Some interactions between operator algebras and dynamical systems

Ian F. Putnam,
University of Victoria

Étale equivalence relations

A *local homeomorphism*, φ, of X is a homeomorphism from one open subset of X to another.

Recall that $\varphi \subset X \times X$, so in usual notation $\varphi : d(\varphi) \to r(\varphi)$, where d, r be the two canonical projections from $X \times X$ to X.

A *local action*, $\mathcal{F}$, is collection of local homeomorphisms such that:

1. $\{U \subset X \mid id_U \in \mathcal{F}\}$ is a neighbourhood base for X.
2. if φ is in $\mathcal{F}$, so is φ^{-1},
3. if φ, ψ are in $\mathcal{F}$, so is $\varphi \circ \psi$,
4. if φ, ψ are in $\mathcal{F}$, so is $\varphi \cap \psi$.

It follows from the first three conditions that

$$R = \cup\mathcal{F} = \{(x, \varphi(x)) \mid \varphi \in \mathcal{F}, x \in d(\varphi)\}$$

is an equivalence relation.

The fourth implies that $\mathcal{F}$ is a basis for a topology of R. We assume that this topology is second countable. As a consequence the equivalence classes are countable.

Such an equivalence relation, with this topology, is called *étale*.

Example 1.

If φ is a free action of G on X, then let

$$\mathcal{F} = \{\varphi^s|U \mid s \in G, U \subset X \text{ open } \}.$$

and

$$(x, s) \in X \times G \to (x, \varphi^s(x)) \in R_\varphi$$

is a homeomorphism.

Example 2.

$X = [0, 1]$ and $R = \Delta_X \cup \{(1, 0), (0, 1)\}$. There is *no* topology on R which makes it étale.

We say that R is *minimal* if every equivalence class is dense. (Does not need topology.)

A measure μ on X is R-*invariant* if

$$\mu(d(\varphi)) = \mu(r(\varphi)),$$

for every φ in $\mathcal{F}$. (Depends only on R).

(X_1, R_1) and (X_2, R_2) are *orbit equivalent*, written $R_1 \sim R_2$, if there is a homeomorphism $h : X_1 \to X_2$ such that $h \times h(R_1) = R_2$ or $h[x]_{R_1} = [h(x)]_{R_2}$ for all x in X_1.

R_1 and R_2 are *isomorphic*, written $R_1 \cong R_2$ if, in addition, $h \times h : R_1 \to R_2$ is a homeomorphism.

C^*-**algebras**

If X is a compact, Hausdorff space, then

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous } \}$$

is a commutative, unital C^*-algebra.

Every commutative, unital is $*$-isomorphic to $C(X)$, for some compact Hausdorff space X.

If (X, R) is an étale equivalence relation, then $C^*(X, R)$ is a natural C^*-algebra which replaces $C(X/R)$. (Usually, X/R is a 'bad' space.)

Consider $C_c(R)$ with the obvious linear structure and product

$$f \cdot g(x,y) = \sum_{(x,z)\in R} f(x,z)g(z,y),$$

and involution $f^*(x,y) = \overline{f(y,x)}$.

Then endow $C_c(R)$ with a norm and complete.

Example 1: X compact, Hausdorff, $R =$ equality. $C^*(X,R) = C(X) = C(X/R)$.

Example 2: $X = \{1,2,\ldots,N\}$, $R = X \times X$.

$C^*(X,R) \cong M_N(\mathbb{C})$.

Generally, $f \in C(X) \to f\chi_\Delta \in C^*(X,R)$ is a homomorphism.

For general X, R, the formula for the product on $C_c(R)$:

$$f \cdot g(x,y) = \sum_{(x,z)\in R} f(x,z)g(z,y),$$

has problems.

Example 3: $X = [0,1]$ $R = \Delta \cup \{(0,1),(1,0)\}$.

$$\chi_{\{(0,1)\}}\chi_{\{(1,0)\}} = \chi_{\{(0,0)\}}$$

which is *not* a continuous function on R.

The condition that R is étale is *exactly* what is needed for this to be well-defined.

Example 5: X compact, φ an action of G.

Fix a φ-invariant measure on X. Hilbert space $L^2(X)$ with operators

$$f \cdot \xi(x) = f(x)\xi(x), f \in C(X),$$

and

$$u_a\xi = \xi \circ \varphi^{-a}, a \in G.$$

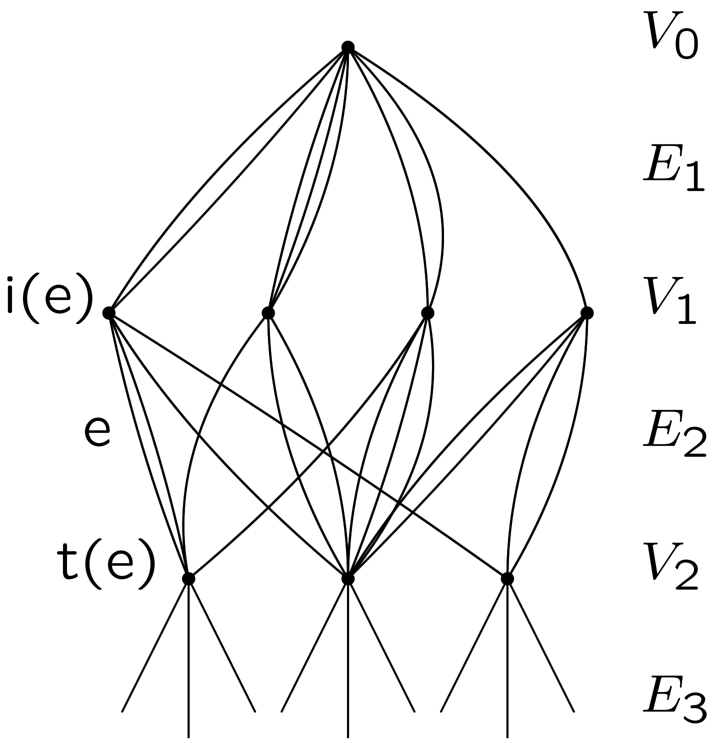
If the action is free and minimal,

$$C^*(X, R) = C^*\{f, u_a \mid f \in C(X), a \in G\}.$$

Identifying R and $X \times G$, $u_a = \chi_{X\times\{a\}}$.

AF equivalence relations

A *Bratteli diagram* is a vertex set $V = V_0 \cup V_1 \cup \ldots$ and an edge set $E = E_1 \cup E_2 \cup \ldots$ with initial and terminal maps $i : E_n \to V_{n-1}, t : E_n \to V_n$. Each V_n and E_n are finite with $V_0 = \{v_0\}$.



Let X be the set of infinite paths from v_0:

$$X = \{(x_1, x_2, \ldots) \mid x_n \in E_n, t(x_n) = i(x_{n+1})\}$$

Relative topology from $X \subset \Pi_n E_n$.
If $p = (p_1, p_2, \ldots, p_N)$ is a finite path, we let

$$C(p) = \{x \in X \mid x_n = p_n, 1 \leq n \leq N\},$$

which is clopen.

For paths p, q of length N, with $t(p_N) = t(q_N)$, define $\varphi : C(p) \to C(q)$ by

$$\begin{aligned} & \varphi(p_1, p_2, \ldots, p_N, x_{N+1}, x_{N+2}, \ldots) \\ = \quad & (q_1, q_2, \ldots, q_N, x_{N+1}, x_{N+2}, \ldots). \end{aligned}$$

The set of all such φ is a local action, $\mathcal{F}$.

R is tail equivalence:

$$(x, y) \in R \Leftrightarrow \exists N, x_n = y_n, n \geq N.$$

For fixed N, let

$$(x, y) \in R_N \Leftrightarrow x_n = y_n, n \geq N.$$

We have

$$R_1 \subset R_2 \subset \cdots, \quad R = \cup_N R_N.$$

and each R_N is a compact, open subequivalence relation.

This makes them tractible, but rich.

Invariant measures for AF-equivalence relations.

We want to assign a measure to each clopen set. Clopen sets are the union of cylinder sets.

If p and q are paths with $t(p) = t(q)$, then there is a local homeomorphism $\varphi : C(p) \to C(q)$ so $\mu(C(p)) = \mu(C(q))$; in other words, $\mu(C(P))$ depends only on $t(p)$.

$$\mu(C(p)) = \omega(t(p)).$$

We require $\omega(v_0) = 1, \omega(v) = \sum_{i(e)=v} \omega(t(e))$.

There is a bijection between R-invariant measures μ and such functions $\omega : V \to [0, 1]$.

K-theory.

A, unital C^*-algebra, $K_0(A)$ abelian group

Flat earth version:

projections p in A: $p^2 = p = p^*$,

$p \sim q$: there exists v in A, $v^*v = p, vv^* = q$, or if there exists u in A, $upu^{-1} = q$,

If $pq = 0$, then $p + q$ is a projection. $p = 0$ is the indentity.

Order $p \geq q$ if $pq = q$.

Example: $X = \{1, \ldots, N\}$, $R = X \times X$, $M_N(\mathbb{C})$.

1. Every projection is similar to one in $C(X)$ (i.e. diagonal).

2. Two projections are similar if and only if they have the same rank, or the same trace.

Thus

$$K_0(M_N(\mathbb{C})) \cong \mathbb{Z}, p \to Rank(p).$$

Example: (X, R) AF.

1. Every projection is similar to one in $C(X)$: χ_U, U clopen. Generated by $\chi_{C(p)}$'s.

2. If $t(p) = t(q)$, $\varphi : C(p) \to C(q)$. Exercise: $\varphi \subset R$ is compact, open and

$$\chi_\varphi^* \chi_\varphi = \chi_{C(q)}, \qquad \chi_\varphi \chi_\varphi^* = \chi_{C(p)}.$$

$[\chi_{C(p)}]$ is determined by $t(p)$.

If μ is an invariant measure, define $\tau : C_c(R) \to \mathbb{C}$ by

$$\tau(f) = \int_X f(x, x) d\mu(x),$$

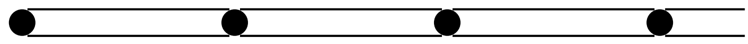
satisfies $\tau(fg) = \tau(gf)$ (trace property) and gives

$$\widehat{\tau} : K_0(C^*(X, R)) \to \mathbb{R}.$$

and the range is

$$\{\mu(U) \mid U \subset X \text{ clopen }\} + \mathbb{Z}.$$

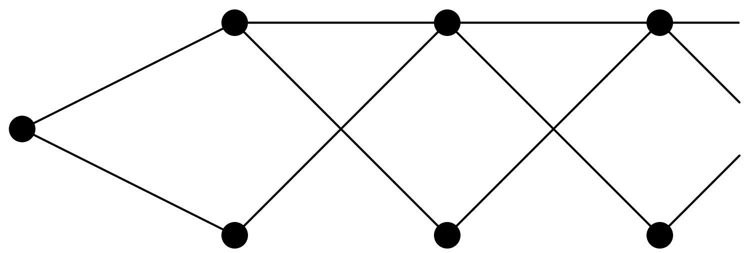
Example 1



First, there is a unique invariant measure given by $\omega(v_N) = 2^{-N}$.

$$\hat{\tau} : K_0(C^*(X,R)) \cong \{p2^{-k} \mid p \in \mathbb{Z}, k \in \mathbb{Z}^+\}.$$

Example 2

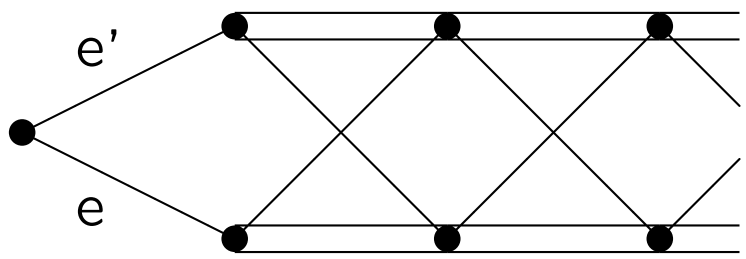


$$\hat{\tau} : K_0(C^*(X,R)) \cong \mathbb{Z} + \gamma\mathbb{Z} \subset \mathbb{R}.$$

where γ is the golden mean.

What is different from the case of $M_N(\mathbb{C})$ is that $\tau(p) = \tau(q)$ does *not* imply $p \sim q$.

Example 3



(X, R)

$$\hat{\tau} : K_0(C^*(X,R)) \to \{p2^{-1}3^{-k} \mid p \in \mathbb{Z}, k \geq 0\}.$$

has kernel $\mathbb{Z}([\chi_{C(e)}] - [\chi_{C(e')}])$, e, e' the two edges in E_1.

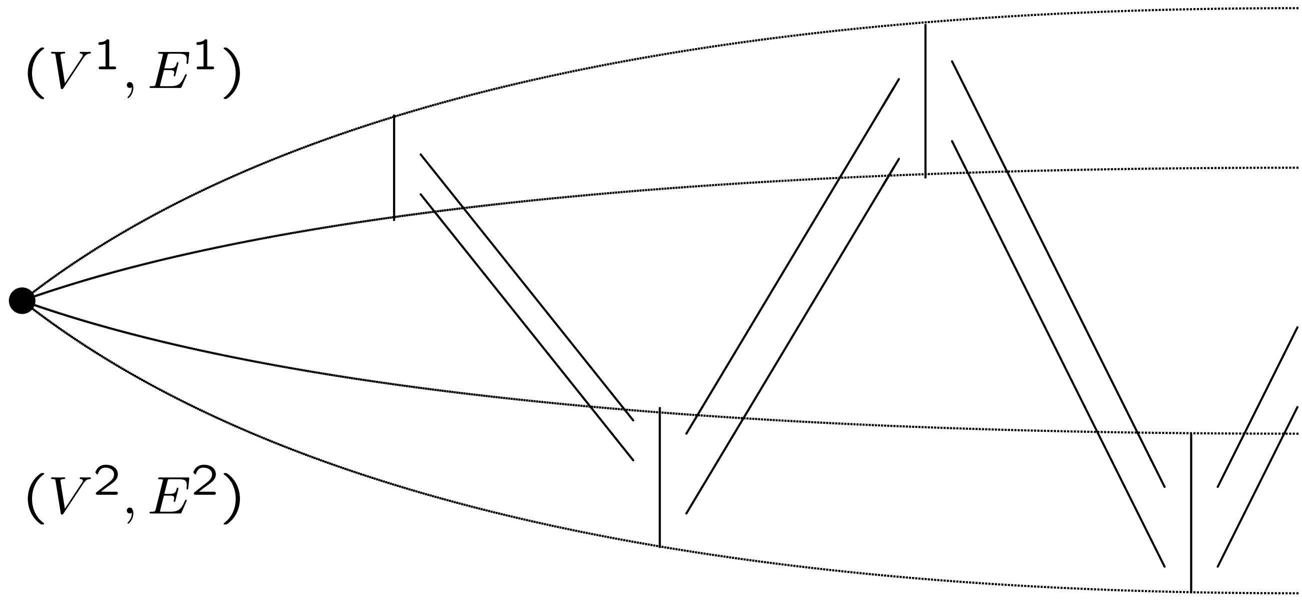
Theorem 1 (Elliott-Krieger). *Let* $(V^i, E^i), i = 1, 2$ *be two Bratteli diagrams with associated AF-relations,* $(X_i, R_i), i = 1, 2$*. TFAE:*

1. *the two diagrams may be "intertwined"*

2. $(X_1, R_1) \cong (X_2, R_2)$

3. $C^*(X_1, R_1) \cong C^*(X_2, R_2)$

4. $K_0(C^*(X_1, R_1)) \cong K_0(C^*(X_2, R_2))$ *as ordered abelian groups with order unit.*


(V^1, E^1)
(V^2, E^2)

Theorem 2 (Absorption Theorem). *Let* (X, R) *be a minimal AF-relation. Suppose that* $Y \subset X$ *and* Q *is an AF-relation on* Y *satisfying:*

1. Y *is closed and* $\mu(Y) = 0$*, for all* R*-invariant* μ.

2. *other technical conditions,*

Then the equivalence relation generated by R *and* Q*,* $\tilde{R} = R \vee Q$*, is orbit equivalent to* R*:*

$$R \vee Q \sim R.$$

In general, if $R \subset \tilde{R}$ are both AF, then $K_0(C^*(X, \tilde{R}))$ is a quotient of $K_0(C^*(X, R))$; both are generated by projections in $C(X)$, but the former has more equivalences.

Also, we have seen

$\hat{\tau} : K_0(C^*(X, R)) \to \{\mu(U) \mid U \subset X \text{ clopen }\} + \mathbb{Z}$

is a quotient.

For R minimal, this can be realized by an $R \subset \tilde{R}$, also AF, such that

$$\tilde{R} = R \vee Q (\text{A.T.} \Rightarrow \tilde{R} \sim R),$$

Combining with the Elliott-Krieger Theorem, we get:

Corollary 3. *For minimal, uniquely ergodic AF-equivalence relations*

$$\{\mu(U) \mid U \subset X \text{ clopen }\} + \mathbb{Z}$$

is a complete invariant for orbit equivalence.

Theorem 4. *Let φ be a minimal $\mathbb{Z}$-action on a Cantor set X. Choose y in X and let $R \subset R_\varphi$ be the equivalence relation generated by $\{(x, \varphi^1(x)) \mid x \neq y\}$. Then R is a minimal AF-relation and*

$$R_\varphi = R \vee (y, \varphi^1(y)).$$

Let

$$Y = \{y, \varphi^1(y)\}, \qquad Q = Y \times Y$$

The Absorption Theorem implies that $R_\varphi \sim R$.

Theorem 5 (Giordano-P-Skau, 1991)**.** *For minimal uniquely ergodic AF-relations and $\mathbb{Z}$-actions, (X, R),*

$$\{\mu(U) \mid U \subset X \text{ clopen } \} + \mathbb{Z}$$

is a complete invariant for orbit equivalence.