C*-algebras and topological dynamics: From dynamical systems to operator algebras

Ian F. Putnam,
University of Victoria

C^*-algebras

A C^*-algebra is a set A:

(1) an algebra over $\mathbb{C}$,
(2) an involution $a \to a^*$ which is conjugate linear, $(ab)^* = b^*a^*$,
(3) a norm, $\| \cdot \|$, in which it is a Banach algebra (i.e. is complete),
(4) $\| a^*a \| = \| a \|^2$, for all a in A.

Often, we will assume the algebra is unital.

Examples:

(1) $\mathbb{C}$, the complex numbers,
(2) $M_n(\mathbb{C})$, the $n \times n$ matrices over $\mathbb{C}$,
(3) $\mathcal{B}(\mathcal{H})$, the bounded linear operators on a Hilbert space $\mathcal{H}$.

Commutative C^*-algebras

Let X be a compact Hausdorff space. Then

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous }\},$$

with pointwise algebraic operations and supremum norm is a commutative, unital C^*-algebra. More generally, if X is locally compact, then $C_0(X)$, the continuous functions vanishing at infinity, is a commutative C^*-algebra.

Theorem 1 (Gelfand-Naimark). *1. Every commutative C^*-algebra is isomorphic to $C_0(X)$, for some locally compact Hausdorff space X.*

2. $C_0(X)$ and $C_0(Y)$ are isomorphic if and only if X and Y are homeomorphic.

Gelfand-Naimark dictionary

X	$C_0(X)$
compact	unital
closed subset $Y \subset X$	closed ideal $I = \{f \mid f\|Y = 0\}$
point x	maximal ideal $I = \{f \mid f(x) = 0\}$
measure μ	linear functional $\phi(f) = \int f d\mu$
topological dimension	stable rank
K-theory	K-theory

The notions on the right may be generalized to all C^*-algebras. For ideals, one or two-sided? Generally, there are too many of the former and perhaps none of the latter.

K-theory

For a C^*-algebra, A (assume unital), there is an abelian group, $K_0(A)$. It is based on projections, elements p such that

$$p^2 = p = p^*.$$

One considers projections in $M_n(A), n \geq 1$, the $n \times n$ matrices over A, with a notion of equivalence, $\sim$. If there is an invertible v such that $vpv^{-1} = q$, then $p \sim q$.

Equivalence classes form a semigroup: $p \in M_n(A), q \in M_m(A)$,

$$p \oplus q = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \in M_{m+n}(A)$$

with identity 0. The set of formal differences

$$K_0(A) = \{[p] - [q] \mid p, q \text{ projections } \},$$

is a group, with a positive cone,

$$K_0(A)^+ = \{[p] - [0] \mid p \text{ a projection } \}$$

and a distinguished positive element [1].

A *trace* $\tau : A \to \mathbb{C}$ is a linear functional which is
1. positive: $\tau(a^*a) \geq 0$, $a \in A$,
2. trace property: $\tau(ab) = \tau(ba)$ (similarity invariant)

It extends to $M_n(A)$ with the same properties ($\tau((a_{i\,j})) = \sum_i \tau(a_{i\,i})$) and induces a positive group homomorphism

$$\widehat{\tau} : K_0(A) \to \mathbb{R}, \quad \widehat{\tau}(K_0(A)^+) \subset [0, \infty).$$

Example: $\mathbb{C}$

Lemma 2. *Two projections p and q in $M_n(\mathbb{C})$ are similar if and only if $Trace(p) = rank(p) = rank(q) = Trace(q)$.*

Proposition 3. *The map $Tr : K_0(\mathbb{C}) \to \mathbb{Z}$*

$$Tr([p] - [q]) = Trace(p) - Trace(q)$$

is an isomorphism. Under this, $K_0(\mathbb{C})^+ = \{0, 1, 2, 3, \ldots\} = \mathbb{Z}^+$.

Construction of non-commutative C^*-algebras from dynamics

Let X be a compact Hausdorff space. Consider homeomorphisms, ϕ, whose domain and range are both open subsets of X. Suppose that $\mathcal{F}$ is collection of such functions such that:
1. if ϕ, ψ are in $\mathcal{F}$, so is $\phi \cap \psi$,
2. if ϕ, ψ are in $\mathcal{F}$, so is $\phi \circ \psi$,
3. if ϕ is in $\mathcal{F}$, so is ϕ^{-1},
4. the collection of open sets U in X such that id_U is in $\mathcal{F}$ generates the topology of X.

It follows that

$$R = \cup\mathcal{F} = \{(x, \phi(x)) \mid \phi \in \mathcal{F}, x \in Dom(\phi)\}$$

is an equivalence relation and $\mathcal{F}$ is a basis for a topology of R. We assume that this topology is second countable and Hausdorff. As a consequence the equivalence classes are countable.

$C_c(R)$ is the linear space of continuous, compactly supported functions on R. Endow it with a product and involution:

$$\begin{aligned} (f \cdot g)(x,y) &= \sum_{z \in [x]_R} f(x,z) g(z,y) \\ f^*(x,y) &= \overline{f(y,x)}, (x,y) \in R. \end{aligned}$$

For x in X and f in $C_c(R)$ consider the operator $\pi_x(f)$ on $l^2[x]_R$:

$$\pi_x(f)\xi(y) = \sum_{z \in [x]_R} f(y,z)\xi(z).$$

$\parallel f \parallel = sup_x \parallel \pi_x(f) \parallel$ is a norm.

The completion of $C_c(R)$ is $C^*(R)$.

Example: $X = \{1, 2, \ldots, N\}$, $R = X \times X$, $C^*(R) \cong M_N(\mathbb{C})$.

Example: X arbitrary, $R =$ equality, $C^*(R) \cong C(X)$.

There is an inclusion $\Delta : C(X) \to C^*(R)$ by $\Delta(f)(x,y) = f(x)$, if $x = y$ and 0 otherwise.

Theorem 4. *1. If $Y \subset X$ is closed and $\mathcal{F}$-invariant, then*

$$\{f \in C_c(R) \mid f|Y \times Y = 0\}^-$$

is a closed two-sided ideal in $C^*(R)$.
2. If μ is a $\mathcal{F}$-invariant measure, then

$$\tau(f) = \int_X f(x,x)d\mu(x)$$

is a trace.
The correspondences above are bijective.

Theorem 5. *If the quotient space X/R is Hausdorff then $C^*(R)$ is Morita equivalent to $C(X/R)$.*

Quote 6 (Connes)**.** *Morita equivalence is more natural than isomorphism.*

Group actions

Let X be a compact, Hausdorff space, G a countable discrete group, ϕ an free action of G on X by homeomorphisms:

$$
\begin{aligned}
\phi^s : X &\rightarrow X \quad s \in G, && (1)\\
\phi^e = &\quad id_X && (2)\\
\phi^{st} = &\ \phi^s \circ \phi^t \quad s,t \in G. && (3)\\
\phi^s(x) = x &\quad \Rightarrow \quad s = e. && (4)
\end{aligned}
$$

Let

$$\mathcal{F} = \{\phi^s|U \mid s \in G, U \subset X \text{ open }\}.$$

and

$$R_\phi = \{(x, \phi^s(x)) \mid x \in X, s \in G\}.$$

Equivalence classes are the orbits.

$C^*(R_\phi)$ is generated by:

$$\Delta(C(X)), u_s, s \in G$$

such that

$$\begin{aligned} u_{st} &= u_s u_t, & s,t \in G \\ u_s^* u_s &= u_s u_s^* = 1, & s \in G \\ u_s \Delta(f) u_s^* &= \Delta(f \circ \varphi^s), & f \in C(X). \end{aligned}$$

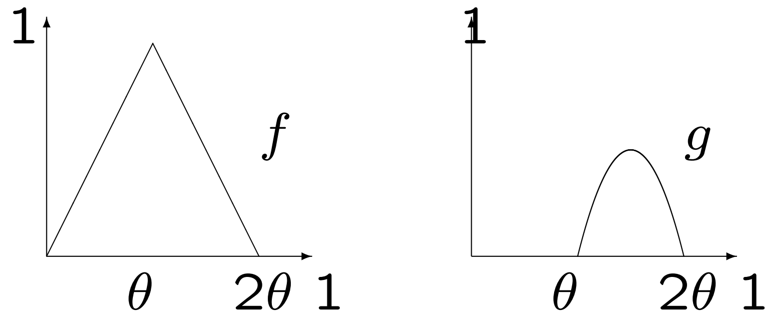
That is, the automorphisms $f \to f \circ \varphi^s$ of $C(X)$ are *inner*.

(Henceforth, we drop Δ.)

Example 1: Irrational rotation on the circle

$X = S^1$, $G = \mathbb{Z}$, $\phi^n(z) = e^{2\pi i n\theta} z$, where θ is irrational. (S^1/R_ϕ disaster!!)

$A_\theta = C(S^1) \times_\phi \mathbb{Z}$ is simple (no closed two-sided ideals) and has unique trace τ. Rieffel:



$$\begin{aligned} p &= u_1^* g + f + g u_1, \\ p &= p^2 = p^*, \\ \tau(p) &= \int f = \theta. \end{aligned}$$

Theorem 7 (Powers-Rieffel-Pimsner-Voiculescu)**.**

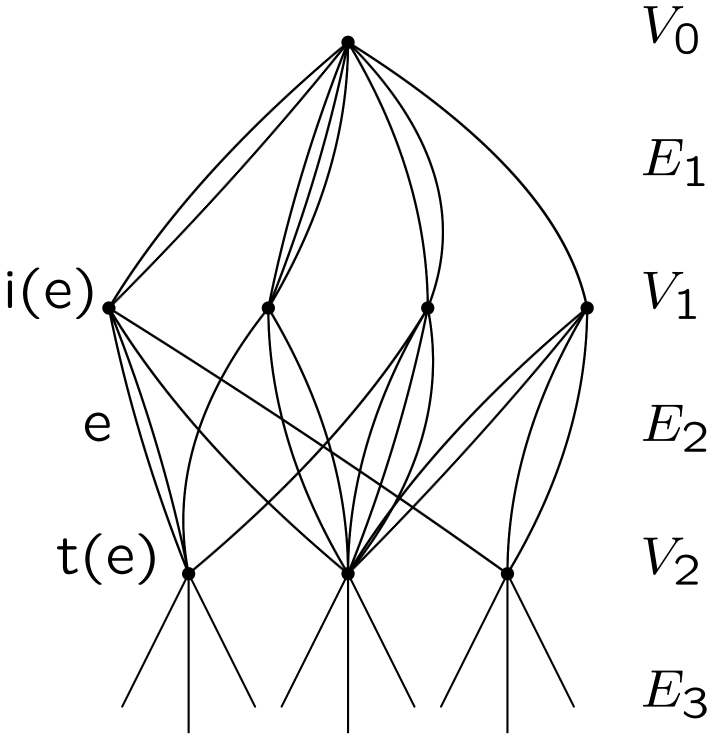
$$\hat{\tau} : K_0(A_\theta) \xrightarrow{\cong} \mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R}.$$

Theorem 8. *If $\phi : S^1 \to S^1$ has no periodic points, then there is an irrational θ such that*

$$K_0(C(S^1) \times_\phi \mathbb{Z}) \cong \mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R}.$$

AF-relations

A *Bratteli diagram* is a vertex set $V = V_0 \cup V_1 \cup \ldots$ and an edge set $E = E_1 \cup E_2 \cup \ldots$ with initial and terminal maps $i : E_n \to V_{n-1}, t : E_n \to V_n$. Each V_n and E_n are finite with $V_0 = \{v_0\}$.



Let X be the set of infinite paths from v_0:

$$X = \{(x_1, x_2, \ldots) \mid x_n \in E_n, t(x_n) = i(x_{n+1})\}$$

Relative topology from $X \subset \Pi_n E_n$.
If $p = (p_1, p_2, \ldots, p_N)$ is a finite path, we let

$$C(p) = \{x \in X \mid x_n = p_n, 1 \leq n \leq N\},$$

which is clopen.

For paths p, q of length N, with $t(p_N) = t(q_N)$, define $\phi : C(p) \to C(q)$ by

$$\begin{aligned} & \phi(p_1, p_2, \ldots, p_N, x_{N+1}, x_{N+2}, \ldots) \\ = \quad & (q_1, q_2, \ldots, q_N, x_{N+1}, x_{N+2}, \ldots). \end{aligned}$$

The set of all such ϕ is $\mathcal{F}$.

R is tail equivalence:

$$(x, y) \in R \Leftrightarrow \exists N, x_n = y_n, n \geq N.$$

Lemma 9. *Let* $e^N_{p,q}$ *be its characteristic function of* $\phi : C(p) \to C(q)$.

$$\begin{aligned} e^N_{p,q} e^N_{p',q'} &= e^N_{p,q'}, \text{ if } q = p', \\ e^N_{p,q} e^N_{p',q'} &= 0, \text{ if } q \neq p' \\ (e^N_{p,q})^* &= e^N_{q,p}. \end{aligned}$$

and, for fixed N,

$$A_N = span\{e^N_{p,q}\} \cong \oplus_{v \in V_N} M_{k(v)}(\mathbb{C}),$$

where $k(v)$ *is the number of paths to* v.

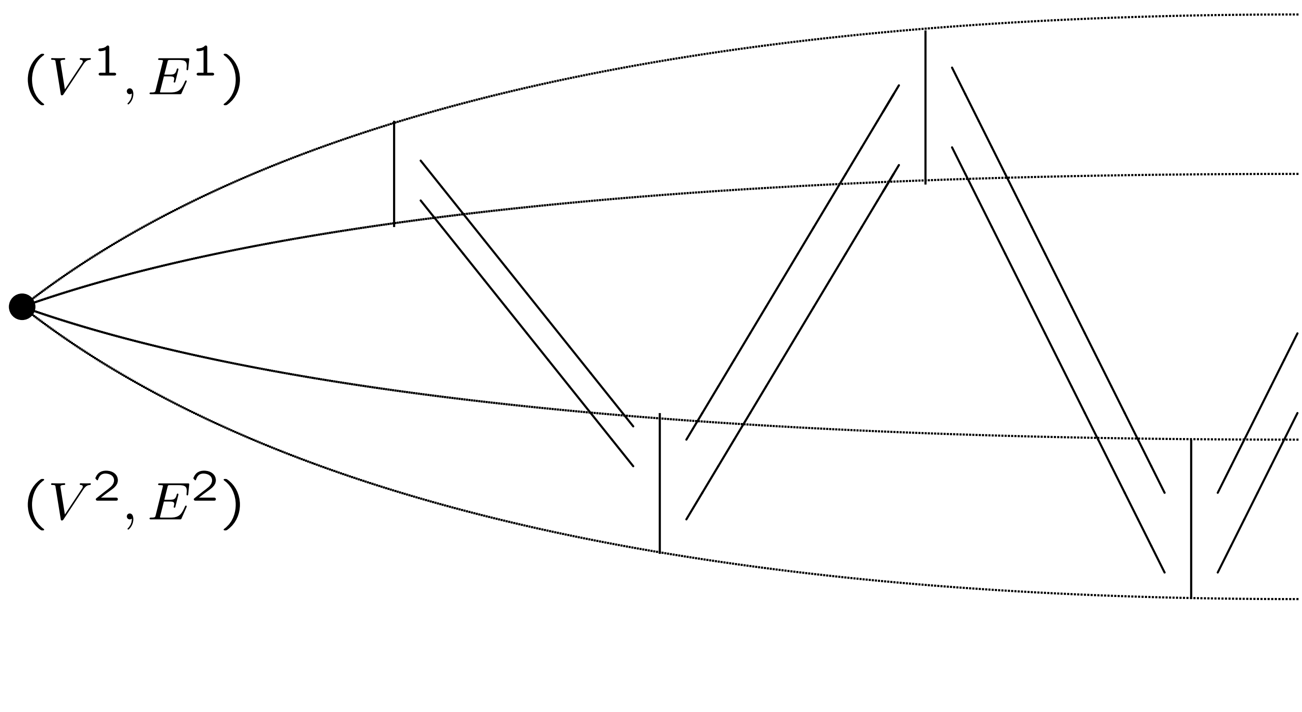
Theorem 10.

$$\begin{aligned} A_1 \subset \ & A_2 \ \subset \cdots, \\ (\cup_N A_N)^- &= C^*(R) \\ K_0(C^*(R)) &\cong \lim_N \oplus_{V_1} \mathbb{Z} \to \oplus_{V_2} \mathbb{Z} \to \cdots \end{aligned}$$

$C^*(R)$ is an AF-algebra (approximately finite dimensional).

Theorem 11 (Elliott-Krieger). *Let* $(V^i, E^i), i = 1, 2$ *be two Bratteli diagrams with associated AF-relations,* $(X_i, R_i), i = 1, 2$. *TFAE:*

1. $(X_1, R_1) \cong (X_2, R_2)$
2. $C^*(R_1) \cong C^*(R_2)$
3. $(K_0(C^*(R_1)), K_0(C^*(R_1))^+, [1]) \cong (K_0(C^*(R_2)), K_0(C^*(R_2))^+, [1])$
4 the two diagrams may be "intertwined":



Chaotic (hyperbolic) systems

Suppose that $f : \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{T}^2$ is induced by the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Its eigenvalues are $\gamma = \frac{1+\sqrt{5}}{2} > 1$ and $-1 < -\gamma^{-1} < 0$.

The two crucial features are:
1. the periodic points are dense in $\mathbb{T}^2$
2. the tangent space splits into contracting and expanding directions for the derivative. (Along the eigenvectors of the matrix.)

Let X be the expanding eigenspace projected into $\mathbb{T}^2$ (topologically it is a line). R is *stable* equivalence;

$$(x, y) \in R \Leftrightarrow \lim_{n \to +\infty} d(f^n(x), f^n(y)).$$

i.e. the points lie on the same contracting eigenspace.

Smale's Axiom A systems

M a Riemannian manifold, f a diffeomorphism. The non-wandering set, $NW(f)$ is the set of all points x such that for any neighbourhood U of x, there is $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. It is closed and contains all periodic points.

Axiom A:
1. $NW(f)$ is the closure of the periodic points,
2. the tangent space restricted to $NW(f)$ has a contractive/expansive splitting.

In the last example, $NW(f) = \mathbb{T}^2$. Typically, $NW(f)$ is a fractal. In the horseshoe, $M = S^2$ and $NW(f)$ consists of one attracting fix-point, one repelling fix-point and a Cantor set where the dynamics is:

$$\{0,1\}^{\mathbb{Z}}, \quad f = \text{ left shift}$$

Stable equivalence is tail equivalence on $\{0,1\}^{\mathbb{N}}$ and $C^*(R)$ is the AF-algebra from the Bratteli diagram with one vertex and two edges at each level.