Orbit equivalence of Cantor minimal systems: Kyoto Winter School 2011

Ian F. Putnam,
Department of Mathematics and Statistics,
University of Victoria,
Victoria, B.C., Canada V8W 3R4,
ifputnam@uvic.ca

Abstract

These notes are based on a series of lectures given at the Winter School in Operator Algebras at RIMS in Kyoto in December of 2011. In fact, the intention is to give a slightly more complete treatment of the main ideas, without being very formal. I have included exercises. These are not meant to be particularly challenging, but I think they may be helpful for young people trying to get a feel for the subject. The current version is being produced December 29, 2011.

Contents

1 Introduction

The papers [8, 9, 10, 13, 20] present a study of orbit equivalence for certain minimal dynamical systems on a Cantor set; i.e. a compact, totally disconnected, metrizable space with no isolated points. This is a parallel program to that initiated by H. Dye in ergodic theory (for example, see [2]) and also that in Borel equivalence relations (for example, see [16]).

The main objects of interest actions are of countable groups on the Cantor set. In fact, all of groups will be abelian, only because of the limitations of the results to this point. We also mostly restrict our attention to *free* actions (as we describe in the next section). To a group action, we associate an equivalence relation, called the orbit relation: two points are equivalent if there is an element of the group whose associated homeomorphism carries one to the other. An orbit equivalence between two such systems is a homeomorphism between the underlying spaces which carries the equivalence classes of one system to those of the other.

In fact, it turns out to be useful to generalize the notion of group action. In section 3, we give a definition of a *local action*. To such an object, we can also associate an analogue of the orbit relation. The key point is that this relation is naturally equipped with a topology, and we refer to such an equivalence relation as *étale*. We shall see that free actions of groups are examples. On the other hand, we will also see there is another class called the AF-equivalence relations which are at once highly complex, yet quite tractible. Here, 'AF' stands for 'approximately finite'. Much of our attention will be devoted to these, at least at first. However, we will see ultimately that they are actually quite close to group actions and this will allow us to extend results from the AF-case to group actions, which are our main interest.

We end this section with a few preliminary notions. First, by a *Cantor set*, we mean a compact, totally disconnected metric space with no isomlated points. Any two such spaces are homeomorphic. Subsets of such a space which are both open and closed will be referred to as *clopen*. If E is such a set, χ_E denotes its characteristic function, which is, of course, continuous. By a *partition* of a Cantor set, we mean a finite collection of clopen sets which are pairwise disjoint and cover the space.

For a Cantor set X, we will frequently consider $C(X, \mathbb{Z})$, which is the set of continuous integer-valued functions on X. Given such a function f, it is easy to see that the sets $F_n = f^{-1}\{n\}, n \in f(X)$, form a partition of X and

$f = \sum_{n \in f(X)} n \chi_{F_n}$.

We will also discuss measures on the Cantor set. All of our measures will be probability measures; i.e. they are finite, positive and normalized so the measure of the space is one. For readers with a good background in measure theory, there is nothing very sophisticated. The reader with little or no background in measure theory may take as the definition of a measure on a Cantor set X: a function μ defined on the clopen subsets of X such that $\mu(X) = 1$ and $\mu(U \cup V) = \mu(U) + \mu(V)$, whenever U, V are clopen and disjoint. We also note that, at least for a function f in $C(X, \mathbb{Z})$, we define

$$\int f d\mu = \sum_{n \in f(X)} n\mu(f^{-1}\{n\}).$$

2 Group actions

In this section, we describe the basic objects of interest: actions of countable abelian groups by homeomorphisms on Cantor sets. In fact, most of this chapter applies equally well to non-abelian groups and some of it to more general spaces. The reason we restrict our attention to abelian groups is first, that we have almost nothing to say about the non-abelian case, and secondly, it is convenient to use the additive notation. Many of the later sections will deal with the case that the group is the group of integers, $\mathbb{Z}$. We review several basic notions: free actions, minimal actions and invariant measures.

Definition 2.1. *Let X be a topological space and G be an abelian group. We say that φ is an* action of G on X *if, for every a in G, we have $\varphi^a : X \to X$, which is a homeomorphism. Moreover, we have*

$$\begin{aligned} \varphi^0(x) &= x, \text{ for all } x \in X, \text{ and} \\ \varphi^a \circ \varphi^b &= \varphi^{a+b}, \text{ for all } a, b \in G. \end{aligned}$$

We denote such an object by (X, G, φ).

If φ is a single homeomorphism of X, we can form an action of the integers by setting φ^n to be the nth iterate of φ, for $n > 0$ and φ^n to be the $-n$th iterate of φ^{-1} for $n < 0$ and φ^0 to be the identity map of X.

Most of our interest will be in group actions which are free in the following sense.

Definition 2.2. *Let φ be an action of the abelian group G on the topological space X. We say that φ is* free *if, whenever x in X and a in G satisfy $\varphi^a(x) = x$, then $a = 0$.*

Remark 2.3. *Let φ be a free action of the abelian group G on the topological space X. Whenever x in X and a,b in G such that $\varphi^a(x) = \varphi^b(x)$ then $a = b$.*

One of our main interests in group actions is the associated orbit relation.

Definition 2.4. *Let φ be an action of the abelian group G on the topological space X. The orbit relation, R_φ, is defined to be*

$$R_\varphi == \{(x, \varphi^a(x)) \mid x \in X, a \in G\}.$$

It is an easy matter to see that R_φ is an equivalence relation on X. The equivalence class of a point x in X is called its orbit and we write it as $O_\varphi(x)$; that is,

$$O_\varphi(x) = \{\varphi^a(x) \mid a \in G\}.$$

With this definition complete, we can define the important notion of a minimal action.

Definition 2.5. *An action φ of an abelian group G on a space X is* minimal *if, for every x in X, $O_\varphi(x)$ is dense in X.*

Exercise 2.6. *Prove that the action φ is minimal if and only if the only closed sets $Z \subset X$ such that $\varphi^a(Z) = Z$, for every a in G, are $\emptyset$ and X.*

Our main interest in group actions will be in determining when two are orbit equivalent in the following sense.

Definition 2.7. *Two free actions (X_1, G_1, φ_1) and (X_2, G_2, φ_2) are* orbit equivalent *if there exists homeomorphism $h : X_1 \to X_2$ such that*

$$h(O_{\varphi_1}(x)) = O_{\varphi_2}(h(x)),$$

for each x in X or, equivalently,

$$h \times h(R_{\varphi_1}) = R_{\varphi_2}.$$

In this case, we write $(X_1, G_1, \varphi_1) \sim (X_2, G_2, \varphi_2)$.

Here is a very simple result, but it will be useful to have it stated this way later.

Proposition 2.8. *Let φ be an action of the abelian group G on the topological space X. The map from $X \times G$ to $X \times X$ which sends (x, a) to $(x, \varphi^a(x))$ has range R_φ and is injective if and only if φ is free.*

One immediate application is that it allows us to put a topology on R_φ, in the case of a free action. We put the usual topology on X, the discrete topology on G, the product topology on $X \times G$ and then transfer this to R_φ by the bijection above. When necessary, we denote this topology by $\mathcal{T}_\varphi$.

We make a short digression to discuss notation. Suppose X is a set and $f : X \to X$ is a function. Recall that the precise definition of a function is a set of ordered pairs, in this case, in $X \times X$, satisfying certain properties. That is, we have $f \subset X \times X$ and we would write $(x, y) \in f$ instead of $f(x) = y$. Usually, this is a formal definition and not how we think about functions. But it turns out here to be rather useful to use this notation. If fact, we invite the reader to check that

$$R_\varphi = \cup_{a \in G} \varphi^a$$

We will let d, r the two canonical projection maps from $X \times X$ to X; that is $d(x, y) = x$ and $r(x, y) = y$.

The final notion of this section is that of an invariant measure for a group action on the Cantor set.

Definition 2.9. *Let μ be a measure on the Cantor set X and let φ be an action of the abelian group G on X. We say that μ is φ-invariant if, for every clopen set $E \subset X$ and every a in G, we have*

$$\mu(\varphi^a(E)) = \mu(E).$$

We let $M(X, \varphi)$ denote the set of all φ-invariant measures on X.

We conclude this section with a couple of simple examples; both are actions of the group $\mathbb{Z}$.

1. **Odometers**

   We consider

$$X = \Pi_{n=1}^{\infty}\{0, 1\} = \{(x_1, x_2, \ldots) \mid x_n = 0, 1\}$$

It is a compact, totally disconnected space with the metric

$$d(x, y) = \inf\{2^{-N} \mid N \geq 0, x_n = y_n, 1 \leq n \leq N\}.$$

We define a map φ on X as adding $(1, 0, 0, \ldots)$. Of course, the addition is done modulo 2, but also with carry over to the right. That is, we have

$$\begin{aligned}
\varphi(0, 0, 0, 1, 1, \ldots) &= (1, 0, 0, 1, 1, \ldots), \\
\varphi(1, 1, 1, 0, 0, 0, 1, \ldots) &= (0, 0, 0, 1, 0, 0, 1, 1, \ldots) \\
\varphi(1, 1, 1, \ldots) &= (0, 0, 0, \ldots)
\end{aligned}$$

In fact, the set X is a compact ring (called the 2-adic integers) and the map φ is simply addition by the unit 1.

We define an action of the group $\mathbb{Z}$ as described earlier. We leave it as an exercise for the reader to verify that this action is minimal. As a hint, check that, for any $N \geq 1$ and x in X, the first N entries of $x, \varphi(x), \ldots \varphi^{2^N - 1}(x)$ comprise all possible sequences of $0, 1$ of length N.

In fact, the number 2 is not particularly important in this example. We may start with any sequence $a_1, a_2, \ldots$ of integers, all at least 2 and form

$$X = \Pi_n\{0, 1, 2, \ldots, a_n - 1\},$$

and carry out the same construction. We refer to these examples as odometers. They are similar to the odometer in a car in the special case $a_n = 10$, for all n, except that the carry-over goes right instead of left and there may be an infinite number of non-zero entries appearing in a sequence.

2. **Denjoy examples**

One of the most basic examples of a dynamical system is a rotation of the circle. We let $\mathbb{S}^1$ denote the circle, which is $\mathbb{R}/\mathbb{Z}$. That is, points are the circle are denoted by real numbers, modulo integers. For any real number α, we define $R_\alpha(x) = x + \alpha$, which is rotation by angle $2\pi\alpha$ in the conventional view of the circle.

It is routine to check that if $\alpha = p/q$ is rational, then $R_\alpha^q(x) = x$, for every x in $\mathbb{S}^1$. That is, every point is periodic of period q. On the other hand, if α is irrational, then R_α is minimal [1]. Of course, the circle is not totally disconnected, but we may create a totally disconnected version of this map as follows.

Let Cut be any countable R_α-invariant subset of $\mathbb{S}^1$ and let $\tilde{Cut}$ be the corresponding set of real numbers. For a start it is probably easiest to begin with $Cut = \{R_\alpha^n(0) \mid n \in \mathbb{Z}\}$ and so $\tilde{Cut} = \mathbb{Z} + \alpha\mathbb{Z}$. Now let $\tilde{\mathbb{R}} = \mathbb{R} \backslash \tilde{Cut} \cup \{x^+, x^- \mid x \in \tilde{Cut}\}$. That is, each point of $\tilde{Cut}$ is removed and replaced by a pair of points. We define a linear order relation on $\tilde{\mathbb{R}}$ as follows. For any x and y distinct, we set $x^+, x^- < y^+, y^-$ if $x < y$. Finally, we set $x^- < x^+$. It is easy to see that intervals of the form $[x^+, y^-] = (x^-, y^+)$ are clopen and form a neighbourhood base for the order topology. We observe that the natural translation of $\mathbb{Z}$ on $\mathbb{R}$ extends to $\tilde{\mathbb{R}}$ and we let X denote the quotient space $\tilde{\mathbb{R}}/\mathbb{Z}$. The action φ of $\mathbb{Z}$ is defined by $\varphi^n(x) = x + n\alpha$ for $x \notin Cut$ and $\varphi^n(x^+) = (x+n\alpha)^+$, $\varphi^n(x^-) = (x + n\alpha)^-$, for x in Cut.

3 Étale equivalence relations

Let us quickly review where we stand after the last section. We have introduced the notion of a group action and some properties such as freeness and minimality. For a group action (X, G, φ), we introduced the orbit relation R_φ and, providing the action is free, a topology, $\mathcal{T}_\varphi$, on R_φ. This equivalence relation, either with or without its topology, will be our main item of interest.

This chapter is aimed at generalizing the notions of the (free) action of a group and the associated equivalence relation. These will be called a local action and an étale equivalence relation, respectively.

By a *local homeomorphism* of a topological space X, we mean a pair of open subsets, U and V, of X and a homeomorphism $\gamma : U \to V$.

Recall our comments from the last section on functions. Here, the function γ as above is actually defined as a subset of $U \times V \subset X \times X$. Also recalling the two coordinate maps, d, r, from $X \times X$ to X, we have $U = d(\gamma)$ and $V = r(\gamma)$. This means that the set $\gamma \subset X \times X$ actually contains the data of its domain, $d(\gamma)$, and range, $r(\gamma)$.

Notice that (in this notation), the inverse of the local homeomorphism γ is

$$\gamma^{-1} = \{(x, y) \mid (y, x) \in \gamma\}$$

which is also a local homeomorphism.

There is some notational trouble when composing functions. (This trouble originates with the fact that we write $f(x)$ instead of $(x)f$, but it is probably too late to change that now.) We define the composition of two local homeomorphisms γ_1 and γ_2 as

$$\gamma_1 \circ \gamma_2 = \{(x, z) \mid \text{ there exists } y, (x, y) \in \gamma_1, (y, z) \in \gamma_2\}$$

which is also a local homeomorphism. Note that with the more usual notation, we have $\gamma_1 \circ \gamma_2(x) = \gamma_2(\gamma_1(x))$. Notice that we do not require that $r(\gamma_1) = d(\gamma_2)$.

This is a good point to observe the empty set is a local homeomorphism as is the identity function on any open set, U, which we denote by id_U. We also observe that if γ is a local homeomorphism and W is an open subset of $d(\gamma)$, then $id_W \circ \gamma$ is simply the restriction of γ to W. Also note that

$$\gamma \circ \gamma^{-1} = id_{d(\gamma)}, \gamma^{-1} \circ \gamma = id_{r(\gamma)}.$$

With these notions in place, it is fairly easy to extend the notion of a group action: we want a collection of local homeomorphisms of a space which is closed under composition and inverses. It turns out to be useful to require a local version of the identity map and one other somewhat more subtle condition.

Definition 3.1. *Let X be a topological space. A* local action *on X is a collection, Γ, of local homeomorphisms of X satisfying:*

1. *if γ is in Γ, then so is γ^{-1},*
2. *if γ_1 and γ_2 are in Γ, then so is $\gamma_1 \circ \gamma_2$,*
3. *the set*

   $$\{U \mid id_U \in \Gamma\}$$

   forms a neighbourhood base for the topology on X.
4. *if γ_1 and γ_2 are in Γ, then so is $\gamma_1 \cap \gamma_2$.*

We also say that the pair (X, Γ) is a local action.

Some remarks are in order regarding the last condition. As a function in the usual sense, the domain of $\gamma_1 \cap \gamma_2$ is the set of points x where both γ_1 and γ_2 are defined and have equal value. The value of $\gamma_1 \cap \gamma_2$ on such an x is this common value. If one considers two continuous functions defined on the same space (to simplify things), the set of points where they agree is easily seen to be closed. As we are requiring $\gamma_1 \cap \gamma_2$ to be a local homeomorphism, the domain must be *open*. This is quite a strong condition which, in some sense, is an analogue of freeness for group actions.

We remark that this definition shares some features with the notion of a pseudogroup.

Suppose that (X, G, φ) is a free action. We would first like to see that this provides an example of a local action. After all, each $\varphi^a, a \in G$, is a homeomorphism of X and hence a local homeomorphism as well. However, this must be done carefully; the collection $\{\varphi^a \mid a \in G\}$ satisfies the first and second conditions of our definition above. There is some difficulty with the fourth since $\varphi^a \cap \varphi^b$ is empty when $a \neq b$. But the more serious problem is with the third condition. However, it is easily verified that

$$\Gamma_\varphi = \{\varphi^a \mid_U \mid U \subset X \text{ open }, a \in G\}$$

is a local action on X.

Notice that, if X is totally disconnected, then instead of restriction to open sets, we could also restrict to clopen sets or compact, open sets.

We can now define the orbit relation of a local action, in much the same way we did for group actions.

Definition 3.2. *Let X be a topological space and let Γ be a local action on X. We define the* orbit relation *of Γ as*

$$R_\Gamma = \cup_{\gamma \in \Gamma} \gamma$$

It is an easy matter to check that R_Γ is an equivalence relation and that, if the local action arises from a free action as described above, then the this definition agrees with the one given earlier.

In fact, one may notice that the fourth condition in the definition is not needed for either of these statements. It is the main ingredient in the following.

Proposition 3.3. *Let X be a topological space and Γ be a local action on X. Then Γ is a neighbourhood base for a topology on R_Γ, which is denoted $\mathcal{T}_\Gamma$.*

We suggest that the reader verify that, in the case of a local action arising from a free action, this topology agrees with the one from the last section.

This brings us to our definition of an étale equivalence relation: it is one which arises from a local action. It will be useful for us to add an extra hypothesis that it is second countable, as a topological space.

Definition 3.4. *Let R be an equivalence relation on a topological space X. We say a topology $\mathcal{T}$ on R is an* étale topology *for R if it is second countable and there is a a neighbourhood base, Γ, for $\mathcal{T}$ which is a local action. Equivalently, we say that $(R, \mathcal{T})$ or $(X, R, \mathcal{T})$ is an étale equivalence relation.*

We will now present some elementary examples of local actions and étale equivalence relations, as well as some non-examples. The first is a little trivial, but worth mentioning.

Example 3.5. *Let $X = \{1, 2, \ldots, N\}$, for some positive integer N. Let Γ consist of the empty set and all subsets of $X \times X$ having a single element. It is easy to see that this is a local action, that $R = X \times X$ and that R is endowed with the discrete topology.*

It is probably worth presenting a simple example of an equivalence relation on a topological space having *no* étale topology.

Example 3.6. *Let $X = [0, 1] \subset \mathbb{R}$ and let $R = \Delta_X \cup \{(0, 1), (1, 0)\}$. We will show that R admits no étale topology. Suppose the contrary: that we may find a local action Γ which gives rise to R. Then there is a local homeomorphism, $\gamma \in \Gamma$, containing $(0, 1)$. Also, we may find $0 \in U \subset [0, 1/2)$ and $1 \in V \subset (1/2, 1]$ such that id_U and id_V are in Γ. It follows that $id_V \circ \gamma \circ id_U$ is in $\mathcal{N}$. On the other hand, $id_V \circ \gamma \circ id_U = \{(0, 1)\}$ by construction. But this is not a local homeomorphism since its domain and range are not open sets.*

The next example is also rather simple, but, in contrast to the last example, it shows how étale equivalence relations have a kind of "openness".

Example 3.7. *Let $X = [0, 1] \cup [2, 3] \subset \mathbb{R}$. Define R to be the equivalence relation whose classes are $\{x, 2 + x\}$, for all $0 < x \leq 1$, $\{0\}$ and $\{2\}$. A local action can be given by all functions of the form $\gamma = \{(x, x) \mid x \in U\}$, where U is open, $\gamma = \{(x, x + 2) \mid x \in U\}$, where $U \subset (0, 1]$ is open, and $\gamma = \{(x, x - 2) \mid x \in U\}$, where $U \subset (2, 3]$ is open. The topology arising from this local action is just the relative topology from $X \times X$.*

The final example shows that some trickery can be used when forming the local action.

Example 3.8. *Let $X = [0,1] \subset \mathbb{R}$ and let R denote the equivalence relation whose classes are of the form $\{x, 1-x\}$. It is tempting to form a local action from local homeomorphisms $\gamma = \{(x,x) \mid x \in U\}$ and $\gamma' = \{(x, 1-x) \mid x \in V\}, x \in [0,1]$, where $U, V \subset [0,1]$ are open. Unfortunately, this fails as the intersection of such a γ and γ' is $\{(\frac{1}{2}, \frac{1}{2})\}$, if $\frac{1}{2}$ is in $U \cap V$. In this case, there is a little trick: only consider γ' for $\frac{1}{2} \notin V \subset [0,1]$ open. This is indeed a local action. The topology does* not *agree with the relative topology from $X \times X$ as the sequence $(\frac{n}{2n+1}, \frac{n+1}{2n+1})$ is convergent in the latter, but not the former*

Let us make a few remarks which we will not prove. First, in an étale equivalence relation, the equivalence classes are always countable. Secondly, although our space X is assumed to be compact, an étale equivalence relation R almost never is. Indeed, it R is compact, then there is a constant M such that every equivalence class has at most M elements. Finally, it is clear that as a subset of $X \times X$, an equivalence relation R obtains a relative topology. If it is étale in this topology, then once again, there is a uniform finite upper bounded on the size of the equivalence classes. Or to put it another way, if $R \subset X \times X$ is an equivalence relation with infinite equivalence classes, then we cannot use the relative topology from $X \times X$ if we want to find an étale topology.

We now extend of our definition of minimality to equivalence relations. Notice this does not require any topology on the equivalence relation itself.

Definition 3.9. *Let R be an equivalence relation on the topological space X. We say that R or (X, R) is* minimal *if every R-equivalence class is dense in X.*

Exercise 3.10. *We say that a set $Y \subset X$ is R-invariant if, for every y in Y, the R-equivalence class of y is also contained in Y. First show that for group actions, this definitions agrees with our earlier one. Then show that R is minimal if and only if the only closed R-invariants sets are X and the empty set.*

We also extend our definition of invariant measures to étale equivalence relations.

Definition 3.11. *Let R be an étale equivalence relation on a Cantor set X. A measure μ on X is R-invariant if and only if*

$$\mu(d(\gamma)) = \mu(r(\gamma)),$$

for every compact, open local homeomorphism $\gamma \subset R$. We let $M(X, R)$ denote the set of all R-invariant probability measures on X.

Recall that in the last section, we introduced the notions of orbit equivalence for group actions. We will generalize this now to equivalence relations, but in fact, we have two properties: orbit equivalence and isomorphism. The former requires no topology, the latter is only for étale equivalence relations.

Definition 3.12. *Let X_1 and X_2 be topological spaces and let R_1 and R_2 be equivalence relations defined on X_1 and X_2, respectively. We say that (X_1, R_1) and (X_2, R_2) are* orbit equivalent *if there is a homeomorphism $h : X_1 \to X_2$ such that*

$$h \times h(R_1) = R_2.$$

We refer to the map h as an orbit equivalence and we write

$$(X_1, R_1) \sim (X_2, R_2)$$

or

$$R_1 \sim R_2.$$

It is trivial to see that orbit equivalence is an equivalence relation, but it is worth noting that it is a slightly unfortunate fact that it is an equivalence relation on equivalence relations. It is also easily seen that this extends our earlier definition of orbit equivalence for group actions 2.7.

Finally, we consider the notion of isomorphism of étale equivalence relations.

Definition 3.13. *Let $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ étale equivalence relations on the spaces X_1 and X_2, respectively. We say that they are* isomorphic *if there is a homeomorphism $h : X_1 \to X_2$ such that*

$$h \times h(R_1) = R_2$$

and so that

$$h \times h : R_1 \to R_2$$

is a homeomorphism. We refer to the map h as an isomorphism and we write

$$(X_1, R_1, \mathcal{T}_1) \cong (X_2, R_2, \mathcal{T}_2),$$

or

$$R_1 \cong R_2.$$

It is clear that isomorphism implies orbit equivalence. The converse turns out to be false, but it is a very deep matter to produce a counter-example. Let us also mention another way of considering this last definition. We need to consider equivalence relations on the Cantor set and endow them with an étale topology. We have seen already that this is not always possible (Example 3.6). If it is, one might equally well ask the question of whether or not the topology is unique. To ask if there exist there exist two distinct topologies on R is really the same as to ask whether there are two étale equivalence relations which are orbit equivalent but not isomorphic.

4 AF-equivalence relations

In the last section, we introduced the notions of local action and étale equivalence relations. We showed that free actions of countable abelian groups gave rise to natural examples. However, we only gave a few other very simple examples. We are going to correct that omission now by discussing AF or approximately finite equivalence relations. Much more than just examples, we will see in subsequent sections that our theory of orbit equivalence is built around AF-equivalence relations. What emerges is that they are rich enough to contain many really deep examples, but at the same time, they have enough structure to be quite tractable. Finally, they are sufficiently close to group actions to be a useful tool in their study.

We begin with the introduction of Bratteli diagrams, a combinatorial object from which we can explicitly construct examples of local actions and étale equivalence relations.

Definition 4.1. *A* Bratteli diagram*, consists of a sequence of finite, pairwise disjoint, non-empty sets, $\{V_n\}_{n=0}^{\infty}$, called the vertices, a sequence of finite non-empty sets $\{E_n\}_{n=1}^{\infty}$ called the edges and maps $i : E_n \to V_{n-1}$ and $t : E_n \to V_n$ called the initial and terminal maps. We let V and E denote the union of these sets and denote the diagram by (V, E). We will assume that V_0 has exactly one element, denoted v_0, that $i^{-1}\{v\}$ is non-empty for every v in V, and that $t^{-1}\{v\}$ is non-empty for every $v \neq v_0$ in V.*

Considered as a directed graph, a Bratteli diagram is a sequence of tiers of vertices with any multitude of edges to the consecutive tier only. We assume there is a unique source, denoted v_0 and no sinks. We may draw the diagram as in Figure 1.

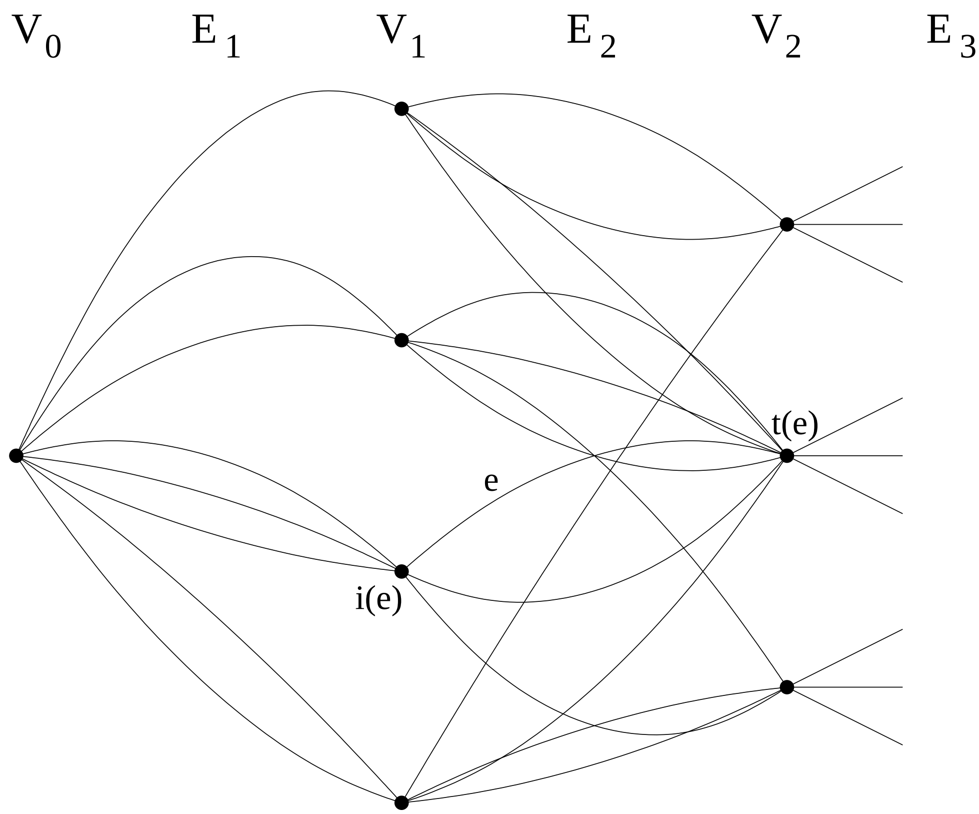


Figure 1: A Bratteli diagram.

Definition 4.2. *Let (V, E) be a Bratteli diagram. An* infinite path *in the diagram is a sequence of edges $(e_1, e_2, \ldots)$ such that e_m is in E_m and $t(e_m) = i(e_{m+1})$ for $m \geq 1$.*

For $M < N$, a finite path *$p = (e_{M+1}, e_{M+2}, \ldots, e_N)$ is a truncation of an infinite path, for which we define $i(p) = i(e_{M+1}) \in V_M$ and $t(p) = t(e_N) \in V_N$*

We define E_M^N to be the set of paths in the diagram from V_M to V_N; that is, we write

$$\begin{aligned} E_M^N &= \{(e_{M+1}, e_{M+2}, \ldots, e_N) \mid e_m \in E_m, M < m \leq N, \\ &\quad t(e_m) = i(e_{m+1}), M < m < N\}. \end{aligned}$$

Note that E_{M-1}^M is just E_M, for $M \geq 1$.

Finally, for v in V_M and w in V_N we define $E_M^N(v, w)$ to be the set of *paths from v to w. Explicitly, we write*

$$E_M^N(v,w) = \{p \in E_M^N \mid i(p) = v \text{ and } t(p) = w\}$$

We remark that paths can be concatenated. If p is in E_L^M and q is in E_M^N, then pq denotes the path in E_L^N obtained by concatenation. (This even makes sense for $N = \infty$.)

We are now ready to take the first step toward a local action: introducing a metric space, called the path space, associated to a Bratteli diagram. The fundamental ideas will be fairly familiar to any reader with a background in symbolic dynamics.

Definition 4.3. *Let (V, E) be a Bratteli diagram. We define a space X_E, to be the collection of infinite paths in the diagram. Explicitly, we write*

$$X_E = \{e = (e_1, e_2, e_3, \ldots) \mid e_n \in E_n, t(e_n) = i(e_{n+1}), \text{ for all } n \geq 1\}.$$

This set is endowed with the metric

$$d(e, f) = \inf\{2^{-N} \mid N \geq 0, e_n = f_n \text{ for } 1 \leq n \leq N\}.$$

We leave it as an exercise to check that d is indeed a metric. The following is actually its most useful characterization.

Exercise 4.4. *For e and f in X_E and $N \geq 1$, $d(e,f) < 2^{1-N}$ if and only if $(e_1, e_2, \ldots, e_N) = (f_1, f_2, \ldots, f_N)$.*

From now on, it will be more convenient to denote elements of X_E as $x = (x_1, x_2, x_3, \ldots)$ rather than $e = (e_1, e_2, e_3, \ldots)$.

For $M < N$ and any p in E_M^N, we define

$$U_p = \{x \in X_E \mid (x_{M+1}, x_{M+2}, \ldots, x_N) = p\}.$$

We refer to such a set as a *cylinder set*. Most frequently, we will use $M = 0$.

We assemble some useful facts about these cylinder sets.

1. For each p in E_M^N, U_p is open.

2. For fixed N, the sets $U_p, p \in E_0^N$ form a (finite) partition on X: that is, they are pairwise disjoint and cover X.

3. For each p in E_M^N, U_p is closed.

4. For fixed N, the partition $U_p, p \in E_0^{N+1}$ is finer than $U_p, p \in E_0^N$: that is, each element of the former is contained in one of the latter.

5. The sets $U_p, p \in E_0^N, N \geq 1$, form a neighbourhood base for the topology on X.

It is an immediate corollary of these facts that the space X_E is compact and totally disconnected.

Having defined our space X_E, we are now ready to provide a local action and, subsequently, an étale equivalence relation on it. The elements of the local action are those maps which are "finite coordinate changes". More specifically, for each $N \geq 1$ and pair p, q in E_0^N, we define

$$\gamma_{p,q} = \{(x, y) \mid x \in U_p, y \in U_q, x_n = y_n, n > N\}.$$

Note that $\gamma_{p,q}$ is non-empty if and only if $t(p) = t(q)$. It is worth stating that, as a local homeomorphism, $\gamma_{p,q} : U_p \to U_q$ is defined by

$$\gamma_{p,q}(p_1, \ldots, p_N, x_{N+1}, x_{N+2}, \ldots) = (q_1, \ldots, q_N, x_{N+1}, x_{N+2}, \ldots).$$

Exercise 4.5. *Let (V, E) be a Bratteli diagram. Prove that the collection of sets*

$$\Gamma_E = \{\gamma_{p,q} \mid p, q \in E_0^N, N \geq 1\}$$

is a local action.

To a Bratteli diagram, (V, E), we have now associated a path space X_E and a local action Γ_E on X_E. This means that we also have an associated étale equivalence relation.

Definition 4.6. *Let (V, E) be a Bratteli diagram. Its associated étale equivalence relation is denoted $(R_E, \mathcal{T}_E)$. That is, we have*

$$R_E = \cup \gamma_{p,q},$$

where the union is over all p, q in E_0^N with $t(p) = t(q)$ and all $N \geq 1$.

In fact, the N in the last definition has a rather subtle part to play. We begin with the following rather easy and innocent looking observation. For each $N \geq 1$, define

$$R_E^N = \cup_{p,q \in E_0^N} \gamma_{p,q}.$$

Then we have

$$R_E^N = \{(x,y) \mid x,y \in X_E, x_n = y_n, n > N\}$$

and is a subequivalence relation of R_E. Moreover, we have

$$R_E^1 \subset R_E^2 \subset R_E^3 \cdots$$

and

$$\cup_N R_E^N = R_E.$$

Exercise 4.7. *Prove that, for each $N \geq 1$, R_E^N is a compact, open subset of R_E.*

For convenience, we also set $R_E^0 = \Delta_{X_E}$ so that R_E^0 is just equality. A pair (x,y) in R_E are usually called 'tail equivalent' or 'cofinal'.

We see in the last result that this equivalence relation R_E has a natural structure as an inductive limit. This brings us to our definition of an AF-equivalence relation.

Definition 4.8. *An étale equivalence relation $(R, \mathcal{T})$ on a compact space X is called an* AF-equivalence relation *if X is totally disconnected and there is a sequence*

$$R^0 \subset R^1 \subset R^2 \subset \cdots \subset R,$$

such that

1. *$\cup_N R^N = R$, and*
2. *for each $N \geq 0$, R^N is a compact, open subequivalence relation of R.*

We have seen that an étale equivalence relation coming from a Bratteli diagram is AF. In fact, the converse is also true. It is slightly messy to prove, and we state it without proof.

Theorem 4.9. *Let $(X, R, \mathcal{T})$ be an AF-equivalence relation. There exists a Bratteli diagram (V, E) such that $(X, R, \mathcal{T}) \cong (X_E, R_E, \mathcal{T}_E)$.*

While AF-relations are quite complex, their inductive limit structure allows us to analyze them quite thoroughly. Of course, the essential tool in this analysis is to represent the relation via a Bratteli diagram. Once this is done, much of the analysis reduces to combinatorial questions about the diagrams.

We next give a simple combinatorial condition on a Bratteli diagram which is necessary and sufficient for its associated AF-equivalence relation to be minimal. We usually call such a diagram *simple*. Recall that an equivalence relation is minimal if and only if every equivalence class is dense. We leave the proof as an exercise.

Theorem 4.10. *Let (V, E) be a Bratteli diagram. The AF-equivalence relation (X_E, R_E) is minimal if and only if for every $M \geq 0$, there exists an $N > M$ such that $E_M^N(v, w)$ is nonempty for every v in V_M and every w in V_N.*

In this section, we have introduced AF-equivalence relations and the suggestion is that they are rather different from the étale equivalence relations which arise from group actions. Let us support that suggestion with the following fairly simple result.

Theorem 4.11. *Let φ be a free action of the group $\mathbb{Z}$ on X. Then R_φ is not isomorphic to an AF-equivalence relation.*

Suppose to the contrary that R_φ is the union of an increasing sequence of compact, open subequivalence relations $R_n, n \geq 1$. Recall that the topology from R_φ comes from $X \times \mathbb{Z}$ and so R_φ is not compact. On the other hand, $\varphi^1 \subset R_\varphi$ is the image of $X \times \{1\}$ and so is compact. The sequence of R_n's form an open cover of φ^1 and so one of them must contain it, say R_k. The equivalence relation generated by φ^1 is R_φ. On the other hand, since R_k is an equivalence relation and contains φ^1, we have $R_\varphi \subset R_k$. We conclude that $R_\varphi = R_k$ which is a contradiction: the second is compact while the first is not.

5 C^*-algebras

In this section, we present a short sketch of the construction of a C^*-algebra from an étale equivalence relation.

Let (X, R) be an étale equivalence relation. (We suppress the topology in our notation. This should cause no confusion, since it will be fixed.) First, we let $C_c(R)$ denote the continuous, complex-valued functions of compact support on R. The first fundamental fact is the following.

Lemma 5.1. *For an étale equivalence relation (X, R), the linear space $C_c(R)$ is a $*$-algebra when given the product*

$$f \cdot g(x, y) = \sum_{(x,z) \in R} f(x, z) g(z, y),$$

for f, g in $C_c(R)$ and (x, y) in R, and involution

$$f^*(x, y) = \overline{f(y, x)},$$

for f in $C_c(R)$ and (x, y) in R.

We leave the proof as an worthwhile exercise for the reader. But let us carefully indicate what is needed. The first task is to see that the product is well-defined. This requires several steps. First, for fixed f in $C_c(R)$ and x in X, the set $\{x\} \times X \cap R$ is discrete (i.e. has no accumulation points in R. This is a consequence of the étale condition. Thus the intersection of this set with a compact set, in this case, the support of f, is finite. It then follows that, for fixed f and g and (x, y), the sum involved in the product has only finitely many non-zero terms. The next task is to prove that $f \cdot g$ is back in $C_c(R)$. This again needs the étale condition. Following all of this are more standard problems, such as showing the product is associative and so on.

We also remark that this product is often denoted by $f * g$, since it is related to the convolution of functions, but this seems a little confusing for a product in a $*$-algebra.

Thus, $C_c(R)$ is a complex $*$-algebra.

Exercise 5.2. *If X is finite and $R = X \times X$ (all with the discrete topology), show that $C_c(R)$ is isomorphic to a familiar $*$-algebra.*

Exercise 5.3. *Let (X, R) be an étale equivalence relation. Prove that the map $\Delta : C(X) \to C_c(R)$ defined by*

$$\Delta(f)(x, y) = \begin{cases} f(x) & x = y, \\ 0 & x \neq y \end{cases}$$

is well-defined, injective and a $*$-homomorphism. In the special case that $R = \Delta_X$, conclude that $C_c(R) \cong C(X)$ (as $*$-algebras).

We now need a norm in $C_c(R)$. There are actually a couple choices and the first is

$$\| f \| = \sup\{\| \pi(f) \| \mid \pi : C_c(R) \to \mathcal{B}(\mathcal{H}), \\ \mathcal{H} \text{ a Hilbert space }, \pi \text{ a *-homomorphism } \}$$

Of course, there is no reason at this point to see that such a finite supremum exists, but it does. We will not prove this. By completing $C_c(R)$ in this norm, we obtain a C^*-algebra which we denote by $C^*(X, R)$.

To describe the other natural norm, we need a specific class of representations of $C_c(R)$ which are called the *regular* representations. For each x in X, let us denote its R-equivalence class by $[x]_R$. We consider the Hilbert space of square-summable, complex-valued functions on $[x]_R$, $l^2[x]_R$. We define

$$\pi_x : C_c(R) \to \mathcal{B}(\mathcal{H})$$

by setting

$$[\pi_x(f)\xi](y) = \sum_{z \in [x]_R} f(y, z)\xi(z),$$

for each f in $C_c(R)$, $\xi \in l^2[x]_R$ and y in $[x]_R$.

Exercise 5.4. *1. Prove that $\pi_x(f)$ is a well-defined bounded operator on $l^2[x]_R$.*

2. Prove that π_x is a $*$-homomorphism.

3. Suppose that $\gamma \subset R$ is a local homeomorphism, f is a continuous function with support contained in γ. Let y be in $[x]_R$ and let δ_y be the function which is 1 at y and zero elsewhere. Compute $\pi_x(f)\delta_y$.

4. Prove that if (x, y) is in R, then $\pi_x = \pi_y$.

Now we can define the *reduced* norm of $C_c(R)$ by

$$\| f \|_r = \sup\{\| \pi_x(f) \| \mid x \in X\},$$

for f in $C_c(R)$. Of course, this norm is less than or equal to the earlier one. Completing in this new norm also yields a C^*-algebra called the *reduced* C^*-algebra, which we denote by $C_r^*(X, R)$. If the étale equivalence relation is *amenable*, then these two norms are equal, and hence the two C^*-algebras coincide. We will not define amenable, but most of our relations will have this property. We refer the reader to [23] for more information.

6 Invariants for étale equivalence relations

In this chapter, we introduce two invariants for étale equivalence relations. Each of these is an ordered abelian group and we begin with a general discussion of this topic.

Definition 6.1. *An* ordered abelian group *is a pair* (G, G^+)*, where* G *is an abelian group and* $G^+ \subset G$ *satisfies:*

1. $G^+ + G^+ \subset G^+$
2. $G^+ - G^+ = G$
3. $G^+ \cap (-G^+) = \{0\}$.

The set $G^+ \subset G$ *is called the* positive cone *of* G*. The elements of* G^+ *are called positive while the non-zero elements of* G^+ *are called strictly positive.*

If (G, G^+) *and* (H, H^+) *are ordered abelian groups, then a group homomorphism* $\alpha : G \to H$ *such that* $\alpha(G^+)$ *is contained in* H^+ *is known as a* positive group homomorphism*.*

The positive cone provides a natural partial order on G, where for any a, b in G, $a \geq b$ if and only if $a - b$ is in G^+. Note that this order is translation invariant in the sense that $a \geq b$ implies $a + c \geq b + c$ for any a, b, c in G.

An *order unit* for $(G, G+)$ is an element u in G^+ such that for every a in G there exists an integer $n \geq 1$ such that $nu \geq a$. In order to identify a particular order unit u in G, we sometimes write the ordered group as (G, G^+, u). Let (G, G^+, u) and (H, H^+, u') be ordered groups with order units. These groups are *isomorphic* if there exists a group isomorphism α from G to H such that $\alpha(G^+) = H^+$ and $\alpha(u) = u'$.

By an *order ideal* in an ordered group (G, G^+) we mean a subgroup H which is generated by $G^+ \cap H$ (so that $H^+ = G^+ \cap H$ makes H itself into an ordered group) and such that, if a is in H^+ and b is in G^+ with $a \geq b$, then b is also in H. We say that the ordered group is *simple* if and only if the only the only order ideals are the two trivial subgroups 0 and G.

The simplest example is the group $\mathbb{Z}$ with positive cone $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. This gives rise to the usual order on $\mathbb{Z}$. Similar examples may be obtained for the groups $\mathbb{Q}$ and $\mathbb{R}$.

The group $\mathbb{Z}^d$ with $d \geq 1$ can actually be given a number of different orders, but the most important is given by $\mathbb{Z}^{d+} = \{(n_1, \ldots, n_d) \mid n_k \geq$

0 for each $k\}$ which we refer to as the standard or simplicial order. Here, any element $(n_1, \ldots, n_d)$ where each n_k is strictly positive is an order unit.

Now we want to see how to construct such objects from étale equivalence relations. In this, we will implicitly assume X is a totally disconnected space.

Let $C(X, \mathbb{Z})$ be the group of continuous functions from X to $\mathbb{Z}$ with the operation of pointwise addition. Define $C(X, \mathbb{Z})^+$ to those functions from X to the non-negative integers. It is easy to show that together these form an ordered group. As X is compact, the range of any f in $C(X, \mathbb{Z})$ is finite and the sets $f^{-1}\{n\}, n \in f(X)$, form a partition of X into clopen sets. The constant function one is an order unit.

Definition 6.2. *Let $(X, R, \mathcal{T})$ be an étale equivalence relation with X totally disconnected. Define $B(X, R)$ as the subgroup of $C(X, \mathbb{Z})$ generated by all functions of the form*

$$\chi_{d(\gamma)} - \chi_{r(\gamma)}$$

where $\gamma \subset R$ is a compact, open local homeomorphism. We define

$$B_m(X, R) = \left\{ f \mid \int f d\mu = 0 \text{ for every } \mu \in M(X, R) \right\}.$$

which is clearly a subgroup of $C(X, \mathbb{Z})$.

If μ is an R-invariant measure on X, then for every compact open local homeomorphism $\gamma \subset R$ we have that

$$\int \chi_{d(\gamma)} - \chi_{r(\gamma)} d\mu = \mu(d(\gamma)) - \mu(r(\gamma)) = 0$$

from which it follows that $B(X, R) \subset B_m(X, R)$.

Definition 6.3. *Let $(X, R, \mathcal{T})$ be an étale equivalence relation with X totally disconnected. We define $D(X, R)$ and $D_m(X, R)$ to be the following groups:*

$$D(X, R) = C(X, \mathbb{Z})/B(X, R)$$

and

$$D_m(X, R) = C(X, \mathbb{Z})/B_m(X, R)$$

The elements of these groups are equivalence classes, denoted $[f]_B$ and $[f]_{B_m}$ respectively. We also define

$$D(X, R)^+ = \{[f]_B \mid f \in C(X, \mathbb{Z})^+\}.$$

$D_m(X, R)^+$ is defined in a similar way.

Provided that the relation R is minimal, $D(X,R)$ and $D_m(X,R)$ are ordered abelian groups and the class of the constant function one is an order unit.

The next result shows that $D(X,R)$ and $D_m(X,R)$ are invariant under isomorphism and orbit equivalence of étale equivalence relations, respectively.

Theorem 6.4.

1. *If the étale equivalence relations* (X,R) *and* (X',R') *are isomorphic, then* $(D(X,R), D(X,R)^+, [1]_B)$ *and* $(D(X',R'), D(X',R')^+, [1]_B)$ *are isomorphic as ordered abelian groups with order unit.*

2. *If the étale equivalence relations* (X,R) *and* (X',R') *are orbit equivalent, then* $(D_m(X,R), D_m(X,R)^+, [1]_{B_m})$ *and* $(D_m(X',R'), D_m(X',R')^+, [1]_{B'_m})$ *are isomorphic as ordered abelian groups with order unit .*

We will not give a proof. The first part is obvious. For the second, one shows that an orbit equivalence actually induces a bijection between the sets of invariant measures for the equivalence relations and the conclusion follows easily from that.

For the remainder of this section, we will restrict our attention to AF-equivalence relations and the computation of their invariants. If (X,R) is tail equivalence on the Bratteli diagram (V,E), then the invariant $D(X,R)$ may be computed directly from the diagram as follows.

For any finite set A, we let $\mathbb{Z}A$ denote the free abelian group on A. That is, a typical element is a formal integral combination of the elements of A. Of course, it is isomorphic as a group to $\mathbb{Z}^n$, where n is the number of elements of A, but our notation allows us to consider A as a subset of the group. We denote by $\mathbb{Z}^+A$ the sub-semigroup with identity generated by the elements of A; that is, it consists of non-negative integral combinations of A. Identifying $\mathbb{Z}A$ with $\mathbb{Z}^n$ as we suggested earlier, this is just the standard order.

Suppose that V and V' are two finite sets of vertices and E is a set of edges between them, meaning that there are initial and terminal maps $i : E \to V$ and $t : E \to V'$. We may define a group homomorphism, $\varepsilon : \mathbb{Z}V \to \mathbb{Z}V'$, by setting

$$\varepsilon(v) = \sum_{i(e)=v} t(e), v \in V.$$

This defines ε on the generators of $\mathbb{Z}V$ and has a unique extension which is a group homomorphism. Equivalently, if we let $E(v,v')$ denote the set of edges

e with $i(e) = v, t(e) = v'$, and $\varepsilon(v, v') = \#E(v, v')$, for any $v \in V, v' \in V'$, then

$$\varepsilon(v) = \sum_{v' \in V'} \varepsilon(v, v')v', v \in V.$$

It is clear this homomorphism is positive in the sense that it maps the positive cone in its domain into the positive cone in the range.

From the Bratteli diagram, (V, E), we may construct a sequence of abelian groups and homomorphisms:

$$\mathbb{Z}V_0 \xrightarrow{\varepsilon_1} \mathbb{Z}V_1 \xrightarrow{\varepsilon_2} \mathbb{Z}V_2 \cdots$$

where ε_n is the group homomorphism obtained as above from the edge set E_n, for $n \geq 1$. For convenience, for any $m < n$, we let

$$\varepsilon_{m,n} = \varepsilon_n \circ \cdots \circ \varepsilon_{m+1} : \mathbb{Z}V_m \to \mathbb{Z}V_n.$$

The inductive limit of such a system, which we denote by $G(V, E)$, is defined as follows. Consider the disjoint union of the groups, which we denote $\bigsqcup_n \mathbb{Z}V_n$. We define an equivalence relation: if a is in $\mathbb{Z}V_m$ and a' is in $\mathbb{Z}V_{m'}$, $a \sim a'$ if there exists $n > m, m'$ such that $\varepsilon_{m,n}(a) = \varepsilon_{m',n}(a')$. Alternately, $\sim$ is the equivalence relation generated by $a \sim \varepsilon_{n+1}(a)$, for $n \geq 0$ and a in $\mathbb{Z}V_n$. Let $G(V, E)$ denote the quotient of $\bigsqcup_n \mathbb{Z}V_n$ by this equivalence relation. If a is in $\mathbb{Z}V_n$, we let $[a, n]$ denote its class in $G(V, E)$. Although $\bigsqcup_n \mathbb{Z}V_n$ is not itself a group, it is easy to see that $G(V, E)$ has a group structure by defining $[a, m] + [a', m'] = [\varepsilon_{m,n}(a) + \varepsilon_{m',n}(a'), n]$, where $n > m, m'$. The group has a positive cone, $G(V, E)^+ = \{[a, n] \mid n \geq 1, a \in \mathbb{Z}^+V_n\}$. A word of warning is in order. It is entirely possible that a is in $\mathbb{Z}V_m$ and is not positive there, yet $\varepsilon_{m,n}(a)$ *is* in $\mathbb{Z}^+V_n$, for some $n > m$. We will see an example in a moment. In this case, $[a, m]$ is in $G(V, E)^+$. We also note that this group has a distinguished positive element, $[v_0, 0]$.

Let us provide a couple of simple examples. First, suppose that each V_n has one vertex, denoted v_n, and each E_n has two edges. In this case, the maps between the groups are all injective. This simplifies things: if two elements lie in the same $\mathbb{Z}V_n$, they are equivalent if and only if they are equal. Using this fact, it is easy to show that

$$G(V, E) \cong \{\frac{k}{2^n} \mid k \in \mathbb{Z}, n \geq 0\}.$$

In fact, the isomorphism sends $[kv_n, n]$ to $\frac{k}{2^n}$.

Next suppose that V_n has two vertices (for $n \geq 1$) which we denoted by v_n and v'_n. Suppose E_n consists of three edges (for $n \geq 2$), one from v_{n-1} to v_n, one from v_{n-1} to v'_n and one from v'_{n-1} to v_n. Here, each of the maps in our sequence ($n \geq 2$) is actually a group isomorphism and it follows that the obvious map from $\mathbb{Z}V_1$ to $G(V, E)$ is an isomorphism. That is at the level of groups. The order is more subtle. For example, $v_1 - v'_1$ is not positive in $\mathbb{Z}V_1$, however its image in $\mathbb{Z}V_2$ is v'_2 which *is* positive. We leave it as an exercise to check that the image of $nv_1 - n'v'_1$ is positive in $G(V, E)$ if and only if $n\gamma - n' \geq 0$, where γ is the golden mean.

The next result gives our combinatorial description of the the invariant $D(X, R)$, when (X, R) is an AF-equivalence relation. From the statement given, the proof is just a matter of checking the claimed map is well-defined and does indeed define an isomorphism.

Theorem 6.5. *Let (V, E) be a Bratteli diagram and (X, R) be its associated AF-equivalence relation. We have*

$$D(X, R) \cong G(V, E) \cong K_0(C^*(X, R)),$$

as ordered abelian groups with order unit. Moreover, for a path p from v_0 to $t(p)$ in V_n, the isomorphism between the first two carries $[\chi_{U_p}]$ in $D(X, R)$ to $[t(p), n]$ in $G(V, E)$.

Next, we turn to the issue of computing $D_m(X, R)$ for an AF-equivalence relation. Here the key issue is identifying the invariant measures and we have the following.

Theorem 6.6. *Let (V, E) be a Bratteli diagram and (X, R) be its associated AF-equivalence relation. Assume that (X, R) is minimal. There is a bijective correspondence between the following.*

1. *The set $\mu \in M(X, R)$, the R-invariant measures on X.*

2. *The set of positive group homomorphisms $\tau : G(V, E) \to \mathbb{R}$ such that $\tau[v_0, 0] = 1$.*

3. *The set of functions $\omega : \cup_n V_n \to [0, 1]$ such that $\omega(v_0) = 1$ and for any v in $\cup_n V_n$, we have*
$$\omega(v) = \sum_{i(e)=v} \omega(t(e)).$$

Let us give a brief sketch of the main ideas.

The first is to observe that any measure μ gives a group homomorphism from $C(X, \mathbb{Z})$ to $\mathbb{R}$ which is positive and takes the order unit to 1. Moreover, if the measure is R-invariant, then this map is zero on all functions of the form $\chi_{d(\gamma)} - \chi_{r(\gamma)}$, where γ is a compact, open local homeomorphism in R. Hence, this map descends to a well-defined positive group homomorphism from $D(X, R)$ to $\mathbb{R}$. By simply identifying $D(X, R)$ with $G(V, E)$ by Theorem 6.5, we obtain a group τ as desired.

If we begin with τ as above, we define $\omega(v) = \tau[v, n]$, for any vertex v in V_n. The property given on ω follows from the fact that $\epsilon_{n+1}(v) = \sum_{i(e)=v} t(e)$ and so $[v, n] = \sum_{i(e)=v}[t(v), n + 1]$.

Given a function ω, we may define a measure μ as follows. For any path p from v_0 to V_n, we define $\mu(U_p) = \omega(t(p))$. It remains to check that this is well-defined and extends to all clopen sets.

Secondly, to define an invariant measure, μ, we must assign a value to each clopen set in X. Each clopen set may be written as a finite union of disjoint cylinder sets. So it suffices for us to determine $\mu(U(p))$, for each finite path p in the diagram. Moreover, if p and q are two such paths with $t(p) = t(q) = v$, then $\gamma_{p,q}$ is a compact, open local homeomorphism with $d(\gamma) = U_p$ and $r(\gamma) = U_q$ and therefore these sets must have the same measure. That is, $\mu(U_p)$ can only depend on $t(p)$. In addition, we know that, for fixed p,

$$U_p = \cup_{i(e)=t(p)} U_{pe}$$

and the sets in this union are pairwise disjoint. It follows that

$$\mu(U_p) = \sum_{i(e)=t(p)} \mu(U_{pe}).$$

This result has an interesting consequence. We know that the groups $D(X, R)$ and $D_m(X, R)$ are defined in terms of the étale equivalence relation (X, R) and we have seen the latter is a quotient of the former. From the result above, it follows that the latter may actually be recovered from the former, as follows.

Corollary 6.7. *Let (X, R) be a minimal AF-equivalence relation. Let H be the subgroup of $D(X, R)$ which is the intersection of all kernels of positive group homomorphisms to $\mathbb{R}$. Then*

$$D(X, R)/H \cong D_m(X, R),$$

as ordered abelian groups, where a coset in the quotient is positive if and only if it contains a positive element.

We have already given a simple combinatorial condition on a Bratteli diagram which is necessary and sufficient for the minimality of its associated AF-equivalence relation. We next note that this property can also be detected through the invariant $D(X, R)$.

Theorem 6.8. *Let (X, R) be an AF-equivalence relation. It is minimal if and only if the only order ideals in $D(X, R)$ are 0 and $D(X, R)$.*

We conclude by mentioning an extremely important result, the Effros-Handelman-Shen Theorem. We know that every AF-equivalence relation arises from a Bratteli diagram and we also now know how, if we are given the Bratteli diagram (V, E), we can compute its invariant as $G(V, E)$. A question which remains is: which ordered abelian groups may arise from this construction? In other words, if (G, G^+) is some ordered abelian group, does it arise from a Bratteli diagram?

Theorem 6.9 (Effros-Handelman-Shen)**.** *Let (G, G^+) be an ordered abelian group. There exists a Bratteli diagram (V, E) such that $G \cong G(V, E)$, as ordered abelian groups, if and only if the following are satisfied:*

1. *G is countable,*
2. *G is* unperforated*. That is, if a is in G and, for some positive integer n, na is in G^+, then a is in G^+,*
3. *G has the* Riesz interpolation property*. That is, if a_1, a_2, b_1, b_2 are in G and satisfy $a_i \leq b_j$, for all i, j, then there exists c in G with $a_1, a_2 \leq c \leq b_1, b_2$.*

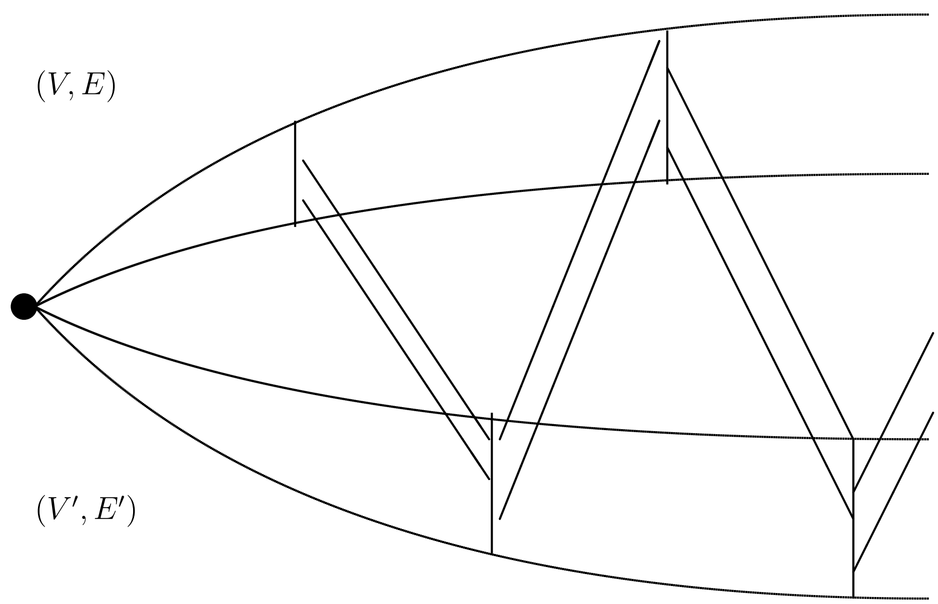
We remark that the properties listed are easily seen to be satisfied for the group $\mathbb{Z}^d$ and it is a nice exercise the see they will hold in the limit group $G(V, E)$ as well, so the 'only if' direction of the result is quite easy. The other direction is highly non-trivial the most useful aspect of the result.

Definition 6.10. *An ordered abelian group (G, G^+) is a* dimension group *if it satisfies the three conditions of the last theorem.*

7 The Bratteli-Elliott-Krieger Theorem

We now come to our first big result which is due to Ola Bratteli, George Elliott and Wolfgang Krieger (all separately). It classifies AF-equivalence relations up to isomorphism.

In order to state the result, we need to introduce an *intertwining* between two Bratteli diagrams. Suppose (V, E) and (V', E') are two Bratteli diagrams. An intertwining between them consists of a monotone sequence of integers $n_0 = 0 < n_1 < n_2 < \cdots$ and a sequence of edge sets $F_1, F_2, \ldots$ along with initial and terminal maps: for j odd, $i : F_j \to V'_{n_{j-1}}, t : F_j \to V_{n_j}$ and for j even, $i : F_j \to V_{n_{j-1}}, t : F_j \to V'_{n_j}$. That is, for j odd, the edges of F_j go from level n_{j-1} in the second diagram to level n_j in the first and the other way for j even. It can be summarized by the following picture:



Notice that we have collapsed the two vertices of V_0 and V'_0 into a single initial vertex. The key requirement is that, for j odd, the number of paths in (V, E) between any vertex v in V_{n_j} and w in $V_{n_{j+2}}$ must be the same as the number of paths from v to w which pass through F_{j+1} and F_{j+2}. Similarly, for j even, the number of paths in (V', E') between any vertex v in V'_{n_j} and w in $V'_{n_{j+2}}$ must be the same as the number of paths from v to w which pass through F_{j+1} and F_{j+2}.

Theorem 7.1 (Bratteli-Elliott-Krieger)**.** *Let (V, E) and (V', E') be two Bratteli diagrams. The following are equivalent.*

1. *There exists an intertwining of the diagrams.*
2. *The AF-equivalence relations (X_E, R_E) and $(X_{E'}), R_{E'})$ are isomorphic.*
3. *The C^*-algebras $C^*(X_E, R_E)$ and $C^*(X_{E'}), R_{E'})$ are isomorphic.*
4. *The invariants $(G(V, E), G(V, E)^+, [v_0])$ and $(G(V', E'), G(V', E')^+, [v'_0])$ are isomorphic; i.e. there is a group isomorphism $\alpha : G(V, E) \to G(V', E')$ such that $\alpha(G(V, E)^+) = G(V', E')^+$ and $\alpha[v_0] = [v'_0]$.*

First, a little historical background on the result. Bratteli introduced the diagrams which now carry his name is his study of AF-algebras. Essentially, he proved the equivalence of the first and third conditions. Elliott introduced the invariant which was essentially the K-theory of the C^*-algebra and realized it could be computed as our $G(V, E)$, as we have explained. Thus Elliott contributed the equivalence of the fourth condition. Krieger realized the dynamical presentation of the C^*-algebra via the equivalence relation and hence contributed the second condition.

Let us give a short sketch of some parts of the proof. If we have such an intertwining, the condition on the numbers of paths involved in the definition ensures that we may define bijections between these sets, one for every j. The collection of these maps for j odd, will transform any infinite path in (V, E) into one in the diagram with edge sets $F_1, F_2, \ldots$. Moreover, these maps for j even, or rather their inverses, will transform that path into one in (V', E'). Together, we have a map from X_E to $X_{E'}$. It is a simple matter to check it is an isomorphism between the equivalence relations.

The implication that the second condition implies the third is immediate. Similarly, knowing that $G(V, E)$ is the K-theory of the C^*-algebra means the third implies the fourth.

The truly subtle part of the argument is that the fourth condition implies the first and this is due to Elliott.

Finally, we state an immediate corollary which is most convenient for our study of AF-equivalence relations.

Corollary 7.2. *AF-equivalence relations (X, R) and (X', R') are isomorphic if and only if $D(X, R)$ and $D(X', R')$ are isomorphic (as ordered abelian groups with order unit).*

8 The absorption theorem

The Bratteli-Elliott-Krieger Theorem of the last section gives us a complete classification of AF-equivalence relations (X, R) up to isomorphism in terms of a highly computable invariant, the ordered abelian group, $D(X, R)$. The two difficulties which now confront us are, first, we are aiming for classification up to orbit equivalence rather than isomorphism and, secondly, we would much prefer to classify group actions rather than AF-equivalence relations.

The key tool in achieving both of these aims is actually the same result which we refer to as the absorption theorem. It is quite technical. Moreover, there are at least three versions in the literature [11, 7, 19]! The reason is that the first handled certain situations, but fell short short for the next objective. Similarly the second wasn't quite adequate for the third. However, it seems that the last version, due to Hiroki Matui, is the most general which could be hoped for. We will first present a particularly simple version which will actually be sufficient for a couple of non-trivial applications later and we will state Matui's version.

We state the main idea, ignoring technical assumptions. Suppose that (X, R) is a minimal AF-equivalence relation. We want to consider a 'small' extension of R. We have a closed subset $Y \subset X$ and an equivalence relation Q on Y. We let $R \vee Q$ be the smallest equivalence relation on X which contains both R and Q. The conclusion of the absorption theorem is that $R \vee Q$ is orbit equivalent to R. That is, R is able to *absorb* the small extension Q. Of course, the trick is in describing exactly what we mean by 'small'.

It is interesting to consider what this result would look like in an ergodic theory context. There, the set Y is small if it has measure zero. In this case, $R \vee Q = R$, since they are the same on a set of full measure.

Here is the simplest (non-trivial) version.

Theorem 8.1. *Let (X, R) be a minimal AF-equivalence relation (with X infinite). If x, y are in X, then $R \vee \{x, y\}^2$ is orbit equivalent to R.*

To state Matui's version of the result, we need two definitions. Let (X, R) be an étale equivalence relation. A closed set $Y \subset X$ is said to be R-étale if the equivalence relation $R|_Y = R \cap (Y \times Y)$, with its relative topology from R, is an étale equivalence relation on Y. Also, Y is said to be R-thin if

$$\inf\{\mu(U) \mid U \text{ clopen }, Y \subset U\} = 0,$$

for every μ in $M(X, R)$. (In usual measure theory terms $\mu(Y) = 0$, for every measure μ in $M(X, R)$.)

In the case of AF-equivalence relations, we may provide examples of R-étale sets as follows. Let (V, E) be a Bratteli diagram. Suppose that F is a subset of E with $i(F) = t(F) \cup \{v_0\}$. Let $W = i(F)$. We call (W, F) (or just F) a *subdiagram* of (V, E). It is clear that the path space $X(W, F)$ is a subset of $X(V, E)$ and it is fairly easy to see that it is both closed and R-étale. (In fact, there is a converse of this result: if (X, R) is an AF-equivalence relation and Y is a closed, R-étale subset of X, then (X, R) and Y may be represented by a Bratteli diagram (V, E) and a subdiagram (W, F) as above. As we will not need this result, see [10], Theorem 3.11 for a precise statement.)

Theorem 8.2 (Matui)**.** *Let (X, R) be a minimal AF-equivalence relation. Let Y be a closed R-étale and R-thin subset of X and Q be an étale equivalence relation on Y such that Q contains $R|_Y$ and the inclusion map is continuous. Then there is a homeomorphism $h : X \to X$ such that*

1. *$h \times h(R \vee Q) = R$,*
2. *$h(Y)$ is R-étale and R-thin,*
3. *$h|_Y \times h|_Y$ is a homeomorphism from Q to $R|_{h(Y)}$*

In particular, $R \vee Q$ is orbit equivalent to R.

Obviously, the first part of the conclusion is the one we are mainly interested in. However, in some actual applications, we will need several applications of the result. That is, we will have a situation where, roughly speaking, we have a finite sequence of closed sets $Y_1, Y_2, \ldots, Y_n$ and equivalence relations $Q_1, Q_2, \ldots, Q_n$ and we want to conclude that R is orbit equivalent to $R \vee Q_1 \vee \cdots \vee Q_n$. The conditions listed in the conclusion allow us to make n applications of the absorption theorem.

9 Classification of minimal AF-equivalence relations

We have already seen that the Bratteli-Elliott-Krieger Theorem provides a complete invariant for isomorphism of AF-equivalence relations. In this section, we discuss the classification of AF-equivalence relations up to orbit equivalence. This will require the added assumption of minimality.

Just to get started, consider the following observation: if $(X, \tilde{R})$ is an étale equivalence relation and $R \subset \tilde{R}$ is an open subequivalence relation, it follows that R is also étale (see 3.12 of [11]). It is clear from the definition 6.2 that $B(X, R) \subset B(X, \tilde{R})$ and hence $D(X, \tilde{R})$ is a quotient of $D(X, R)$.

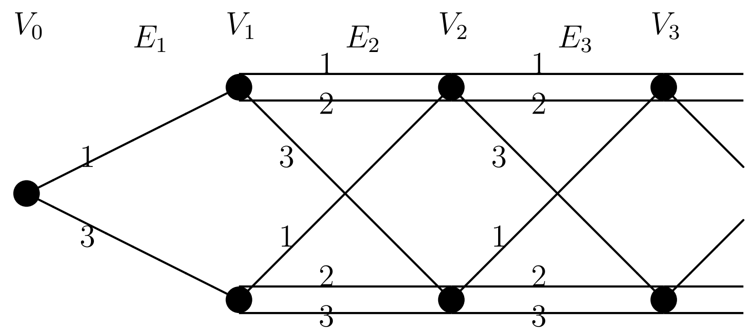
It is then natural to ask the question: if we are given an étale equivalence relation (X, R) and a quotient of $D(X, R)$, is it possible the realize this quotient by an extension $R \subset \tilde{R}$? (We should make it clear that $\tilde{R}$ is also étale and the relative topology from it on R agrees with the given one on R.) Although there is a general answer to this given in [22], let us concentrate on a more specific version: is it possible to realize the group $D_m(X, R)$ as an extension $R \subset \tilde{R}$? The answer, at least for minimal AF-equivalence relation, is not only yes, but we can say a good deal more.

Theorem 9.1. *Let (X, R) be a minimal AF-equivalence relation. There exists a minimal AF-equivalence relation, $\tilde{R}$, on X, containing R, a closed set $Y \subset X$ and a compact étale equivalence relation Q on Y such that*

1. *$(X, R), Y, Q$ satisfy the hypotheses of the absorption theorem 8.2,*
2. *$\tilde{R} = R \vee Q$,*
3. *$D(X, \tilde{R}) \cong D_m(X, R)$, as ordered abelian groups with distinguished order units.*

In particular, (X, R) and $(X, \tilde{R})$ are orbit equivalent.

We will not give a proof, but it will probably be useful to have an example. Consider the following Bratteli diagram:



We let (X, R) denote the associated AF-equivalence relation. The reader will note that we have added labels to the edges of the diagram. This is a matter of convenience, for we can now see that the path space X is homeomorphic to $\{1, 3\} \times \{1, 2, 3\}^{\mathbb{N}}$ and the map is just reading the labels on the edges.

Suppose that $V_n = \{v_n, v'_n\}$, for any $n \geq 1$. Using Theorem 6.6, it is fairly easy to see that there is a unique R-invariant measure which corresponds to the function $\omega(v_n) = \omega(v'_n) = 2^{-1} \cdot 3^{1-n}$, for $n \geq 1$. The subgroup of infinitesimals in $G(V, E)$ is isomorphic to $\mathbb{Z}$; its generator is $[v_n - v'_n, n]$, for any $n \geq 1$. It is fairly easy to check that $D_m(X, R) \cong \mathbb{Z}[1/3]$.

The AF-equivalence relation $\tilde{R}$ is just tail equivalence on the sequences of $\{1, 3\} \times \{1, 2, 3\}^{\mathbb{N}}$. The closed set Y consists of the two sequences $x = (1, 2, 2, \ldots)$ and $y = (3, 2, 2, \ldots)$. We leave the reader with the amusing tasks of proving $R \subset \tilde{R}$ and $\tilde{R} = R \vee \{x, y\}^2$.

The example is quite revealing, but is far from the general case. It is particularly simple for two reasons: the first is that the kernel of the natural map from $D(X, R)$ to $D_m(X, R)$ is represented at each stage by elements of the form $[v_n - v'_n, n]$ and, secondly, that these are all the same for different n. The second fact simply does not hold in general (the kernel of the map may not be the infinite cyclic group). But most of the work in the proof involves showing that the first property can be arranged through a careful choice of the Bratteli diagram for (X, R). In general, the set Y will not be finite.

This rather technical result gives us enough to prove our classification for minimal AF-equivalence relations up to orbit equivalence.

Corollary 9.2. *Two minimal AF-equivalence relations* (X_1, R_1) *and* (X_2, R_2) *are orbit equivalent if and only if* $D_m(X_1, R_1)$ *and*

$D_m(X_2, R_2)$ are isomorphic as ordered abelian groups with distinguished order units.

The 'only if' statement follows from the second part of Theorem 6.4. We will explain the proof of the 'if' direction.

We apply Theorem 9.1 (twice) to find AF-equivalence relations $\tilde{R}_1 \supset R_1$ on X_1 and $\tilde{R}_2 \supset R_2$ on X_2 such that

$$D(X_i, \tilde{R}_i) \cong D_m(X_i, R_i),$$

as ordered abelian groups with distinguished order unit, for $i = 1, 2$. It follows from Theorem 9.1 that $\tilde{R}_1 \sim R_1$ and $\tilde{R}_2 \sim R_2$. Moreover, from the hypothesis and the Bratteli-Elliott-Krieger Theorem 7.1 we have $\tilde{R}_1 \cong \tilde{R}_2$. This completes the proof.

10 Minimal actions of $\mathbb{Z}$ and $\mathbb{Z}^d, d \geq 2$

We discuss the structure of minimal actions of the groups $\mathbb{Z}^d, d \geq 1$ on the Cantor set and their classification up to orbit equivalence. We start with the case $d = 1$, which is by far the simplest.

Let us begin with a fairly simple result.

Lemma 10.1. *Let φ be a free minimal action of the group $\mathbb{Z}$ on the Cantor set X. Let x be in X and let $Y \subset X$ be a non-empty clopen set. There exists $n \geq 1$ such that $\varphi^n(x)$ is in Y. Similarly, there exists $m \leq 0$ such that $\varphi^n(x)$ is in Y.*

To see this, consider the set Z of accumulation points of the sequence $\varphi^n(x), \not\geq 1$. That is z is in Z if there exists an increasing sequence n_i such that $\lim_{i\to\infty} \varphi^{n_i}(x) = z$. Since X is compact, this set is non-empty. It is also closed and it is quite easy to see $\varphi^n(Z) = Z$, for any $n \in \mathbb{Z}$. By minimality and 2.6, Z must be all of X and hence contains Y. Since Y is open, the conclusion follows.

For any closed subset $Y \subset X$, we define R_Y to be the subequivalence relation of R_φ which is generated by $\{(x, \varphi^{-1}(x)) \mid x \notin Y\}$. We observe:

1. $R_\emptyset = R_\varphi$,
2. If $Y \supset Y'$, then $R_Y \subset R_{Y'}$.

More subtly, we have the following.

Lemma 10.2. *If Y is non-empty, then R_Y is compact*

We sketch the key idea. For y in Y, let $\lambda(y)$ be the first positive integer n such that $\varphi^n(y)$ is in Y. This is well-defined from the Lemma above. Moreover, it is easy to see that

$$\lambda^{-1}\{n\} = Y \cap \varphi^{-1}(X \setminus Y) \cap \cdots \cap \varphi^{-n+1}(X \setminus Y) \cap \varphi^{-n}(Y),$$

which is clearly both closed and open and it follows that the function λ is continuous. It is fairly clear then that the R_Y-equivalence class of a point y in Y is simply $\{y, \varphi^1(y), \ldots, \varphi^{\lambda(y)-1}(y)\}$. From these facts, the conclusion is fairly easy.

Now we fix a point y in X and choose a sequence of clopen sets $Y_1 \supset Y_2 \supset \cdots$ such that $\cap_{n=1}^{\infty} Y_n = \{y\}$. Then, each R_{Y_n} is a compact, open subequivalence relation of R_φ and their union is just R_y. Hence R_y is an AF-equivalence relation. It is fairly easy to see that R_y is minimal and clearly $R_y \vee \{(y, \varphi^{-1}(y)\}^2 = R_\varphi$. By applying the Absorption Theorem 8.1, we have proved:

Theorem 10.3. *If φ is a minimal action of the group of integers on the Cantor set X, then (X, R_φ) is orbit equivalent to a minimal AF-equivalence relation.*

At this point, we can make a general observation: we have an invariant $(D_m(X,R))$ and a class (minimal AF-equivalence relations) for which it is a complete invariant. We also have another class (minimal actions of $\mathbb{Z}$) and each of these is equivalent to one in the first class. It follows at once, that our invariant is complete for the union of the two classes.

Corollary 10.4. *For the class of all minimal AF-equivalence relations and all equivalence relations arising from minimal $\mathbb{Z}$-actions on a Cantor set, the invariant $D_m(X,R)$ is a complete invariant for orbit equivalence.*

Let us introduce a bit of terminology. Above we showed that if φ is a minimal action of $\mathbb{Z}$, then R_φ is orbit equivalent to a minimal AF-equivalence relation, (X', R'). That means that there is a homeomorphism $h : X \to X'$ such that $h \times h(R') = R_\varphi$. We know from Theorem 4.11 that $h \times h$ cannot be a homeomorphism between R' and R. But if we simply transfer the topology of R' using $h \times h$, we see that R_φ does possess another topology in which it is AF. We say that an étale equivalence relation R is *affable* if it is orbit equivalent to an AF-equivalence relation. (It can be given an AF-topology or is 'AF-able'.)

The same result above also holds for actions of $\mathbb{Z}^d, d \geq 2$:

Theorem 10.5. *Let φ be a minimal free action of the group $\mathbb{Z}^d, d \geq 2$. Then the orbit relation R_φ is affable. In consequence, $D_m(X, R)$ is a complete invariant for orbit equivalence for the class of all minimal AF-equivalence relations and all orbit relations R_φ arising from the groups $\mathbb{Z}^d, d \geq 1$.*

The proof in the case $d = 2$ [8] is substantially more complicated than $d = 1$, while $d > 2$ [9] is even more so. They are similar to the proof above in that the main idea is to find a large AF-equivalence relation R inside of R_φ and by application of the Absorption Theorem, prove that R_φ is orbit equivalent to R and hence affable. The essential increased complexity comes from the geometry of $\mathbb{R}^d$ as d increases. In these cases, the set Y is (presumably never) finite. Moreover, the proof for $\mathbb{Z}^d$ requires d applications of the absorption theorem.

We close with a word on the history of the results since our presentation is slightly different. The first classification was given in [10] for minimal $\mathbb{Z}$-actions. In the same paper, it was extended to minimal AF-equivalence relations by arguments which are not dissimilar to those we used in this section to go the other way. That argument for $\mathbb{Z}$-actions was rather ad-hoc. If we compare it to what we have discussed here for the classification for minimal AF-equivalence relations (which appears in [22]), the latter is not particularly easier, but it is much more conceptual. Secondly, the argument for $\mathbb{Z}$-actions relied on a non-trivial fact from homological algebra, which seemed a little out of place in the proof. Finally, the argument we have given here from [22] emphasizes the importance of the absorption theorem. Basically, it is used first to establish the classification of minimal AF-equivalence relations and then also to show that the actions of various groups are affable.

References

[1] M. Brin and G. Stuck, *Introduction to dynamical systems*, Cambridge U. Press, Cambridge, 2002.

[2] H.A. Dye, *On groups of measure preserving transformations I*, Amer. J. Math. **81**(1959), 119-159.

[3] E.G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. in Math. **46**, 1981.

[4] E.G. Effros, D. Handelman and C.-L. Shen, *Dimension groups and their affine representations*, Amer. J. Math. **102** (1980), 385-407.

[5] G.A. Elliott, *On the classification of inductive limits of sequences of semi-simple finite dimensional algebras*, J. Algebra **38** (1976), 29-44.

[6] A. Forrest and J. Hunton, *Cohomology and K-theory of commuting homeomorphisms of the Cantor set*, Ergodic Theory Dynam. Systems **19** (1999), 611-625.

[7] T. Giordano, H. Matui, I.F. Putnam and C.F. Skau, *The absorption theorem for affable equivalence relations*, Ergodic Theory Dynam. Sys. 28 (2008), 1509-1531.

[8] T. Giordano, H. Matui, I.F. Putnam and C.F. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^2$-systems*, J. Amer. Math. Soc. 21 (2008), 863-892.

[9] T. Giordano, H. Matui, I.F. Putnam and C.F. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems*, preprint.

[10] T. Giordano, I.F. Putnam and C.F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. **469** (1995), 51-111.

[11] T. Giordano, I.F. Putnam and C.F. Skau, *Affable equivalence relations and orbit structure of Cantor dynamical systems*, Ergodic Theory Dynam. Systems **23** (2004), 441-475.

[12] T. Giordano, I.F. Putnam and C.F. Skau, *The orbit structure of Cantor minimal $\mathbb{Z}^2$-systems*, Proceedings of the first Abel Symposium, O. Bratteli, S. Neshveyev and C. Skau, Eds., Springer-Verlag, Berlin, 2006.

[13] E. Glasner and B. Weiss, *Weak orbit equivalence of Cantor minimal systems*, Internat. J. Math. **6** (1995), 559-579.

[14] K.R. Goodearl, *Partially ordered abelian groups with interpolation*, Mathematical Surveys and Monographs **20**, Amer. Math. Soc., Providence, RI, 1986.

[15] R.H. Herman, I.F. Putnam and C.F. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. **3** (1992), 827-864.

[16] A.S. Kechris and B.D. Miller, *Topics in Orbit Equivalence*, Lecture Notes in Mathematics, **1852**, Springer, Berlin, 2004.

[17] S. Kerov, Thesis Leningrad State University.

[18] W. Krieger, *On a dimension for a class of homeomorphism groups*, Math. Ann. **252** (1980), 87-95.

[19] H. Matui, *An absorption theorem for minimal AF equivalence relations on Cantor sets*, J. Math. Soc. Japan 60(2008), 1171-1185.

[20] N. Ormes, *Strong orbit realization for minimal homeomorphisms*, J. d'Anal. Math., 71 (1997), 103–133.

[21] I.F. Putnam, *The C^*-algebras associated with minimal homeomorphisms of the Cantor set*, Pacif. J. Math. **136** (1989), 329-353.

[22] I.F. Putnam, *Orbit equivalence of Cantor minimal systems: A survey and a new proof*, Expos. Math., **28** (2010), 101-131.

[23] J. Renault, *A Groupoid Approach to C^*-algebras*, Lecture Notes in Mathematics **793**, Springer, Berlin, 1980.