Aperiodic order with a little topology and dynamical systems

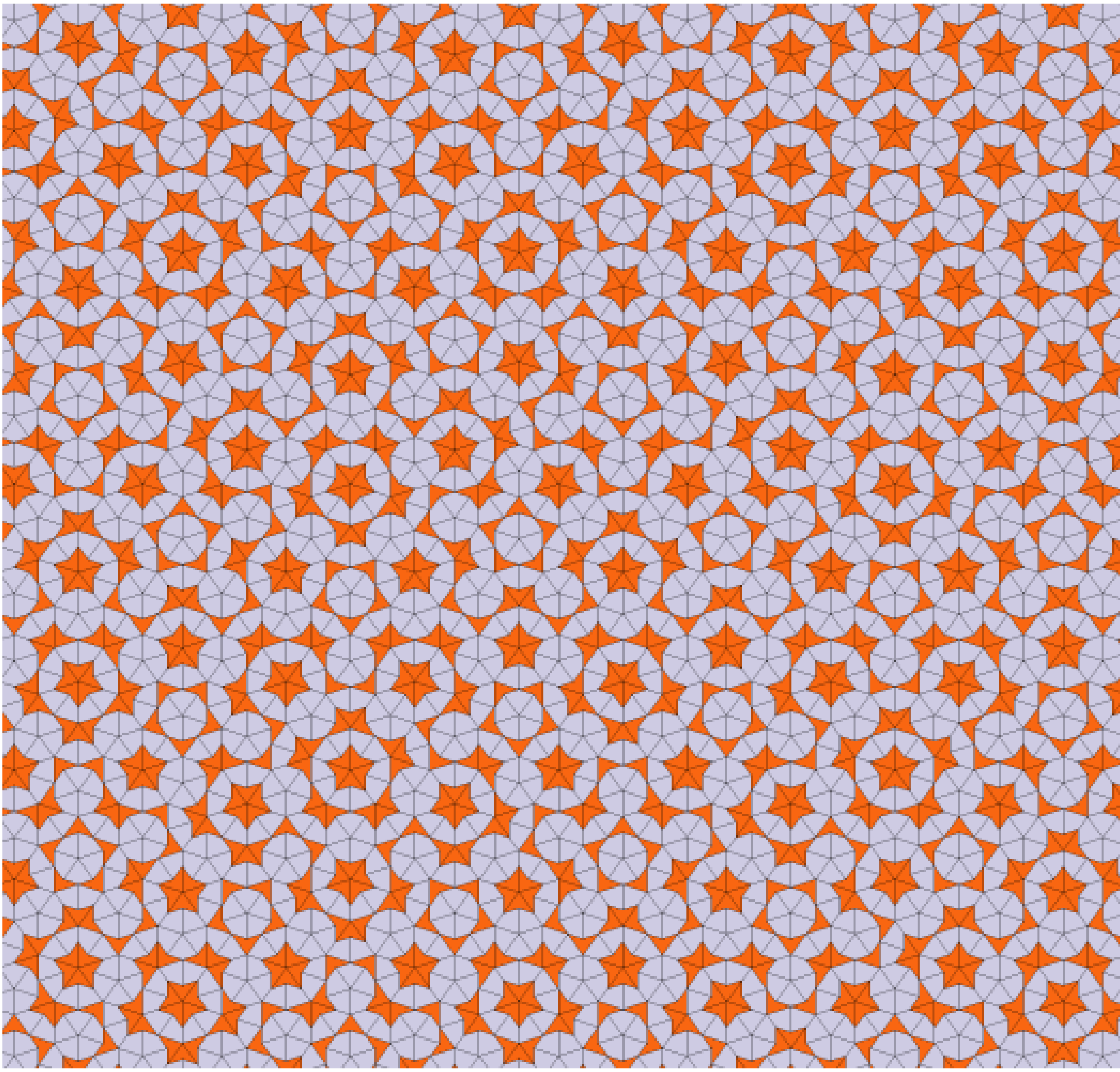
Ian F. Putnam,
University of Victoria

- Introduction to aperiodic order

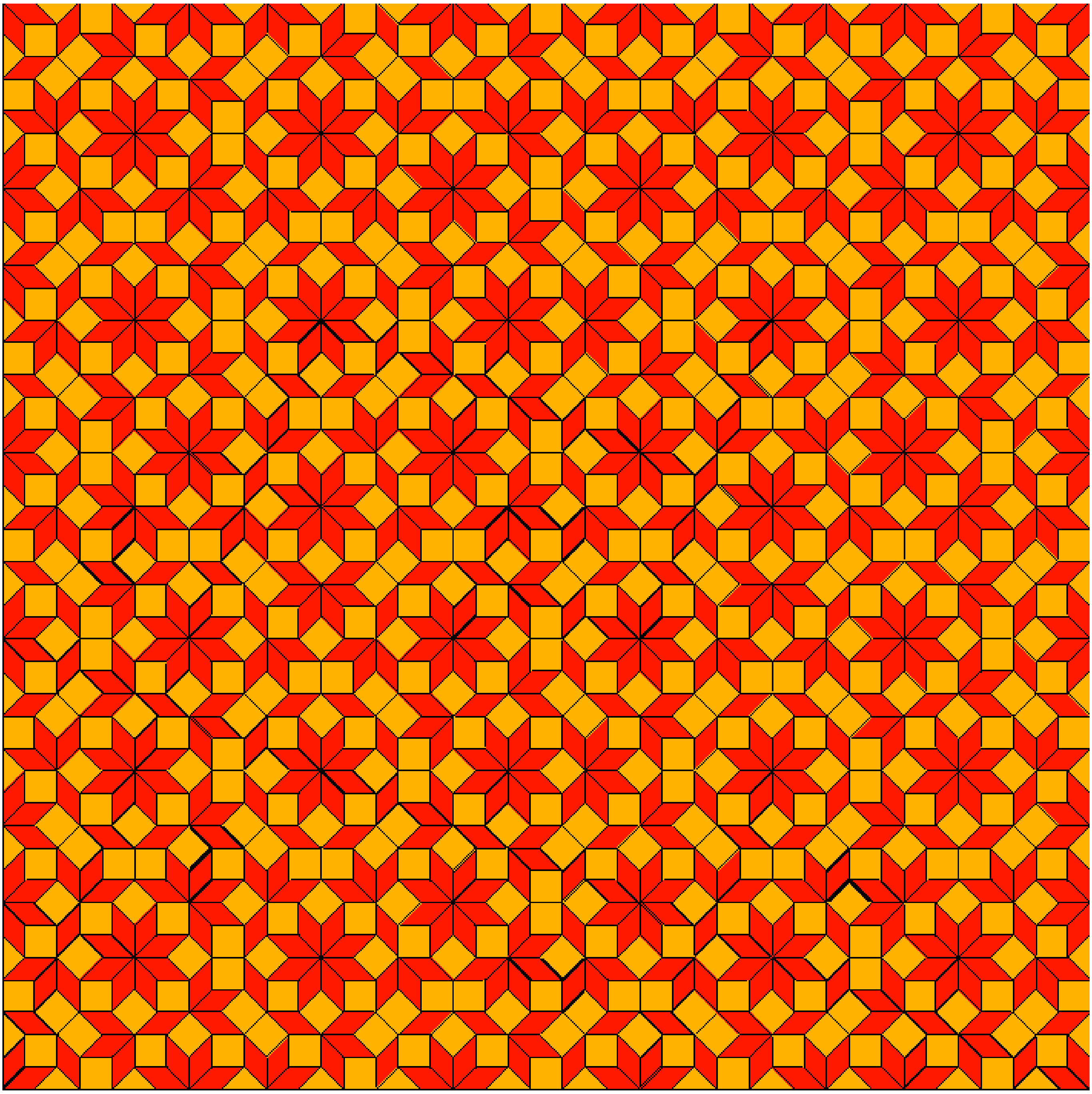
- Constructing examples

- Some basic topology which is helpful

- Some more advanced topology that is helpful



A Penrose tiling



An Octagonal or Ammann-Beenker tiling

Work in Euclidean space $\mathbb{R}^d$; $B(x, r)$ denotes the usual open ball at x, radius r.

A *tile* is a polyhedron in $\mathbb{R}^d$.

A *tiling* is a collection of tiles with disjoint interiors that cover $\mathbb{R}^d$. Usually we assume there are only finitely many tiles up to rigid motions (or translations).

A *patch* in a tiling is a finite subcollection of the tiles.

Tiles may be translated: $t + x$, $t \subset \mathbb{R}^d, x \in \mathbb{R}^d$.

A tiling T may be translated: $T + x = \{t + x \mid t \in T\}$.

T is periodic if $T + x = T$, for some $x \neq 0$.

Periodicity is pleasant.

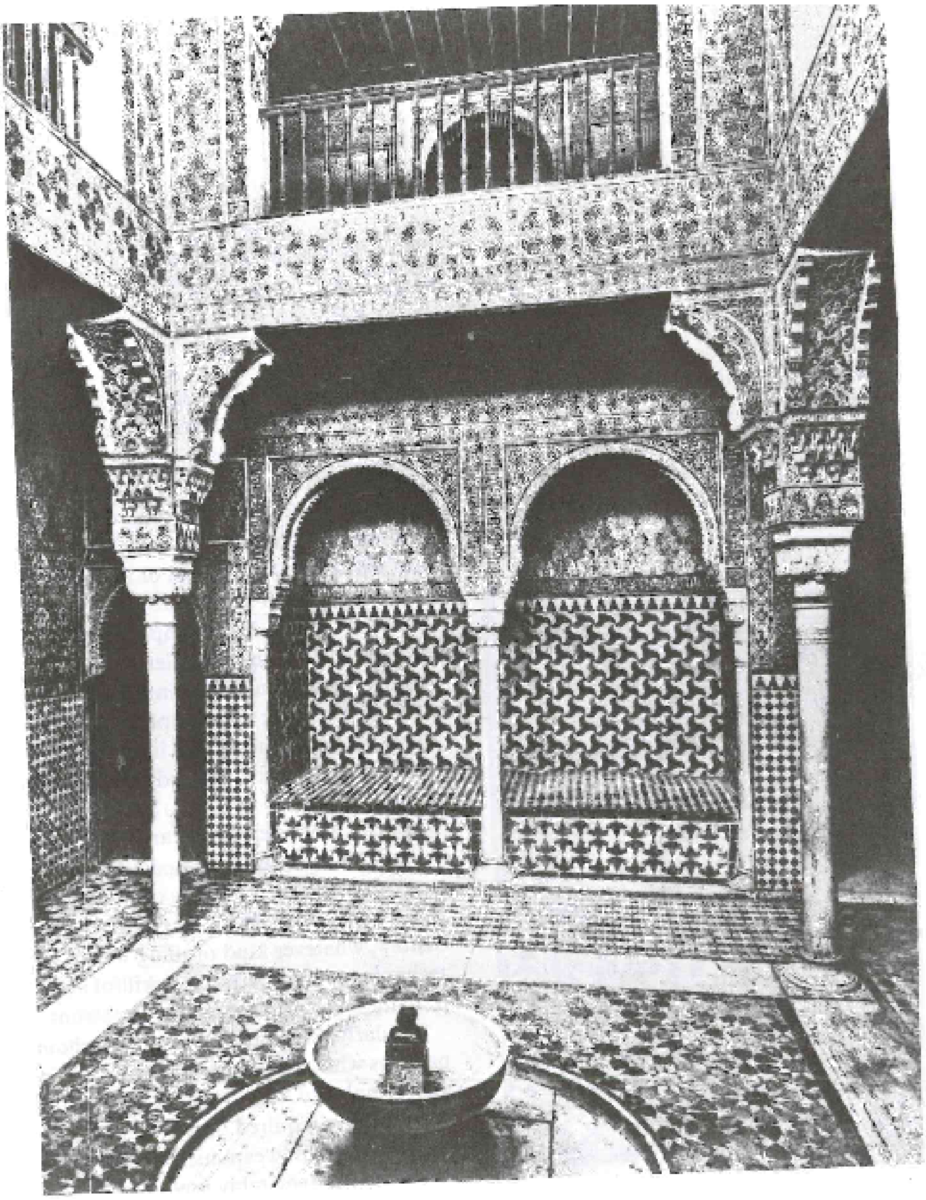


Figure 2
A view in the Alhambra showing the wealth of tilings used by its Moorish builders (Saladin [1926, plate 21]).

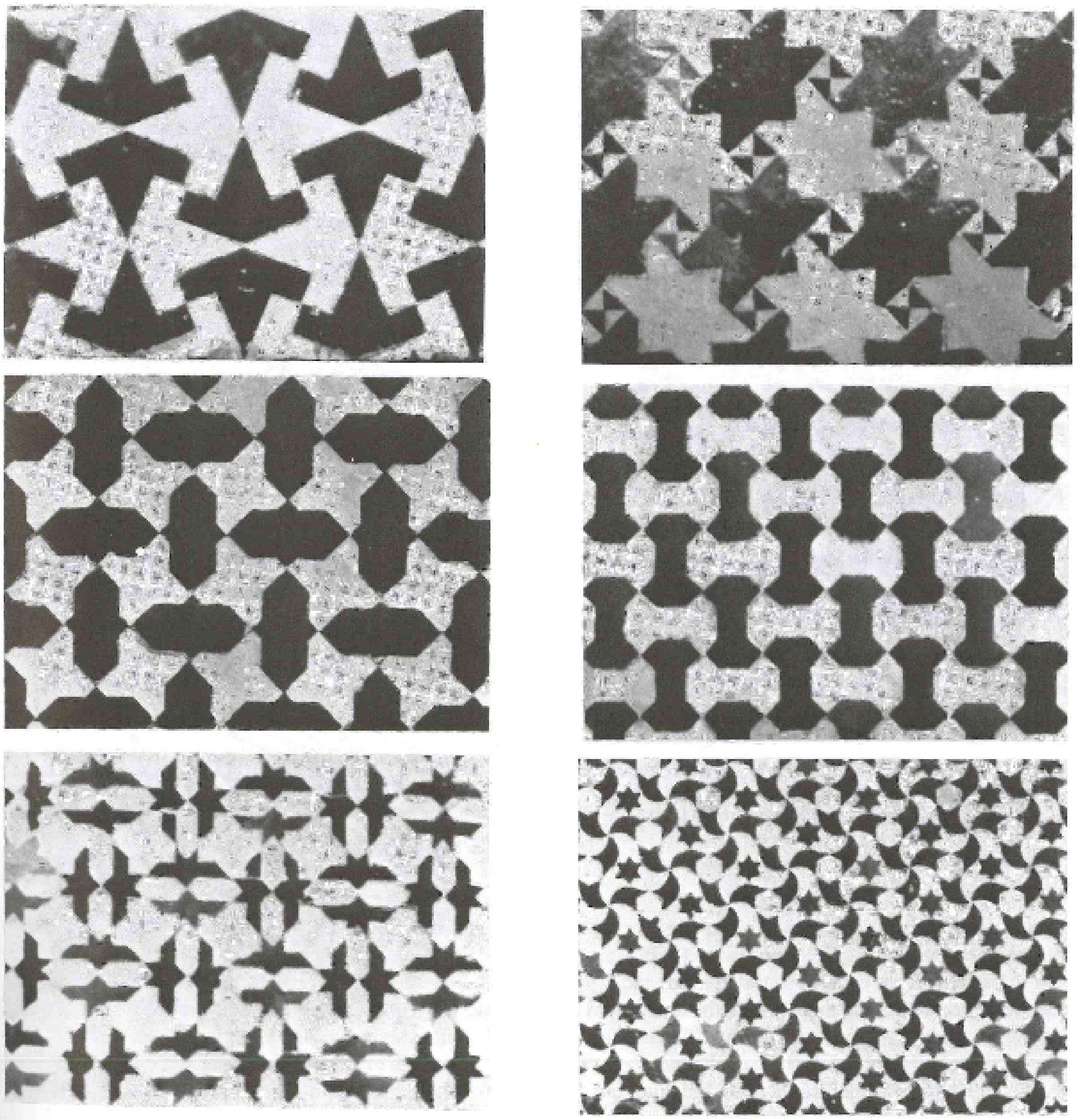
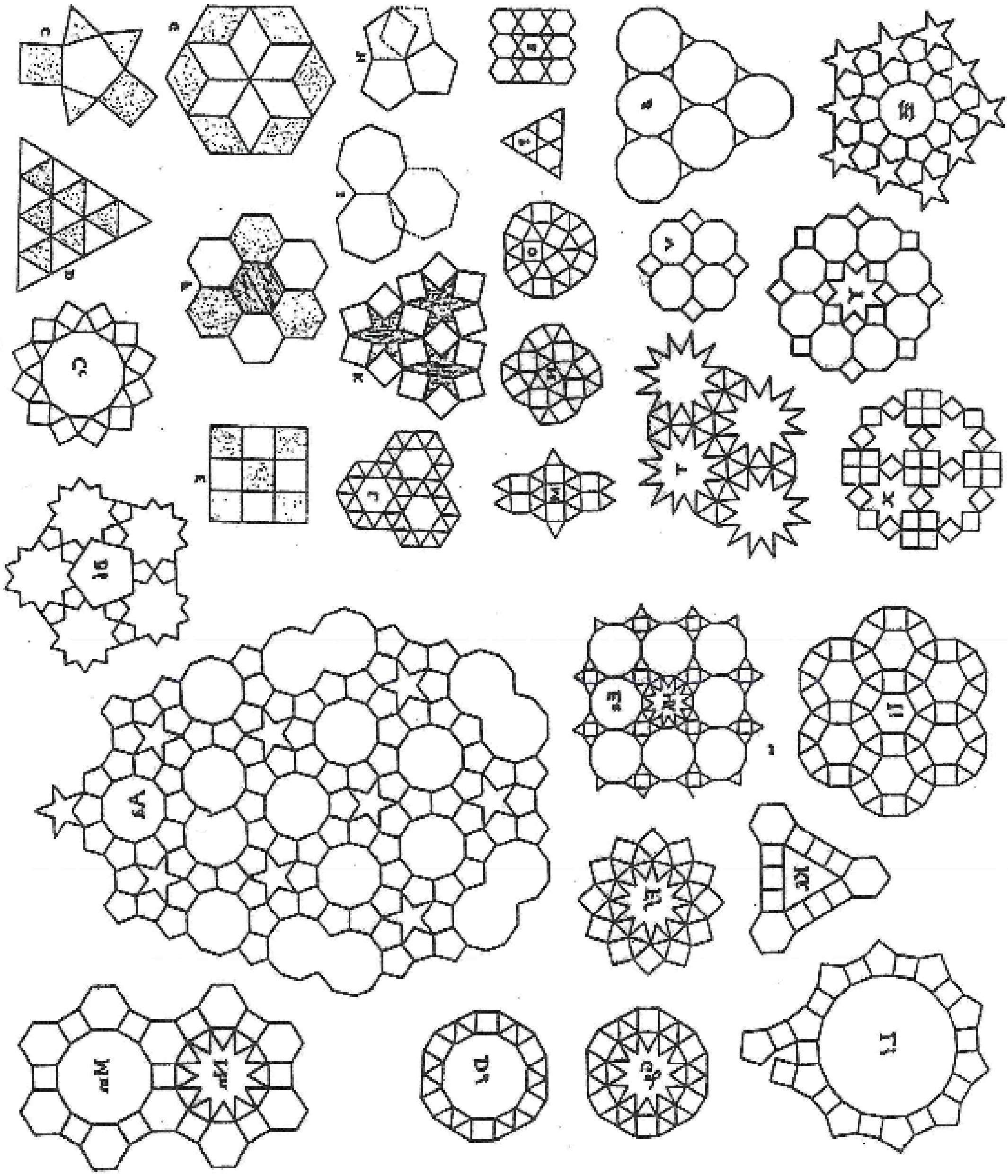


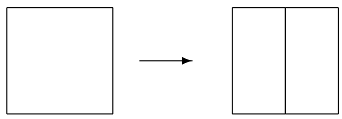
Figure 3
Details of several tilings from the Alhambra. Such tilings are widely known, in part due to sketches made in 1936 by the Dutch artist M. C. Escher (see Escher [1971, plates 83, 84]).



Aperiodicity is easy:

Consider a tiling of the plane by unit squares, matching edge to edge and vertex to vertex.

Replace one square by two rectangles:



Or more drastically, at each square, randomly choose whether to leave the square as it is or replace it with the pair of rectangles.

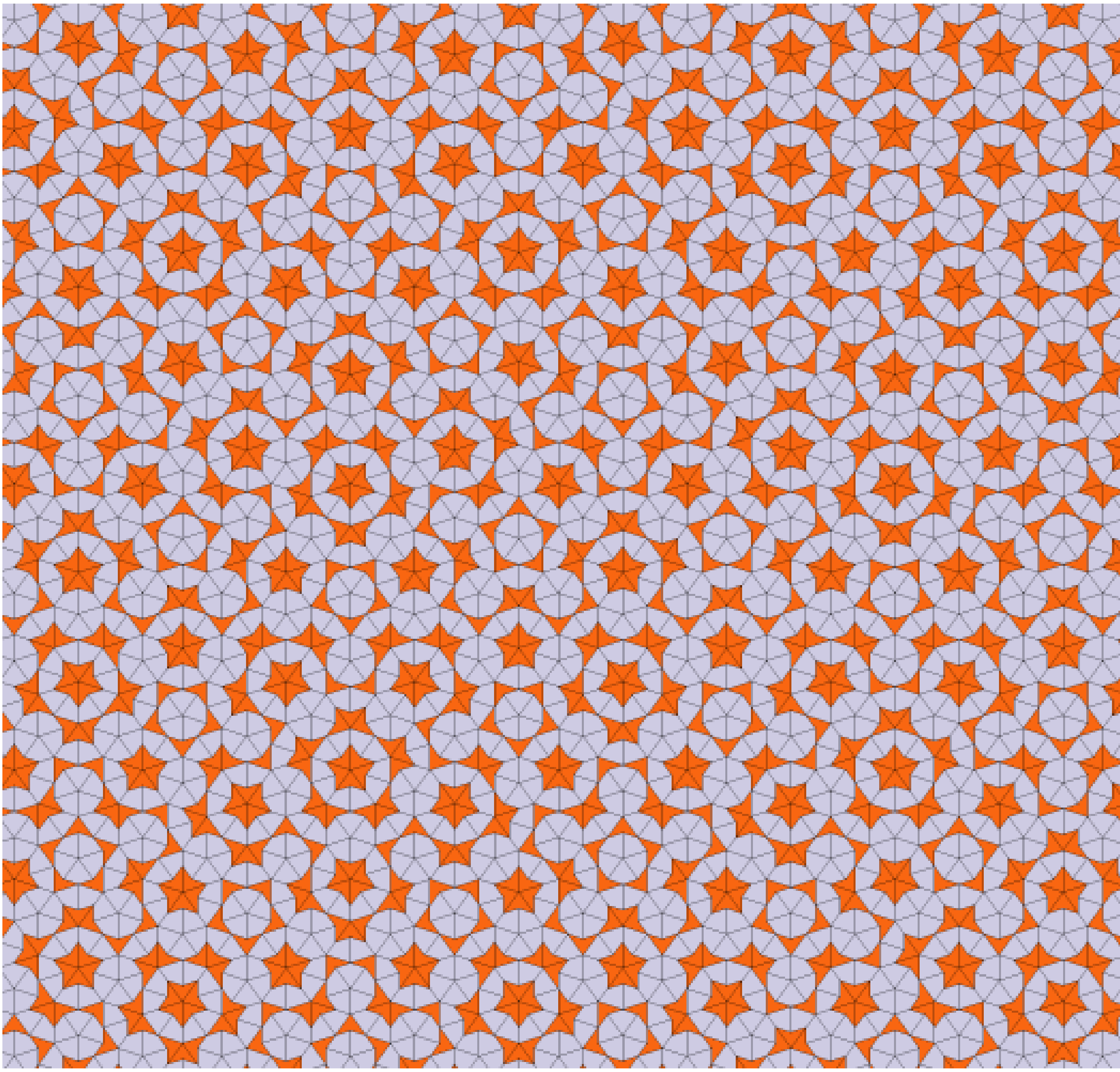
The latter destroys order.

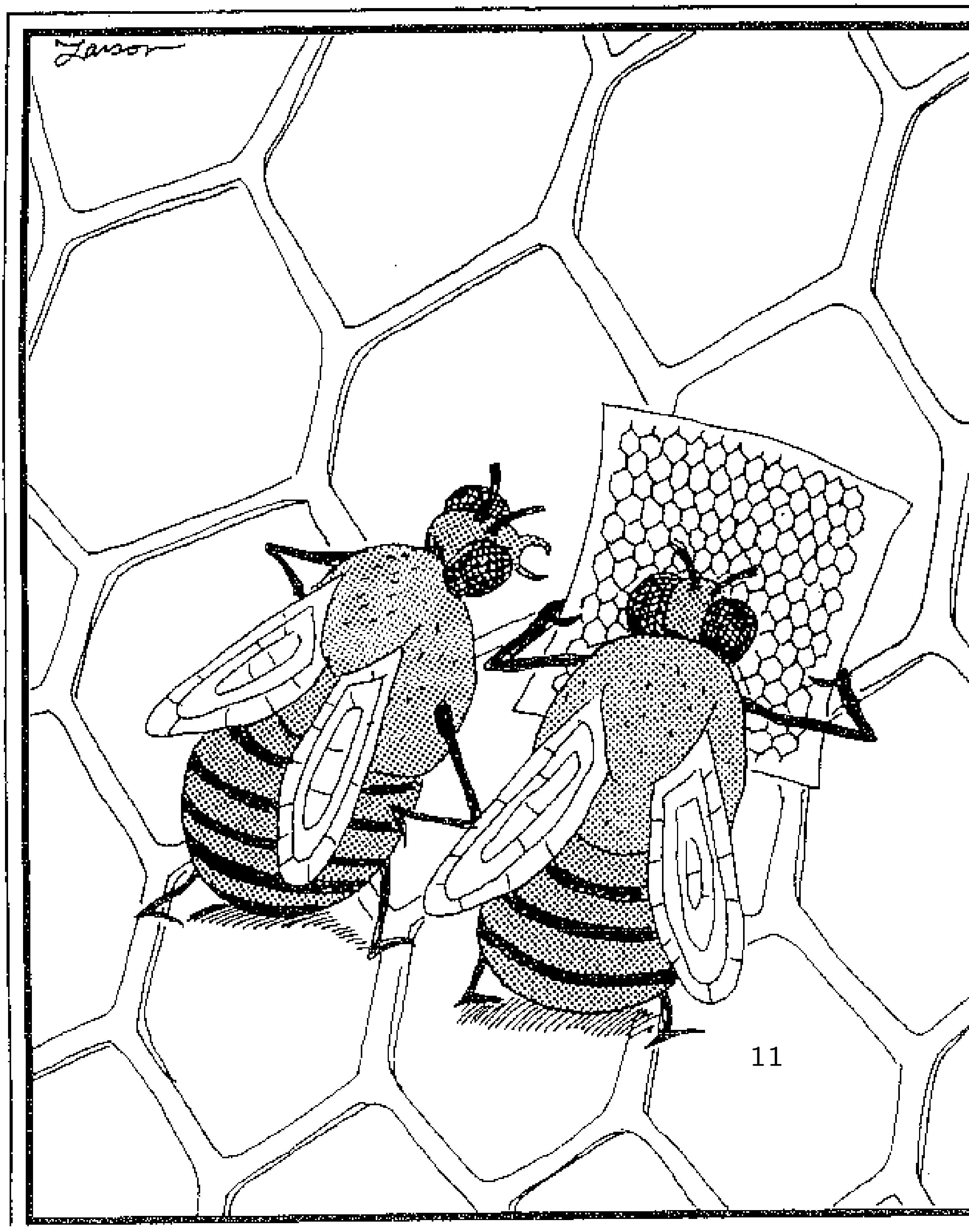
“Aesthetic delight lies somewhere between boredom and confusion” - E.H. Gombrich.

Aperiodic Order

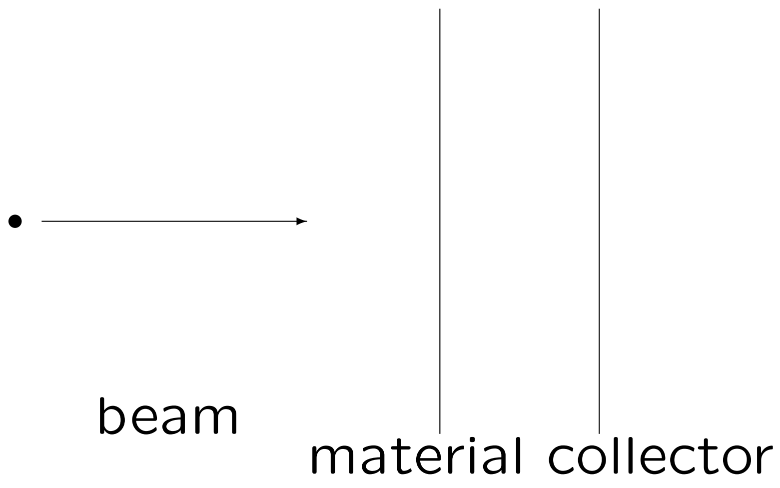
In the 1960's and 70's, various geometric patterns in Euclidean space were discovered which displayed a high degree of regularity, but not periodicity. The most famous are Roger Penrose' tilings. There are an uncountable number of such tilings (even after ignoring translations), but they all have the same highly regular local structure.

As a specific example, the Penrose tilings are *repetitive*: given any finite patch in any Penrose tiling, there is a constant R such that the same patch will appear in any ball of radius R in any other Penrose tiling,





"Face it, Fred—you're lost!"



In 1980, Shechtman et. al. discovered physical materials whose diffraction pattern showed:

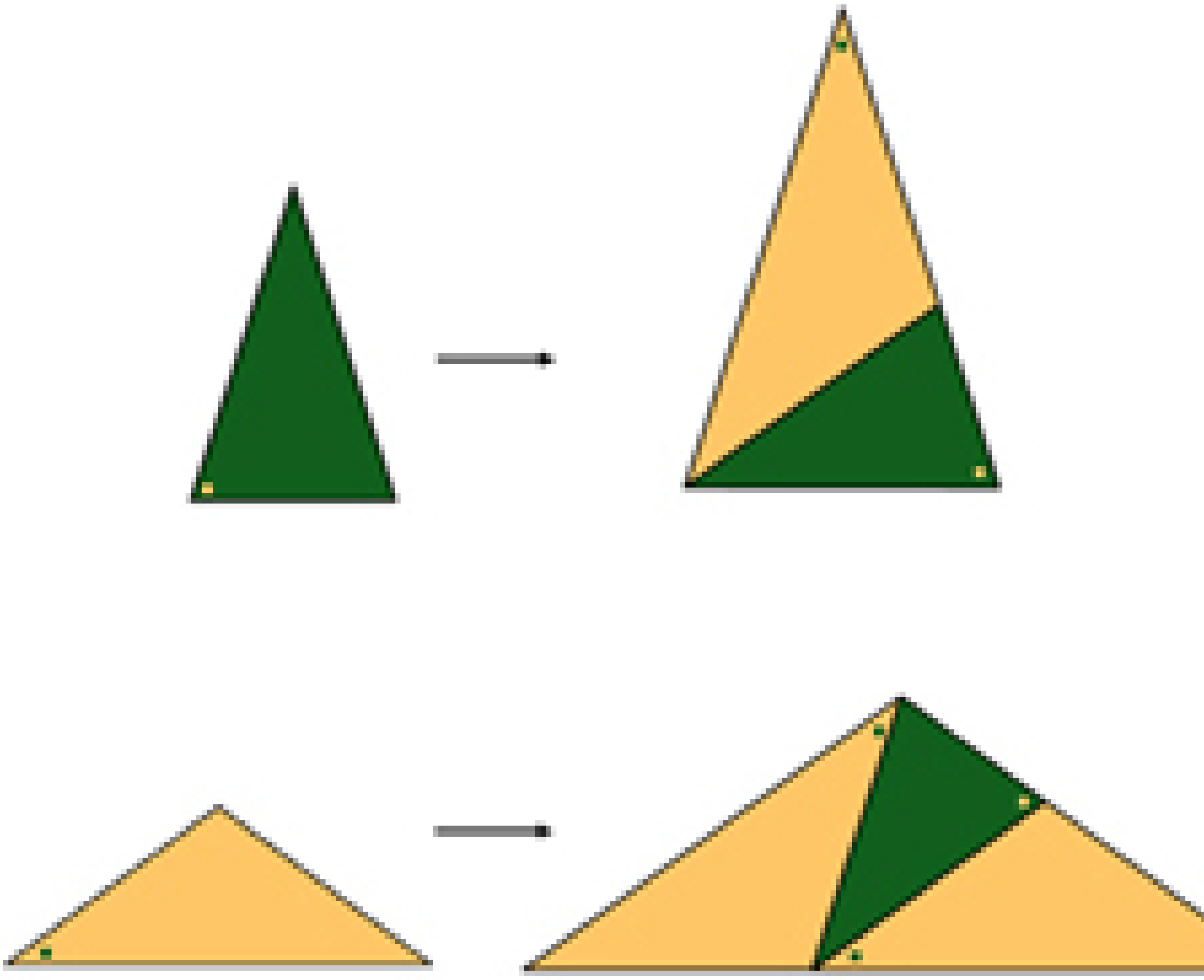
- pure-point diffraction: concentration at a countable point-set, indicating a highly ordered atomic arrangement

- five-fold rotational symmetry, impossible for periodic patterns.

There are by now several hundred of such materials, now called quasicrystals.

Constructions of aperiodic order: by substitution

The Penrose substitution:



The substitution rule defines, for each prototile p, $\omega(p)$ a patch with union λp, $\lambda > 1$. This can be extended:

to translates: $\omega(p + x) = \omega(p) + \lambda x$,

to patches: $\omega(\{t_1, \ldots, t_N\} = \cup_n \omega(t_i)$

These can be used to find growing collections of consistent patches whose union is a tiling.

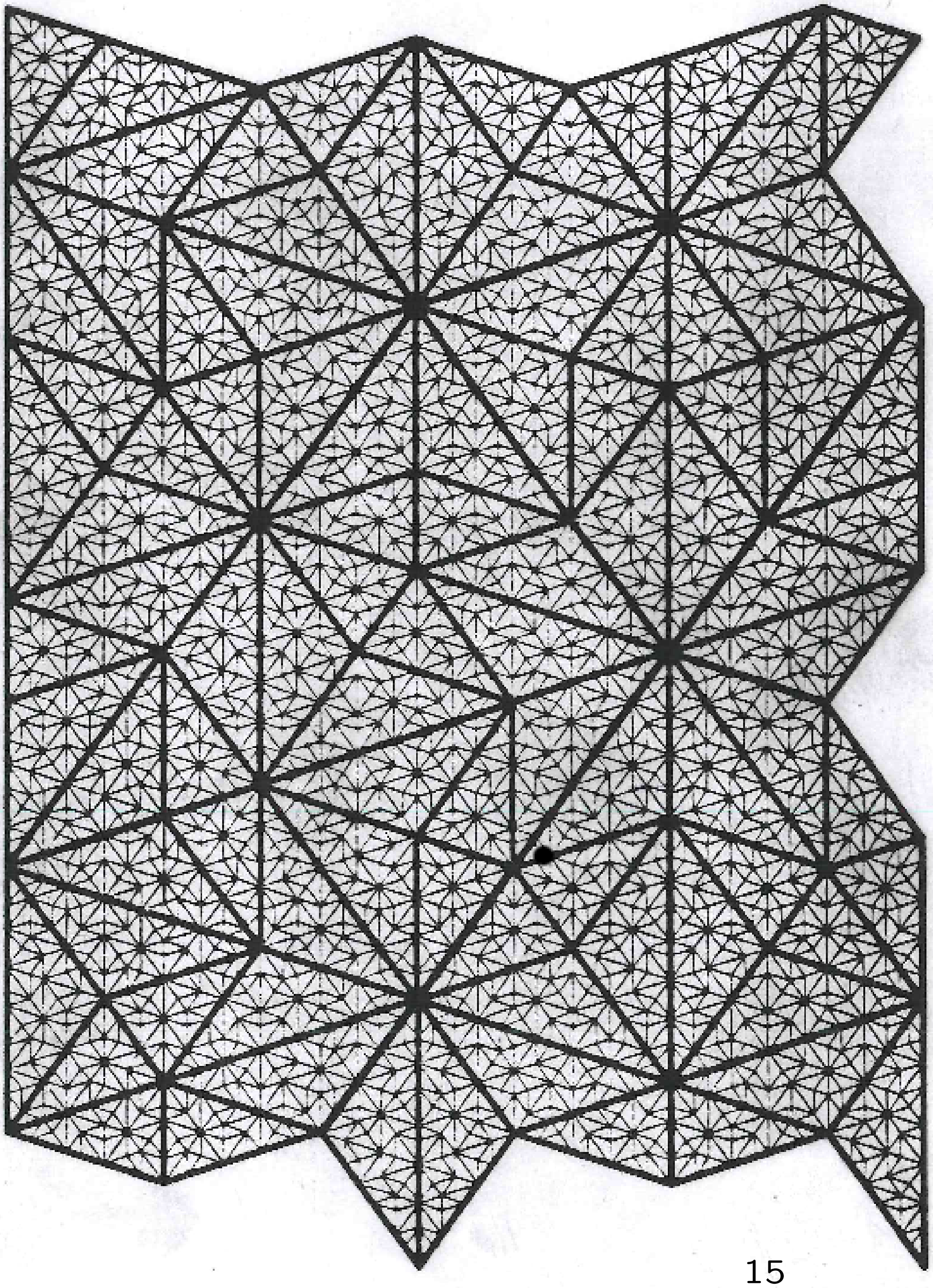
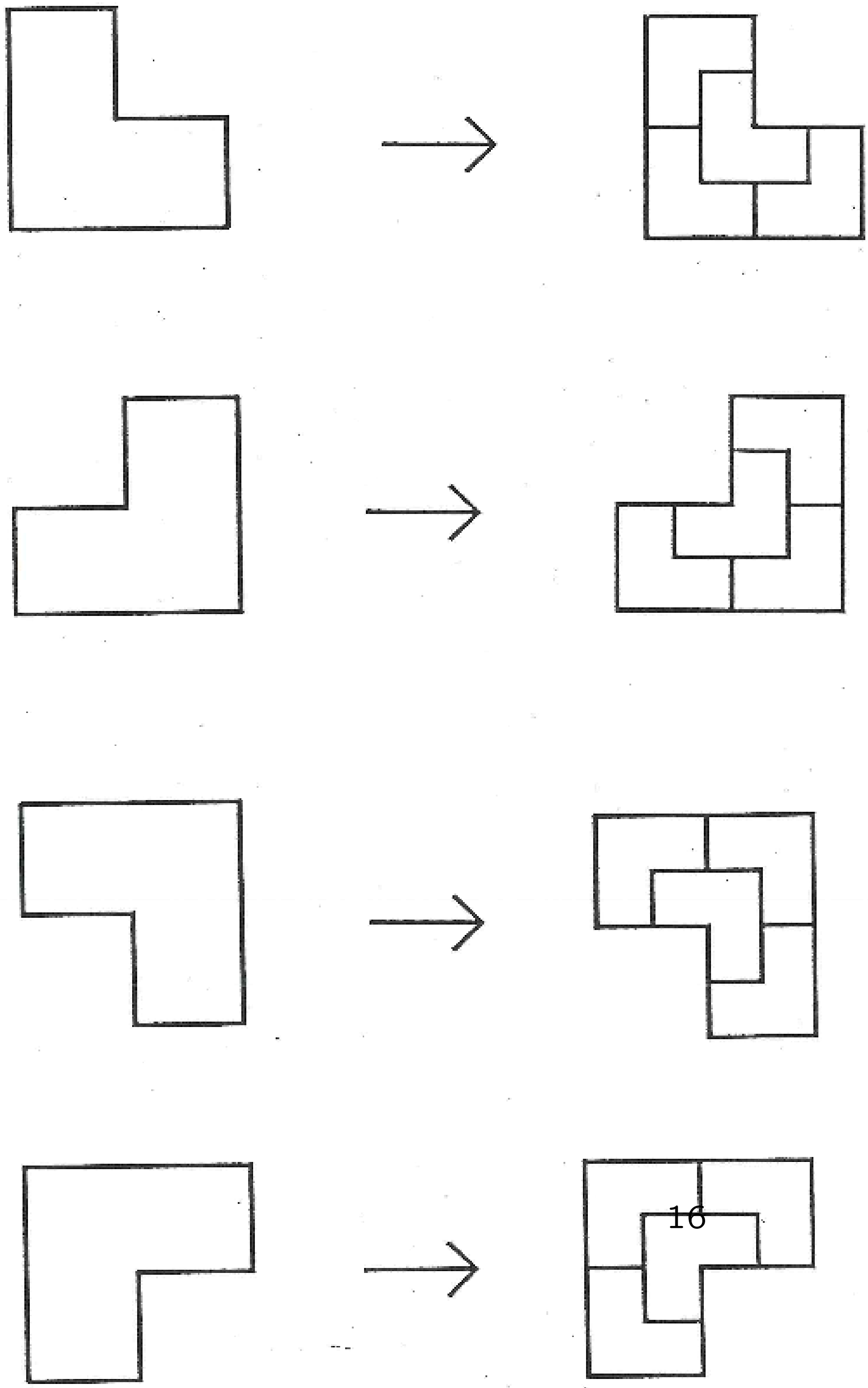


Figure 5: A Penrose Tiling



Pinwheel Conway, Radin, Sadu

$\lambda = \sqrt{5}$

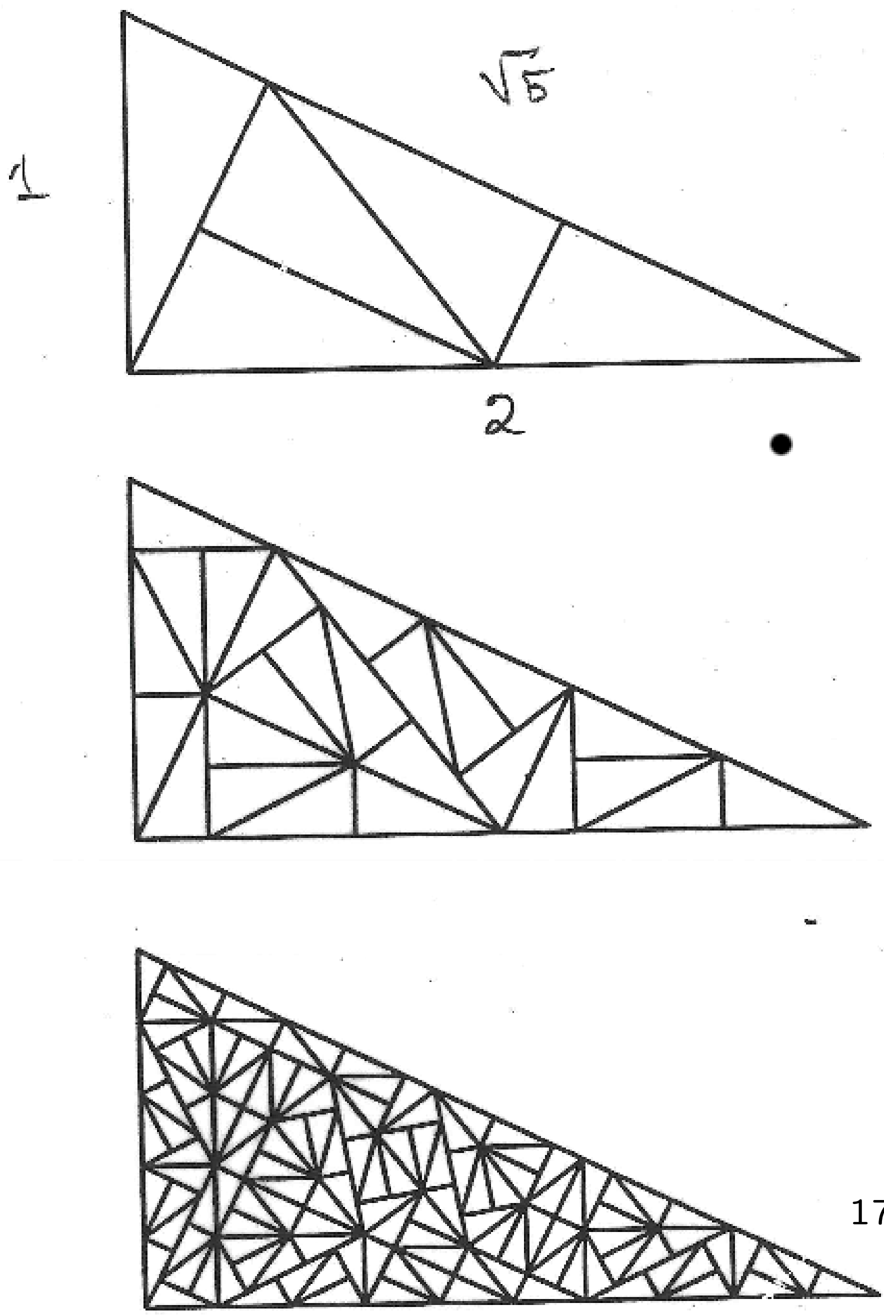


Fig. 7.10 Constructing the pinwheel tiling.

The projection method

Begin with the lattice $\mathbb{Z}^N \subset \mathbb{R}^N$. Let C be the unit cube in $\mathbb{R}^N$.

Select a d-dimensional subspace $E \subset \mathbb{R}^N$ satisfying $E \cap \mathbb{Z}^N = \{0\}$ and $E + \mathbb{Z}^N$ dense in R^N. Let π_E be the orthogonal projection of $\mathbb{R}^N$ onto E.

Define

$$\Lambda = \pi_E((E + C) \cap \mathbb{Z}^N).$$

This is an aperiodic discrete set in E and can be made into a tiling.

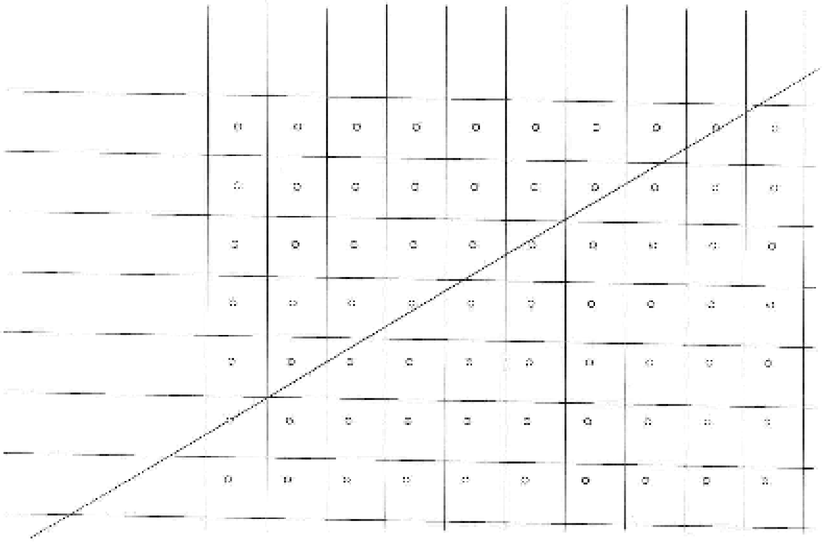


Fig. 4.2 The line is a totally irrational subspace of $\mathbb{Z}_2$; X is the set of lattice points whose Voronoï cells are cut by the line.

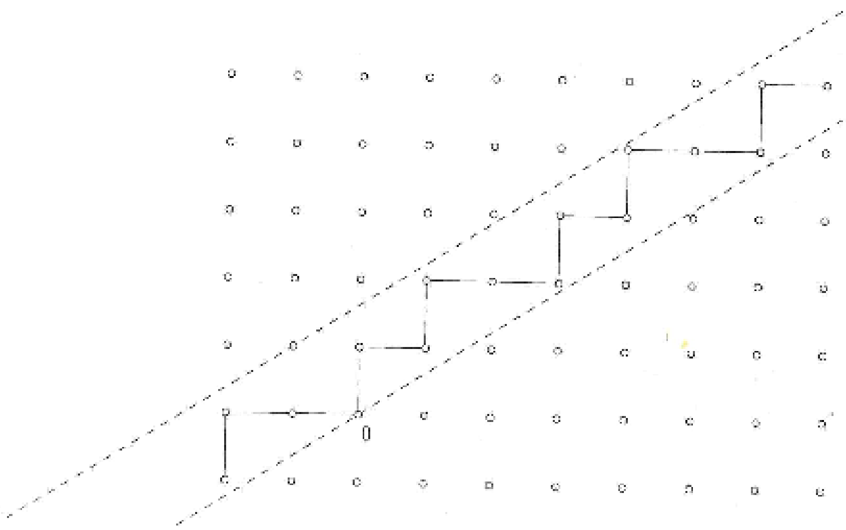


Fig. 4.3 The mth point of X is the vector $\vec{p}_m$ and the $(m+1)$st is $\vec{p}_{m+1} = \vec{p}_m + \vec{e}_j$, where $j = 1$ or $j = 2$.

Local rules or shifts of finite type

A a finite set (alphabet). Partition $A \times A$ into $\mathcal{A}$ (for allowed) and $\mathcal{F}$ (for forbidden).

$$X_{\mathcal{A}} = \{(a_n)_{n\in\mathbb{Z}} \mid a_n \in A, (a_n, a_{n+1}) \in \mathcal{A}, \text{all } n\}.$$

Imagine a graph with vertices A and an edge from a to b if (a, b) is in $\mathcal{A}$. Then $X_{\mathcal{A}}$ consists of infinite paths in the graph.

Problem: Is $X_{\mathcal{A}}$ empty?

1. It is easy to find an algorithm to see whether or not the graph has a cycle: $(a_1, a_2, \ldots, a_N, a_1)$ with (a_n, a_{n+1}) and (a_N, a_1) in $\mathcal{A}$.

2. If it has a cycle, $X_{\mathcal{A}}$ contains a periodic sequence

$$(\ldots, a_N, a_1, a_2, \ldots, a_N, a_1 \ldots)$$

3. If there is no cycle then $X_{\mathcal{A}}$ is empty. (Proof: if we have an element, then $(a_1, a_2, \ldots, a_N)$ contains a repeated entry if $N > \#A$.)

In dimension 2: Same A, divide the set of all 2×2 arrays from A into $\mathcal{A}$ and $\mathcal{F}$.

$$X_{\mathcal{A}} = \{(a_{m,n})_{(m,n)\in\mathbb{Z}^2} \mid \begin{matrix} a_{m,n+1} & a_{m+1,n+1} \\ a_{m,n} & a_{m+1,n} \end{matrix} \in \mathcal{A}\}$$

Same question: is $X_{\mathcal{A}}$ empty?

1. H. Wang: If there is an algorithm which determines an answer, it does so by finding a periodic array.

2. Berger: there exists an example in which $X_{\mathcal{A}}$ is non-empty, but contains only aperiodic elements.

The bad news: $\#A \approx 20,000$.

R. Robinson (1960's):

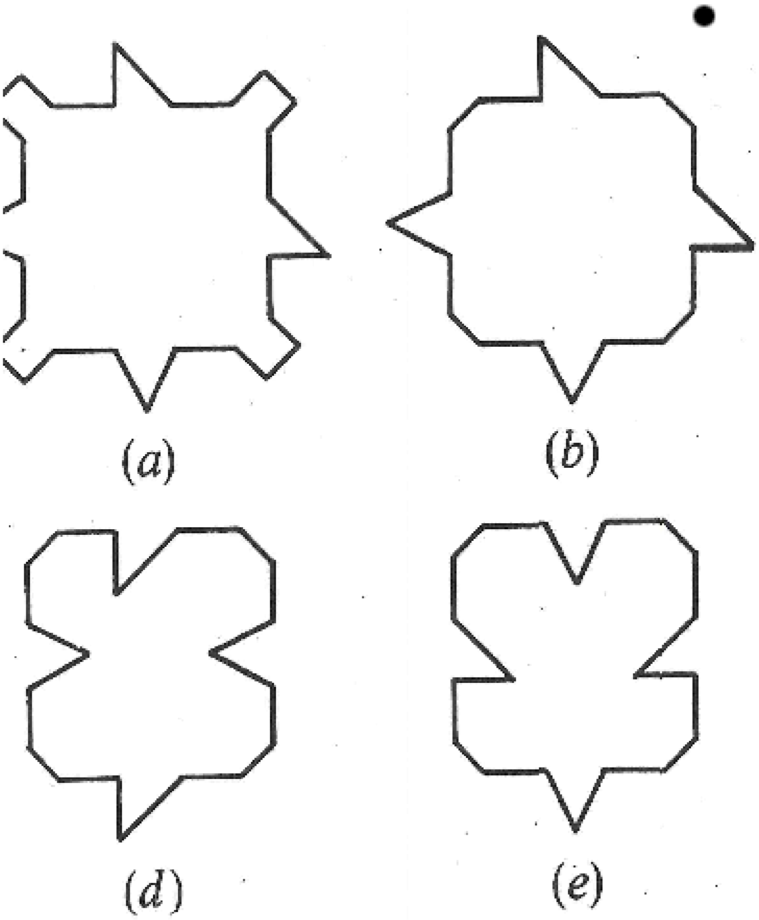


Figure 10.2.1
Robinson's aperiodic set R1 of six tiles.

Topology for tilings

Fix a finite set of prototiles which are polygons. We look at all tilings made of tiles which are translates of these, meeting vertex to vertex and edge to edge.

Define a metric: $d(T, T') < \epsilon$ if there exist y, y' in $B(0, \epsilon)$ with $(T - y) = (T' - y')$ on $B(0, \epsilon^{-1})$.

Notice $d(T-x, T-y)$ is small when the patches in T at x and y agree (up to small translation).

If we start with a single square, once you decide where to put a square covering the origin, the rest is determined uniquely. Putting the bottom edge on the origin is the same as the corresponding point on the bottom edge; i.e. a 2-torus.

Theorem 1. *With hypotheses as above, the set of all tilings with this metric is compact. Moreover, for each* x *in* $\mathbb{R}^2$*, the map* $T \to T - x$ *is a homeomorphism; this is an action of* $\mathbb{R}^2$ *on this space.*

Usually we consider a *hull*, Ω, which is a subset described by a substitution rule or as the closure of $\{T_0 - x \mid x \in \mathbb{R}^2\}$ for some fixed tiling T_0.

Some facts about Ω in the aperiodic case.

- For fixed T in Ω, the set

$$\Xi_T = \{T' \mid T = T' \text{ on } B(0,1)\}$$

  is totally disconnected with no isolated points and is homeomorphic to $\{0,1\}^{\mathbb{N}}$.

- the map sending (T', x) in $\Xi_T \times B(0,r)$ to $T' + x$ is a homeomorphism to a neighbourhood of T (matchbox manifold).

- Ω is connected, but not path connected.

- The path connected component of T is its $\mathbb{R}^2$-orbit: $\{T - x \mid x \in \mathbb{R}^2\}$.

- If T_0 is repetitive, then every $\mathbb{R}^2$-orbit is dense.

Cohomology of the hull

Topologists have a variety of algebraic tools for the study of topological spaces: homotopy groups, homology groups, cohomology groups (Cech and de Rham), etc.

The tiling space Ω can be analyized this way. There are effective techniques for computing the Cech cohomology $H^*(\Omega)$ for substitution tilings (Anderson -P) and for projection method tilings (Forrest-Hunton-Kellendonk). F. Gähler even has software that will be the computations.

E.g. Ω the space of all Penrose tilings:

$$H^0(\Omega) = \mathbb{Z}, H^1(\Omega) = \mathbb{Z}^5, H^2(\Omega) = \mathbb{Z}^8.$$

Why compute $H^*(\Omega)$? Short answer: $H^*(\Omega)$ is (alleged to be) a quantitative measure of aperiodicity.

Homology vs. cohomology and the periodic case

Suppose that T periodic tiling of $\mathbb{R}^d$. Let

$$Per(T) = \{x \in \mathbb{R}^d \mid T + x = T\}.$$

Ω is all translations of T and is $\mathbb{R}^d/Per(T)$.

The homology group $H_1(\Omega)$ consists of loops in Ω. How do you find a loop of tilings? Suppose x is in $Per(T)$. Then

$$T^x(t) = T + tx, 0 \leq t \leq 1,$$

is a loop of tilings since $T^x(0) = T^x(1)$. In fact,

$$x \in Per(T) \to T^x \in H_1(\Omega)$$

is an isomorphism.

What happens if T is aperiodic? Remember the path components look like $\mathbb{R}^d$, which is contractible, but $H^*(\Omega)$ is still interesting.

Shouldn't invariants be geometric?

For the Penrose tilings, $H^1(\Omega) \cong \mathbb{Z}^5$; doesn't look like a quantitative measure of aperiodicity.

Let T be a tiling of $\mathbb{R}^d$. A function $f : \mathbb{R}^d \to A$ is T-equivariant if, there is a constant $R > 0$ such that, the value of f at x depends only on the pattern of T in an R-ball around x.

That is, for any x, y in $\mathbb{R}^d$,

$$\begin{array}{rrcl} & (T-x) \cap B(0,R) & = & (T-y) \cap B(0,R) \\ \Rightarrow & f(x) & = & f(y). \end{array}$$

Let C_T^k denote the set of all smooth differential forms of degree k on $\mathbb{R}^N$ which are T-equivariant.

$$\begin{array}{lll} C_T^0(\mathbb{R}^2) & = \{ & f(x,y), & T-\text{equiv}\} \\ C_T^1(\mathbb{R}^2) & = \{ & P(x,y)dx + Q(x,y)dy, & T-\text{equiv}\} \\ C_T^2(\mathbb{R}^2) & = \{ & g(x,y)dxdy, & T-\text{equiv}\} \end{array}$$

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ d(Pdx + Qdy) &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \end{aligned}$$

Define $H_T^k(\mathbb{R}^d) = \ker(d)/Im(d) =$ closed / exact

Theorem 2 (Kellendonk -P.)**.**

$$H_T^*(\mathbb{R}^d) \cong H^*(\Omega, \mathbb{R}).$$

If $\omega = Pdx + Qdy$ is in C^1_T, we can take

$$\tau(\omega) = \lim_R vol(R)^{-1} \int_{|x| \le R} (P, Q) dx \in \mathbb{R}^2$$

We get, in particular,

$$H^1(\Omega) \to H^1(\Omega, \mathbb{R}) \cong H^1_T \xrightarrow{\tau} \mathbb{R}^2.$$

In the Penrose case, the image is generated by the fifth-roots of 1. (This subgroup of $\mathbb{R}^2$ is rank 4, so the map has $\mathbb{Z}$ as a kernel.)

If T is completely periodic, then the image of $H^1(\Omega)$ is $\{x \in \mathbb{R}^d \mid < x, Per(T) > \subset \mathbb{Z}\}$, the dual lattice.

Periodic $\Rightarrow$ lattice. Aperiodic $\Rightarrow$ dense in $\mathbb{R}^2$?

References:

- *Quasicrystals and Geometry*, Marjorie Senechal, Cambridge U. Press.

- *The topology of tiling spaces*, Lorenzo Sadun, AMS

- *The Tilings Encyclopedia*, U. of Bielefeld.