C^*-algebras for hyperbolic dynamical systems, COSy & Barcelona worshop

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Outline:

I. Smale spaces

II. C^*-algebras from Smale spaces

III. Homology for Smale spaces

Part I: Smale spaces (D. Ruelle)

(X, d) compact metric space,

$\varphi : X \to X$ homeomorphism with canonical coordinates: there is a constant $0 < \lambda < 1$, and for x in X and $\epsilon > 0$ and small, there are sets $X^s(x, \epsilon)$ and $X^u(x, \epsilon)$:

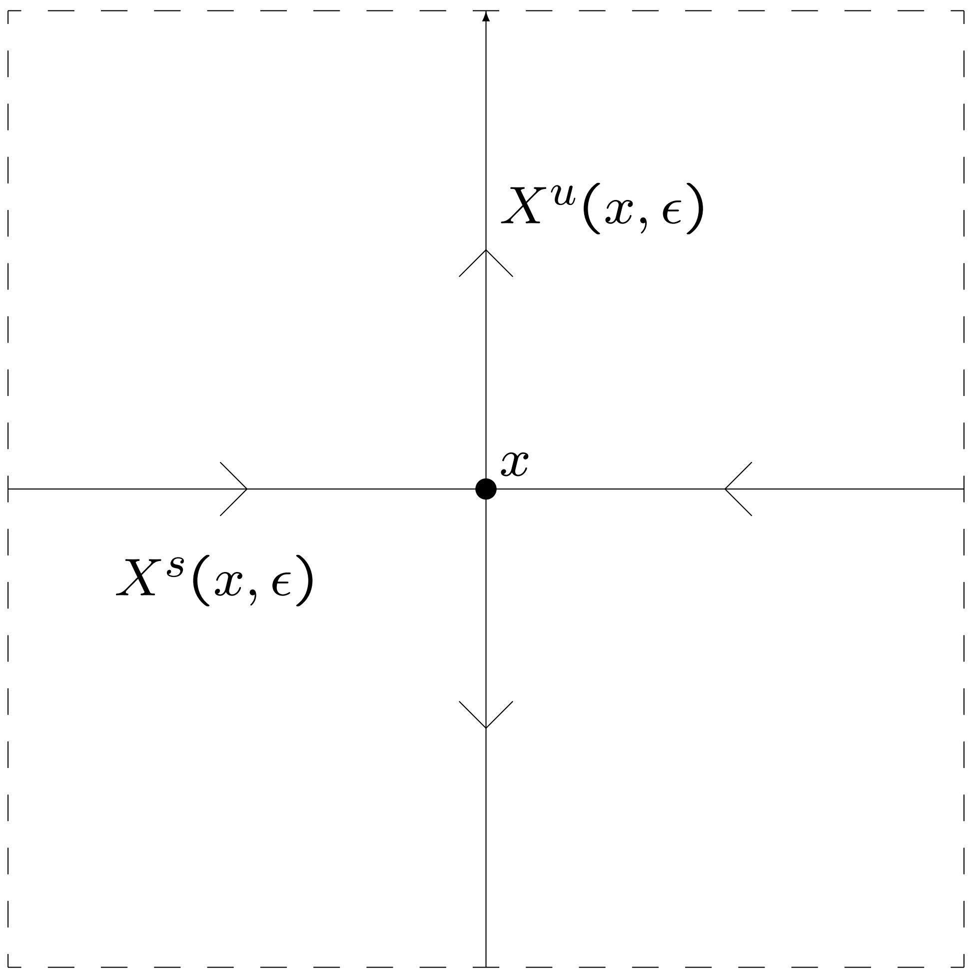
1. $X^s(x, \epsilon) \times X^u(x, \epsilon)$ is homeomorphic to a neighbourhood of x,

2.

$$\begin{aligned} d(\varphi(y), \varphi(z)) &\leq \lambda d(y, z), \quad y, z \in X^s(x, \epsilon), \\ d(\varphi^{-1}(y), \varphi^{-1}(z)) &\leq \lambda d(y, z), \quad y, z \in X^u(x, \epsilon), \end{aligned}$$

3. φ-invariance

That is, we have a local picture:



The definition is aimed at giving a purely topological axiomatization of the dynamics of the basic sets of Smale's Axiom A systems.

If X, φ is a Smale space, we define stable and unstable equivalence relations:

$$\begin{aligned} E^s &= \{(x,y) \mid \lim_{n\to+\infty} d(\varphi^n(x), \varphi^n(y)) = 0\} \\ E^u &= \{(x,y) \mid \lim_{n\to+\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0\}. \end{aligned}$$

Note that

$$\begin{aligned} X^s(x,\epsilon) &\subset E^s(x), \\ X^u(x,\epsilon) &\subset E^u(x), \end{aligned}$$

but their global structure is much more complicated.

Example: Hyperbolic toral automorphisms

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Notice that $\det A = -1$. Moreover its eigenvalues are $\gamma > 1$ and $-\gamma^{-1}$.

$$\begin{aligned} X &= \mathbb{R}^2/\mathbb{Z}^2 \\ \varphi(x + \mathbb{Z}^2) &= Ax + \mathbb{Z}^2 \end{aligned}$$

The local coordinates of contracting and expanding directions are given by the eigenspaces for eigenvalues $|-\gamma^{-1}| < 1$ and $\gamma > 1$.

E^s, E^u are Kronecker foliations.

Example: Solenoids

Let

$$\begin{aligned} X &= \{(z_n)_{n=0}^{\infty} \mid z_n \in \mathbb{T}, \\ & \qquad z_{n+1}^2 = z_n, n \geq 0\} \\ \varphi(z_0, z_1, \ldots) &= (z_0^2, z_1^2, \ldots) \end{aligned}$$

Let $\pi : X \to \mathbb{T}$ be

$$\pi((z_n)_{n=0}^{\infty}) = z_0$$

Then, for a small open set $U \subset \mathbb{T}$,

$$\pi^{-1}(U) \cong U \times C,$$

where C is totally disconnected. This is the local product structure:

$$X^s(z, \epsilon) = C, \qquad X^u(z, \epsilon) = U.$$

Example: Substitution tilings

Example: Basic sets for an Axiom A system

Example: Shifts of finite type

Let $G = (G^0, G^1, i, t)$ be a finite directed graph. Then

$$\begin{aligned} \Sigma_G &= \{(e_n)_{n=-\infty}^{\infty} \mid e_n \in G^1, \\ &\quad\; i(e_{n+1}) = t(e_n), \text{ for all } n\} \\ \sigma(e)_n &= e_{n+1}, \text{ ''left shift''} \end{aligned}$$

The local product structure is given by

$$\Sigma^s(e,1) = \{(\ldots, *, *, *, e_0, e_1, e_2, \ldots)\}$$
$$\Sigma^u(e,1) = \{(\ldots, e_{-2}, e_{-1}, e_0, *, *, *, \ldots)\}$$

Theorem 1. *Shifts of finite type are precisely the zero-dimensional Smale spaces.*

Theorem 2 (Bowen)**.** *Every irreducible Smale space is the image of an irreducible shift of finite type under a finite-to-one factor map.*

C^*-algebras from Smale spaces

Let P denote a set of periodic points of (X, φ), $\varphi(P) = P$. For each p in P, look at $E^u(p)$.

The sets $X^u(x, \epsilon)$ provide a nbhd base for a new (better) topology. This space is then transverse to to stable equivalence.

Let E^s_P denote the equivalence relation E^s restricted to the set $\cup_{p \in P} E^u(p)$. We define E^u_P analogously. These groupoids are étale and we define

$$\begin{aligned} S(X, \varphi, P) &= C^*(E^s_P) \\ U(X, \varphi, P) &= C^*(E^u_P). \end{aligned}$$

The maps $\varphi \times \varphi$ and $\varphi^{-1} \times \varphi^{-1}$ define automorphisms of E_P^s and E_P^u and hence of $S(X, \varphi, P)$, $U(X, \varphi, P)$, respectively.

The Ruelle algebras are defined as

$$\begin{aligned} R^s &= S(X, \varphi, P) \times_{\alpha^s} \mathbb{Z}, \\ R^u &= U(X, \varphi, P) \times_{\alpha^u} \mathbb{Z}, \end{aligned}$$

Define a countable set:

$$H(P) = \cup_{p,q \in P} E^s(p) \cap E^u(q).$$

Hilbert space $l^2(H(P))$, basis $\delta_x, x \in H(P)$.

Define u in $\mathcal{B}(l^2(H(P)))$

$$u\delta_x = \delta_{\varphi(x)}, x \in H(P).$$

x, y in $E^u(p)$,

y in $X^s(x, \epsilon)$,

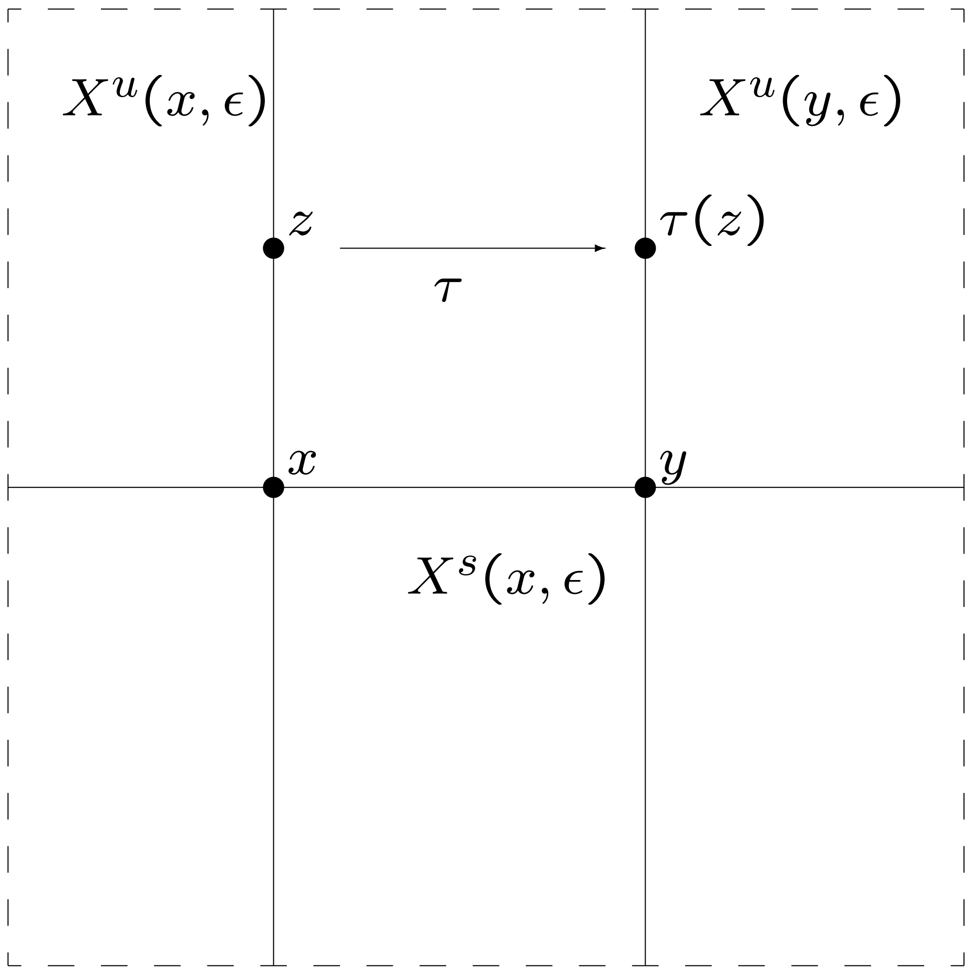
$a_0 \in C_c(X^u(x, \epsilon))$

$z \in X^u(x, \epsilon) \to \tau(z) \in X^u(y, \epsilon)$ defined by

$$\tau(z) \in X^s(z, \epsilon) \cap X^u(y, \epsilon).$$

Define a in $\mathcal{B}(l^2(H(P)))$

$$a\delta_x = a_0(x)\delta_{\tau(x)}, x \in H(P).$$



$$a\delta_z = a_0(z)\delta_{\tau(z)}, z \in H(P).$$

$$S(X,\varphi,P) = span\{u^{-n}au^n \mid n \in \mathbb{Z}, a\}^-$$

$$R_s = C^*\{au^n \mid n \in \mathbb{Z}, a\}$$

Example: Shifts of finite type(W. Krieger)

Let G be a graph and (Σ_G, σ) be the associated shift of finite type. We can take advantage of two nice facts:

- the topologies of $\Sigma_G, E^u(p), E^s_P$ are generated by compact open sets,

- E^s = right tail equivalence is the union of E^s_N = equality to the right of N.

We can construct a sequence of finite dimensional C^*-subalgebras

$$S_1 \subset S_2 \subset \cdots \subset S(\Sigma_G, \sigma, P),$$

whose union is dense in $S(\Sigma_G, \sigma, P)$. So $S(\Sigma_G, \sigma, P)$ is an AF-algebra.

The result for $U(\Sigma_G, \sigma, P)$ is analogous.

Let $N = \#G^0$, A be the (N by N) adjacency matrix for G.

$$D(G) = \lim \mathbb{Z}^N \xrightarrow{A} \mathbb{Z}^N \xrightarrow{A} \cdots$$

$D^*(G)$ is obtained by replacing A by A^T.

Theorem 3.

$$\begin{aligned} K_0(S(\Sigma_G, \sigma, P)) &\cong D^*(G), \\ K_0(U(\Sigma_G, \sigma, P)) &\cong D(G) \end{aligned}$$

Theorem 4.

$$\begin{aligned} R_s &\cong O_{A^T} \otimes \mathcal{K}, \\ R_u &\cong O_A \otimes \mathcal{K}, \end{aligned}$$

where O_A is the Cuntz-Krieger algebra associated with the matrix A.

Theorem 5 (P.-Spielberg)**.** *For a general irreducible Smale space (X, φ), we have*

- $S(X, \varphi, P)$ *is amenable,*
- $S(X, \varphi, P)$ *has a densely defined faithful trace, which is scaled by the automorphism* α^s,
- $S(X, \varphi, P)$ *is simple if and only if* (X, φ) *is mixing.*

We also have

- R_s *is amenable*
- R_s *is purely infinite and simple.*

Functoriality.

A factor map

$$\pi : (Y, \psi) \to (X, \varphi)$$

is *strongly u-resolving* if, for every y in Y,

$$\pi : E^u(y) \to E^u(\pi(y))$$

is bijective. It implies π is a local homeomorphism from $Y^u(y, \epsilon)$ to $X^u(\pi(y), \epsilon)$.

Such a map induces $*$-homomorphisms

$$\begin{aligned} \pi_* : &\quad S(Y, \psi, P) &\to S(X, \varphi, \pi(P)) \\ \pi^* : &\quad U(X, \varphi, \pi(P)) &\to U(Y, \psi, P) \end{aligned}$$

A strongly s-resolving map π induces

$$\begin{aligned} \pi_* : &\quad U(Y, \psi, P) &\to U(X, \varphi, \pi(P)) \\ \pi^* : &\quad S(X, \varphi, \pi(P)) &\to S(Y, \psi, P) \end{aligned}$$

Recall $S(X, \varphi, P), U(X, \varphi, P), R_s, R_u$ are all represented on $l^2(H(P))$. Their relative positions are rather special:

Lemma 6. *For any a in $S(X, \varphi, P)$, b in $U(X, \varphi, P)$, we have*

- *ab is compact,*

- *$\| (u^n a u^{-n})b - b(u^n a u^{-n}) \| \to 0$ as $n \to +\infty$.*

The facts above can be used to define E-theory classes (i.e. asymptotic morphisms). These in turn provide a type of duality.

Theorem 7 (Kaminker-P.). *Let* (X, φ) *be an irreducible Smale space. The* C^**-algebras* R_s *and* R_u *are K-theoretically dual. In particular, there are natural isomorphisms*

$$\begin{aligned} K_i(R_s) &\cong K^{i+1}(R_u), i = 0, 1 \\ K_i(R_u) &\cong K^{i+1}(R_s), i = 0, 1 \end{aligned}$$

Example:

$$K_0(O_A) \cong \mathbb{Z}^N/(I - A^T)\mathbb{Z}^N \cong K^1(O_{A^T}).$$

Homology for Smale spaces

For a Smale space (X, φ) we define two homology theories, $H_*^s(X, \varphi)$, $H_*^u(X, \varphi)$.

Theorem 8. *There exists a spectral sequence with E^2 term $H_*^s(X, \varphi)$ converging to $K_*(S(X, \varphi, P))$.*

Proof in progress.

G a graph

$\mathbb{Z}G^0$ - free abelian group on G^0 (or $\mathbb{Z}^N$)

$\gamma(v) = \sum_{i(e)=v} t(e)$ (or $n \to nA$)

$$D(G) = \lim \mathbb{Z}G^0 \xrightarrow{\gamma} \mathbb{Z}G^0 \xrightarrow{\gamma} \cdots$$

Theorem 9 (Bowen)**.** *Let* (X, φ) *be an irreducible Smale space. Then there exists an irreducible shift of finite type,* (Σ_G, σ)*, and a map*

$$\pi : (\Sigma_G, \sigma) \to (X, \varphi),$$

which is continuous, surjective and finite-to-one.

For $N \geq 0$,

$$\begin{aligned} \Sigma_N &= \{(e_0, e_1, \ldots, e_N) \mid \\ &\quad \pi(e_n) = \pi(e_0), \\ &\quad 0 \leq n \leq N\}. \end{aligned}$$

(Σ_N, σ) is also a shift of finite type. Moreover, $\Sigma_N = \Sigma_{G_N}$, $G_N \subset \prod_0^N G$. It also has an action of S_{N+1}.

$\mathbb{Z}(G_N^0, S_{N+1})$:

Generators $v = (v_0, \ldots, v_N) \in G_N^0$,

Relations $< v >= 0$, if $v_i = v_j$, some $i \neq j$,

$< v_{\alpha(0)}, \ldots, v_{\alpha(N)} >= sgn(\alpha) < v_0, \ldots, v_N >$

$D_N(G_N) = \lim \mathbb{Z}(G_N^0, S_{N+1}) \xrightarrow{\gamma_N} \mathbb{Z}(G_N^0, S_{N+1}) \xrightarrow{\gamma_N} \cdots$

$D_N^*(G_N)$ is obtained replacing G_N by G_N^{op}.

We want a boundary map:

$$\partial_N^s : D_N(G_N) \to D_{N-1}(G_{N-1})$$

and there is an obvious choice from using:

$$\partial_N : \mathbb{Z}(G_N, S_{N+1}) \to \mathbb{Z}(G_{N-1}, S_N)$$

given by

$$\partial_N(< v_0, v_1, \ldots, v_N >) = \\ \sum_{n=0}^{N} (-1)^n < \Delta_n(v_0, v_1, \ldots, v_N) >,$$

where Δ_n = delete entry n.

This does **not** commute with the inductive limits.

Instead, for $K \geq 0$, define

$$\partial_N^K(< v >) = \sum_{n=0}^{N} \sum_p (-1)^n < t(p) >,$$

where the sum is taken over paths of length K:

$$p \in \Delta_n(G_N^K \cap i^{-1}\{v\}).$$

Lemma 10. *If π is strongly u-resolving, then for K sufficiently large,*

$$\partial_N^K \circ \gamma_N = \gamma_{N-1} \circ \partial_N^K = \partial_N^{K+1}.$$

Define, for K large, $[a, k]$ in $D_N(G_N)$:

$$\partial_N^s[a, k] = [\partial_N^K(a), k + K].$$

Lemma 11. *For K sufficiently large,*

$$\partial_N^K \circ \partial_{N+1}^K = 0.$$

The hypothesis is rather strong: it requires $dim(X^u(x,e)) = 0$. We will try to ammend this in a moment, but first note the other case:

If π is s-resolving: define ∂_N^{*K} by interchanging i and t.

Lemma 12. *If π is strongly s-resolving, then for K sufficiently large*

$$\partial_N^{*K} \circ \gamma_N^* = \gamma_{N-1}^* \circ \partial_N^{*K} = \partial_N^{*K+1}.$$

Then define

$$\partial_N^s[a,k] = [Hom(\partial_{N+1}^{*K})(a), k+K].$$

which maps

$$D_N(G_N) \to D_{N+1}(G_{N+1}).$$

Let (X, φ) be a Smale space. We look for a Smale space (Y, ψ) and a factor map $\pi^u : (Y, \psi) \to (X, \varphi)$ satisfying:

1. $dim(Y^s(y, \epsilon)) = 0$,

2. π^u is strongly u-resolving.

That is, $Y^s(y, \epsilon)$ is totally disconnected, $Y^u(y, \epsilon) \sim X^u(\pi^u(y), \epsilon)$.

Similarly, we look for a Smale space (Z, η) and a factor map π^s satisfying:

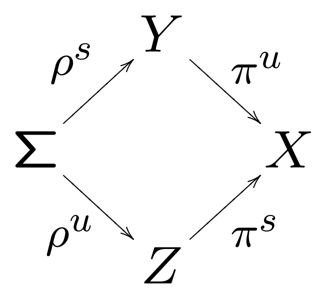
1. $dim(Z^u(z, \epsilon)) = 0$,

2. π^s is strongly s-resolving.

We call $\pi = (\pi^u, \pi^s)$ a resolving pair for (X, φ).

Theorem 13. *For (X, φ) irreducible, resolving pairs exist.*

Let (Σ, σ) be the fibred product:



Then Σ is a SFT. $\Sigma = \Sigma_G$, for some graph G.

For $L, M \geq 0$,

$$\begin{aligned} \Sigma_{L,M} &= \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid \\ &\quad y_l \in Y, z_m \in Z, \\ &\quad \pi^u(y_l) = \pi^s(z_m)\}. \end{aligned}$$

For each $L, M \geq 0$, $\Sigma_{L,M}$ is a shift of finite type. The graph $G_{L,M}$ presenting $\Sigma_{L,M}$ can be viewed as $L+1$ by $M+1$ arrays over G.

Incorporating $S_{L+1} \times S_{M+1}$ actions, we get inductive limit groups $D_{L,M}(G_{L,M})$ and a double complex:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
D_{0,2}(G_{0,2}) & \longleftarrow & D_{1,2}(G_{1,2}) & \longleftarrow & D_{2,2}(G_{2,2}) & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D_{0,1}(G_{0,1}) & \longleftarrow & D_{1,1}(G_{1,1}) & \longleftarrow & D_{2,1}(G_{2,1}) & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D_{0,0}(G_{0,0}) & \longleftarrow & D_{1,0}(G_{1,0}) & \longleftarrow & D_{2,0}(G_{2,0}) & \longleftarrow &
\end{array}$$

$$\begin{aligned}
\partial^s_N : \quad & \oplus_{L-M=N} D_{L,M}(G_{L,M}) \\
\rightarrow \quad & \oplus_{L-M=N-1} D_{L,M}(G_{L,M})
\end{aligned}$$

$$H^s_N(\pi) = \ker(\partial^s_N)/Im(\partial^s_{N+1}).$$

Properties

Theorem 14. *The groups* $H^s_N(\pi)$ *do not depend on the choice of resolving pair* $\pi = (\pi^u, \pi^s)$.

From now on, we write $H^s_N(X, \varphi)$.

Theorem 15. *The functor* $H^s_*(X, \varphi)$ *is covariant for strongly* u*-resolving maps, contravariant for strongly* s*-resolving maps.*

We can regard $\varphi : (X, \varphi) \to (X, \varphi)$, which is both s and u-resolving and so induces an automorphism of the invariants.

Theorem 16. *(Lefschetz Formula) Let* (X, φ) *be any Smale space having a resolving pair and let* $p \geq 1$.

$$\begin{aligned}\sum_{N \in \mathbb{Z}} (-1)^N \quad Tr[\varphi_*^p : \ & H_N^s(X, \varphi) \otimes \mathbb{Q} \\ \to \quad & H_N^s(X, \varphi) \otimes \mathbb{Q}] \\ = \quad & \#\{x \in X \mid \varphi^p(x) = x\}\end{aligned}$$

Question: Relation between $H^s_*(X, \varphi)$ and $\check{H}^*(BR^s)$?

Question: Axiomatic definition of $H^s(X, \varphi)$?

Dimension axiom becomes the dimension group axiom:

For a shift of finite type,

$$H^s_N(\Sigma_G, \sigma) = \begin{cases} D(G) & N = 0 \\ 0 & N \neq 0 \end{cases}$$