Orbit equivalence for Cantor minimal systems

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Cantor minimal systems

Let X be a Cantor set; compact, totally disconnected, metrizable, no isolated points.

Let G be a countable, discrete abelian group with an action φ on X: for s in G,

$$\varphi^s : X \to X$$

is a homeomorphism,

$$\begin{aligned} \varphi^0 &= id_X, \\ \varphi^s \circ \varphi^t &= \varphi^{s+t}, \end{aligned}$$

s, t in G.

- The action is *free* if, $\varphi^s(x) = x$ only if $s = 0$,
- The *orbit* of x in X is $\{\varphi^s(x) \mid s \in G\}$,
- The action is *minimal* if, for every x in X, its orbit is dense in X.

2^∞-odometer

Let $X = \{0,1\}^{\mathbb{N}}$ and define φ to be addition of $(1,0,0,\ldots)$, mod 2, with carry over to the right. For example:

$$\begin{aligned} \varphi(0,0,1,0,1,1,\ldots) &= (1,0,1,0,1,1,\ldots) \\ \varphi(1,1,1,0,0,1,\ldots) &= (0,0,0,1,0,1,\ldots) \\ \varphi(1,1,1,1,1,1,\ldots) &= (0,0,0,0,0,0,\ldots) \end{aligned}$$

$\mathbb{Z}$ action, φ^n is the nth iterate of φ, $n \geq 1$, or the $-n$th iterate of φ^{-1}, $n < 0$.

X is also the ring of 2-adic integers and the map is addition of 1.

More generally:

Let X be a compact Hausdorff space. Consider homeomorphisms, φ, whose domain and range are both open subsets of X. Suppose that $\mathcal{F}$ is collection of such functions such that:
1. if φ, ψ are in $\mathcal{F}$, so is $\varphi \cap \psi$,
2. if φ, ψ are in $\mathcal{F}$, so is $\varphi \circ \psi$,
3. if φ is in $\mathcal{F}$, so is φ^{-1},
4. the collection of open sets U in X such that id_U is in $\mathcal{F}$ generates the topology of X.

It follows that

$$R = \cup\mathcal{F} = \{(x, \varphi(x)) \mid \varphi \in \mathcal{F}, x \in Dom(\varphi)\}$$

is an equivalence relation and $\mathcal{F}$ is a basis for a topology of R. We assume that this topology is second countable and Hausdorff. As a consequence the equivalence classes are countable.

Such an equivalence relation, with this topology, is called *étale*.

If φ is a free action of G on X, then let

$$\mathcal{F} = \{\varphi^s|U \mid s \in G, U \subset X \text{ open }\}.$$

and

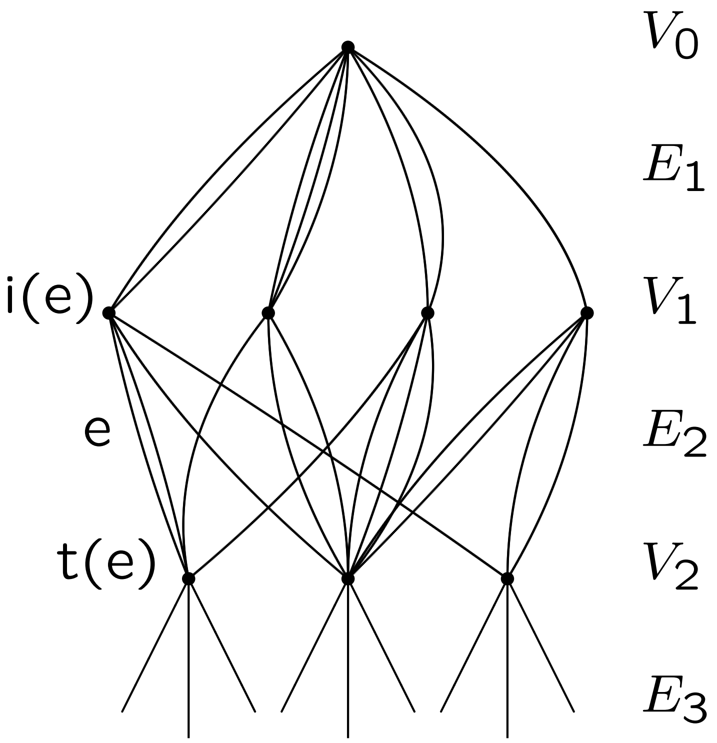
$$R_\varphi = \{(x, \varphi^s(x)) \mid x \in X, s \in G\}.$$

Equivalence classes are the orbits.

In general, R is minimal if every equivalence class is dense.

AF-relations

A *Bratteli diagram* is a vertex set $V = V_0 \cup V_1 \cup \ldots$ and an edge set $E = E_1 \cup E_2 \cup \ldots$ with initial and terminal maps $i : E_n \to V_{n-1}, t : E_n \to V_n$. Each V_n and E_n are finite with $V_0 = \{v_0\}$.



Let X be the set of infinite paths from v_0:

$$X = \{(x_1, x_2, \ldots) \mid x_n \in E_n, t(x_n) = i(x_{n+1})\}$$

Relative topology from $X \subset \Pi_n E_n$.
If $p = (p_1, p_2, \ldots, p_N)$ is a finite path, we let

$$C(p) = \{x \in X \mid x_n = p_n, 1 \leq n \leq N\},$$

which is clopen.

For paths p, q of length N, with $t(p_N) = t(q_N)$, define $\varphi : C(p) \to C(q)$ by

$$\begin{aligned} & \varphi(p_1, p_2, \ldots, p_N, x_{N+1}, x_{N+2}, \ldots) \\ = \quad & (q_1, q_2, \ldots, q_N, x_{N+1}, x_{N+2}, \ldots). \end{aligned}$$

The set of all such φ is $\mathcal{F}$.

R is tail equivalence:

$$(x, y) \in R \Leftrightarrow \exists N, x_n = y_n, n \geq N.$$

For fixed N, let

$$(x, y) \in R_N \Leftrightarrow x_n = y_n, n \geq N.$$

We have

$$R_1 \subset R_2 \subset \cdots, \quad R = \cup_N R_N.$$

Definition 1. *An étale equivalence relation R on X is* AF *if X is totally disconnected and R is the union of an increasing sequence of compact, open subequivalence relations.*

Theorem 2. *Every AF-relation can be presented by a Bratteli diagram.*

Recall: 2^∞-odometer

Let $X = \{0,1\}^{\mathbb{N}}$ and define φ to be addition of $(1,0,0,\ldots)$, with carry over to the right. For example:

$$
\begin{aligned}
\varphi(0,0,1,0,1,1,\ldots) &= (1,0,1,0,1,1,\ldots)\\
\varphi(1,1,1,0,0,1,\ldots) &= (0,0,0,1,0,1,\ldots)\\
\varphi(1,1,1,1,1,1,\ldots) &= (0,0,0,0,0,0,\ldots)
\end{aligned}
$$

Let $R \subset R_\varphi$ be the equivalence relation generated by $\{(x,\varphi(x)) \mid x \neq (1,1,1,\ldots)\}$. Consider the Bratteli diagram with one vertex and two edges at every level (labelled 0 and 1). Then R is just tail equivalence.

Theorem 3. *Let φ be a minimal $\mathbb{Z}$-action on a Cantor set X. Choose y in X and let $R \subset R_\varphi$ be the equivalence relation generated by $\{(x, \varphi^1(x)) \mid x \neq y\}$. Then R is a minimal AF-relation and*

$$R_\varphi = R \vee (y, \varphi^1(y))$$

($\vee$ means the equivalence relation generated by).

Proof. Choose $Y_1 \supset Y_2 \supset \cdots$, clopen sets with intersection $\{y\}$ and let R_N be the equivalence relation generated by $\{(x, \varphi^1(x)) \mid x \notin Y_N\}$. Then

$$R_1 \subset R_2 \subset \cdots \quad , \cup_N R_N = R,$$

and each R_N is compact and open. □

Consequence: every minimal homeomorphism of a Cantor can be presented as a map on a Bratteli diagram. The edges are ordered and the map is to take successor under a type of reverse lexicographic order. The Bratteli-Vershik model.

Orbit equivalence and isomorphism

Definition 4. *For $i = 1, 2$, let R_i be an equivalence relation on the topological space X_i. R_1 and R_2 are* orbit equivalent, *written $R_1 \sim R_2$ if there is a homeomorphism $h : X_1 \to X_2$ such that $h \times h(R_1) = R_2$ or $h[x]_{R_1} = [h(x)]_{R_2}$ for all x in X_1.*

Definition 5. *For $i = 1, 2$, let R_i be an étale equivalence relation on the topological space X_i. R_1 and R_2 are* isomorphic, *written $R_1 \cong R_2$ if there is a homeomorphism $h : X_1 \to X_2$ such that $h \times h : R_1 \to R_2$ is a homeomorphism.*

Remark 1. *It follows from a result of Sierpinski that for $R_i, i = 1, 2$ arising from actions of discrete groups on* connected *spaces $X_i, i = 1, 2$, orbit equivalence is equivalent to conjugacy of the actions. Hence, we restrict to totally disconnected spaces.*

Invariants

X, Cantor set, R, an étale equivalence relation.

Definition 6. *A probability measure* μ *on* X *is* R-invariant *if*

$$\mu(\varphi(U)) = \mu(U),$$

for all $\varphi \in \mathcal{F}$, $U \subset Dom(\varphi)$, *Borel. Let* $M(R)$ *denote the set of all such measures.* R *is* uniquely ergodic *if there is a unique* R*-invariant measure.*

$$\begin{aligned} C(X,\mathbb{Z}) &= \{f : X \to \mathbb{Z} \mid f \text{ continuous } \} \\ B_m(X,R) &= \{f \in C(X,\mathbb{Z}) \mid \int_X f d\mu = 0, \\ &\qquad \text{for all } \mu \in M(R)\} \\ B(X,R) &= < \{\chi_U - \chi_{\varphi(U)} \mid \varphi \in \mathcal{F}, \\ &\qquad U \subset Dom(\varphi), clopen\} > \end{aligned}$$

$$B(X,R) \subset B_m(X,R) \subset C(X,\mathbb{Z}).$$

We define

$$\begin{aligned} D(R) &= C(X,\mathbb{Z})/B(X,R) \\ D_m(R) &= C(X,\mathbb{Z})/B_m(X,R) \end{aligned}$$

Notice that $D_m(R)$ is a quotient of $D(R)$.

These are abelian groups and have an *order*:

$$\begin{aligned} D(R)^+ &= \{[f] \mid f \geq 0\} \\ D_m(R)^+ &= \{[f] \mid f \geq 0\} \end{aligned}$$

and a distinguished positive element: [1].

Theorem 7. *1.* $(D(R), D(R)^+, [1])$ *is an invariant of isomorphism.*

2. $(D_m(R), D_m(R)^+, [1])$ *is an invariant of orbit equivalence.*

Theorem 8. *If* $M(R) = \{\mu\}$ *(*R *is uniquely ergodic), then*

$$D_m(R) = \{\mu(E) \mid E \subset X \text{ clopen }\} + \mathbb{Z} \subset \mathbb{R}.$$

$D(R)$ and $D_m(R)$ for AF-relations R

Theorem 9. *Let (V, E) be a Bratteli diagram and (X, R) its AF-relation. $(D(R), D(R)^+, [1])$ is isomorphic to the inductive limit*

$$(\mathbb{Z}V_0, \mathbb{Z}^+V_0) \stackrel{\gamma_1}{\to} (\mathbb{Z}V_1, \mathbb{Z}^+V_1) \stackrel{\gamma_2}{\to} (\mathbb{Z}V_2, \mathbb{Z}^+V_2) \stackrel{\gamma_3}{\to}$$

where

$$\gamma_n(v) = \sum_{i(e)=v} t(e),$$

or

$$(\mathbb{Z}, \mathbb{Z}^+) \stackrel{A_1}{\to} (\mathbb{Z}^{n_1}, (\mathbb{Z}^+)^{n_1}) \stackrel{A_2}{\to} (\mathbb{Z}^{n_2}, (\mathbb{Z}^+)^{n_2}) \stackrel{A_3}{\to}$$

where $n_k = \#V_k$ and A_k is the adjacency matrix of E_k. The element v_0 is mapped to $[1]$.

The inductive limit of groups $G_1 \stackrel{\alpha_1}{\to} G_2 \stackrel{\alpha_2}{\to} \cdots$ is

$$\cup_n G_n / \{g \sim \alpha_n(g) \mid g \in G_n\}.$$

Idea of proof: $D(R) = C(X, \mathbb{Z})/B(X, R)$. For a path p of length N, $C(p)$ is clopen, $\chi_{C(p)} \in C(X, \mathbb{Z})$:

$$[\chi_{C(p)}] \in D(R) \to t(p_N) \in \mathbb{Z}V_N.$$

Notice that if $t(p_N) = t(q_N)$, then $\chi_{C(p)} - \chi_{C(q)} \in B(X, R)$.

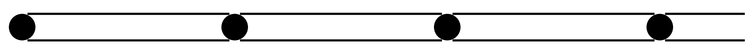
Theorem 10. *Let (V, E) be a Bratteli diagram and (X, R) be the associated AF-relation. There is a bijection between $f : \cup_N V_N \to [0, 1]$ such that*

$$f(v_0) = 1, f(v) = \sum_{i(e)=v} f(t(e))$$

and R-invariant probability measures given by:

$$\mu(C(p)) = f(t(p_N)).$$

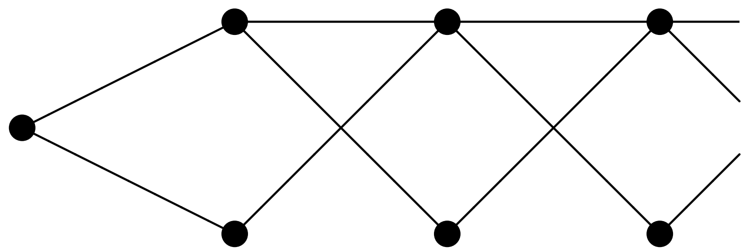
Example 1



$$\mathbb{Z} \quad \xrightarrow{2} \quad \mathbb{Z} \quad \xrightarrow{2} \quad \mathbb{Z} \quad \xrightarrow{2} \quad \cdots$$

$$D(R) \quad = \quad D_m(R) \quad = \quad \{p2^{-k} \ \mid p \in \mathbb{Z}, \ \ k \in \mathbb{Z}^+\}.$$

Example 2



$$\mathbb{Z} \ \xrightarrow{(1,1)} \ \mathbb{Z}^2 \ \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \ \mathbb{Z}^2 \ \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \ \cdots$$

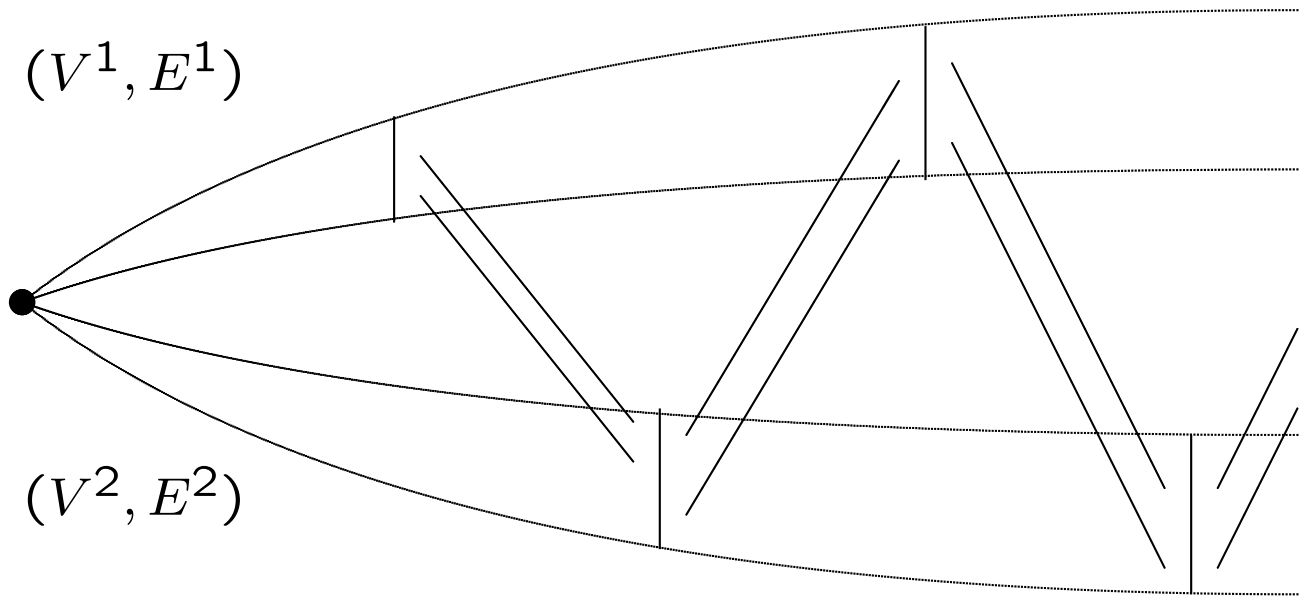
$$D(R) = D_m(R) = \{m + \left(\frac{1+\sqrt{5}}{2}\right) n \mid m, n \in \mathbb{Z}\}$$

Theorem 11 (Elliott-Krieger). *Let* $(V^i, E^i), i = 1, 2$ *be two Bratteli diagrams with associated AF-relations,* $(X_i, R_i), i = 1, 2$. *TFAE:*

1. $(X_1, R_1) \cong (X_2, R_2)$

2. $(D(R_1), D(R_1)^+, [1]) \cong (D(R_2), D(R_2)^+, [1])$

3. *the two diagrams may be "intertwined":*



Our main technical result for the study of orbit equivalence is:

Theorem 12 (Absorption Theorem)**.** *Let* (X, R) *be a minimal AF-relation. Suppose that* $Y \subset X$ *and* Q *is an AF-relation on* Y *satisfying:*

1. Y *is closed and* $\mu(Y) = 0$*, for all* μ *in* $M(R)$*,*

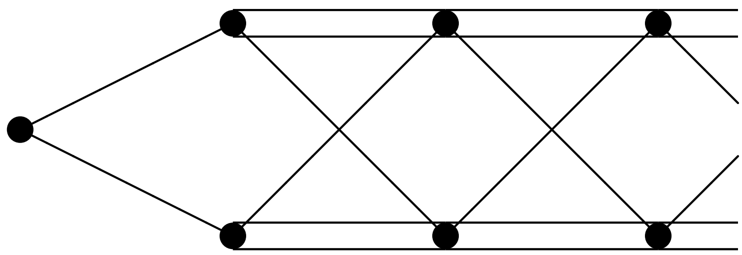
2. *other technical conditions,*

Then the equivalence relation generated by R *and* Q*,* $\tilde{R} = R \vee Q$*, is orbit equivalent to* R*:*

$$R \vee Q \sim R.$$

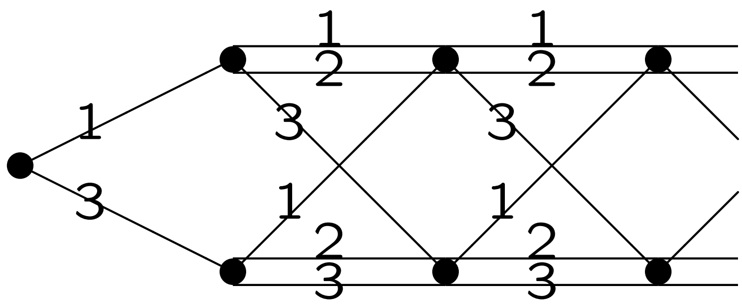
Absorption Thm: Application 1

Consider the AF-equivalence relation for following Bratteli diagram



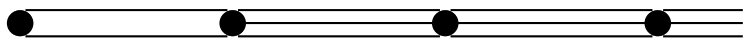
(X, R)

$$0 \to \mathbb{Z} \to D(R) \to \frac{1}{2}\mathbb{Z}[1/3] = D_m(R) \to 0.$$



(X, R)

$$X = \{1,3\} \times \{1,2,3\}^{\mathbb{N}} = \text{path space of}$$



$(X, \tilde{R})$

1. $D(\tilde{R}) = D_m(\tilde{R}) = \frac{1}{2}\mathbb{Z}[1/3]$,

2. $R \subset \tilde{R}$,

3. $\tilde{R} = R \vee ((1,2,2,2\ldots),(3,2,2,\ldots))$.

Apply the absorption theorem with

$$\begin{aligned} Y &= \{(1,2,2,2\ldots),(3,2,2,\ldots)\}, \\ Q &= Y \times Y. \end{aligned}$$

to conclude that

$$R \sim \tilde{R}.$$

Theorem 13. *Let* (X, R) *be a minimal AF-relation. There exists an AF-relation* $R \subset \tilde{R}$ *such that*

$$\tilde{R} = R \vee Q(\text{A.T.} \Rightarrow \tilde{R} \sim R),$$

and

$$(D(\tilde{R}), D(\tilde{R})^+, [1]) \cong (D_m(R), D_m(R)^+, [1]).$$

Corollary 14. *For minimal AF-relations* (X, R)*,* $(D_m(R), D_m(R)^+, [1])$ *is a complete invariant for orbit equivalence.*

Proof. $i = 1, 2$, (X_i, R_i) minimal AF. Let $\tilde{R}_i, i = 1, 2$ be as above. If $D_m(R_1) \cong D_m(R_2)$, then

$$D(\tilde{R}_1) \cong D_m(R_1) \cong D_m(R_2) \cong D(\tilde{R}_2).$$

Elliott-Krieger implies

$$R_1 \overset{A.T.}{\sim} \tilde{R}_1 \overset{E-K}{\cong} \tilde{R}_2 \overset{A.T.}{\sim} R_2.$$

□

Absorption Thm: Application 2

Theorem 15 (Revisited). *Let φ be a minimal $\mathbb{Z}$-action on a Cantor set X. Choose y in X and let $R \subset R_\varphi$ be the equivalence relation generated by $\{(x, \varphi^1(x)) \mid x \neq y\}$. Then R is a minimal AF-relation and*

$$R_\varphi = R \vee (y, \varphi^1(y))$$

($\vee$ means the equivalence relation generated by).

Let

$$Y = \{y, \varphi^1(y)\}, \quad Q = Y \times Y$$

The Absorption Theorem implies that $R_\varphi \sim R$.

Theorem 16 (Giordano-P-Skau, 1991). *For minimal AF-relations and minimal $\mathbb{Z}$-actions, (X, R), $(D_m(R), D_m(R)^+, [1])$ is a complete invariant for orbit equivalence.*

Theorem 17 (Giordano-Matui-P-Skau, 2005). *For minimal AF-relations, minimal* $\mathbb{Z}$*-actions and minimal* $\mathbb{Z}^2$*-actions,* (X, R), $(D_m(R), D_m(R)^+, [1])$ *is a complete invariant for orbit equivalence.*

Theorem 18 (Giordano-Matui-P-Skau, 2008). *For minimal AF-relations and minimal* $\mathbb{Z}^d$*-actions,* $d \geq 1$, (X, R), $(D_m(R), D_m(R)^+, [1])$ *is a complete invariant for orbit equivalence.*