# THE TWO-PRIME ANALOGUE OF THE HECKE $C^*$-ALGEBRA OF BOST AND CONNES

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Abstract. Let $p$ and $q$ be distinct odd primes. We analyse a semigroup crossed product $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ similar to the semigroup crossed product which models the Hecke $C^*$-algebra of Bost and Connes. We describe a composition series of ideals in $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$, and show that the structure of one of the subquotients reflects interesting number-theoretic information about the multiplicative orders of $q$ in the rings $\mathbb{Z}/p^l\mathbb{Z}$.

In [3], Bost and Connes introduced and studied a Hecke $C^*$-algebra $\mathcal{C}_\mathbb{Q}$ which has many fascinating connections with number theory. It was shown in [11] that $\mathcal{C}_\mathbb{Q}$ can be realised as a crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ by an endomorphic action $\alpha$ of the multiplicative semigroup $\mathbb{N}^*$ of positive integers, and this realisation gives a great deal of insight into the Bost-Connes analysis (see [9]). Here we fix two odd primes $p$ and $q$, and analyse the semigroup crossed product $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ associated to the subgroup $G_{p,q} := \{n/p^k q^l : n \in \mathbb{Z}\}/\mathbb{Z}$ of $\mathbb{Q}/\mathbb{Z}$ and the restriction of $\alpha$ to the subsemigroup $\{p^k q^l\} \subset \mathbb{N}^*$, which is isomorphic to the additive semigroup $\mathbb{N}^2$. This crossed product still exhibits rich connections with number theory, though of a somewhat different nature: it has a subquotient, for example, whose ideal structure encodes the multiplicative orders of $q$ in the rings $\mathbb{Z}/p^l\mathbb{Z}$.

We begin our analysis by passing to the Fourier transform of our dynamical system, which involves the algebras of continuous functions on the spaces of $p$-adic and $q$-adic integers. We describe our dynamical system $(C^*(G_{p,q}), \mathbb{N}^2, \alpha)$ and its Fourier transform in §1. Next we construct a composition series for $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ using general results about invariant ideals and tensor products of semigroup crossed products which have been worked out in [13]. Our main structure theorem is Theorem 2.2, which is proved in §2 and §3. Theorem 3.1, which gives a detailed description of an ordinary crossed product $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes \mathbb{Z}$ arising in our analysis, is interesting in its own right: it shows, for example, that $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes \mathbb{Z}$ is simple if and only if $q$ is a primitive root modulo $p^l$ for all $l$, which happens if and only if it is primitive modulo $p^l$ for any single $l > 1$ (see Remark 3.8). In the last section, we describe the topology

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on the primitive ideal space of $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$, thus completely determining its ideal structure.

## 1. The dynamical system and its Fourier transform

Let $p$ and $q$ be distinct odd primes. We consider the additive group

$$\mathbb{Z}[p^{-1}, q^{-1}] = \{rp^{-k}q^{-l} : r, k, l \in \mathbb{Z}\}$$

and its quotient $G_{p,q} := \mathbb{Z}[p^{-1}, q^{-1}]/\mathbb{Z}$. We write $\alpha$ for the action of $\mathbb{N}^2$ by endomorphisms of the group $C^*$-algebra $C^*(G_{p,q})$ which is characterised on the canonical generating unitaries $\{\delta_r : r \in G_{p,q}\}$ by

$$\alpha_{m,n}(\delta_r) = \frac{1}{p^m q^n} \sum_{\{s \in G_{p,q} : p^m q^n s = r\}} \delta_s; \tag{1.1}$$

we can see that there is such an action either by modifying [11, Proposition 2.1] or by applying the general method of [14, §1] to the action of $\mathbb{N}^2$ on $\mathbb{Z}$ defined by $\eta_{m,n}(k) = p^m q^n k$ (see [14, Example 1.2]). As in [10, Proposition 2.1], the action satisfies

$$\alpha_{k,l}(1)\alpha_{m,n}(1) = \alpha_{k\vee m,\, l\vee n}(1). \tag{1.2}$$

A covariant representation of the dynamical system $(C^*(G_{p,q}), \mathbb{N}^2, \alpha)$ consists of a nondegenerate representation $\pi$ of $C^*(G_{p,q})$ and a representation $V$ of $\mathbb{N}^2$ by isometries on the same space such that

$$\pi(\alpha_{m,n}(a)) = V_{m,n}\pi(a)V_{m,n}^* \quad \text{for } a \in C^*(G_{p,q}) \text{ and } (m,n) \in \mathbb{N}^2; \tag{1.3}$$

the relation (1.2) then implies that the isometric representation $V$ is Nica covariant, in the sense that $V_{k,l}V_{k,l}^*V_{m,n}V_{m,n}^* = V_{k\vee m,\, l\vee n}V_{k\vee m,\, l\vee n}^*$. One can see that the system has nontrivial covariant representations by modifying the constructions in [11], or by applying [14, Lemma 1.7]. Thus there is a crossed product $\big(C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2, i_A, i_S\big)$, which is a universal $C^*$-algebra for covariant representations of the system (see [10, Proposition 2.1]). (To avoid complicated notation, we always write $i_A$ and $i_S$ for the algebra and semigroup components of the universal covariant representation.) This crossed product carries a dual action $\widehat{\alpha}$ of $\mathbb{T}^2$ which leaves $i_A(C^*(G_{p,q}))$ invariant and satisfies $\widehat{\alpha}_{w,z}(i_S(m,n)) = w^m z^n i_S(m,n)$.

To compute the Fourier transform of the system, we need a description of the dual group $\widehat{G}_{p,q}$. Note that with $G_p := \mathbb{Z}[p^{-1}]/\mathbb{Z}$, the map $(r,s) \mapsto r+s$ is an isomorphism of $G_p \times G_q$ onto $G_{p,q}$, and, dually, we have $\widehat{G}_{p,q} \cong \widehat{G}_p \times \widehat{G}_q$. To describe $\widehat{G}_p$, note that $\mathbb{Z}[p^{-1}] = \bigcup_{l=1}^\infty p^{-l}\mathbb{Z}$, so $G_p = (\bigcup p^{-l}\mathbb{Z})/\mathbb{Z}$ has a natural description as a direct limit $\varinjlim p^{-l}\mathbb{Z}/\mathbb{Z}$, and $\widehat{G}_p$ is an inverse limit $\varprojlim (p^{-l}\mathbb{Z}/\mathbb{Z})^\wedge$ of finite groups. The usual pairing $\langle t, n\rangle = \exp 2\pi i t n$ of $\mathbb{Z}$ with $\mathbb{R}/\mathbb{Z}$ induces an isomorphism of $\mathbb{Z}/p^l\mathbb{Z}$ onto $(p^{-l}\mathbb{Z}/\mathbb{Z})^\wedge$, and it is easy to check that the dual of the inclusion $p^{-l}\mathbb{Z}/\mathbb{Z} \hookrightarrow p^{-(l+1)}\mathbb{Z}/\mathbb{Z}$ is the map of $\mathbb{Z}/p^{l+1}\mathbb{Z}$ onto $\mathbb{Z}/p^l\mathbb{Z}$ given by reduction mod $p^l$. Thus $\widehat{G}_p$ is naturally identified as a compact group with the inverse limit $\varprojlim \mathbb{Z}/p^l\mathbb{Z}$.

Each $\mathbb{Z}/p^l\mathbb{Z}$ is a ring, and the reduction maps are ring homomorphisms, so $\varprojlim \mathbb{Z}/p^l\mathbb{Z}$ is a compact topological ring $\mathbb{Z}_p$, which is called the ring of $p$-adic integers; in the previous paragraph, we identified $\widehat{G}_p$ with the additive group of $\mathbb{Z}_p$. However, the multiplicative structure of $\mathbb{Z}_p$ plays a crucial role in our analysis, for two reasons. First, we can use it to describe the action $\alpha$: the reduction maps $\mathbb{Z} \to \mathbb{Z}/p^l\mathbb{Z}$ induce an embedding of $\mathbb{Z}$ in $\mathbb{Z}_p$, and $\alpha_{m,n}$ is, loosely speaking, division by $p^m q^n$ (see Lemma 1.1 below). Second, the group $\mathcal{U}(\mathbb{Z}_p)$ of units in $\mathbb{Z}_p$ (the multiplicatively invertible elements) appears in our theorems. We need to know that there is a natural identification of $\mathcal{U}(\mathbb{Z}_p)$ with $\varprojlim \mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$, and that an integer $m$ is a unit in $\mathbb{Z}_p$ precisely when $m$ is coprime to $p$. For these and other properties of $\mathbb{Z}_p$, we refer to [16, Chapter II].

We are now ready to describe the Fourier-transform system. The dual of $G_{p,q}$ is $\mathbb{Z}_p \times \mathbb{Z}_q$; if $\pi_l$ denotes the canonical map of $\mathbb{Z}_p$ onto $\mathbb{Z}/p^l\mathbb{Z}$, then the pairing is given by

$$\langle r+s,(x,y)\rangle = \exp 2\pi i(r\pi_l(x)+s\pi_l(y)) \quad \text{for } r \in \mathbb{Z}[p^{-1}],\ s \in \mathbb{Z}[q^{-1}] \text{ and } l \text{ large.} \tag{1.4}$$

**Lemma 1.1.** *The Fourier transform $C^*(G_{p,q}) \cong C(\mathbb{Z}_p \times \mathbb{Z}_q)$ carries the action defined by* (1.1) *into the action given by*

$$\alpha_{m,n}(f)(x,y) = \begin{cases} f(p^{-m}q^{-n}x, p^{-m}q^{-n}y) & \text{if } x \in p^m q^n \mathbb{Z}_p \text{ and } y \in p^m q^n \mathbb{Z}_q, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

*Proof.* We aim to apply [13, Proposition 4.5]. To do this, note that $\alpha_{m,n}$ is defined by averaging over the solutions $s$ of $\beta_{m,n}(s) = r$, where $\beta_{m,n}$ is the endomorphism of $G_{p,q}$ defined by $\beta_{m,n}(s) = p^m q^n s$. From the pairing (1.4), we see that the endomorphism $\widehat{\beta}_{m,n}$ of $\mathbb{Z}_p \times \mathbb{Z}_q$ is given in terms of the ring structure by $\widehat{\beta}_{m,n}(x,y) = (p^m q^n x, p^m q^n y)$. Thus the Lemma follows directly from [13, Proposition 4.5]. □

## 2. The structure theorem

Our main theorem describes the structure of $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ — or, equivalently, of the crossed product $C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes_\alpha \mathbb{N}^2$ of the Fourier-transform system described in Lemma 1.1. To state it, we need a number-theoretic lemma. If $k$ and $m$ are coprime integers, so that $m$ is a unit in $\mathbb{Z}/k\mathbb{Z}$, we write $o_k(m)$ for the order of $m$ in $\mathcal{U}(\mathbb{Z}/k\mathbb{Z})$.

**Lemma 2.1.** *Let $p$ and $q$ be distinct odd primes. Then there is a positive integer $L = L_p(q)$ such that*

$$o_{p^l}(q) = \begin{cases} o_p(q) & \text{if } 1 \le l \le L \\ p^{l-L} o_p(q) & \text{if } l > L. \end{cases} \tag{2.1}$$

This lemma is presumably well-known; certainly some of its immediate consequences are (see Remark 3.8). We are not going to prove it now, because we shall prove a slightly more general result in Theorem 3.1. However, we want to use the integers $L_p(q)$ from this lemma in the statement of our main theorem.

**Theorem 2.2.** *Let $p$ and $q$ be distinct odd primes. Then there are $\widehat{\alpha}$-invariant ideals $I_1$ and $I_2$ in $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ such that $I_1 \subset I_2$,*

$$I_1 \cong \mathcal{K}(l^2(\mathbb{N}^2)) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)), \tag{2.2}$$

$$I_2/I_1 \cong (\mathcal{K}(l^2(\mathbb{N})) \otimes C) \oplus (\mathcal{K}(l^2(\mathbb{N})) \otimes D), \text{ and} \tag{2.3}$$

$$(C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2)/I_2 \cong C(\mathbb{T}^2), \tag{2.4}$$

*where $C$ is the direct sum of $(p-1)p^{L_p(q)-1}/o_p(q)$ Bunce-Deddens algebras with supernatural number $o_p(q)p^\infty$ and $D$ is the direct sum of $(q-1)q^{L_q(p)-1}/o_q(p)$ Bunce-Deddens algebras with supernatural number $o_q(p)q^\infty$.*

The algebra $C^*(G_{p,q}) \cong C(\mathbb{Z}_p \times \mathbb{Z}_q)$ decomposes as a tensor product $C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)$, and the action $\alpha$ given by (1.5) decomposes as a tensor product of two actions of $\mathbb{N}^2$. At this point, we cannot separate the actions of the two copies of $\mathbb{N}$ (as Bost and Connes say, the two primes interact), but there is a large invariant ideal $C_0(\mathbb{Z}_p \setminus \{0\})$ in $C(\mathbb{Z}_p)$ where the action does split as a tensor product of two actions of $\mathbb{N}$. The ideals $I_1$ and $I_2$ will be crossed products of different invariant ideals in $C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)$ built from $C_0(\mathbb{Z}_p \setminus \{0\})$ and its twin.

For ordinary crossed products $A \rtimes G$ by group actions, invariant ideals in $A$ give rise to short exact sequences

$$0 \longrightarrow I \rtimes G \longrightarrow A \rtimes G \longrightarrow (A/I) \rtimes G \longrightarrow 0.$$

For semigroup crossed products $A \rtimes_\alpha S$, one has to know that the ideal $I$ is *extendibly invariant*, in the sense that each endomorphism $\alpha_s$ extends to endomorphisms of $M(I)$ and $M(A)$ in such a way that $\alpha_s(1_{M(I)}) = \alpha_s(1_{M(A)})$ as multipliers of $I$ (see [1, 13]). Since the endomorphism $x \mapsto p^m q^n x$ of $\mathbb{Z}_p$ leaves both $\mathbb{Z}_p \setminus \{0\}$ and $\{0\}$ invariant, it follows from Lemma 1.1 and [13, Theorem 4.3] that $I := C_0(\mathbb{Z}_p \setminus \{0\})$ and $J := C_0(\mathbb{Z}_q \setminus \{0\})$ are extendibly invariant ideals in $A := C(\mathbb{Z}_p)$ and $B := C(\mathbb{Z}_q)$. We can therefore apply [13, Theorem 3.1] to deduce that the ideals $I_1 := (I \otimes J) \rtimes \mathbb{N}^2$ and $I_2 := (I \otimes B + A \otimes J) \rtimes \mathbb{N}^2$ form a composition series in which

$$I_1 \cong (I \otimes J) \rtimes_\alpha \mathbb{N}^2, \tag{2.5}$$

$$I_2/I_1 \cong ((A/I) \otimes J) \rtimes \mathbb{N}^2 \oplus (I \otimes (B/J)) \rtimes \mathbb{N}^2, \text{ and} \tag{2.6}$$

$$(A \otimes B) \rtimes_\alpha \mathbb{N}^2/I_2 \cong ((A/I) \otimes (B/J)) \rtimes \mathbb{N}^2. \tag{2.7}$$

Notice that because the ideals are crossed products, they are $\widehat{\alpha}$-invariant. To prove Theorem 2.2, therefore, we have to identify the subquotients.

We begin by noting that the maps $f \mapsto f(0)$ induce isomorphisms $A/I \cong \mathbb{C}$ and $B/J \cong \mathbb{C}$, so $(A/I) \otimes (B/J) \cong \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$. Thus (2.7) is $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{N}^2$. When the action is unital, as the identity action id certainly is, the covariance relation (1.3) implies that the isometries are all unitary; thus $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{N}^2$ is the universal $C^*$-algebra generated by a unitary representation of $\mathbb{Z}^2$. In other words, $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{N}^2 = C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$, and we have proved (2.4).

For the other two parts, we need the promised decomposition of the action of $\mathbb{N}^2$ on $I = C_0(\mathbb{Z}_p \setminus \{0\})$.

**Lemma 2.3.** *The map $(n, x) \mapsto p^n x$ is a homeomorphism of $\mathbb{N} \times \mathcal{U}(\mathbb{Z}_p)$ onto $\mathbb{Z}_p \setminus \{0\}$.*

*Proof.* Since every nonzero $p$-adic number can be uniquely written as a power of $p$ times a unit (by Proposition 2 of [16, Chapter II], for example), the map is a bijection. It is a homeomorphism because it carries the basic open sets $\{n\} \times V$ for the topology on $\mathbb{N} \times \mathcal{U}(\mathbb{Z}_p)$ into the basic open sets $p^n V$ for the topology on $\mathbb{Z}_p \setminus \{0\}$. □

The lemma implies that $I = C_0(\mathbb{Z}_p \setminus \{0\}) \cong c_0(\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p))$. To describe what happens to the action $\alpha$ under this isomorphism, we need some notation. We let $\tau$ denote the action of $\mathbb{N}$ on $c_0(\mathbb{N})$ by forward shifts; if we think of elements of $c_0(\mathbb{N})$ as functions on $\mathbb{N}$, then

$$\tau_m(f)(k) = \begin{cases} f(k-m) & \text{if } k \geq m \\ 0 & \text{if } k < m. \end{cases}$$

Since $(q, p) = 1$, $q$ is a unit in $\mathbb{Z}_p$, and division by powers of $q$ defines an action $\sigma = \sigma^{p,q}$ of $\mathbb{Z}$ by automorphisms of $C(\mathcal{U}(\mathbb{Z}_p))$: $\sigma_n(f)(x) = f(q^{-n}x)$. We now have the following immediate corollary of Lemma 2.3:

**Corollary 2.4.** *The isomorphism $C_0(\mathbb{Z}_p \setminus \{0\}) \cong c_0(\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p))$ induced by the homeomorphism of Lemma 2.3 carries $\alpha$ into the tensor product action $\tau \otimes \sigma$ : $(m, n) \mapsto \tau_m \otimes \sigma_n$.*

**Lemma 2.5.** *There is an isomorphism*

$$I_2/I_1 \cong \mathcal{K}(l^2(\mathbb{N})) \otimes \big(C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma^{p,q}} \mathbb{Z}\big) \oplus \mathcal{K}(l^2(\mathbb{N})) \otimes \big(C(\mathcal{U}(\mathbb{Z}_q)) \rtimes_{\sigma^{q,p}} \mathbb{Z}\big). \tag{2.8}$$

*Proof.* First, recall that $A/I \cong \mathbb{C}$ and $B/J \cong \mathbb{C}$, so from (2.6) we have

$$I_2/I_1 \cong (I \rtimes_\alpha \mathbb{N}^2) \oplus (J \rtimes_\alpha \mathbb{N}^2). \tag{2.9}$$

Next, we use the decomposition of Corollary 2.4 and [13, Theorem 2.6] (which applies because our action satisfies (1.2)), to see that

$$I \rtimes_\alpha \mathbb{N}^2 \cong (c_0(\mathbb{N}) \rtimes_\tau \mathbb{N}) \otimes \big(C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma^{p,q}} \mathbb{N}\big). \tag{2.10}$$

Because $\sigma^{p,q}$ consists of automorphisms, the isometries in any covariant representation of $(C(\mathcal{U}(\mathbb{Z}_p)), \mathbb{N}, \sigma)$ are unitary, and $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{N}$ is the usual crossed product $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}$.

To handle the other factor in (2.10), recall that $c \rtimes_\tau \mathbb{N} = B_{\mathbb{N}} \rtimes_\tau \mathbb{N}$ is the Toeplitz algebra, and $c_0(\mathbb{N}) \rtimes_\tau \mathbb{N}$ is the ideal of compact operators. More precisely, let $M$ denote the representation of $c$ by multiplication operators on $l^2(\mathbb{N})$, and let $S$ be the unilateral shift on $l^2(\mathbb{N})$. Then $(M, S)$ is a covariant representation of $(c, \mathbb{N}, \tau)$

such that $M \times S$ is an isomorphism of $c \rtimes_\tau \mathbb{N}$ onto the $C^*$-algebra generated by $S$. (This formulation of Coburn's Theorem is described in [2], for example.) It is easy to check that $M \rtimes S$ carries the ideal $c_0 \rtimes_\tau \mathbb{N}$ onto $\mathcal{K}(l^2(\mathbb{N}))$. Thus (2.10) implies that $I \rtimes_\alpha \mathbb{N}^2 \cong \mathcal{K} \otimes (C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z})$. Swapping $p$ and $q$ gives an analogous description of $J \rtimes_\alpha \mathbb{N}^2$, and the Lemma follows from (2.9). $\square$

The description of $I_2/I_1$ in (2.3) will follow from this lemma and Theorem 3.1.

To describe $I_1 := (I \otimes J) \rtimes_\alpha \mathbb{N}^2$, we use two applications of Corollary 2.4 to get an isomorphism

$$I \otimes J = C_0(\mathbb{Z}_p \setminus \{0\}) \otimes C_0(\mathbb{Z}_q \setminus \{0\}) \cong C_0(\mathbb{N} \times \mathbb{N} \times \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))$$

which carries the endomorphism $\alpha_{m,n}$ into $\tau_m \otimes \tau_n \otimes \sigma_n^{p,q} \otimes \sigma_m^{q,p}$. We now borrow another idea from the theory of ordinary crossed products: recall that $(C_0(G) \otimes A) \rtimes_{\tau\otimes\beta} G \cong (C_0(G) \rtimes_\tau G) \otimes A$ for any action $\beta$. Because $q \in \mathcal{U}(\mathbb{Z}_p)$ and $p \in \mathcal{U}(\mathbb{Z}_q)$, the endomorphism $\phi$ of $C_0(\mathbb{N} \times \mathbb{N} \times \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))$ defined by

$$\phi(f)(k, l, x, y) = f(k, l, q^l x, p^k y)$$

is an automorphism. A quick calculation shows that

$$\phi \circ (\tau_m \otimes \tau_n \otimes \sigma_n^{p,q} \otimes \sigma_m^{q,p}) = \tau_m \otimes \tau_n \otimes \mathrm{id} \otimes \mathrm{id},$$

so $\phi$ induces an isomorphism

$$(I \otimes J) \rtimes_\alpha \mathbb{N}^2 \cong \big(c_0(\mathbb{N} \times \mathbb{N}) \rtimes_{\tau\otimes\tau} (\mathbb{N} \times \mathbb{N})\big) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)).$$

To finish off the proof of (2.2), either note that

$$c_0(\mathbb{N}^2) \rtimes_{\tau\otimes\tau} \mathbb{N}^2 \cong (c_0 \rtimes_\tau \mathbb{N}) \otimes (c_0 \rtimes_\tau \mathbb{N}) \cong \mathcal{K}(l^2(\mathbb{N})) \otimes \mathcal{K}(l^2(\mathbb{N})) = \mathcal{K}(l^2(\mathbb{N}^2)),$$

or check directly that the natural covariant representation of $B_{\mathbb{N}^2} \rtimes_\tau \mathbb{N}^2$ on $l^2(\mathbb{N}^2)$ restricts to an isomorphism of $c_0(\mathbb{N}^2) \rtimes \mathbb{N}^2$ onto $\mathcal{K}(l^2(\mathbb{N}^2))$.

To prove Theorem 2.2, therefore, it remains to prove Lemma 2.1 and to identify $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}$ with the appropriate number of Bunce-Deddens algebras. We do this in Theorem 3.1.

## 3. The crossed products $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}$

Our analysis of $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma^{p,q}} \mathbb{Z}$ does not require that $q$ is prime, only that it is coprime to $p$. We therefore fix an odd prime $p$ and an integer $m$ coprime to $p$, and consider the action $\sigma = \sigma^{p,m}$ of $\mathbb{Z}$ on $C(\mathcal{U}(\mathbb{Z}_p))$ defined by

$$\sigma_n^{p,m}(f)(x) = f(m^{-n}x). \tag{3.1}$$

**Theorem 3.1.** *Suppose that $p$ is an odd prime and $(m, p) = 1$, and denote by $o_{p^l}(m)$ the order of $m$ in $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$. Then there is a positive integer $L$ such that*

$$o_{p^l}(m) = \begin{cases} o_p(m) & \text{if } 1 \le l \le L \\ p^{l-L} o_p(m) & \text{if } l > L, \end{cases} \tag{3.2}$$

*and $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma^{p,m}} \mathbb{Z}$ is the direct sum of $p^{L-1}(p-1)/o_p(m)$ Bunce-Deddens algebras with supernatural number $o_p(m)p^\infty$.*

We begin by establishing the number-theoretic statements. Because $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ is cyclic of order $(p-1)p^{l-1}$ (see Theorem 2 of [8, Chapter 4], for example), we can apply the following elementary lemma about cyclic groups.

**Lemma 3.2.** *Suppose that $(n,p)=1$ and $G$, $H$ are cyclic groups of orders $p^l n$, $p^{l-1}n$, respectively. If $\pi : G \to H$ is a surjective homomorphism and $g$ is a generator of $G$, then the order of $\pi(g^r)$ is given by*

$$o(\pi(g^r)) = \begin{cases} |G|/(r,|G|) & \text{if } p^l \text{ divides } r \\ |G|/p(r,|G|) & \text{if } p^l \text{ does not divide } r. \end{cases}$$

*Proof.* Since $\pi(g)$ is a generator of $H$, we have

$$o(\pi(g^r)) = o(\pi(g)^r) = \frac{|H|}{(r,|H|)} = \frac{|G|}{p(r,|H|)}.$$

If $p^l$ divides $r$, say $r = sp^l$, then

$$p(r,|H|) = p(p^l s, p^{l-1}n) = p^l(ps,n) = p^l(s,n) = (r,p^l n) = (r,|G|),$$

as claimed. If $p^l$ does not divide $r$, then $(r,|G|) = (r,p^l n) = (r,p^{l-1}n) = (r,|H|)$. ☐

**Corollary 3.3.** *Suppose $p$ is prime and $(p,m)=1$. Then*

$$o_{p^l}(m) = \begin{cases} o_{p^{l+1}}(m) & \text{if } p \text{ does not divide } o_{p^{l+1}}(m) \\ o_{p^{l+1}}(m)/p & \text{if } p \text{ does divide } o_{p^{l+1}}(m). \end{cases}$$

*Proof.* Since a number is coprime to $p^l$ iff it is coprime to $p^{l+1}$, the reduction map $\pi$ is a homomorphism of $\mathcal{U}(\mathbb{Z}/p^{l+1}\mathbb{Z})$ onto $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$, and Lemma 3.2 applies. Indeed, there is a generator $g$ such that $m = g^r$ where $r := (p-1)p^l/o_{p^{l+1}}(m)$. Then

$$o_{p^l}(m) = o(\pi(g^r)) = \begin{cases} o_{p^{l+1}}(m) & \text{if } p^l \text{ divides } (p-1)p^l/o_{p^{l+1}}(m) \\ o_{p^{l+1}}(m)/p & \text{if } p^l \text{ does not divide } (p-1)p^l/o_{p^{l+1}}(m), \end{cases}$$

which translates into what we want. ☐

**Corollary 3.4.** *There is a positive integer $L$ such that (3.2) holds.*

*Proof.* We first note that the sequence $\{o_{p^l}(m) : l \in \mathbb{N}\}$ must be unbounded: for fixed $N$, $m^N$ is eventually less than $p^l$, and then $o_{p^l}(m) > N$. In particular, $\{o_{p^l}(m)\}$ is certainly not constant. Let $L$ be the first integer such that $o_{p^L}(m) < o_{p^{L+1}}(m)$. Then $o_{p^l}(m) = o_p(m)$ for $1 \le l \le L$, and by Corollary 3.3, we have $o_{p^{L+1}}(m) = po_p(m)$, and $p$ divides $o_{p^{L+1}}(m)$. Since $o_{p^{L+1}}(m)$ divides $o_{p^l}(m)$ for all $l > L$, it follows that $p$ divides $o_{p^l}(m)$ for all $l > L$, and $l-L$ applications of Corollary 3.3 show that $o_{p^l}(m) = p^{l-L}o_{p^L}(m) = p^{l-L}o_p(m)$. ☐

*Remark* 3.5. The referee has pointed out that one can also deduce Corollary 3.4 from the isomorphism of $\mathcal{U}(\mathbb{Z}/p\mathbb{Z}) \times p\mathbb{Z}_p^+$ onto $\mathcal{U}(\mathbb{Z}_p)$ provided by sending elements of $\mathcal{U}(\mathbb{Z}/p\mathbb{Z})$ to their Teichmüller representatives and the exponential isomorphism of the additive group $p\mathbb{Z}_p^+$ onto $1 + p\mathbb{Z}_p$ (see [7, Corollary 4.5.10], for example). This isomorphism is compatible with the inverse limit decompositions of $\mathcal{U}(\mathbb{Z}_p)$ and $p\mathbb{Z}_p^+$, and hence it suffices to prove the analogous properties of additive orders in $p\mathbb{Z}_p^+$.

Let $H$ be the closed subgroup of $\mathcal{U}(\mathbb{Z}_p)$ generated by $m$. Then $H$ is invariant under multiplication by powers of $m$, and the formula (3.1) also defines an action $\sigma$ of $\mathbb{Z}$ on $C(H)$. This is where the Bunce-Deddens algebras come from:

**Proposition 3.6.** *The crossed product $C(H) \rtimes_\sigma \mathbb{Z}$ is a Bunce-Deddens algebra with supernatural number $o_p(m)p^\infty$.*

The *Bunce-Deddens algebras* were originally defined to be the $C^*$-algebras generated by certain weighted shifts on $l^2$ [5, §V.3], but we shall recognise them as crossed products associated to odometer actions. Let $\{n_k\}$ be a sequence of integers each of which is at least 2, and let $X_k = \{0, 1, \dots, n_k - 1\}$. The *odometer action* $\tau$ of $\mathbb{Z}$ on $\prod_{k\geq 1} X_k$ is given by addition with carry over: let $N_1 = 1$, $N_k := \prod_{i<k} n_i$ for $k > 1$, and then

$$\tau_n(\{a_k\}) = \{b_k\} \text{ where } \textstyle\sum_{k\geq 1}^{l} b_k N_k :\equiv n + \sum_{k\geq 1}^{l} a_k N_k \pmod{N_{l+1}}.$$

The crossed product $C(\prod_{k\geq 1} X_k) \rtimes_\tau \mathbb{Z}$ is then a Bunce Deddens algebra with supernatural number $\prod_{k\geq 1} n_k$ [5, Theorem VIII.4.1]. In general, Bunce-Deddens algebras are simple [5, Theorem V.3.3], and are determined up to isomorphism by their supernatural number [5, Theorem V.3.5].

*Proof.* Write $d$ for $o_p(m)$, and let

$$\mathcal{O} := \{0, 1, \cdots, d-1\} \times \{0, 1, \cdots, p-1\}^{\mathbb{N}}.$$

For $l > L$, we define $h_l : \mathcal{O} \to \mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ by

$$h_l(\{a_n\}) = m^{a_0 + da_1 + dpa_2 + \cdots + dp^{l-L-1}a_{l-L}} \pmod{p^l\mathbb{Z}};$$

because the order of $m$ in $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ is $dp^{l-L}$, the maps $h_l$ satisfy $h_{l+1}(\{a_n\}) = h_l(\{a_n\}) \pmod{p^l\mathbb{Z}}$. Since the $h_l$ are continuous by definition of the product topology, they induce a continuous map $h : \mathcal{O} \to \mathcal{U}(\mathbb{Z}_p) = \varprojlim \mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$, which is an injection because $h_l(\{a_n\})$ determines $a_0, \dots, a_{l-L}$ uniquely. The range of $h$ is a compact subgroup, and contains the positive powers of $m$, which are the images of the sequences in $\mathcal{O}$ which are eventually zero; since such sequences are dense in $\mathcal{O}$, their images generate the range. In other words, $h$ is a continuous injection of $\mathcal{O}$ onto $H$, and is therefore a homeomorphism. Since $h(\tau\{a_n\}) = mh(\{a_n\})$ for all $\{a_n\}$, we deduce that the Bunce-Deddens algebra $C(\mathcal{O}) \rtimes_\tau \mathbb{Z}$ is isomorphic to $C(H) \rtimes_\sigma \mathbb{Z}$. □

To finish the proof of our theorem, we need to decompose the dynamical system $(C(\mathcal{U}(\mathbb{Z}_p)), \mathbb{Z}, \sigma)$ as a sum of copies of $(C(H), \mathbb{Z}, \sigma)$. This needs a simple group-theoretic lemma.

**Lemma 3.7.** *Suppose $G = \varprojlim G_n$ is a compact group which is the inverse limit of finite groups $G_n$, and suppose that the canonical maps $\pi_n : G \to G_n$ are surjective. If $H$ is a closed subgroup of $G$ and there is an integer $k$ such that $|G_n/\pi_n(H)| = k$ for all $n$, then $|G/H| = k$.*

*Proof.* Certainly $|G/H| \geq |\pi_n(G)/\pi_n(H)| = k$. Suppose $g_1H, \cdots, g_{k+1}H$ are cosets in $G/H$; we shall prove that two must be the same. The hypothesis implies that for each $n$, two of $\pi_n(g_iH)$ coincide. Since there are only finitely many possibilities, we can assume by passing to a subsequence that the same two coincide in each $G_n/\pi_n(H)$; say $\pi_n(g_1H) = \pi_n(g_2H)$ for all $n$. Then $\pi_n(g_1g_2^{-1}) \in \pi_n(H)$; say $\pi_n(g_1g_2^{-1}) = \pi_n(h_n)$. By definition of the topology on the inverse limit, we have $h_n \to g_1g_2^{-1}$ in $G$, so that $g_1g_2^{-1} \in H$ and $g_1H = g_2H$. □

*End of the proof of Theorem 3.1.* Since $\pi_l(H)$ is the subgroup of $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ generated by $m$, we have

$$|\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})/\pi_l(H)| = (p-1)p^{l-1}/o_{p^l}(m) = (p-1)p^{L-1}/o_p(m) \text{ for all } l \geq L.$$

We can therefore apply Lemma 3.7 to $\mathcal{U}(\mathbb{Z}_p) = \varprojlim(\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z}), l \geq L)$ to deduce that $H$ has index $N := (p-1)p^{L-1}/o_p(m)$ in $\mathcal{U}(\mathbb{Z}_p)$.

Next, note that because $H$ is a closed subgroup of finite index, it is also open: its complement is the finite union of cosets of $H$, and hence closed. Since $H$ is by definition invariant under multiplication by powers of $m$, it follows that $\mathcal{U}(\mathbb{Z}_p)$ is the disjoint union of $N$ open and closed invariant sets of the form $xH$, and $C(\mathcal{U}(\mathbb{Z}_p))$ is the direct sum of $\sigma$-invariant ideals of the form $C(xH)$. The dynamical systems $(C(xH), \mathbb{Z}, \sigma)$ are all conjugate to $(C(H), \mathbb{Z}, \sigma)$. Thus the Theorem follows from Proposition 3.6. □

*Remark* 3.8. An integer $m$ which generates $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ is called a *primitive root modulo* $p^l$. If $m$ is a primitive root modulo $p^l$ for one $l > 1$, then (3.2) implies that $L_p(m) = 1$ and $o_p(m) = p - 1$, and hence that $m$ is a primitive root modulo $p^k$ for all $k$. (This is known; see [6, §17, Exercise VI.4], for example.) Theorem 3.1 gives a curious $C^*$-algebraic characterisation of primitive roots: $m$ is primitive modulo $p^l$ for all $l$ if and only if $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}$ is simple. More generally, the cardinality of $\operatorname{Prim} C(\mathcal{U}(\mathbb{Z}_p)) \rtimes \mathbb{Z}$ determines the orders $o_{p^l}(m)$ of $m$ in $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$.

The relations (3.2) are the only restrictions on the possible values of $o_p(m)$. Indeed, given an odd prime $p$, a divisor $d$ of $p-1$, and an integer $L \geq 1$, there are infinitely many primes $q$ with $o_p(q) = d$ and $L_p(q) = L$. To see this, choose $k$ such that $o_{p^{L+1}}(k) = pd$. Then every integer $q$ in the arithmetic progression $k + np^{L+1}$ has $o_p(q) = d$ and $o_{p^{L+1}}(q) = pd$, and it follows from (3.2) that $o_{p^l}(q) = p^{l-L}d$ for all $l > L$. Now our assertion follows from Dirichlet's Theorem: every arithmetic progression $k + nr$ with $(k, r) = 1$ contains infinitely many primes [8, §16.1].

## 4. The primitive ideal space

Since $\operatorname{Prim} C(X,\mathcal{K})$ is homeomorphic to $X$ [15, Example A.24] and Bunce-Deddens algebras are simple [5, Theorem V.3.3], Theorem 2.2 gives us a setwise description of the primitive ideal space of the algebra $C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes_\alpha\mathbb{N}^2$. It consists of a copy $\{I_{x,y}\}$ of $\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q)$ embedded as an open subset, a copy $\{L_{w,z}\}$ of $\mathbb{T}^2$ embedded as a closed subset, and two finite sets $\{J_{xH_p}\}$, $\{K_{yH_q}\}$ parametrised by the quotients $\mathcal{U}(\mathbb{Z}_p)/H_p = \mathcal{U}(\mathbb{Z}_p)/\overline{q^{\mathbb{Z}}}$ and $\mathcal{U}(\mathbb{Z}_q)/H_q = \mathcal{U}(\mathbb{Z}_q)/\overline{p^{\mathbb{Z}}}$ whose cardinalities determine the number of Bunce-Deddens algebras in the subquotients. The topology on $\operatorname{Prim}(C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes_\alpha\mathbb{N}^2)$ is then given by:

**Theorem 4.1.** *The maps $(x,y)\mapsto I_{x,y}$, $xH_p\mapsto J_{xH_p}$, $yH_q\mapsto J_{yH_q}$ and $(w,z)\mapsto L_{w,z}$ combine to give a bijection of the disjoint union*

$$\big(\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q)\big)\ \sqcup\ \mathcal{U}(\mathbb{Z}_p)/\overline{q^{\mathbb{Z}}}\ \sqcup\ \mathcal{U}(\mathbb{Z}_q)/\overline{p^{\mathbb{Z}}}\ \sqcup\ \mathbb{T}^2 \tag{4.1}$$

*onto* $\operatorname{Prim}(C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes_\alpha\mathbb{N}^2)$. *Write $\pi_p$ for the map $\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q)\to\mathcal{U}(\mathbb{Z}_p)\to\mathcal{U}(\mathbb{Z}_p)/\overline{q^{\mathbb{Z}}}$. Then the hull-kernel closure of a nonempty subset $F$ of* (4.1) *is*

(a) *the usual closure of $F$ in $\mathbb{T}^2$ if $F\subset\mathbb{T}^2$;*
(b) *$F\cup\mathbb{T}^2$ if $F\subset\mathcal{U}(\mathbb{Z}_p)/\overline{q^{\mathbb{Z}}}\ \sqcup\ \mathcal{U}(\mathbb{Z}_q)/\overline{p^{\mathbb{Z}}}$;*
(c) *the usual closure of $F$ in $\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q)$ together with $\pi_p(F)\cup\pi_q(F)\cup\mathbb{T}^2$ if $F\subset\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q)$.*

We shall prove this by writing down irreducible representations of $C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes_\alpha\mathbb{N}^2$ realising each of these primitive ideals, identifying their kernels as crossed products of invariant ideals in $C(\mathbb{Z}_p\times\mathbb{Z}_q)$ using results from [13], and then reading off the topology from standard properties of the topology on $\mathbb{Z}_p\times\mathbb{Z}_q$.

The ideals $L_{w,z}$ are lifted from the quotient $(C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes\mathbb{N}^2)/I_2=\mathbb{C}\rtimes_{\mathrm{id}}\mathbb{N}^2$, and are the kernels of the characters $\gamma_{w,z}:(m,n)\mapsto w^mz^n$; more precisely, $L_{w,z}=\ker(\varepsilon_{0,0}\times\gamma_{w,z})$, where $\varepsilon_{0,0}(f):=f(0,0)$. Because $\operatorname{Prim}(\mathbb{C}\rtimes_{\mathrm{id}}\mathbb{N}^2)$ is a closed subset of $\operatorname{Prim}(C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes\mathbb{N}^2)$, this also proves part (a) of Theorem 4.1.

The ideals $J_{xH_p}$ are lifted from the image of the surjection $(\mathrm{id}\otimes\varepsilon_0)^*$ of $C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes\mathbb{N}^2$ onto $C(\mathbb{Z}_p)\rtimes\mathbb{N}^2$ induced by $\mathrm{id}\otimes\varepsilon_0: C(\mathbb{Z}_p\times\mathbb{Z}_q)\to C(\mathbb{Z}_p)$, and are determined in the image by their intersections with the ideal $C_0(\mathbb{Z}_p\setminus\{0\})\rtimes\mathbb{N}^2$. Recall that the homeomorphism $h_p:(k,x)\mapsto p^kx$ induces an isomorphism

$$h_p^*: C_0(\mathbb{Z}_p\setminus\{0\})\rtimes\mathbb{N}^2\cong C(\mathcal{U}(\mathbb{Z}_p),c_0(\mathbb{N})\rtimes_\tau\mathbb{N})\rtimes_{\sigma\otimes\mathrm{id}}\mathbb{Z}. \tag{4.2}$$

Because $M\times T$ is an isomorphism of $c_0(\mathbb{N})\rtimes_\tau\mathbb{N}$ onto $\mathcal{K}(l^2(\mathbb{N}))$ and $\mathbb{Z}$ acts freely on $\mathcal{U}(\mathbb{Z}_p)=\operatorname{Prim} C(\mathcal{U}(\mathbb{Z}_p),\mathcal{K})$, the primitive ideals of the right-hand side of (4.2) are induced from the ideals $\ker(M\times T)\circ\varepsilon_x$. In particular, we have

$$J_{xH_p}\cap\big(C_0(\mathbb{Z}_p\setminus\{0\})\rtimes\mathbb{N}^2\big)=\ker\Big(\big(\operatorname{Ind}_{\{0\}}^{\mathbb{Z}}(M\times T)\circ\varepsilon_x\big)\circ h_p^*\circ(\mathrm{id}\otimes\varepsilon_0)^*\Big).$$

We can now use the standard form $\widetilde{\pi}\times\lambda$ of the induced representation to see that we can realise $J_{xH_p}$ as the kernel of the representation $\rho_x\times(T\otimes\lambda)$ of $C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes_\alpha\mathbb{N}^2$ on $l^2(\mathbb{N}\times\mathbb{Z})$, where

$$(\rho_x(f)\xi)(k,l) := f(p^kq^lx,0)\xi(k,l)$$

Similarly, with $\sigma_y : C(\mathbb{Z}_p\times\mathbb{Z}_q)\to B(l^2(\mathbb{Z}\times\mathbb{N}))$ defined by

$$(\sigma_y(f)\xi)(k,l) = f(0,p^kq^ly)\xi(k,l),$$

we have $\ker(\sigma_y\times(\lambda\otimes T)) = K_{yH_q}$.

The ideals $I_{x,y}$ are determined by their intersection with $I_1$, and $I_{x,y}\cap I_1$ is pulled back under the isomorphism (2.2) from the kernel of the evaluation map $\varepsilon_{x,y}$ : $C(\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q),\mathcal{K})\to\mathcal{K}$. This isomorphism is induced by the homeomorphism $h:(l,k,x,y)\mapsto p^kq^l(x,y)$ of $\mathbb{N}^2\times\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q)$ onto $(\mathbb{Z}_p\setminus\{0\})\times(\mathbb{Z}_q\setminus\{0\})$, and the Toeplitz representation $M\times T$ of $c_0(\mathbb{N}^2)\rtimes_\tau\mathbb{N}^2$ onto $\mathcal{K}(l^2(\mathbb{N}^2))$. The representation

$$(\pi_{x,y}(f)\xi)(k,l) = f(p^kq^lx,p^kq^ly)\xi(k,l)$$

satisfies $\pi_{x,y}|_{C_0((\mathbb{Z}_p\setminus\{0\})\times(\mathbb{Z}_q\setminus\{0\}))} = M\circ\varepsilon_{x,y}\circ(h^{-1})^*$, and it follows that $(\pi_{x,y},T)$ is a covariant representation of $(C(\mathbb{Z}_p\times\mathbb{Z}_q),\mathbb{N}^2,\alpha)$ with $I_{x,y}=\ker(\pi_{x,y}\times T)$.

To identify the kernels of these representations, we shall use the following analogue of the standard characterisations of faithful representations.

**Lemma 4.2.** *Let $(\eta,T)$ be a covariant representation of a semigroup dynamical system $(A,\mathbb{N}^k,\alpha)$ with extendible endomorphisms. Suppose that $\ker\eta$ is an extendibly $\alpha$-invariant ideal, and that there is a unitary representation $W$ of $\mathbb{T}^k$ such that $(\eta\times T,W)$ is a covariant representation of the dual system $(A\rtimes\mathbb{N}^k,\mathbb{T}^k,\widehat{\alpha})$. Then*

$$\ker(\eta\times T) = (\ker\eta)\rtimes\mathbb{N}^k = \overline{\operatorname{span}}\{i_S(m)^*i_A(a)i_S(n) : m,n\in\mathbb{N}^k,\ a\in\ker\eta\}.$$

*Proof.* We know from [13, Theorem 1.8] that $(\ker\eta)\rtimes\mathbb{N}^k$ is naturally isomorphic to the ideal

$$\overline{\operatorname{span}}\{i_S(m)^*i_A(a)i_S(n) : m,n\in\mathbb{N}^k,\ a\in\ker\eta\}\subset A\rtimes\mathbb{N}^k,$$

and that the quotient map $\pi : A\to A/(\ker\eta)$ induces a homomorphism $\pi\times\mathrm{id}$ of $A\rtimes\mathbb{N}^k$ onto $(A/\ker\eta)\rtimes\mathbb{N}^k$ with kernel $(\ker\eta)\rtimes\mathbb{N}^k$. There is a faithful representation $\zeta$ of $A/\ker\eta$ such that $\eta=\zeta\circ\pi$, and then $(\zeta,T)$ and $(\zeta\times T,W)$ are covariant. It suffices to prove that $\zeta\rtimes T$ is faithful, for then $\eta\times T=(\zeta\times T)\circ(\pi\times\mathrm{id})$, and

$$\ker(\eta\times T) = \ker(\zeta\times T)\circ(\pi\times\mathrm{id}) = \ker(\pi\times\mathrm{id}) = (\ker\eta)\rtimes\mathbb{N}^k.$$

To prove $\zeta\times T$ faithful, we follow the standard procedure of [4, Lemma 2.2]. Write $C=A/\ker\eta$, and let $\theta : C\rtimes\mathbb{N}^k\to C\rtimes\mathbb{N}^k$ be the expectation obtained by averaging over the dual action $\widehat{\alpha}$ on $C\rtimes\mathbb{N}^k$, which is faithful on positive elements by [10, Remark 3.6]. Because $S=\mathbb{N}^k$ is abelian, $C\rtimes\mathbb{N}^k$ is spanned by the elements $i_S(m)^*i_C(c)i_S(n)$ [13, Lemma 1.3], and hence $\theta(C\rtimes\mathbb{N}^k)$ is spanned by the elements $i_S(m)^*i_C(c)i_S(m)$; because every finite set of elements in $\mathbb{N}^k$ has an upper bound, we can imitate the proof of [2, Lemma 1.5] to see that $\zeta \times T$ is faithful on $\theta(C \rtimes \mathbb{N}^k)$. Now we can use the covariance of $(\zeta \times T, W)$ to get an estimate

$$\begin{aligned} \|(\zeta \times T)(\theta(f))\| &= \Big\| \int_{\mathbb{T}^k} W_z^*(\zeta \times T)(f) W_z \, dz \Big\| \\ &\leq \int_{\mathbb{T}^k} \|W_z^* \zeta \times T(f) W_z\| \, dz \\ &= \|\zeta \times T(f)\|, \end{aligned}$$

and follow the argument of [4, Lemma 2.2] to see that $\zeta \times T$ is faithful. □

The ideal $\ker \pi_{x,y}$ consists of the functions which vanish on the closure of the orbit $p^{\mathbb{N}} q^{\mathbb{N}}(x, y)$; to check that $\ker \pi_{x,y}$ is extendibly invariant, we need to know exactly what this closure is.

**Lemma 4.3.** *Let $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_q$. Then $q^{\mathbb{N}} x$ has the same closure in $\mathbb{Z}_p$ as $q^{\mathbb{Z}} x$, and the closure of $p^{\mathbb{N}} q^{\mathbb{N}}(x, y)$ in $\mathbb{Z}_p \times \mathbb{Z}_q$ is*

$$p^{\mathbb{N}} q^{\mathbb{N}}(x, y) \;\cup\; \big(\overline{p^{\mathbb{N}} q^{\mathbb{Z}} x} \times \{0\}\big) \;\cup\; \big(\{0\} \times \overline{p^{\mathbb{Z}} q^{\mathbb{N}} y}\big). \tag{4.3}$$

*Proof.* Since $q \in \mathcal{U}(\mathbb{Z}_p)$, multiplication by $q$ is a homeomorphism of $\mathcal{U}(\mathbb{Z}_p)$, and defines a free and minimal action of $\mathbb{Z}$ on $\overline{q^{\mathbb{Z}} x}$. The sequence $\{q^k x : k \in \mathbb{N}\}$ has a convergent subsequence, $q^{k_n} x \to x_0$, say, and then $\overline{q^{\mathbb{Z}} x_0} = \overline{q^{\mathbb{Z}} x}$ by minimality. Thus every element of $\overline{q^{\mathbb{Z}} x}$ can be approximated first by $q^n x_0$, and then by elements $q^{n+k_n} x$ of $q^{\mathbb{N}} x$. Thus $\overline{q^{\mathbb{N}} x} = \overline{q^{\mathbb{Z}} x}$. This argument also shows that every element of $\overline{q^{\mathbb{Z}} x}$ is the limit of a sequence $q^{m_n} x$ in which $m_n \to \infty$.

Since $(0, 0) = \lim_n p^n q^n(x, y)$, it certainly belongs to the orbit closure. Suppose $p^{k_n} q^{l_n} x \to s$ and $s \neq 0$. Write $s = p^i s_0$ for $s_0 \in \mathcal{U}(\mathbb{Z}_p)$. Then $p^i \mathcal{U}(\mathbb{Z}_p)$ is an open neighbourhood of $s$, so $k_n = i$ for large $n$, and $q^{l_n} x \to p^{-i} s$. As observed above, we may as well suppose $l_n \to \infty$; but then $q^{l_n} y \to 0$, and $p^{k_n} q^{l_n}(x, y) \to (s, 0)$. Thus $\overline{p^{\mathbb{N}} q^{\mathbb{Z}} x} \times \{0\}$ is contained in the orbit closure, and, by symmetry, so is $\{0\} \times \overline{p^{\mathbb{Z}} q^{\mathbb{N}} y}$.

For the other inclusion, suppose $(w, z) \in \mathbb{Z}_p \times \mathbb{Z}_q$ and $p^{k_n} q^{l_n}(x, y) \to (w, z)$. It is obvious that $(w, z)$ belongs to (4.3) if one of $w$ or $z$ is 0, so suppose $w$ and $z$ are both nonzero. We can write $(w, z) = (p^i w_0, q^j z_0)$ for units $w_0, z_0$ and $i, j \in \mathbb{N}$, and then $p^i \mathcal{U}(\mathbb{Z}_p) \times q^j \mathcal{U}(\mathbb{Z}_q)$ is a neighbourhood of $(w, z)$. Thus $(k_n, l_n) = (i, j)$ for large $n$, and $(w, z) = p^i q^j(x, y)$ belongs to $p^{\mathbb{N}} q^{\mathbb{N}}(x, y)$, as required. □

**Lemma 4.4.** *Let $(x, y) \in \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)$. Then*

(a) $J_{xH_p} = \overline{\operatorname{span}}\big\{i_S(i, j)^* i_A(f) \iota_S(m, n) : f \equiv 0$ *on* $\overline{p^{\mathbb{N}} q^{\mathbb{Z}} x} \times \{0\}\big\}$;

(b) $K_{yH_q} = \overline{\operatorname{span}}\big\{i_S(i, j)^* i_A(f) i_S(m, n) : f \equiv 0$ *on* $\{0\} \times \overline{p^{\mathbb{Z}} q^{\mathbb{N}} y}\big\}$; *and*

(c) $I_{x,y} = \overline{\operatorname{span}}\big\{i_S(i, j)^* i_A(f) i_S(m, n) : f \equiv 0$ *on* $\overline{p^{\mathbb{N}} q^{\mathbb{N}}(x, y)}\big\}$.

*Proof.* For part (a), we want to apply Lemma 4.2 with $\eta = \rho_x$, and we therefore need to know that $\ker \rho_x$ is extendibly invariant. We have $\rho_x(f) = 0$ iff $f \equiv 0$ on

$\overline{p^{\mathbb{N}}q^{\mathbb{N}}x} \times \{0\}$, which is equivalent by Lemma 4.3 to $f \equiv 0$ on $\overline{p^{\mathbb{N}}q^{\mathbb{Z}}x} \times \{0\}$. Thus it is enough by [13, Theorem 4.3] to prove that $\overline{p^{\mathbb{N}}q^{\mathbb{N}}x} \times \{0\}$ and its complement are invariant under multiplication by $p^k q^l$. This is trivially true for $\overline{p^{\mathbb{N}}q^{\mathbb{N}}x} \times \{0\}$. Suppose $(w, z) \notin \overline{p^{\mathbb{N}}q^{\mathbb{N}}x} \times \{0\}$. If $z \neq 0$, then $p^k q^l(w, z)$ is certainly not in $\overline{p^{\mathbb{N}}q^{\mathbb{N}}x} \times \{0\}$. So we consider the case $z = 0$, and suppose $p^{k_n}q^{l_n}x \to p^k q^l w$. Since $w$ cannot be 0, we can write $w = p^i w_0$ for $w_0 \in \mathcal{U}(\mathbb{Z}_p)$. Eventually $p^{k_n}q^{l_n}x \in p^{k+i}\mathcal{U}(\mathbb{Z}_p)$, so $k_n = k + i$ for large $n$, and $q^l w = \lim p^{k_n-k}q^{l_n}x$ belongs to $p^i\big(\overline{q^{\mathbb{N}}x}\big)$. Since $\overline{q^{\mathbb{N}}x} = \overline{q^{\mathbb{Z}}x}$, this implies that $w \in p^i\big(\overline{q^{\mathbb{N}}x}\big)$, and hence that $(w, z) \in \overline{p^{\mathbb{N}}q^{\mathbb{N}}x} \times \{0\}$, which is a contradiction. So $p^k q^l(w, z) \notin \overline{p^{\mathbb{N}}q^{\mathbb{N}}x} \times \{0\}$ for all $k, l \in \mathbb{N}$, and we have shown that $\ker \rho_x$ is extendibly invariant.

Next we observe that $W_{w,z}\xi(k, l) := w^k z^l \xi(k, l)$ defines a unitary representation $W$ of $\mathbb{T}^2$ on $l^2(\mathbb{N} \times \mathbb{Z})$ such that $(\rho_x \times (T \otimes \lambda), W)$ is covariant for the dual action. Thus we can deduce from Lemma 4.2 that $J_{xH_p} = \ker(\rho_x \times (T \otimes \lambda))$ has the required form. This gives (a), and of course (b) is exactly the same.

For (c), we apply the same argument to

$$\ker \pi_{x,y} = \big\{f \in C(\mathbb{Z}_p \times \mathbb{Z}_q) : f \equiv 0 \text{ on } \overline{p^{\mathbb{N}}q^{\mathbb{N}}(x, y)}\big\};$$

as above, the crux is to prove that if $p^k q^l(w, z)$ belongs to the closure of $p^{\mathbb{N}}q^{\mathbb{N}}(x, y)$, then so does $(w, z)$. So suppose $(w, z) \in \mathbb{Z}_p \times \mathbb{Z}_q$ and $p^{k_n}q^{l_n}(x, y) \to p^k q^l(w, z)$. If $w$ or $z$ is 0, we are in the situation covered by the first paragraph. So suppose $w$ and $z$ are both nonzero: say $w = p^i w_0$ and $z = q^j z_0$ for units $w_0$, $z_0$. By Lemma 4.3, we must have $p^k q^l(w, z) = p^m q^n(x, y)$ for some $m, n \in \mathbb{N}$. Then $p^{k+i}q^l w_0 = p^m q^n x$ and $p^k q^{l+j} z_0 = p^m q^n y$. The first of these equations implies that $k + i = m$, so $k \leq m$, and the second that $l \leq n$. Thus $(w, z) = p^{m-k}q^{n-l}(x, y)$ belongs to $p^{\mathbb{N}}q^{\mathbb{N}}(x, y)$. This proves that $\ker \pi_{x,y}$ is extendibly invariant. Part (c) follows from an application of Lemma 4.2 with $W$ given by the same formula as before. □

*Proof of Theorem 4.1.* We have already observed that (a) is easy. For (b), notice that for any $xH_p \in \mathcal{U}(\mathbb{Z}_p)/H_p$, the spanning elements $i_S(i, j)^* i_A(f) i_S(m, n)$ of $J_{xH_p}$ go to $f(0, 0)i_S(i, j)^* i_S(m, n)$ in the quotient $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{N}^2$, and hence $J_{xH_p} \subset L_{w,z}$ for all $(w, z) \in \mathbb{T}^2$.

For (c), we observe that $\overline{F} \cap (\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))$ is the usual closure because $(x, y) \mapsto I_{x,y}$ is a homeomorphism of $\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)$ onto the open set $\operatorname{Prim} I_1$. That the closure contains the other points follows from Lemma 4.4: $f \in \ker \pi_{x,y}$ implies $f \in \ker \rho_x$, so all the generators for $I_{x,y}$ described in Lemma 4.4 belong to $J_{xH_p}$, and $(x, y) \in F$ implies

$$J_{xH_p} \in \overline{F} = \big\{P \in \operatorname{Prim}(C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{N}^2) : \textstyle\bigcap_{Q \in F} Q \subset P\big\}.$$

To see that $J_{x_0H_p}$ does not belong to $\overline{F}$ when $x_0H_p \notin \pi_p(F)$, let $F_1$ be the union of the cosets in $\pi_p(F)$. Choose $g \in C(\mathcal{U}(\mathbb{Z}_p))$ such that $g(x_0) = 1$ and $g \equiv 0$ on $F_1$, extend $g$ to a continuous function on $\mathbb{Z}_p$ by taking it to be zero outside $\mathcal{U}(\mathbb{Z}_p)$, and define $f(x, y) = g(x)$. Now we can see from Lemma 4.3 that $f$ vanishes on the closure of

$p^{\mathbb{N}}q^{\mathbb{N}}(x,y)$ for every $(x,y)\in F$, and hence $i_A(f)$ belongs to $\bigcap\{I_{x,y} : x,y\in F\}$ but not to $J_{x_0H_p}$. Thus $\overline{F}\cap\mathcal{U}(\mathbb{Z}_p)/H_p$ is precisely $\pi_p(F)$, and part (c) follows from (b). □

*Remark* 4.5. It is interesting to compare our description of $\operatorname{Prim} C^*(G_{p,q})\rtimes\mathbb{N}^2$ with that obtained for the Bost-Connes algebra $\mathcal{C}_{\mathbb{Q}}$ in [12]. In $\operatorname{Prim}\mathcal{C}_{\mathbb{Q}}$, the finite sets coming from $\operatorname{Prim} I_2/I_1$ do not appear; loosely speaking, we believe this happens because $\mathcal{C}_{\mathbb{Q}}$ contains all the primes, and some of these will act minimally on any given $\mathcal{U}(\mathbb{Z}_p)$ (see Remark 3.8). So the numbers $o_{p^l}(q)$ cannot be recovered from $\operatorname{Prim}\mathcal{C}_{\mathbb{Q}}$. Of course this information is still buried somewhere in $\mathcal{C}_{\mathbb{Q}}$: it follows from Theorem 2.1 of [14] that the inclusion of $G_{p,q}$ in $\mathbb{Q}/\mathbb{Z}$ induces an isomorphism of $C^*(G_{p,q})\rtimes\mathbb{N}^2$ into $C^*(\mathbb{Q}/\mathbb{Z})\rtimes\mathbb{N}^*=\mathcal{C}_{\mathbb{Q}}$.

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