.e. expanding direction) of y. From the bracket, the local stable and unstable sets of x are given by

$$\begin{aligned} V^s(x,\epsilon) &= \{y \mid d(x,y) < \epsilon, [y,x] = x\} \\ V^u(x,\epsilon) &= \{y \mid d(x,y) < \epsilon, [x,y] = x\}, \end{aligned}$$

for any $\epsilon \leq \epsilon_X$. The global stable and unstable sets of x are given by

$$\begin{aligned} V^s(x) &= \cup_{n=0}^{\infty} \phi^{-n}(V^s(\phi^n(x),\epsilon)) \\ V^u(x) &= \cup_{n=0}^{\infty} \phi^{n}(V^u(\phi^{-n}(x),\epsilon)). \end{aligned}$$

We consider the principal groupoids [Ren] of stable and unstable equivalence

$$\begin{aligned} G_s(X,\phi) &= \{(x,y) \mid y \in V^s(x)\} \\ G_u(X,\phi) &= \{(x,y) \mid y \in V^u(x)\}, \end{aligned}$$

respectively. These are locally compact Hausdorff groupoids with Haar systems and we may build their groupoid C^*-algebras, which are denoted $S(X,\phi)$ and $U(X,\phi)$ respectively [Put1].

One property which has not been considered is the question of the functoriality of this construction. A *factor map*, π, between two such systems (X,ϕ) and (Y,ψ) is a continuous surjection from X onto Y satisfying $\pi \circ \phi = \psi \circ \pi$. These are the natural morphisms between such dynamical systems. We want to understand whether such maps induce *-homomorphisms of the C^*-algebras.

The importance of this issue can hardly be understated. A basic result in dynamics asserts that such a system, (X,ϕ), if is also topologically transitive, admits a Markov partition [KH, Bo] and therefore there is a very well-behaved factor map

$$\pi : (\Sigma_A, \sigma_A) \to (X,\phi),$$

where A is some square, non-negative integer matrix and (Σ_A, σ_A) the associated shift of finite type. It isn't too important for the moment exactly what a shift of finite type is (see [LM] for the definition), or in exactly what sense this map is well-behaved. The main point is that these form a particularly well-understood class of Smale space. Usually in dynamics, this is the starting point for an analysis of (X, ϕ). From our point of view, the C^*-algebras $S(\Sigma_A, \sigma_A)$ and $U(\Sigma_A, \sigma_A)$ are both AF-algebras and are particularly tractable. (See [Put2, Put1] for more details.) Some sort of map at the level of C^*-algebras would be particularly useful.

Unfortunately, such an algebra homomorphism does not exist in general. The aim of the paper is to show that there is functoriality for a special class of factor maps which are called s-resolving. A map is s-resolving if it is injective on each stable set. (Of course, there is an analogous notion of u-resolving.) Basically, we will show that the construction of $S(X, \phi)$ is contravariant for finite-to-one, s-resolving maps, while the construction of $U(X, \phi)$ is covariant.

That statement is not exactly what we will show. One of the disadvantages of the groupoids $G_s(X, \phi)$ and $G_u(X, \phi)$ is that they are not r-discrete. In [PS], a technique was developed to introduce r-discrete groupoids which are equivalent in the sense of Muhly, Renault and Williams [MRW] to the ones of interest. The basic technique is to reduce on a transversal. For stable equivalence, the natural choice for such a transversal is the unstable set of some point, x, of X. There is a subtlety here since the relative topology of such a subset is rather horrid. In [PS], it is shown how to introduce a new topology and how to adapt the methods of [MRW] to this situation. The result is to produce groupoids, denoted $G_s(X, \phi, x)$ and $G_u(X, \phi, x)$ which are equivalent to our earlier pair and are r-discrete. We denote their C^*-algebras by $S(X, \phi, x)$ and $U(X, \phi, x)$, respectively. In the next section, we also discuss generalizing this situation to allow a finite or countable set instead of a single point x. Our main results show the functoriality of the constructions of the C^*-algebras $S(X, \phi, x)$ and $U(X, \phi, x)$.

It is worth noting that stable equivalence for (X, ϕ) is unstable equivalence for (X, ϕ^{-1}) and that if π is a factor from (X, ϕ) to (Y, ψ), then it is also a factor from (X, ϕ^{-1}) to (Y, ψ^{-1}). With these observations, it is a simple matter to state analogous results for u-resolving maps instead of s-resolving. Our only real excuse in concentrating on the case of s-resolving is that they arise naturally if the context of substitution tiling systems [AP].

It is interesting to view the results of [BMT] in the context of our result. Suppose that A is a primitive, non-negative integer matrix. Associated to A are a shift of finite type (Σ_A, σ_A) (which is a Smale space) and a dimension group G_A. Given two such matrices, A and B, [BMT] shows that if there is an s-resolving factor map between the systems, then there is a surjective positive group homomorphisms between the dimnsion groups. In fact, these dimension groups are the K-zero groups of the associated C^*-algebras. The construction of [BMT] is just the composition of our functor with K-theory, which is well-known to be functorial. (Here, "right closing" is u-resolving and the C^*-algebra under consideration is $S(\Sigma_A, \sigma_A)$.)

There is still one other issue which we address in this paper. The construction of the C^*-algebras $S(X, \phi)$ and $U(X\phi)$, to this point, has been given only for mixing Smale spaces. In the next section we will discuss more general situations. We show how, using basic structure results for the dynamics, we can extend the definition easily to any Smale space in which each point is non-wandering.

The third section describes the functoriality results.

2 Non-mixing Smale spaces

So far, the construction of C^*-algebras from a Smale space, (X, d, ϕ), has only been given in the case of a mixing Smale space. Here, we will discuss more general situations. Basically, we can reduce to the mixing case by applying Smale's spectral decomposition theorem.

Theorem 2.1. *Let (X, ϕ) be any Smale space in which every point is non-wandering. Then there is a partition of the space X into a finite number of pairwise disjoint closed sets $X_1, \ldots, X_n$, and a permutation, σ, of $1, \ldots, n$ such that*

$$\phi(X_i) = X_{\sigma(i)}$$

for all $i = 1, \ldots, n$. Moreover, for any i and k such that $\sigma^k(i) = i$, the system (X_i, ϕ^k) is a mixing Smale space.

We will not give a proof. Notice that the local product structure on a Smale space means that it has the shadowing property. It can then be seen, just as in the case of the non-wandering set for a hyperbolic diffeomorphism, that the periodic points are dense. The basic outline of the rest of the proof is exactly for basic sets for Axiom A systems, as given in [KH].

We can define the stable and unstable groupoids of (X_i, ϕ^k), where $\sigma^k(i) = i$, as before and then define stable and unstable groupoids for (X, ϕ) by

$$G_s(X, \phi) = \cup_{i=1}^n G_s(X_i, \phi^{k_i})$$

and

$$G_u(X,\phi) = \cup_{i=1}^{n} G_u(X_i, \phi^{k_i}),$$

where k_i is the least integer such that $\sigma^{k_i}(i) = i$, and each being given the disjoint union topology.

To define the Haar systems, we proceed as follows. For each cycle, $(i_1 \cdots i_k)$ in σ, we choose an element, say i_1, of the cycle. We let μ_s be the Haar system on $G_s(X_{i_1}, \phi^k)$. Then for each $1 < j \leq k$, we define the Haar system on $G_s(X_{i_j}, \phi^k)$ to be $\lambda^{(1-j)/k} \mu \circ (\phi \times \phi)^{-j+1}$, where $\log(\lambda)$ is the entropy of (X_{i_1}, ϕ^k). The unstable groupoid is handled in a similar way.

This provides groupoids for stable and unstable equivalence with almost all the properties of the mixing case. The most important difference, of course, is that the equivalence classes are no longer dense. We have

$$S(X,\phi) \cong \bigoplus_{i=1}^{n} S(X_i, \phi^{k_i}).$$

The automorphism $\alpha_s^{\phi} = \alpha_s$ of [Put1] permutes these summands according to σ.

We also want to remark that the techniques of [PS] are still essentially valid. Let us describe the main ideas. Consider a mixing Smale space, (X,ϕ). If we select any point, x_0, of X, we let $T = V^u(x_0)$ be its unstable set. This set can be regarded as an abstract transversal to the groupoid $G_s(X,\phi)$ and the reduction of the groupoid is denoted by $G_s(X,\phi,x_0)$. The important subtlety here is that we must endow T and this groupoid with a new topology, which is finer than the relative topologies of X and $G_s(X,\phi)$. In this context, the automorphism α_s becomes an isomorphism

$$\alpha_s^{\phi} : S(X,\phi,x_0) \to S(X,\phi,\phi(x_0)).$$

Although it is not stated there in this generality, this situation can be extended. Suppose that A is any finite or countable subset of X with no two points of A being unstably equivalent. Then we let

$$T = V^u(A) = \cup_{x \in A} V^u(x)$$

and we can reduce the groupoid $G_s(X,\phi)$ on this abstract transversal and the result we denote by $G_s(X,\phi,A)$. Here, we give the set T the disjoint union topology, using the same topology on each $V^u(x)$ as before. We will let $S(X, \phi, A)$ to denote the C^*-algebra of $G_s(X, \phi, A)$. Now our automorphism becomes an isomorphism

$$\alpha_s^\phi : S(X, \phi, A) \to S(X, \phi, \phi(A)).$$

There are obvious analogous constructions with the groupoid of unstable equivalence, using stable sets as abstract transversals.

Now all of these techniques may be extended to the case that every point of (X, ϕ) is non-wandering. The only important issue is that one must choose a finite or countable set A such that T contains at least one point from each of the sets X_i as above, so that every stable (or unstable) equivalence class will meet $V^u(A)$ ($V^s(A)$, respectively).

3 The main results

Let (X, ϕ) and (Y, ψ) denote mixing Smale spaces as described in [Put1]. A *factor map* π from (X, ϕ) to (Y, ψ) is a continuous surjection

$$\pi : X \to Y$$

such that

$$\pi \circ \phi = \psi \circ \pi.$$

We will usually write $\pi : (X, \phi) \to (Y, \psi)$ for such a factor map.

We will say that such a map is *finite-to-one* if there is a constant B such that $\#\pi^{-1}\{y\} \leq B$, for all y in Y. (Here $\#A$ denotes the number of elements of a set A.)

Definition 3.1 (Fried [Fr]). *A factor map $\pi : (X, \phi) \to (Y, \psi)$ is* s-resolving *(*u-resolving*) if, for every x in X, $\pi \mid V^s(x)$ ($\pi \mid V^u(x)$, respectively) is injective.*

We begin our analysis of factor maps with the following simple lemma.

Lemma 3.2. *Let $\pi : (X, \phi) \to (Y, \psi)$ be a factor map between two Smale spaces. Then there is $\epsilon_\pi > 0$ such that, for all x, x' in X with $d(x, x') < \epsilon_\pi$, we have*

$$\pi([x, x']) = [\pi(x), \pi(x')].$$

Proof. Let $\epsilon_Y > 0$ be such that $[y, y']$ is defined for all y, y' in Y with $d(y, y') < \epsilon_Y$. As π is continuous and X is compact, π is uniformly continuous. So there is an $\epsilon > 0$ such that $d(\pi(x), \pi(x')) < \epsilon_Y$ for all x, x' in X with $d(x, x') < \epsilon$. We may also choose this $\epsilon < \epsilon_X$ so that $[x, x']$ is defined for all x, x' in X with $d(x, x') < \epsilon$. Now, for any x, x' in X with $d(x, x') < \epsilon$, both sides of the equation above are defined. That fact that they are equal follows easily from the choice of ϵ and the description of $[x, x']$ as

$$[x, x'] = V^s(x, \epsilon) \cap V^u(x', \epsilon)$$

and

$$\begin{aligned} V^s(x, \epsilon) &= \{z \in X \mid d(\phi^n(z), \phi^n(x)) < \epsilon, \text{ for all } n \geq 0\} \\ V^u(x', \epsilon) &= \{z \in X \mid d(\phi^{-n}(z), \phi^{-n}(x')) < \epsilon, \text{ for all } n \geq 0\} \end{aligned}$$

and a similar description of $[\pi(x), \pi(x')]$. We omit the details. $\square$

The next result is easy, but will be used frequently.

Lemma 3.3. *Let $\pi : (X, \phi) \to (Y, \psi)$ be an s-resolving factor map between two Smale spaces. Let ϵ_π be as in Lemma 3.2. If x, x' are in X with $d(x, x') < \epsilon_\pi$ and $\pi(x) = \pi(x')$, then x and x' are unstably equivalent; that is, $x' \in V^u(x)$.*

Proof. From Lemma 3.2, each of the following is defined and we have

$$\begin{aligned} \pi([x, x']) &= [\pi(x), \pi(x')] \\ &= [\pi(x), \pi(x)] \\ &= \pi(x). \end{aligned}$$

Now the points x and $[x, x']$ are both in $V^s(x)$ and since π is s-resolving, it is injective on this set. So we have $[x, x'] = x$ and this implies the result. $\square$

We are now prepared to prove the following result.

Theorem 3.4. *Let $\pi : (X, \phi) \to (Y, \psi)$ be a finite-to-one, s-resolving factor map between two Smale spaces in which each point is non-wandering. Let x be any point of X. Then $\pi \mid V^s(x)$ is a homeomorphism from $V^s(x)$ to $V^s(\pi(x))$.*

Proof. By applying our spectral decomposition result in the last section and the discussion which follows it, we may restrict to the case when (X, ϕ) is mixing.

First, we let $r = min\{\#\pi^{-1}\{y\} \mid y \in Y\}$ and choose a point y in Y such that $\pi^{-1}\{y\}$ has exactly r elements, which we denote by $x_1, \dots, x_r$. For each i, choose an open set U_i containing x_i with diameter less than ϵ_π. We also assume that these sets are chosen to be pairwise disjoint.

We claim that there is an open neighbourhood W of y in Y with the following property. Every periodic point in W has exactly r pre-images under π, one in each of the sets U_i. If there is no such set, we may find a sequence of periodic points, y_k, converging to y where this property fails. Now the factor map π must map orbits to orbits. And since it is finite-to-one, it cannot map an infinite set onto a finite one. This means that every point in the pre-image of a periodic point must itself be periodic. No two distinct periodic points can be unstably equivalent, hence no two distinct points in the pre-image of a periodic point can be within ϵ_π, by Lemma 3.3. Therefore there is at most one point in the pre-image of any y_k in any U_i. Next we argue that there must be at least one point in the pre-image of y_k in each U_i. By the minimality of r and choice of y, we know that the pre-image of y_k must contain at least r points. If there is a U_i not containing any of them, then there must be one, say z_k, in $X - (U_1 \cup \dots \cup U_r)$. By passing to a subsequence, we may assume that the z_k converge to some point z in X. Then we have

$$\pi(z) = \pi(\lim_k z_k) = \lim_k \pi(z_k) = \lim y_k = y$$

and since $X - (U_1 \cup \dots \cup U_r)$ is closed, it contains z. But this contradicts that fact that we chose the U_i to contain the pre-image of y. This establishes the existence of W with the desired property.

Next, let x in a periodic point in any one of the sets U_i be such that $\pi(x)$ is in W and let U be an open neighbourhood of x contained in U_i. We claim that there is an open neighbourhood Z of $\pi(x)$, such that

$$Z \cap V^s(\pi(x), \epsilon_Y) \subset \pi(U \cap V^s(x, \epsilon_\pi)). \tag{1}$$

Begin by choosing open sets $V_1 \subset V^s(x, \epsilon_X)$ and $V_2 \subset V^u(x, \epsilon_X)$ and letting $V = [V_2, V_1]$. Such open sets form a neighbourhood base for X at x. We choose such a V so that its closure is contained in U. Now we repeat exactly the same argument we used above to produce W, replacing y with $\pi(x)$

and the one set U_i containing x with V. We obtain in this way an open neighbourhood Z of $\pi(x)$ such that each periodic point in Z has a unique pre-image in V. To establish the containment we want, we let y be in $Z \cap V^s(\pi(x), \epsilon_Y)$ Since Z is open and the periodic points of Y are dense, we choose a sequence of periodic points, y_n, in Z converging to y. Each has a pre-image in V, say x_n. By passing to a subsequence, we may assume that the x_n converge to a point x' in $\bar{V} \subset U$. Now the points $[x, x_n]$ are in $V^s(x, \epsilon_X)$ and also in V, since we chose V to be a rectangle. They converge to $[x, x']$ which is in $U \cap V^s(x, \epsilon_\pi)$. It is easy to check that $\pi([x, x']) = y$ as desired.

We now choose a periodic point x in U_1 with $\pi(x)$ in W. Let m denote the period of x. We choose ϵ_0 sufficiently small so that the closure of $V^s(x, \epsilon_0)$ is contained in U_1 and so that its image under π is contained in W. It follows from 1 that this image is an open set in $V^s(\pi(x))$ and that the map π is open on this set.

The set

$$\begin{aligned} V^s(x) &= \cup_{n=0}^{\infty} \phi^{-nm}(V^s(\phi^{nm}(x), \epsilon_0)) \\ &= \cup_{n=0}^{\infty} \phi^{-nm}(V^s(x, \epsilon_0)) \end{aligned}$$

is dense in X. It follows that we may choose a positive integer n such that, for every x' in X, $V^u(x', \epsilon_\pi/2)$ meets $\phi^{-nm}(V^s(x, \epsilon_0))$.

We claim that, given $\epsilon > 0$, there is a $\delta > 0$ such that, for any x' in X, we have

$$V^s(\pi(x'), \delta) \subset \pi(V^s(x', \epsilon)). \tag{2}$$

We know this holds for x' in $V^s(x, \epsilon_0)$. The fact that the δ may be chosen uniformly for all x' in this set follows from the pre-compactness. Next, if y and z are any two points of X with $[y, z] = z$ and $d(y, z) < \epsilon_X/2$, then the map sending w in $V^s(y, \epsilon_X/2)$ to $[z, w]$ is a local homeomorphism. It satisfies the same condition, and it is easy to show, again using compactness, that for a given ϵ the δ may be chosen independent of y and z. This same conclusion holds in Y. Finally, the maps ϕ and ψ also have the same property using the uniform continuity of their inverses.

Now for any x' in X, the map π on $V^s(x', \epsilon)$ may be written as a composition of these maps. First we choose x'' in $V^u(x', \epsilon_X/2) \cap \phi^{-nm}(V^s(x, \epsilon_0))$. Then we write π as the composition of the map $y \to [x'', y]$, followed by ϕ^{nm}, followed by π, followed by ψ^{-nm}, followed by the map $y \to [y, \pi(x')]$. Each of maps has the desired property, hence so does π. We omit the rather tedious details.

To complete the proof, we let x be any point in X and note that π maps $V^s(x)$ onto $V^s(\pi(x))$ as follows

$$\begin{aligned}
\pi(V^s(x)) &= \pi\left(\cup_{n\geq 0}\phi^{-n}(V^s(\phi^{-n}(x),\epsilon)\right) \\
&= \cup_{n\geq 0}\pi\left(\phi^{-n}(V^s(\phi^{-n}(x),\epsilon)\right) \\
&= \cup_{n\geq 0}\left(\psi^{-n}(\pi(V^s(\phi^{-n}(x),\epsilon))\right) \\
&\supset \cup_{n\geq 0}\left(\psi^{-n}(V^s(\pi(\phi^{-n}(x)),\delta))\right) \\
&= \cup_{n\geq 0}\left(\psi^{-n}(V^s(\psi^{-n}(\pi(x)),\delta))\right) \\
&= V^s(\pi(x))
\end{aligned}$$

So we see that π is a bijection from $V^s(x)$ to $V^s(\pi(x))$. Moreover, it is clearly continuous and open by 2. Since both of these spaces are locally compact, it is a homeomorphism. This completes the proof.

□

We are now in a position to prove the first of our main results on functoriality.

Corollary 3.5. *Let $\pi : (X,\phi) \to (Y,\psi)$ be a finite-to-one, s-resolving factor map between two mixing Smale spaces and let x be any point of X. Then*

$$\pi \times \pi : G_u(X,\phi,x) \to G_u(Y,\psi,\pi(x))$$

is a homeomorphism to an open subgroupoid of $G_u(Y,\psi,\pi(x))$.

*Moreover, this inclusion induces a *-homomorphism of C^*-algebras*

$$\pi_* : U(X,\phi,x) \to U(Y,\psi,\pi(x)).$$

Finally, the map π_* satisfies

$$\pi_* \circ \alpha_u^{\phi} = \alpha_u^{\psi} \circ \pi_*.$$

Proof. We use the fact that the range and source maps for both groupoids are local homeomorphisms to their respective unit spaces. Then the result follows easily from the last theorem. The final equation also follows easily from the definitions. We omit the details. □

The above result is in the context of mixing Smale spaces. As we discussed in the last section, these ideas can be generalized to the non-mixing case in the following way. The proof is an easy consequence of the mixing case and the discussion of the last section.

Corollary 3.6. *Let $\pi : (X, \phi) \to (Y, \psi)$ be a finite-to-one, s-resolving factor map between two Smale spaces in which each point is non-wandering. Also suppose that (X, ϕ) (and hence (Y, ψ)) has a dense orbit. Suppose that A is any finite or countable subset of X such that no two points of A are stably equivalent after applying π. Also assume that $V^s(A)$ meets each mixing component of (X, ϕ). Then*

$$\pi \times \pi : G_u(X, \phi, A) \to G_u(Y, \psi, \pi(A))$$

is a homeomorphism to an open subgroupoid of $G_u(Y, \psi, \pi(A))$.

*Moreover, this inclusion induces a *-homomorphism of C^*-algebras*

$$\pi_* : U(X, \phi, A) \to U(Y, \psi, \pi(A)).$$

Finally, the map π_* satisfies

$$\pi_* \circ \alpha_u^\phi = \alpha_u^\psi \circ \pi_*.$$

We next begin to look at the situation for the stable C^*-algebras. Again, we begin with a technical result.

Lemma 3.7. *Let $\pi : (X, \phi) \to (Y, \psi)$ be an s-resolving factor map between two Smale spaces. Then there is a constant, M, with the following property. If $\{x_1, \ldots, x_m\}$ is any finite set in X such that no two elements are unstably equivalent, and there is a y in Y such that $\pi(x_i) \in V^u(y)$, for all $i = 1, \ldots, m$, then $m \leq M$.*

Proof. We will find M with the property that, if $\{x_1, \ldots, x_m\}$ is any finite set in X with $m > M$ and $\pi(x_i) \in V^u(y)$, for some y in Y and for all i, then for some $i \neq j$, $x_i \in V^u(x_j)$. The conclusion follows from this.

Since X is a compact metric space, we may find M such that, any finite set in X with more than M elements will contain two distinct points within distance ϵ_π. Now let $\{x_1, \ldots, x_m\}$ be any finite set. Since the set is finite, if all points $\pi(x_i)$ are unstably equivalent to each other, we may find a negative integer n such that

$$d(\psi^n(\pi(x_i)), \psi^n(\pi(x_j))) < \epsilon_Y,$$

for all i, j. Now if $m > M$, two points of the set $\{\phi^n(x_1), \ldots, \phi^n(x_m)\}$ must be within distance ϵ_π. Let $\phi^n(x_i)$ and $\phi^n(x_j)$ be two such points. Now we consider the point $[\phi^n(x_i), \phi^n(x_j)]$, which is well-defined since the distance is less than ϵ_π. This point is clearly stably equivalent to $\phi^n(x_i)$. We apply π to this point, using Lemma 3.2 and the fact that π is a factor, and obtain

$$\begin{aligned}
\pi([\phi^n(x_i), \phi^n(x_j)]) &= [\pi(\phi^n(x_i)), \pi(\phi^n(x_j))] \\
&= [\psi^n(\pi(x_i)), \psi^n(\pi(x_j))] \\
&= \psi^n(\pi(x_i)) \\
&= \pi(\phi^n(x_i)).
\end{aligned}$$

Since π is s-resolving and the points $[\phi^n(x_i), \phi^n(x_j)]$ and $\phi^n(x_i)$ are stably equivalent, we see that there are actually equal. This means that $\phi^n(x_i)$ and $\phi^n(x_j)$ are unstably equivalent and hence, so are x_i and x_j. This completes the proof. □

Theorem 3.8. *Let $\pi : (X, \phi) \to (Y, \psi)$ be an s-resolving factor between two Smale spaces and let y be any point of Y. Then there is a finite subset $\{x_1, \ldots, x_m\}$ of X such that*

$$\pi^{-1}(V^u(y)) = \bigcup_{i=1}^{m} V^u(x_i)$$

and the sets in the union are pairwise disjoint.

Moreover, the map

$$\pi : \bigcup_{i=1}^{m} V^u(x_i) \to V^u(y)$$

is continuous and proper.

Proof. It is clear that the image under π of an unstable equivalence class in X is contained in an unstable class in Y. So the pre-image of $V^u(y)$ under π can be written as a union of disjoint unstable classes in X. The fact this union is finite follows from Lemma 3.7. So the first statement is established. As for the second, it is fairly clear from the definitions that π is continuous. To show it is proper, it suffices to prove that if $\{z_k\}$ is any sequence in $\cup_{i=1}^{m} V^u(x_i)$ such that $\{\pi(z_k)\}$ is convergent in $V^u(y)$, then it has a convergent subsequence.

Since X itself is compact, the sequence has a convergent subsequence in X. Passing to yet another subsequence, we may assume that all the terms are contained in one of the sets, $V^u(x_i)$. We let $\{z_{k_l}\}$ denote this subsequence and z its limit. Then since the topology on $V^u(y)$ is finer than the relative one from X, we see that the limit of $\{\pi(z_{k_l})\}$ must be $\pi(z)$. This means that the sequence converges in the usual topology on Y and that, for sufficiently large l,

$$[\pi(z_{k_l}), \pi(z)] = \pi(z_{k_l}).$$

Now we consider the points $[z_{k_l}, z]$ which are well-defined for l sufficiently large and are stably equivalent to z_{k_l}. Moreover, we have

$$\pi([z_{k_l}, z]) = [\pi(z_{k_l}), \pi(z)] = \pi(z_{k_l}).$$

Since π is s-resolving and $[z_{k_l}, z]$ and z_{k_l} are stably equivalent we conclude that, for sufficiently large l, $[z_{k_l}, z] = z_{k_l}$ and this means that the subsequence is actually converging to z in $V^u(z)$. This completes the proof. □

We are now ready to state our other main result on functoriality.

Corollary 3.9. *Let $\pi : (X, \phi) \to (Y, \psi)$ be a finite-to-one, s-resolving factor map between two mixing Smale spaces and let y be in Y. Let $\{x_1, \dots, x_m\}$ be as in Theorem 3.8. Then the map*

$$\pi \times \pi : G_s(X, \phi, \{x_1, \dots, x_m\}) \to G_s(Y, \psi, y)$$

*is continuous, proper and surjective. Moreover, it induces a *-homomorphism of C^*-algebras*

$$\pi^* : S(Y, \psi, y) \to S(X, \phi, \{x_1, \dots, x_m\}).$$

Finally, the map π^* satisfies

$$\pi^* \circ \alpha_s^{\psi} = \alpha_s^{\phi} \circ \pi^*$$

Proof. Let us begin with some observations. First, if x is in X and z' is in Y and is stably equivalent to $\pi(x)$, then there is a unique point x' in X which is stably equivalent to x and and has $\pi(x') = z'$. This follows from Theorem 3.4. If, in addition, z' is in $V^u(y)$, then both x and x' lie in $V^u(\{x_1, \dots, x_m\})$.

The facts that $\pi \times \pi$ is continuous and proper follow immediately from the definitions and the last theorem. Let us prove it is surjective. In fact, we will show a little more. Given (z, z') in $V^u(y)$, we can find x in $V^u(\{x_1, \ldots, x_m\})$ such that $\pi(x) = z$. Now for this given x, there is a *unique* x' which is stably equivalent to x and has $\pi(x') = z'$. As noted above, this x' is also in $V^u(\{x_1, \ldots, x_m\})$. This means that $\pi \times \pi$ is surjective.

Now we must check that $\pi \times \pi$ induces a *-homomorphism of C^*-algebras. We first consider the dense subalgebras of continuous, compactly supported functions on our groupoids, $C_c(G_s(X, \phi, \{x_1, \ldots, x_m\}))$ and $C_c(G_s(Y, \psi, y))$. Since $\pi \times \pi$ is continuous and proper, it induces a linear map, denoted π^*, from the latter to the former. It is clear that this map is also *-preserving. It remains to check that it is multiplicative. Let f, g be in $C_c(G_s(Y, \psi, y))$ and let (x, x') be in $G_s(X, \phi, \{x_1, \ldots, x_m\})$. Then we have

$$
\begin{aligned}
\pi^*(f \cdot g)(x, x') &= (f \cdot g)(\pi(x), \pi(x')) \\
&= \sum f(\pi(x), z) g(z, \pi(x')),
\end{aligned}
$$

where the sum is over all z in $V^u(y)$ which are stably equivalent to $\pi(x)$. But we know from above that for each such z there is a unique x'' in $V^u(\{x_1, \ldots, x_m\})$ which is stably equivalent to x and with $\pi(x'') = z$. Thus, we have

$$
\begin{aligned}
\pi^*(f \cdot g)(x, x') &= \sum f(\pi(x), \pi(x'')) g(\pi(x''), \pi(x')) \\
&= \sum \pi^*(f)(x, x'') \pi^*(g)(x'', x') \\
&= (\pi^*(f) \cdot \pi^*(g))(x, x')
\end{aligned}
$$

where the sums above are over all x'' stably equivalent to x in $V^u(\{x_1, \ldots, x_m\})$. This shows that π^* is a homomorphism. Finally, we must argue that π^* will extend continuously to the C^*-algebras which are the completions of the spaces of functions above. Here, we can use the fact that these groupoids are amenable [PS] and r-discrete. Furthermore, the map $\pi \times \pi$ maps the unit space onto the unit space. So if λ is the left regular representation induced from a point mass on the unit space of $G_s(X, \phi, \{x_1, \ldots, x_m\})$, then $\lambda \circ \pi^*$ is the direct sum of representations induced from point masses on the unit space of $G_s(Y, \psi, y)$. Continuity follows from these observations. The proof of the final equality follows at once from the definitions. This completes the proof. $\square$

Again, we want to observe that this result extends to the non-mixing case. Again, the proof follows easily from the mixing case and the discussion of the last section.

Corollary 3.10. *Let $\pi : (X, \phi) \to (Y, \psi)$ be a finite-to-one, s-resolving factor map between two Smale spaces, having each point non-wandering. Also suppose that (X, ϕ) (and hence (Y, ψ)) has a dense orbit. Let B be any subset of Y such that no two points of B are stably equivalent. For each point y in B, let $\{x_1, \dots, x_m\}$ be as in Theorem 3.8 and let A denote the union of these sets as y runs over B. Also assume that A meets each mixing component of (X, ϕ). Then the map*

$$\pi \times \pi : G_s(X, \phi, A) \to G_s(Y, \psi, B)$$

*is continuous, proper and surjective. Moreover, it induces a *-homomorphism of C^*-algebras*

$$\pi^* : S(Y, \psi, B) \to S(X, \phi, A).$$

Finally, the map π^* satisfies

$$\pi^* \circ \alpha_s^{\psi} = \alpha_s^{\phi} \circ \pi^*$$

We conclude this section with a discussion of functoriality at the level of K-theory of our C^*-algebras. For simplicity, we will begin with the unstable C^*-algebras, although an analogous situation exists for the stable ones.

Let $\pi : (X, \phi) \to (Y, \psi)$ be a finite-to-one, s-resolving factor between mixing Smale spaces and let A be a finite set in X such that $V^s(A)$ meets each unstable equivalence class and so that no two points in A are stably equivalent. From [PS], we know that the C^*-algebras $U(X, \phi)$ and $U(X, \phi, A)$ are strongly Morita equivalent. In fact, there is a canonical equivalence bimodule and this implements an isomorphism at the level of K-theory, which we denote by

$$\kappa_A : K_*(U(X, \phi)) \to K_*(U(X, \phi, A)).$$

Similarly, is we assume that no two points of A are stably equivalent after applying π, there is an isomorphism

$$\kappa_{\pi(A)} : K_*(U(Y, \psi)) \to K_*(U(Y, \psi, \pi(A))).$$

Now, we claim that

$$\kappa_{\pi(A)}^{-1} \circ (\pi_*)_* \circ \kappa_A : K_*(U(X,\phi)) \to K_*(U(X,\psi))$$

is independent of the choice of A. To see this, suppose that A and A' are two such sets. Let B be a subset of $A \cup A'$ such that each point in the union is stably equivalent to exactly one point in B. Then, $U(X,\phi,A)$ and $U(X,\phi,A')$ are both full corners in $U(X,\phi,B)$. Denote the inclusion maps by i, i', respectively. Similarly, $U(Y,\psi,\pi(A))$ and $U(Y,\psi,\pi(A'))$ are full corners in $U(Y,\psi,\pi(B))$. We use the same i, i' to denote the inclusions. Then there is a a *-homomorphism

$$\pi_* : U(X,\phi,B) \to U(Y,\psi,\pi(B))$$

such that $(\pi_*)_* \circ i_* = i_* \circ (\pi_*)_*$ and $(\pi_*)_* \circ i'_* = i'_* \circ (\pi_*)_*$. The conclusion follows from these observations. We omit the details.

It seems most convenient (although rather an abuse of notation) to denote $\kappa_{\pi(A)}^{-1} \circ (\pi_*)_* \circ \kappa_A$ by

$$\pi_* : K_*(U(X,\phi)) \to K_*(U(Y,\psi)).$$

In an analogous way, there is a homomorphism

$$\pi^* : K_*(S(Y,\psi)) \to K_*(S(X,\phi)).$$

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