Topological orbit equivalence

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Cantor minimal systems

Let X be a Cantor set; compact, totally disconnected, metrizable, no isolated points.

Consider homeomorphisms, ϕ, whose domain and range are both open subsets of X. Suppose that $\mathcal{F}$ is collection of such functions such that:
1. if ϕ, ψ are in $\mathcal{F}$, so is $\phi \cap \psi$,
2. if ϕ, ψ are in $\mathcal{F}$, so is $\phi \circ \psi$,
3. if ϕ is in $\mathcal{F}$, so is ϕ^{-1},
4. the collection of open sets U in X such that id_U is in $\mathcal{F}$ generates the topology of X.

It follows that

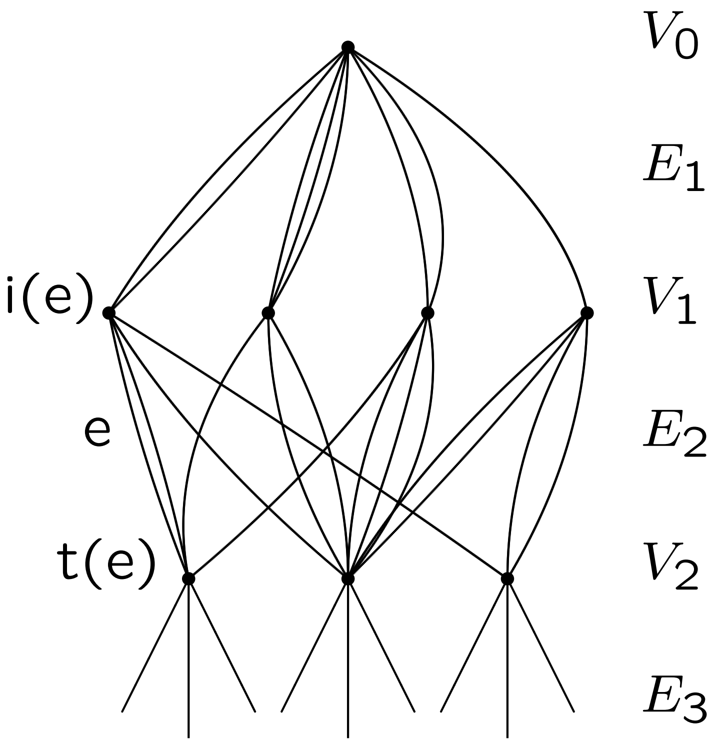
$$R = \cup\mathcal{F} = \{(x, \phi(x)) \mid \phi \in \mathcal{F}, x \in Dom(\phi)\}$$

is an equivalence relation and $\mathcal{F}$ is a basis for a topology of R. Such a topology is called *étale*. We assume that this topology is second countable and Hausdorff. As a consequence the equivalence classes are countable.

We say that R is *minimal* there are no non-trivial closed R-invariants sets, or equivalently, if every R-equivalence class is dense in X.

AF-relations

A *Bratteli diagram* is a vertex set $V = V_0 \cup V_1 \cup \ldots$ and an edge set $E = E_1 \cup E_2 \cup \ldots$ with initial and terminal maps $i : E_n \to V_{n-1}, t : E_n \to V_n$. Each V_n and E_n are finite with $V_0 = \{v_0\}$.



Let X be the set of infinite paths from v_0:

$$X = \{(x_1, x_2, \ldots) \mid x_n \in E_n, t(x_n) = i(x_{n+1})\}$$

Relative topology from $X \subset \Pi_n E_n$.
If $p = (p_1, p_2, \ldots, p_N)$ is a finite path, we let

$$C(p) = \{x \in X \mid x_n = p_n, 1 \leq n \leq N\},$$

which is clopen.

For paths p, q of length N, with $t(p_N) = t(q_N)$, define $\phi : C(p) \to C(q)$ by

$$\begin{aligned} & \phi(p_1, p_2, \ldots, p_N, x_{N+1}, x_{N+2}, \ldots) \\ = \quad & (q_1, q_2, \ldots, q_N, x_{N+1}, x_{N+2}, \ldots). \end{aligned}$$

The set of all such ϕ is $\mathcal{F}$.

R is tail equivalence:

$$(x, y) \in R \Leftrightarrow \exists N, x_n = y_n, n \geq N.$$

Definition 1. *An étale equivalence relation* R *on* X *is* AF *if* X *is totally disconnected and* R *is the union of an increasing sequence of compact, open subequivalence relations.*

Theorem 2. *Every AF-relation can be presented by a Bratteli diagram.*

Group actions

Let G be a discrete, abelian group with a free action φ on X: for s in G,

$$\varphi^s : X \to X$$

is a homeomorphism,

$$\begin{aligned} \varphi^0 &= id_X, \\ \varphi^s \circ \varphi^t &= \varphi^{s+t}, \\ \varphi^s(x) &= x, \text{ only if } s = 0, \end{aligned}$$

s, t in G.

The equivalence relation is:

$$R_\varphi = \{(x, \varphi^s(x)) \mid x \in X, s \in G\}$$

and

$$\mathcal{F} = \{\varphi^s|U \mid, s \in G, U \subset X \text{ open } \}.$$

2^∞-odometer

Let $X = \{0,1\}^{\mathbb{N}}$ and define φ to be addition of $(1,0,0,\ldots)$, with carry over to the right. For example:

$$\begin{aligned}
\varphi(0,0,1,0,1,1,\ldots) &= (1,0,1,0,1,1,\ldots) \\
\varphi(1,1,1,0,0,1,\ldots) &= (0,0,0,1,0,1,\ldots) \\
\varphi(1,1,1,1,1,1,\ldots) &= (0,0,0,0,0,0,\ldots)
\end{aligned}$$

$\mathbb{Z}$ action, φ^n is the nth iterate of φ, $n \geq 1$, or the $-n$th iterate of φ^{-1}, $n < 0$.

X is also the ring of 2-adic integers and the map is addition of 1.

Let $R \subset R_\varphi$ be the equivalence relation generated by $\{(x,\varphi(x)) \mid x \neq (1,1,1,\ldots)\}$. Then R is just tail equivalence on X; or rather the Bratteli diagram with one vertex and two edges (0 and 1) at every level.

Theorem 3. *Let φ be any minimal $\mathbb{Z}$-action on a Cantor set X. Choose y in X and let $R \subset R_\varphi$ be the equivalence relation generated by $\{(x, \varphi^1(x)) \mid x \neq y\}$. Then R is a minimal AF-relation and*

$$R_\varphi = R \vee (y, \varphi^1(y))$$

($\vee$ means the equivalence relation generated by).

Proof. Choose $Y_1 \supset Y_2 \supset \cdots$, clopen sets with intersection $\{y\}$ and let R_N be the equivalence relation generated by $\{(x, \varphi^1(x)) \mid x \notin Y_N\}$. Then

$$R_1 \subset R_2 \subset \cdots \quad , \cup_N R_N = R,$$

and each R_N is compact and open. □

Consequence: every minimal homeomorphism of a Cantor can be presented as a map on a Bratteli diagram. The edges are ordered and the map is to take successor under a type of reverse lexicographic order. The Bratteli-Vershik model.

Orbit equivalence and isomorphism

Definition 4. *For* $i = 1, 2$, *let* R_i *be an equivalence relation on the topological space* X_i. R_1 *and* R_2 *are* orbit equivalent, *written* $R_1 \sim R_2$ *if there is a homeomorphism* $h : X_1 \to X_2$ *such that* $h \times h(R_1) = R_2$ *or* $h[x]_{R_1} = [h(x)]_{R_2}$ *for all* x *in* X_1.

Definition 5. *For* $i = 1, 2$, *let* R_i *be an étale equivalence relation on the topological space* X_i. R_1 *and* R_2 *are* isomorphic, *written* $R_1 \cong R_2$ *if there is a homeomorphism* $h : X_1 \to X_2$ *such that* $h \times h : R_1 \to R_2$ *is a homeomorphism.*

Remark 1. *It follows from a result of Sierpinski that for* $R_i, i = 1, 2$ *arising from actions of discrete groups on* connected *spaces* $X_i, i = 1, 2$, *orbit equivalence is equivalent to conjugacy of the actions. Hence, we restrict to totally disconnected spaces.*

Invariants

X, Cantor set, R, an étale equivalence relation with neighbourhood base $\mathcal{F}$.

Definition 6. *A probability measure* μ *on* X *is* R-invariant *if*

$$\mu(U) = \mu(\phi(U)),$$

for all ϕ *in* $\mathcal{F}$, $U \subset Dom(\phi)$. *Let* $M(R)$ *denote the set of all such measures.* R *is* uniquely ergodic *if there is a unique* R*-invariant measure.*

$$\begin{aligned}
C(X,\mathbb{Z}) &= \{f : X \to \mathbb{Z} \mid f \text{ continuous } \} \\
B_m(X,R) &= \{f \in C(X,\mathbb{Z}) \mid \int_X f d\mu = 0, \\
&\qquad \text{for all } \mu \in M(R)\} \\
B(X,R) &= < \{\chi_K - \chi_{\phi(K)} \mid \phi \in \mathcal{F}, \\
&\qquad K \subset Dom(\phi) \text{ compact, open } \} > \\
B(X,R) &\subset B_m(X,R) \subset C(X,\mathbb{Z}).
\end{aligned}$$

We define

$$\begin{aligned} D(R) &= C(X,\mathbb{Z})/B(X,R) \\ D_m(R) &= C(X,\mathbb{Z})/B_m(X,R) \end{aligned}$$

Notice that $D_m(R)$ is a quotient of $D(R)$.

These are abelian groups and have an *order*:

$$\begin{aligned} D(R)^+ &= \{[f] \mid f \geq 0\} \\ D_m(R)^+ &= \{[f] \mid f \geq 0\} \end{aligned}$$

and a distinguished positive element: [1].

Theorem 7. *1.* $(D(R), D(R)^+, [1])$ *is an invariant of isomorphism.*

2. $(D_m(R), D_m(R)^+, [1])$ *is an invariant of orbit equivalence.*

Theorem 8. *If* $M(R) = \{\mu\}$ *(*R *is uniquely ergodic), then*

$$D_m(R) = \{\mu(E) \mid E \subset X \text{ clopen }\} + \mathbb{Z} \subset \mathbb{R}.$$

$D(R)$ and $D_m(X,R)$ for AF-relations R

Theorem 9. *Let (V,E) be a Bratteli diagram and (X,R) its AF-relation. $(D(R), D(R)^+, [1])$ is isomorphic to the inductive limit*

$$(\mathbb{Z}V_0, \mathbb{Z}^+V_0) \xrightarrow{\gamma_1} (\mathbb{Z}V_1, \mathbb{Z}^+V_1) \xrightarrow{\gamma_2} (\mathbb{Z}V_2, \mathbb{Z}^+V_2) \xrightarrow{\gamma_3}$$

where

$$\gamma_n(v) = \sum_{i(e)=v} t(e),$$

or

$$(\mathbb{Z}, \mathbb{Z}^+) \xrightarrow{A_1} (\mathbb{Z}^{n_1}, (\mathbb{Z}^+)^{n_1}) \xrightarrow{A_2} (\mathbb{Z}^{n_2}, (\mathbb{Z}^+)^{n_2}) \xrightarrow{A_3}$$

where $n_k = \#V_k$ and A_k is the adjacency matrix of E_k. The element v_0 is mapped to [1].

The inductive limit of groups

$$G_1 \stackrel{\alpha_1}{\to} G_2 \stackrel{\alpha_2}{\to} \cdots$$

is

$$\cup_n G_n / \{g \sim \alpha_n(g) \mid g \in G_n\}.$$

Idea of proof: For a path p of length N,

$$[\chi_{C(p)}] \in D(R) \to t(p_N) \in \mathbb{Z}V_N.$$

Notice that if $t(p_N) = t(q_N)$, then
$\phi : C(p) \to C(q)$ and
$\chi_{C(p)} - \chi_{C(q)} \in B(X, R)$.

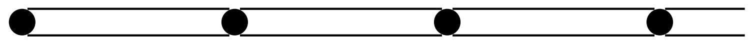
Invariant measure μ arises from
$f : \cup_N V_N \to [0, 1]$ such that

$$f(v_0) = 1, f(v) = \sum_{i(e)=v} f(t(e))$$

via

$$\mu(C(p)) = f(t(p_N)).$$

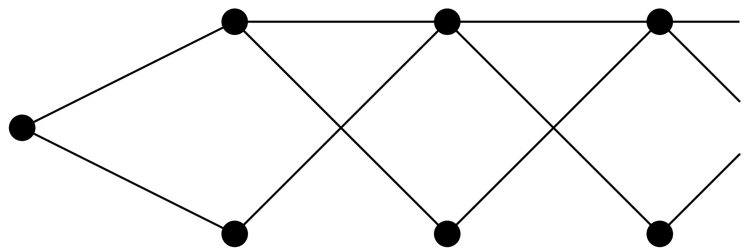
Example 1



$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots$$

$$D(R) = D_m(R) = \{p2^{-k} \mid p \in \mathbb{Z},\ k \in \mathbb{Z}^+\}.$$

Example 2



$$\mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \cdots$$

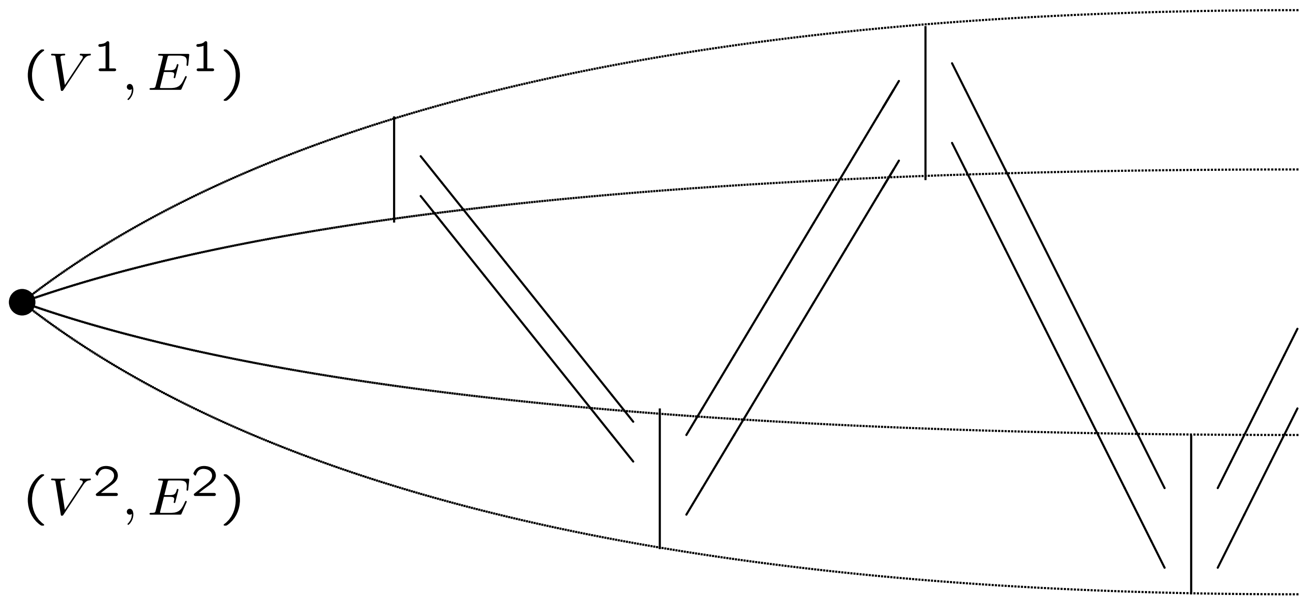
$$D(R) = D_m(R) = \{m + \left(\frac{1+\sqrt{5}}{2}\right) n \mid m, n \in \mathbb{Z}\}$$

Theorem 10 (Elliott-Krieger). *Let* $(V^i, E^i), i = 1, 2$ *be two Bratteli diagrams with associated AF-relations,* $(X_i, R_i), i = 1, 2$*. TFAE:*

1. $(X_1, R_1) \cong (X_2, R_2)$

2. $(D(R_1), D(R_1)^+, [1]) \cong (D(R_2), D(R_2)^+, [1])$

3. *the two diagrams may be "intertwined":*



Theorem 11 (Absorption Theorem). *Let* (X, R) *be a minimal AF-relation. Suppose that* $Y \subset X$ *and* Q *is an AF-relation on* Y *satisfying:*

1. Y *is closed and* $\mu(Y) = 0$*, for all* μ *in* $M(R)$*,*

2. $R \cap Y \times Y$ *is an étale relation on* Y*,*

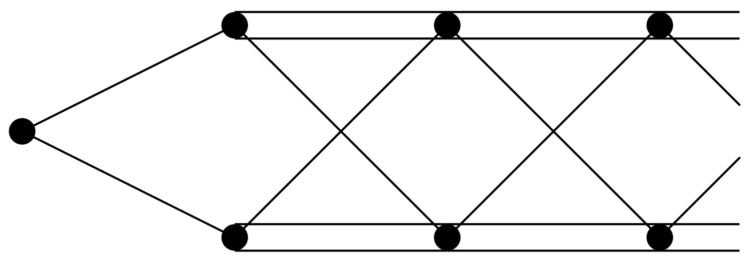
Then the equivalence relation generated by R *and* Q*,* $\tilde{R} = R \vee Q$ *is orbit equivalent to* R*:*

$$R \vee Q \sim R.$$

(Warning: the statement has been simplified!)

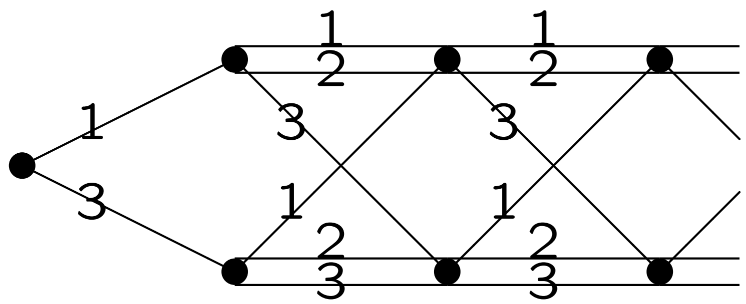
Absorption Thm: Application 1

Consider the AF-equivalence relation for following Bratteli diagram



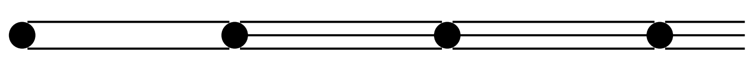
(X, R)

$$0 \to \mathbb{Z} \to D(R) \to \frac{1}{2}\mathbb{Z}[1/3] = D_m(R) \to 0.$$



(X, R)

$$X = \{1,3\} \times \{1,2,3\}^{\mathbb{N}} = \text{path space of}$$



$(X, \tilde{R})$

1. $D(\tilde{R}) = D_m(\tilde{R}) = \frac{1}{2}\mathbb{Z}[1/3]$,

2. $R \subset \tilde{R}$,

3. $\tilde{R} = R \vee ((1, 2, 2, 2 \ldots), (3, 2, 2, \ldots))$.

Apply the absorption theorem with $Y = \{(1, 2, 2, 2 \ldots), (3, 2, 2, \ldots)\}$, $Q = Y \times Y$ to conclude that

$$R \sim \tilde{R}.$$

Theorem 12. *Let* (X,R) *be a minimal AF-relation. There exists an AF-relation* $R \subset \tilde{R}$ *such that*

$$\tilde{R} = R \vee Q \quad (\textit{A.T.} \;\Rightarrow \tilde{R} \sim R),$$

$$\begin{aligned}(D(\tilde{R}), D(\tilde{R})^+, [1]) &\cong (D_m(\tilde{R}), D_m(\tilde{R})^+, [1]) \\ &\cong (D_m(R), D_m(R)^+, [1]).\end{aligned}$$

Corollary 13. *For minimal AF-relations* (X,R), $(D_m(R), D_m(R)^+, [1])$ *is a complete invariant for orbit equivalence.*

Proof. $i = 1, 2$, (X_i, R_i) minimal AF. Let $\tilde{R}_i, i = 1, 2$ be as above. If $D_m(R_1) \cong D_m(R_2)$, then

$$D(\tilde{R}_1) \cong D_m(R_1) \cong D_m(R_2) \cong D(\tilde{R}_2).$$

Elliott-Krieger implies

$$R_1 \sim \tilde{R}_1 \cong \tilde{R}_2 \sim R_2.$$

□

Absorption Thm: Application 2

φ, a minimal $\mathbb{Z}$-action, $R \subset R_\varphi$, minimal AF with $R_\varphi = R \vee (y, \varphi^1(y))$.

$Y = \{y, \varphi^1(y)\}$, $Q = Y \times Y$, A.T. implies $R_\varphi \sim R$.

Theorem 14 (Giordano-P-Skau, 1991)**.** *For minimal AF-relations and minimal $\mathbb{Z}$-actions, (X, R), $(D_m(R), D_m(R)^+, [1])$ is a complete invariant for orbit equivalence.*

Theorem 15 (Giordano-Matui-P-Skau, 2005)**.** *For minimal AF-relations, minimal $\mathbb{Z}$-actions and minimal $\mathbb{Z}^2$-actions, (X, R), $(D_m(R), D_m(R)^+, [1])$ is a complete invariant for orbit equivalence.*

Theorem 16 (Giordano-Matui-P-Skau, 2008)**.** *For minimal AF-relations and minimal $\mathbb{Z}^d$-actions, $d \geq 1$, (X, R), $(D_m(R), D_m(R)^+, [1])$ is a complete invariant for orbit equivalence.*