A homology theory for basic sets

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Smale spaces: hyperbolic topological systems

1. basic sets for Axiom A systems
2. hyperbolic toral automorphisms
3. solenoids (R. Williams)
4. shifts of finite type (SFT's)

Shifts of finite type

1. zero-dimensional Smale spaces
2. universal property (Bowen's Theorem)
3. Krieger's dimension group invariant

Goal: Extend Krieger's invariant to Smale spaces

Smale spaces (D. Ruelle)

(X, d) compact metric space,

$\varphi : X \to X$ homeomorphism with canonical coordinates: there is a constant $0 < \lambda < 1$, and for x in X and $\epsilon > 0$ and small, there are sets $X^s(x, \epsilon)$ and $X^u(x, \epsilon)$:

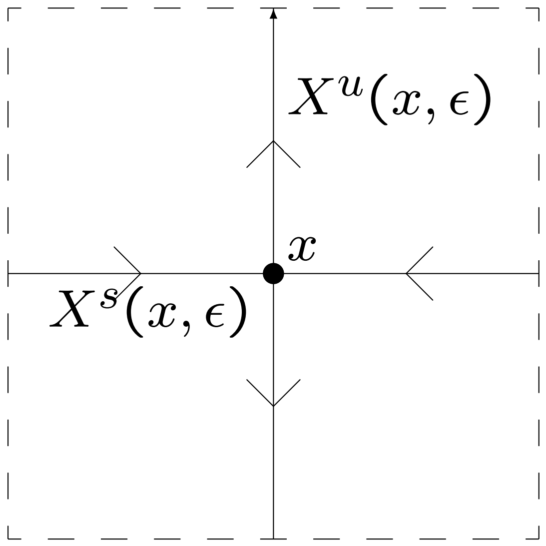
1. $X^s(x, \epsilon) \times X^u(x, \epsilon)$ is homeomorphic to a neighbourhood of x,

2. φ-invariance,

3.

$$\begin{aligned} d(\varphi(y), \varphi(z)) &\leq \lambda d(y, z), \quad y, z \in X^s(x, \epsilon), \\ d(\varphi^{-1}(y), \varphi^{-1}(z)) &\leq \lambda d(y, z), \quad y, z \in X^u(x, \epsilon), \end{aligned}$$

That is, we have a local picture:



Actual definition: existence of $[,]$ satisfying some axioms. $[x,y]$ is the intersection of $X^s(x,\epsilon)$ and $X^u(y,\epsilon)$.

Stable and unstable equivalence:

$$R^s = \{(x,y) \mid \lim_{n\to+\infty} d(\varphi^n(x),\varphi^n(y)) = 0\}$$
$$R^u = \{(x,y) \mid \lim_{n\to+\infty} d(\varphi^{-n}(x),\varphi^{-n}(y)) = 0\}$$

$R^s(x), R^u(x)$ denote equivalence classes.

Factor maps

Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between Smale spaces. For every y in Y, $\pi : R^s(y) \to R^s(\pi(y))$.

- π is s-*resolving* if $\pi : R^s(y) \to R^s(\pi(y))$ is injective, for all y.

- π is s-*bijective* if $\pi : R^s(y) \to R^s(\pi(y))$ is bijective, for all y.

We remark:

- Y irreducible, s-resolving $\Rightarrow$ s-bijective

- s-resolving $\Rightarrow$ finite-to-one

- s-bijective $\Rightarrow \pi : Y^s(y, \epsilon) \to X^s(\pi(y), \epsilon')$ is a local homeomorphism

Shifts of finite type

Let $G = (G^0, G^1, i, t)$ be a finite directed graph. Then

$$\begin{aligned} \Sigma_G &= \{(e^k)_{k=-\infty}^{\infty} \mid e^k \in G^1, \\ &\quad i(e^{k+1}) = t(e^k), \text{ for all } n\} \\ \sigma(e)^k &= e^{k+1}, \text{ "left shift"} \end{aligned}$$

The local product structure is given by

$$\Sigma^s(e, 1) = \{(\ldots, *, *, *, *, e^1, e^2, \ldots)\}$$
$$\Sigma^u(e, 1) = \{(\ldots, e^{-2}, e^{-1}, e^0, *, *, *, \ldots)\}$$

A shift of finite type is any system conjugate to (Σ_G, σ), for some G.

Dimension groups

Motivation: For $dim(X) = 0$, the Cech cohomology of X is $C(X, \mathbb{Z})$. Or, the free abelian group on the collection of clopen sets with relation

$$E \cup F = E + F, \text{ if } E \cap F = \emptyset.$$

Let (Σ, σ) be a shift of finite type. $\mathcal{D}^s(\Sigma, \sigma)$ denotes the set of all $E \subset \Sigma^s(e, \epsilon)$ which are compact and open.

Equivalence relation $\sim$:

$$\begin{aligned} [E, F] = F, [F, E] = E &\Rightarrow E \sim F \\ E \sim F &\Leftrightarrow \sigma(E) \sim \sigma(F) \end{aligned}$$

$D^s(\Sigma, \sigma)$ is the free abelian group generated by equivalence classes of $\mathcal{D}^s(\Sigma, \sigma)$ modulo the relation:

$$[E \cup F] = [E] + [F], \text{ if } E \cap F = \emptyset.$$

G a finite directed graph.

$$\mathbb{Z}G^0 = \text{ free abelian group on } G^0$$

or $\mathbb{Z}^N, N = \# G^0$.

Define $\gamma^s : \mathbb{Z}G^0 \to \mathbb{Z}G^0$, by

$$\gamma^s(v) = \sum_{t(e)=v} i(e).$$

and A is the adjacency matrix of G,

$$A : \mathbb{Z}^N \to \mathbb{Z}^N$$

Then

$$\begin{aligned} D^s(\Sigma_G, \sigma) &\cong \lim \mathbb{Z}G^0 \xrightarrow{\gamma^s} \mathbb{Z}G^0 \xrightarrow{\gamma^s} \cdots \\ &\cong \lim \mathbb{Z}^N \xrightarrow{A} \mathbb{Z}^N \xrightarrow{A} \cdots \end{aligned}$$

D^s as a functor

Let $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ be a factor map. If π is s-bijective, then there is a map

$$\pi^s : D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma).$$

(The idea is that $\pi^s[E] = [\pi(E)]$.)

If π is u-bijective, then there is a map

$$\pi^{s*} : D^s(\Sigma', \sigma) \to D^s(\Sigma, \sigma)$$

(The idea is that $\pi^{s*}[E'] = [\pi^{-1}(E')]$.)

(Kitchens, Boyle, Marcus, Trow)

Homology: First attempt

(X, φ) a Smale space. What is $H^s(X, \varphi)$?

Bowen: For (X, φ) irreducible, there exists

$$\pi : (\Sigma, \sigma) \to (X, \varphi),$$

continuous, surjective and finite-to-one.

For $N \geq 0$, define

$$\begin{aligned} \Sigma_N(\pi) &= \{(e_0, e_1, \ldots, e_N) \mid \\ & \quad \pi(e_n) = \pi(e_0), \\ & \quad 0 \leq n \leq N\}. \end{aligned}$$

For all $N \geq 0$, $(\Sigma_N(\pi), \sigma)$ is also a shift of finite type.

Idea: Compute homology of (X, φ) from that of $(\Sigma_N(\pi), \sigma), N \geq 0$.

For $0 \leq n \leq N$, let $\delta_n : \Sigma_N(\pi) \to \Sigma_{N-1}(\pi)$ be the map which deletes entry n.

Lemma 1. *If π is s or u-bijective, then so is δ_n.*

Definition 2. *If π is s-bijective, define $\partial_N^s(\pi) : D^s(\Sigma_N) \to D^s(\Sigma_{N-1})$ by*

$$\partial_N^s(\pi) = \sum_{n=0}^{N} (-1)^n (\delta_n)^s.$$

If π is u-bijective, define $\partial_N^{s*}(\pi) : D^s(\Sigma_N) \to D^s(\Sigma_{N+1})$ by

$$\partial_N^{s*}(\pi) = \sum_{n=0}^{N+1} (-1)^n (\delta_n)^{s*}.$$

If π is s-bijective, we get a chain complex; u-bijective, we get a cochain complex.

But to get either, we would need $X^s(x, \epsilon)$ or $X^u(x, \epsilon)$ totally disconnected.

Homology: Second attempt

(X, φ) a Smale space, what is $H^s(X, \varphi)$?

Let (X, φ) be a Smale space. We look for a Smale space (Y, ψ) and a factor map $\pi_s : (Y, \psi) \to (X, \varphi)$ satisfying:

1. $dim(Y^u(y, \epsilon)) = 0$,

2. π_s is s-bijective.

That is, $Y^u(y, \epsilon)$ is totally disconnected, while $Y^s(y, \epsilon)$ is homeomorphic to $X^s(\pi_s(y), \epsilon)$.

This is a "one-coordinate" version of Bowen's Theorem.

Similarly, we look for a Smale space (Z, ζ) and a factor map π_u satisfying:

1. $dim(Z^s(z, \epsilon)) = 0$,

2. π_u is u-bijective.

We call $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ a *resolving pair* for (X, φ).

Theorem 3. *For (X, φ) irreducible, resolving pairs exist.*

Consider the fibred product:

$$\Sigma = \{(y, z) \in Y \times Z \mid \pi_s(y) = \pi_u(z)\}$$

with

$$\begin{array}{ccccc} & & \Sigma & & \\ & \rho_u \swarrow & & \searrow \rho_s & \\ Y & & & & Z \\ & \pi_s \searrow & & \swarrow \pi_u & \\ & & X & & \end{array}$$

$\rho_s(y, z) = z$ is s-bijective, $\rho_u(y, z) = y$ is u-bijective. Hence, Σ is a SFT, $\Sigma = \Sigma_G$, for some graph G.

For $L, M \geq 0$,

$$\begin{aligned} \Sigma_{L,M}(\pi) \;=\; & \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid \\ & y_l \in Y, z_m \in Z, \\ & \pi_s(y_l) = \pi_u(z_m)\}. \end{aligned}$$

Moreover, the maps

$$\delta_{l,} : \Sigma_{L,M} \to \Sigma_{L-1,M}, \delta_{,m} : \Sigma_{L,M} \to \Sigma_{L,M-1}$$

are s-bijective and u-bijective, respectively.

We get a double complex:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,2}) & \longleftarrow & D^s(\Sigma_{1,2}) & \longleftarrow & D^s(\Sigma_{2,2}) & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,1}) & \longleftarrow & D^s(\Sigma_{1,1}) & \longleftarrow & D^s(\Sigma_{2,1}) & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,0}) & \longleftarrow & D^s(\Sigma_{1,0}) & \longleftarrow & D^s(\Sigma_{2,0}) & \longleftarrow &
\end{array}$$

$$\begin{aligned}
\partial_N^s : \quad & \oplus_{L-M=N} D^s(\Sigma_{L,M}) \\
\rightarrow \quad & \oplus_{L-M=N-1} D^s(\Sigma_{L,M})
\end{aligned}$$

$$\partial_N^s = \sum_{l=0}^{L}(-1)^l \delta_{l,}^s + \sum_{m=0}^{M+1}(-1)^{m+M}\delta_{,m}^{*s}$$

$$H_N^s(\pi) = \ker(\partial_N^s)/Im(\partial_{N+1}^s).$$

Five basic theorems

Recall: beginning with (X, φ), we select a resolving pair $\pi = (Y, \pi_s, Z, \pi_u)$ and compute $H^s_N(\pi)$.

Theorem 4. *The groups $H^s_N(\pi)$ do not depend on the choice of resolving pair π.*

From now on, we write $H^s_N(X, \varphi)$.

Theorem 5. *The functor $H^s_*(X, \varphi)$ is covariant for s-bijective maps, contravariant for u-bijective maps.*

Theorem 6. *For (X, φ) irreducible, the group $H^s_0(X, \varphi)$ has a natural order structure.*

The invariant $D^s(\Sigma_N)$ is computed as an inductive limit

$$\mathbb{Z}G_N^0 \to \mathbb{Z}G_N^0 \to \cdots$$

and the generators of $\mathbb{Z}G_N^0$ are $N+1$-tuples of vertices from G.

Instead, define $\mathbb{Z}_a G_N^0$ as the quotient by the relations

$$\begin{aligned} (v_0, \ldots, v_N) &= 0, \\ &\quad \text{if } v_i = v_j, i \neq j, \\ (v_{\alpha(0)}, \ldots, v_{\alpha(N)}) &= sgn(\alpha)(v_0, \ldots, v_N), \\ &\alpha \in S_{N+1} \end{aligned}$$

with limit $D_a^s(\Sigma_N)$. $D_a^s(\Sigma_N) \neq 0$ for only finitely many N.

Theorem 7. *The homologies obtained from $D^s(\Sigma_N)$ and $D_a^s(\Sigma_N)$ are the same.*

There is also a two-variable version.

We can regard $\varphi : (X, \varphi) \to (X, \varphi)$, which is both s and u-bijective and so induces an automorphism of the invariants.

Theorem 8. *(Lefschetz Formula) Let (X, φ) be any Smale space having a resolving pair and let $p \geq 1$.*

$$\begin{aligned} \sum_{N \in \mathbb{Z}} (-1)^N \quad Tr[(\varphi^s)^p : \quad & H^s_N(X, \varphi) \otimes \mathbb{Q} \\ \to \qquad & H^s_N(X, \varphi) \otimes \mathbb{Q}] \\ \\ = \qquad & \#\{x \in X \mid \varphi^p(x) = x\} \end{aligned}$$

Example 1: Shifts of finite type

If $(X, \varphi) = (\Sigma, \sigma)$, then $Y = \Sigma = Z$ is a resolving pair.

The double complex D_a^s is:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow &
\end{array}$$

and $H_0^s(\Sigma, \sigma) = D^s(\Sigma)$ and $H_N^s(\Sigma, \sigma) = 0, N \neq 0$.

Example 2: $\mathbf{dim(X^s(x, \epsilon)) = 0}$ and (X, φ) irred.

We may find a SFT and s-bijective map

$$\pi_s : (\Sigma, \sigma) \to (X, \varphi).$$

The $Y = \Sigma, Z = X$ is a resolving pair and the double complex D^s_a is:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s_a(\Sigma_0) & \longleftarrow & D^s_a(\Sigma_1) & \longleftarrow & D^s_a(\Sigma_2) & \longleftarrow &
\end{array}$$

Example 3: $(\mathbf{X}, \varphi) = \mathbf{m}^{\infty}$**-solenoid** (Bazett-P.)

A resolving pair is $Y = \{0, 1, \ldots, m-1\}^{\mathbb{Z}}$, the full m-shift, $Z = X$ and the double complex D_a^s is

$$
\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}[1/m] & \longleftarrow & \mathbb{Z} & \longleftarrow & 0 & \longleftarrow &
\end{array}
$$

and we get $H_0^s(X, \varphi) \cong \mathbb{Z}[1/m]$, $H_1^s(X, \varphi) \cong \mathbb{Z}$, $H_N^s(\Sigma_G, \sigma) = 0, N \neq 0, 1$.

D. Pollock considering Williams-Yi 1-dimensional solenoids.

Example 4: A hyperbolic toral automorphism (Bazett-P.):

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

The double complex D_a^s looks like:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}^2 & \leftarrow & \mathbb{Z} & \leftarrow & 0 & \leftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}^3 & \leftarrow & \mathbb{Z}^2 & \leftarrow & 0 & \leftarrow &
\end{array}$$

and

N	$H_N^s(X, \varphi)$	φ^s
-1	$\mathbb{Z}$	1
0	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
1	$\mathbb{Z}$	$-1.$

As an ordered group, $H_0^s(X, \varphi) \cong \mathbb{Z} + \frac{1+\sqrt{5}}{2}\mathbb{Z}$.