K-THEORETIC DUALITY FOR SHIFTS OF FINITE TYPE

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Preprint # 96-2

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February 1996

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K-THEORETIC DUALITY FOR SHIFTS OF FINITE TYPE

JEROME KAMINKER AND IAN PUTNAM

ABSTRACT. We will describe a notion of K-theoretic duality for C^*-algebras which generalizes Spanier-Whitehead duality in topology. A criterion for checking that it holds is presented. As an application it is shown that the Ruelle algebras which are associated to subshifts of finite type satisfy this criterion and hence are dual.

1. INTRODUCTION

This paper is part of a study of certain C^*-algebras which can be associated to a hyperbolic homeomorphism of a compact space, (X, f). They are called the the stable and unstable Ruelle algebras, $\mathcal{R}^u$ and $\mathcal{R}^s$, and are higher dimensional generalizations of Cuntz-Krieger algebras. This means that if the dimension of X is zero, then $\mathcal{R}^u \cong O_{A^t}$ and $\mathcal{R}^s \cong O_A$. One of the basic results of the theory is a duality relation between $\mathcal{R}^u$ and $\mathcal{R}^s$. In the present paper we prove this explicitely in the zero dimensional case. Our reason for doing this is to bring out the use of Fock space to construct the K-theory class implementing the duality in the zero dimensional case.

The notion of Spanier-Whitehead duality in topology has a very natural generalization to K-theory of C^*-algebras. Briefly, it says that two algebras, A and B, are dual if there are duality classes $\Delta \in KK^i(A \otimes B, \mathbb{C})$ and $\delta \in KK^i(\mathbb{C}, A \otimes \mathcal{B})$ which induce an isomorphism between the K-theory of A and the K-homology of B via Kasparov product. It is closely related to the notion of Poincaré duality used by Connes in his study of the standard model of particle physics, [**?**] . We will describe it in more detail in Section 2. A useful result proved in

The first author was supported in part by NSF Grant DMS-9104636.
The second author was supported in part by an NSERC Grant.

the paper is a criterion, presented in Section 3, for deciding when one has a duality between two algebras. It is applicable when two duality classes such as Δ and δ are given and one wants to show that they induce duality isomorphisms. Section 4 and Section 5 apply this criterion to the case of two algebras associated to a hyperbolic dynamical system. If A is an $n \times n$ aperiodic matrix then one can associate to it the subshift of finite type, (Σ_A, σ_A). There are two C^*-algebras that can be constructed from this data–the Cuntz-Krieger algebras O_A and O_{A^t}. We show that these algebras are dual. In Section 6 we will discuss some further applications and make some concluding remarks.

In a later paper we will establish duality for the stable and unstable Ruelle algebras associated to any hyperbolic homeomorphism of a compact space, (a Smale space). Ruelle algebras were introduced by the second author in [**?**]. They can be thought of as higher dimensional generalizations of Cuntz-Krieger algebras. They are constructed by defining two equivalence relations on the Smale space, stable and unstable equivalence. One takes the C^*-algebras associated to them and then takes the crossed products by the automorphism induced by the homeomorphism.

The stable and unstable equivalence classes in a Smale space behave very much like transverse foliations. Because of that, and the fact that the homeomorphism is contracting along the stable leaves, one obtains a duality in K-theory for the algebras.

Cuntz-Krieger algebras are special cases of Ruelle algebras, so the duality established here would follow for the more general theory. However, there is an intriguing aspect to this which as yet has no analogue in the general case. Namely, the duality classes have representatives constructed using Fock space. These classes are obtained in a natural manner following work of D. Evans, [**?**] and D. Voiculescu, [**?**]. This provides potential connections with physics (c.f. [**?**, **?**, **?**]) and Voiculescu's work on free products which we hope to pursue in

the future. We would like to thank Dan Voiculescu and Marius Dadarlat for very helpful conversations.

It should be noted that the general duality theory for Smale spaces requires a different approach which is based on the notion of asymptotic morphism. The final version of the duality theorem is be based on these methods, [?].

2. K-theoretic duality for C^*-algebras

In this paper we will be describing an example of some C^*-algebras that are dual with respect to K-theory. This notion of duality has appeared several times in the past, [?, ?, ?] and recently was used by Connes [?]. We present the definitions here and list some basic facts. More details can be found in [?].

We will use the following conventions. Let $\mathcal{S}$ denote $C_0(\mathbb{R})$. Then $KK^1(A, B)$ will be, by definition, equal to $KK(\mathcal{S} \otimes A, B)$. For A and B separable, and A nuclear one has that $KK^1(A, B) \cong Ext(A, B)$. We establish some additional notation. If σ is a permutation then we will use the same letter to denote the isomorphism

$$A_1 \otimes \cdots \otimes A_n \to A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)}.$$

If σ is a transposition interchanging i and j, we will write σ_{ij} for the homomorphism induced by σ on the first variable and σ^{ij} for that induced on the second variable in KK. Let $\tau_D \colon KK^i(A, B) \to KK^i(A \otimes D, B \otimes D)$ and $\tau^D \colon KK^i(A, B) \to KK^i(D \otimes A, D \otimes B)$ denote the standard maps, [?].

Also, we will have need of the following version of Bott periodicity. Let

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\sigma\tau} C(S^1) \longrightarrow 0 \tag{1}$$

be the Toeplitz extension. We will denote it by $\mathcal{T} \in KK^1(C(S^1), \mathbb{C})$ and its restriction to $\mathcal{S}$ by $\mathcal{T}_0 \in KK(\mathcal{S} \otimes \mathcal{S}, \mathbb{C})$. Let $\beta \in KK(\mathbb{C}, \mathcal{S} \otimes \mathcal{S})$ be the Bott element. Then the following holds. (c.f. [**?**])

Theorem 2.1. *One has* $\beta \otimes_{\mathcal{S}\otimes\mathcal{S}} \mathcal{T}_0 = 1_{\mathbb{C}}$ *and* $\mathcal{T}_0 \otimes \beta = 1_{\mathcal{S}\otimes\mathcal{S}}$, .

We describe next the notion of duality we will be using.

Definition 2.2. *Let* A *and* B *be* C^**-algebras. Suppose that, for* $n = 0$ *or* 1*, two classes,* $\Delta \in KK^n(A \otimes B, \mathbb{C})$ *and* $\delta \in KK^n(\mathbb{C}, A \otimes B)$ *, are given. Define homomorphisms* $\Delta_i \colon K_i(A) \to K^{i+n}(B)$ *and* $\delta_i \colon K^{i+n}(B) \to K_i(A)$ *in the following way. In* $n = 1$ *set*

$$\Delta_i(x) = \begin{cases} x \otimes_A \Delta & \text{if } i = 0, \\ \beta \otimes_{\mathcal{S}\otimes\mathcal{S}} (\sigma_{12}(x \otimes_A \Delta)) & \text{if } i = 1 \end{cases} \tag{2}$$

and let

$$\delta_i(y) = \begin{cases} \beta \otimes_{\mathcal{S}\otimes\mathcal{S}} (\delta \otimes_B y) & \text{if } i = 0, \\ \delta \otimes_B y & \text{if } i = 1, \end{cases} \tag{3}$$

If $n = 0$ *set*

$$\Delta_i(x) = \begin{cases} x \otimes_A \Delta & \text{if } i = 0, \\ \sigma_{12}(x \otimes_A \Delta) & \text{if } i = 1 \end{cases} \tag{4}$$

and let

$$\delta_i(y) = \begin{cases} \delta \otimes_B y & \text{if } i = 0, \\ \beta \otimes_{\mathcal{S}\otimes\mathcal{S}} (\delta \otimes_B y) & \text{if } i = 1, \end{cases} \tag{5}$$

We say that A *and* B *are dual if*

$$\Delta_i : K_i(A) \to K^{i+n}(B)$$

and

$$\delta_i : K^{i+n}(B) \to K_i(A)$$

are inverse isomorphisms. Given A, if such an algebra B exists it is called a dual of A and it is denoted $\mathcal{D}A$.

In this generality a dual is not unique, so care must taken with the notation $\mathcal{D}A$. We will only use it if a specific dual is in hand. However, it is easy to see that a dual is unique up to KK-equivalence. Indeed, $\sigma^{12}(\delta') \otimes_A \Delta \in KK(B', B)$ and $\sigma^{12}(\delta) \otimes_A \Delta' \in KK(B, B')$ yield the required KK-equivalence.

The form of the definition of the homomorphisms Δ_* and δ_* is forced by our convention that $KK^1(A, B) = KK(A \otimes \mathcal{S}, B)$. It is an interesting point that when dealing with an odd type duality one must bring in some form of Bott periodicity explicitely. Either one can incorporate it into the definition of the homomorphisms as we have done, one one can modify the definitions of the K-theory groups. As the reader will see, our choice is the most convenient one for the proofs we are giving. Note also that we are working only in the odd case, (i.e. $n = 1$), in this paper.

For a specific algebra A it is not clear whether a dual, $\mathcal{D}A$, exists. In general, the existence of $\mathcal{D}A$ with prescribed properties, such as separability, is a strong condition. If one can take $\mathcal{D}A$ equal to A then this agrees with what Connes has developed as Poincaré duality in [**?**].

If one requires only the existence of Δ and the fact that it yields an isomorphism in the definition above, then there is no guarantee that a class δ exists to give the inverse isomorphism. If A was $C(X)$, with X a finite complex, then the existence of δ would follow from that of Δ. However, in general this need not hold.

The origin of this notion is in Spanier-Whitehead duality in topology, [**?**]. Recall that if X is a finite complex then there is a dual complex, DX, along with class $\Delta \in H_m(X \wedge DX)$

satisfying that $\backslash\Delta : H^i(X) \to H_{m-i}(DX)$ is an isomorphism. The space DX is called the Spanier-Whitehead dual of X. It is unique up to stable homotopy. If M is a closed manifold of dimension n embedded in $\mathbb{R}^m$, then DM can be taken to be $(\nu M)^+$, the Thom space of the normal bundle of M. It is interesting to note that there is a relation between Spanier-Whitehead duality, the Thom isomorphism, ϕ, and Poincaré duality

$$\begin{array}{ccc} H_{n-i}(M) & \xleftarrow[\backslash\Delta]{} & H^{i+m-n}((\nu M)^+) \\ {\scriptstyle \cap[M]}\uparrow & & \uparrow{\scriptstyle \phi} \\ H^i(M) & \longrightarrow & H^i(M), \end{array}$$

where $[M] = U\backslash\Delta$, U the Thom class. Of course, $\mathcal{D}(C(X)) = C(D(X))$ for X a finite complex.

If one works in the class $\mathcal{N}$ introduced by Rosenberg and Schochet in their study of the Universal Coefficient Theorem, [**?**], the theory simplifies and there is a strong analogy with the commutative case. (However, in general, the restriction that the algebras lie in $\mathcal{N}$ is too strong. In several important examples this does not hold.) Recall that $\mathcal{N}$ is defined to be the smallest class of separable, nuclear C^*-algebras containing $\mathbb{C}$ and closed under forming extensions, direct limits, and KK-equivalence. The Universal Coefficient Theorem for KK-theory holds for $KK(A, B)$ if B is separable and $A \in \mathcal{N}$. Let $\mathcal{DN}$ be the subclass of $\mathcal{N}$ consisting of algebras A in $\mathcal{N}$ for which a dual $\mathcal{D}A$ exists and is also in $\mathcal{N}$. For algebras in $\mathcal{DN}$ the following facts are easy consequences of the properties of the Kasparov product and the Universal Coefficient Theorem.

i) If A is dual to B, then B is dual to A.

ii) If $A \in \mathcal{DN}$, then $\mathcal{D}(\mathcal{D}A)$ is KK-equivalent to A.

iii) If $A \in \mathcal{N}$, then $A \in \mathcal{DN}$ if and only if $K_*(A)$ is finitely generated.

iv) Let $E, D \in \mathcal{N}$ and $A \in \mathcal{DN}$. Then

$$\Delta_* \colon KK^*(E, D \otimes A) \to KK^{*+n}(E \otimes \mathcal{D}A, D)$$

and

$$\delta_* \colon KK^*(E \otimes \mathcal{D}A, D) \to KK^{*+n}(E, D \otimes A)$$

are inverse isomorphisms.

v) If A has a dual, and A' is KK-equivalent to A, then A' has a dual which is KK-equivalent to the dual of A.

For details and further development, see [**?**]. It is not apparent if an algebra has a dual or not. Indeed, the main goal of this paper is to exhibit an example of a class of algebras with specific types of duals which have a geometric and dynamical origin. However, one can start to build up a class of algebras which have duals in an elementary way. For example, if X is a finite complex, then $\mathcal{D}X$ exists. If $A \in \mathcal{N}$ and $K_*(A)$ is finitely generated, then A is KK-equivalent to $C(X)$ where X is a finite complex, and hence A has a dual. Moreover, Connes has shown that $\mathcal{A}_\theta$ is self-dual for θ irrational.

The largest subclass of $\mathcal{N}$ for which $\mathcal{D}$ is involutive modulo KK-equivalence is $\mathcal{DN}$. This can be compared with a result of M. Boardman, [**?**], which states that the largest category on which Spanier-Whitehead duality is involutive is the homotopy category of finite complexes. Thus, $\mathcal{DN}$ has a formal similarity with the homotopy category of finite complexes. This itself does not clarify the issue of which C^*-algebras should play the role of non-commutative finite complexes. This will be discussed further in [**?**].

One may view duality as being of even or odd type depending on whether Δ belongs to $KK^n(A \otimes \mathcal{D}A, \mathbb{C})$ for n even or odd. We will discuss the odd type of duality here. However, in connections to the Novikov Conjecture, [**?**], and physics, [**?**], the even type naturally appears.

3. Criterion for duality classes

In this section we will present a technical result, Proposition **??**, which gives a criterion for when two classes Δ and δ yield duality isomorphisms. This is essentially the same as the condition given by Connes, [**?**, p. 588], except that our duality is in the odd case and this requires adjusting the arguments for Bott periodicity. This technicality is actually what allows us to obtain the duality isomorphisms in the case of shifts of finite type.

Thus, we shall give useable conditions under which $\Delta_* \circ \delta_* = 1$ and $\delta_* \circ \Delta_* = 1$. This breaks into two parts. The first is an uncoupling step and the second is a type of cancellation. In the following sections we apply this to the case of Cuntz-Krieger algebras.

We will first prove in detail that $\delta_0 \Delta_0 = 1_{K_0(A)}$. The statement, $\delta_1 \Delta_1 = 1_{K_1(A)}$, follows in a similar manner. We then sketch the proof that $\Delta_0 \delta_0 = 1_{K_0(A)}$. To start with we will perform the uncoupling step. Let $x \in K_0(A) = KK(\mathbb{C}, A)$. Then we have

$$\delta_0 \Delta_0(x) = \beta \otimes_{\mathcal{S} \otimes \mathcal{S}} (\delta \otimes_B (x \otimes_A \Delta)).$$

Consider, first, the factor $(\delta \otimes_B (x \otimes_A \Delta))$. We have

$$\begin{aligned}(\delta \otimes_B (x \otimes_A \Delta)) &= \tau_{\mathcal{S}}(\delta) \otimes (\tau^A \tau_{\mathcal{S}} \tau_B(x) \otimes \tau^A(\Delta)) \\ &= (\tau_{\mathcal{S}}(\delta) \otimes (\tau^A \tau_{\mathcal{S}} \tau_B(x)) \otimes \tau^A(\Delta).\end{aligned}$$

Now, a direct computation yields that

$$\begin{aligned}(\tau_{\mathcal{S}}(\delta) \otimes (\tau^A \tau_{\mathcal{S}} \tau_B(x)) \otimes \tau^A(\Delta) &= (\tau_{\mathcal{S}} \tau^{\mathcal{S}}(x) \otimes \sigma_{12} \sigma^{24} \tau^A \tau^{\mathcal{S}}(\delta)) \otimes \tau^A(\Delta) \\ &= \tau_{\mathcal{S}} \tau^{\mathcal{S}}(x) \otimes (\sigma_{12} \sigma^{24} \tau^A \tau^{\mathcal{S}}(\delta)) \otimes \tau^A(\Delta)).\end{aligned}$$

Putting β back into the product and simplifying, one obtains

$$\beta \otimes_{\mathcal{S} \otimes \mathcal{S}} (x \otimes_A (\delta \otimes_B \Delta)) = x \otimes_A (\beta \otimes_{\mathcal{S} \otimes \mathcal{S}} (\delta \otimes_B \Delta)).$$

This accomplishes the uncoupling.

Proposition 3.1. *One has* $\delta_0\Delta_0(x) = x \otimes_A (\beta \otimes_{\mathcal{S}\otimes\mathcal{S}} (\delta \otimes_B \Delta))$.

What one would hope is that

$$\beta \otimes_{\mathcal{S}\otimes\mathcal{S}} (\delta \otimes_B \Delta) = 1_A \in KK(A, A), \tag{6}$$

thus yielding

$$\delta_0\Delta_0(x) = x.$$

Indeed, if $\delta \otimes_B \Delta = \tau_A(\mathcal{T}_0)$, then $\beta_{\mathcal{S}\otimes\mathcal{S}}\tau^A(\mathcal{T}_0) = 1_A$ by Bott periodicity. However, this need not be the case. This is because δ and Δ behave like K-theory fundamental classes and may differ by a unit from ones which would yield (**??**). There is a way to compensate for this which we address next.

Proposition 3.2. *Suppose that there are automorphisms* $\Theta_A \colon A\otimes\mathcal{S} \to A\otimes\mathcal{S}$ *and* $\Theta_B \colon B \otimes \mathcal{S} \to B \otimes \mathcal{S}$ *such that*

$$(\Theta_A)_i \colon K_i(A \otimes \mathcal{S}) \to K_i(A \otimes \mathcal{S})$$
$$(\Theta_B)_i \colon K_i(B \otimes \mathcal{S}) \to K_i(B \otimes \mathcal{S})$$

are the identity map, for $i = 0, 1$, *and, further,*

$$\sigma_{12}(\delta \otimes_B \sigma_{12}(\Delta)) = \Theta_A \otimes_{A\otimes\mathcal{S}} \tau^A(\mathcal{T}_0) \tag{7}$$
$$\sigma_{12}(\delta \otimes_A \sigma_{12}(\Delta)) = \Theta_B \otimes_{B\otimes\mathcal{S}} \tau^B(\mathcal{T}_0) \tag{8}$$

Then,

$$\delta_i\Delta_i \colon K_i(A) \to K_i(A)$$

is the identity for $i = 0, 1$

Proof. We will give the proof for $\delta_0\Delta_0$, the other case being similar. Condition (**??**) states that

$$\sigma_{12}\tau_{\mathcal{S}}\tau_A(\delta)\otimes(\tau_A(\sigma_{12}(\Delta)))=\tau^{\mathcal{S}}(\Theta_A)\otimes\tau^A(\mathcal{T}_0).$$

Thus, one has

$$\beta\otimes_{\mathcal{S}\otimes\mathcal{S}}(\delta\otimes_B(\sigma_{12}(\Delta)))=\tau^A(\beta)\otimes\tau_{\mathcal{S}}(\Theta_A)\otimes\tau^A(\mathcal{T}_0).$$

Now,

$$\begin{aligned}\tau^A(\beta)\otimes\tau_{\mathcal{S}}(\Theta_A)&=(\Theta_A)_*(\tau^A(\beta))\\&=\tau^A(\beta),\end{aligned}$$

so we obtain

$$\begin{aligned}\beta\otimes_{\mathcal{S}\otimes\mathcal{S}}(\delta\otimes_B\sigma_{12}(\Delta)&=\tau^A(\beta)\otimes\tau^A(\mathcal{T}_0)\\&=1_A,\end{aligned}$$

which yields the desired result. □

For the composition $\Delta_*\delta_*$ we have a similar result.

Proposition 3.3. *Under the hypothesis of Proposition* **??**, *we have that*

$$\Delta_i\delta_i\colon K^{i+1}(B)\to K^{i+1}(B) \tag{9}$$

is the identity for $i=0,1$.

Proof. The proof is obtained from the previous one by making obvious changes. □

The other cases follow in the same way. Thus, showing that one has a duality between algebras reduces to constructing the maps Θ_A and Θ_B satisfying the conditions above. In the next two sections we will do this for the case of the stable and unstable Ruelle algebras associated to a subshift of finite type.

4. Construction of duality classes for shifts of finite type

In this section we will construct the classes in KK-theory needed to exhibit the duality between $O_A \otimes \mathcal{K}$ and $O_{A^t} \otimes \mathcal{K}$. Let A be an $n \times n$ matrix with entries which are all zero or one. We assume that A has no row or column consisting entirely of zeros and that the associated shift space is a Cantor set.

The Cuntz-Krieger algebra, O_A, is the universal C^*-algebra generated by partial isometries $s_1, \ldots, s_n$ satisfying

i) the projections $s_1 s_1^*, \ldots, s_n s_n^*$ are pairwise orthogonal and add up to the identity of O_A,

ii) for $k = 1, \ldots, n$ one has

$$s_k^* s_k = \sum_i A_{ki} s_i s_i^*. \tag{10}$$

The condition above, that the shift space be a Cantor set, guarantees that the algebra described does not depend on the choice of the partial isometries, [**?**]. If $A_{ij} = 1$ for all i, j, then the algebra O_A is denoted O_n.

In a similar manner we consider O_{A^T}, with generators $t_1, \ldots, t_n$ satisfying

$$t_k^* t_k = \sum_i A_{ik} t_i t_i^* \tag{11}$$

for $k = 1, \ldots, n$.

Our aim in this section is to explicitely construct the elements

$$\delta \in KK^1(\mathbb{C}, O_A \otimes O_{A^T})$$

and

$$\Delta \in KK^1(O_A \otimes O_{A^T}, \mathbb{C})$$

which are needed to show that O_A and O_{A^T} are dual.

The construction of δ is the easier of the two. Let

$$w = \sum_{i=1}^{n} s_i^* \otimes t_i \in O_A \otimes O_{A^t}. \tag{12}$$

Then one has

$$w^*w = ww^* = \sum_{i,j} A_{ij} s_j s_j^* \otimes t_i t_i^*.$$

We let $\bar{w}\colon C(S^1) \to O_A \otimes O_{A^t}$ denote both the (non-unital) map defined by

$$\bar{w}(z) = w \tag{13}$$

as well as its restriction to $C_0(\mathbb{R}) \subseteq C(S^1)$.

Definition 4.1. *Let $\delta \in KK^1(\mathbb{C}, O_A \otimes O_{A^t})$ be the element determined by the homomorphism $\bar{w}$.*

The element Δ is constructed using the Fock space of a finite dimensional Hilbert space. (For related constructions see the papers of D. Evans and D. Voiculescu, [**?**, **?**].)

Let $\mathcal{H}$ denote an n-dimensional Hilbert space with orthonormal basis $\xi_1, \ldots, \xi_n$. Let $\mathcal{H}^{\otimes m} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ be the m-fold tensor product of $\mathcal{H}$ and let $\mathcal{H}_0$ be a one dimensional Hilbert space with unit vector Ω. Then the Fock space of $\mathcal{H}$, $\mathcal{F}$, is defined to be

$$\mathcal{F} = \mathcal{H}_0 \oplus (\bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n})$$

There is a natural orthonormal basis for $\mathcal{F}$,

$$\{\Omega, \xi_{i_1} \otimes \cdots \otimes \xi_{i_m} | m = 1, 2, \ldots, \qquad 1 \leq i_j \leq n\}.$$

Define the left and right creation operators, $L_1, \ldots, L_n$ and $R_1, \ldots, R_n$, on $\mathcal{F}$ by

$$L_k\Omega = \xi_k = R_k\Omega$$

and

$$L_k(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}) = \xi_k \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_m} \tag{14}$$

$$R_k(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}) = \xi_{i_1} \otimes \cdots \otimes \xi_{i_m} \otimes \xi_k \tag{15}$$

Next, we bring in the matrix A. Let $\mathcal{F}_A \subseteq \mathcal{F}$ denote the closed linear span of the vectors Ω and those $\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}$ satisfying the condition that $A_{i_j,i_{j+1}} = 1$ for all $j = 1, \ldots, m-1$. Let P_A denote the orthogonal projection of $\mathcal{F}$ onto $\mathcal{F}_A$. Let

$$L_k^A = P_A L_k P_A \in \mathcal{B}(\mathcal{F}_A)$$

$$R_k^A = P_A R_k P_A \in \mathcal{B}(\mathcal{F}_A)$$

for $k = 1, \ldots, n$.

It is easily checked that one has the following formulas.

$$L_k^A \xi_{i_1} \otimes \cdots \otimes \xi_{i_m} = A_{k,i_1} \xi_k \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_m}$$

$$R_k^A \xi_{i_1} \otimes \cdots \otimes \xi_{i_m} = A_{i_m,k} \xi_{i_1} \otimes \cdots \otimes \xi_{i_m} \otimes \xi_k$$

$$(L_k^A)^* \xi_{i_1} \otimes \cdots \otimes \xi_{i_m} = A_{k,i_1} \xi_{i_2} \otimes \cdots \otimes \xi_{i_m}$$

$$(R_k^A)^* \xi_{i_1} \otimes \cdots \otimes \xi_{i_m} = A_{i_m,k} \xi_{i_1} \otimes \cdots \otimes \xi_{i_{m-1}}.$$

From this one easily obtains the following result.

Proposition 4.2. *The operators R_k^A and L_k^A are partial isometries and satisfy*

i) $(L_k^A)^* L_k^A = \sum_i A_{ki} L_i^A (L_i^A)^* + P_\Omega$

ii) $(R_k^A)^* R_k^A = \sum_i A_{ik} R_i^A (R_i^A)^* + P_\Omega$

iii) $[L_k^A, R_l^A] = 0$

iv) $[(L_k^A)^*, R_l^A] = \delta_{kl} P_\Omega$

We are now able to construct the element Δ. Let $\mathcal{E} \subseteq \mathcal{B}(\mathcal{F}_A)$ be the C^*-algebra generated by $\{R_1^A, \ldots, R_n^A, L_1^A, \ldots, L_n^A\}$. By Proposition **??** the operator P_Ω, which is compact, is in $\mathcal{E}$. It is easy to check that there is no non-trivial $\mathcal{E}$-invariant subspace of $\mathcal{F}_A$. Thus, $\mathcal{E}$ contains the compact operators, $\mathcal{K}(\mathcal{F}_A)$.

Modulo the ideal $\mathcal{K}(\mathcal{F}_A)$ the elements $L_1^A, \ldots, L_n^A$ and $R_1^A, \ldots, R_n^A$ satisfy the relations for O_A and O_{A^t} respectively. Moreover, the L_i^A's and the R_j^A's commute modulo $\mathcal{K}(\mathcal{F}_A)$. It follows that the C^*-algebra $\mathcal{E}/\mathcal{K}(\mathcal{F}_A)$ is a quotient of $O_A \otimes O_{A^t}$. In fact they are isomorphic. This follows since both O_A and O_{A^t} are nuclear and the ideal structure of their tensor product may be completely described in terms of the ideals of O_A and O_{A^t}. These, in turn, have been completely described in [**?**]. It is then straightforward to verify that the generators of the ideals of $O_A \otimes O_{A^t}$ give rise to non-compact operators (via the L_k^A and R_k^A) and thus $\mathcal{E}/\mathcal{K}(\mathcal{F}_A) \cong O_A \otimes O_{A^t}$.

Definition 4.3. *Let $\Delta \in KK^1(O_A \otimes O_{A^t}, \mathbb{C})$ be the class determined by the exact sequence*

$$0 \longrightarrow \mathcal{K}(\mathcal{F}_A) \longrightarrow \mathcal{E} \xrightarrow{\pi_A} \to O_A \otimes O_{A^t} \to 0. \tag{16}$$

Note that one has

$$\pi_A(R_k^A) = 1 \otimes t_k$$

and

$$\pi_A(L_k^A) = s_k \otimes 1.$$

5. Duality for Cuntz-Krieger algebras

In this section we will show that the duality classes constructed in the previous section actually implement a duality isomorphism for the algebras O_A and O_{A^t}. According to Proposition **??**, it will be sufficient to construct homomorphisms

$$\Theta_{O_A} : O_A \otimes \mathcal{S} \to O_A \otimes \mathcal{S} \tag{17}$$

$$\Theta_{O_{A^t}} : O_{A^t} \otimes \mathcal{S} \to O_{A^t} \otimes \mathcal{S} \tag{18}$$

which satisfy the conditions stated there. That is, we must show that Θ_{O_A} and $\Theta_{O_{A^t}}$ induce the identity homomorphism on K-theory and satisfy the second condition in Proposition **??** which states

$$\sigma_{12}(\delta \otimes_{O_{A^t}} \sigma_{12}(\Delta)) = \Theta_{O_A} \otimes_{O_A \otimes \mathcal{S}} \tau^{O_A}(\mathcal{T}_0)$$

$$\sigma_{12}(\delta \otimes_{O_A} \sigma_{12}(\Delta)) = \Theta_{O_{A^t}} \otimes_{O_{A^t} \otimes \mathcal{S}} \tau^{O_{A^t}}(\mathcal{T}_0)$$

We will work out the details only for Θ_{O_A}, the other case being similar.

To define Θ_{O_A} we first set

$$\bar{\Theta} : O_A \otimes C(S^1) \to O_A \otimes C(S^1)$$

by

$$\bar{\Theta}(1 \otimes z) = 1 \otimes z$$

$$\bar{\Theta}(s_i \otimes 1) = s_i \otimes z.$$

Then $\bar{\Theta}$ extends to an automorphism of $O_A \otimes C(S^1)$, as follows from the universal property of O_A. The diagram

$$\begin{array}{ccc} O_A \otimes C(S^1) & \xrightarrow{\bar{\Theta}} & O_A \otimes C(S^1) \\ {\scriptstyle 1_{O_A} \otimes \pi} \downarrow & & {\scriptstyle 1_{O_A} \otimes \pi} \downarrow \\ O_A & \xrightarrow{id} & O_A \end{array}$$

commutes, where $\pi\colon C(S^1) \to \mathbb{C}$ is defined by $\pi_1(z) = 1$. It follows that we may define $\Theta_{O_A} = \bar{\Theta}|\ker(1_{O_A} \otimes \pi)$. It is an automorphism of $O_A \otimes \mathcal{S}$. We now must show that Θ_{O_A} satisfies the necessary conditions.

Theorem 5.1. *The maps*

$$\Theta_{O_A *}\colon K_i(O_A \otimes \mathcal{S}) \to K_i(O_A \otimes \mathcal{S})$$

are the identity for $i = 0, 1$

Proof. Recall that

$$O_A \otimes \mathcal{K} \cong \bar{F}_A \rtimes_{\sigma_A} \mathbb{Z}$$

where $\bar{F}_A$ is a stable AF-algebra with automorphism σ_A. In this situation, O_A is actually a full corner in $\bar{O}_A$ and compressing $\bar{F}_A$ to this corner yields $\bar{F}_A \subseteq O_A$ which is the closure of the "balanced words" in the s_i's as described in [**?**]. Observe that the restriction of $\bar{\Theta}$ to $F_A \otimes C(S^1)$ is the identity. We will apply the Pimsner-Voiculescu exact sequence to compute $K_*(O_A \otimes \mathcal{S})$, making necessary modifications since $\bar{F}_A$ is not unital and then study $\Theta_{O_A *}$.

Let B denote the multiplier algebra of $\bar{F}_A \otimes \mathcal{S} \otimes \mathcal{K}$ where $\mathcal{K} = \mathcal{K}(l^2(\mathbb{N}))$. Let e_{ij} denote the standard matrix units in $\mathcal{K}$. Define $\rho\colon \bar{F}_A \otimes \mathcal{S} \to B$ by

$$\rho(a \otimes b) = \sum_{i \in \mathbb{N}} \sigma_A^i(a) \otimes f \otimes e_{ii}$$

where the sum is taken in the strict topology. Let S denote the unilateral shift on $\ell^2(\mathbb{N})$. Let D denote the C^*-algebra generated by $\bar{F}_A \otimes \mathcal{S} \otimes \mathcal{K}$, $1 \otimes 1 \otimes S$ and $\{\rho(a \otimes f) | f \in \mathcal{S}, a \in \bar{F}_A\}$. Let D_0 be the ideal in D generated by $\bar{F}_A \otimes \mathcal{S} \otimes \mathcal{K}$ and $\{\rho(a \otimes f) | f \in \mathcal{S}, a \in \bar{F}_A\}$. There is an exact sequence

$$0 \to \bar{F}_A \otimes \mathcal{S} \otimes \mathcal{K} \to D_0 \to \mathcal{S} \otimes (\bar{F}_A \rtimes \mathbb{Z}) \to 0.$$

Moreover, the two maps

$$j\colon \bar{F}_A \otimes \mathcal{S} \to \bar{F}_A \otimes \mathcal{S} \otimes \mathcal{K}$$

defined by $j(a \otimes f) = a \otimes f \otimes e_{11}$ and

$$\rho\colon \bar{F}_A \otimes \mathcal{S} \to D$$

both induce isomorphisms on K-theory.

Finally, we have

$$K_0(\bar{F}_A \otimes \mathcal{S}) \cong K_1(\bar{F}_A) = 0 \tag{19}$$

since $\bar{F}_A$ is an AF-algebra. Putting this together, we obtain the Pimsner-Voiculescu sequence for the $\bar{O}_A$'s:

$$0 \to K_1(\bar{O}_A \otimes \mathcal{S}) \to K_1(\bar{F}_A \otimes \mathcal{S}) \to K_1(\bar{F}_A \otimes \mathcal{S}) \to K_0(\bar{O}_A \otimes \mathcal{S}) \to 0$$

We define an automorphism $\tilde{\Theta}$ of D by

$$\tilde{\Theta} = ad(\sum_{i \in \mathbb{N}} 1 \otimes z^i \otimes e_{ii}) \tag{20}$$

where, again, the sum is in the strict topology. Notice that $\tilde{\Theta} \circ \rho = \rho$ and $\tilde{\Theta}|(\bar{F}_A \otimes \mathcal{S} \otimes \mathcal{K})$ is approximately inner and hence trivial on K-theory. Also observe that $\tilde{\Theta}|(\bar{F}_A \otimes \mathcal{S} \otimes \mathcal{K}) = \bar{F}_A \otimes \mathcal{S} \otimes \mathcal{K}$ and that the automorphism of the quotient of D_0 by $\bar{F}_A \otimes \mathcal{S} \otimes \mathcal{K}$ induced by $\tilde{\Theta}$ is precisely Θ_{O_A}, after identifying this quotient with $\bar{O}_A \otimes \mathcal{S}$ and restricting to $O_A \otimes \mathcal{S} \subseteq \bar{O}_A \otimes \mathcal{S}$. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(\bar{O}_A \otimes \mathcal{S}) & \longrightarrow & K_1(\bar{F}_A \otimes \mathcal{S}) & \longrightarrow & K_1(\bar{F}_A \otimes \mathcal{S}) & \longrightarrow & K_1(\bar{O}_A \otimes \mathcal{S}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Theta_{O_A *}} & & \downarrow{\scriptstyle \tilde{\Theta}_*} & & \downarrow{\scriptstyle \tilde{\Theta}_*} & & \downarrow{\scriptstyle \Theta_{O_A *}} & & \\
0 & \longrightarrow & K_0(\bar{O}_A \otimes \mathcal{S}) & \longrightarrow & K_1(\bar{F}_A \otimes \mathcal{S}) & \longrightarrow & K_1(\bar{F}_A \otimes \mathcal{S}) & \longrightarrow & K_1(\bar{O}_A \otimes \mathcal{S}) & \longrightarrow & 0
\end{array}$$

From the observations above, we have both maps $\tilde{\Theta}_* = id$, and it follows that $\Theta_{O_A *}$ is the identity. $\square$

It remains for us to verify that condition (??) is satisfied. To that end we observe first that

$$\sigma_{12}(\delta \otimes_{O_{A^t}} \sigma_{12}(\Delta)) = \Theta_{O_A} \otimes_{O_A \otimes \mathcal{S}} \tau^{O_A}(\mathcal{T}_0)$$

is equivalent to

$$\tau_{\mathcal{S}}((\Theta_{O_A})^{-1}) \otimes \sigma_{12}\tau_{\mathcal{S}}\tau_{O_A}(\delta) \otimes \tau^{O_A}(\sigma_{12}(\Delta)) = \tau^{O_A}(\mathcal{T}_0). \tag{21}$$

Thus, we will prove the latter statement.

Now, $\tau^{O_A}(\sigma_{12}(\Delta)) \in KK^1(O_A \otimes O_{A^t} \otimes O_A, O_A)$ was obtained from the extension

$$0 \to \mathcal{K} \otimes O_A \to \mathcal{E} \otimes O_A \xrightarrow{\pi_A \otimes 1_{O_A}} O_A \otimes O_{A^t} \otimes O_A \to 0.$$

Moreover, the remaining term

$$\tau_{\mathcal{S}}((\Theta_{O_A})^{-1}) \otimes \sigma_{12}\tau_{\mathcal{S}}\tau_{O_A}(\delta)$$

actually yields a $*$-homomorphism from $O_A \otimes \mathcal{S} \otimes \mathcal{S}$ to $O_A \otimes O_{A^t} \otimes O_A \otimes \mathcal{S}$. Thus, the left side of (??) is represented by applying $\tau_{\mathcal{S}}$ to the element represented by the top row of the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} \otimes O_A & \longrightarrow & \mathcal{E}' & \longrightarrow & O_A \otimes \mathcal{S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1_{O_A} \otimes i} & & \\
0 & \longrightarrow & \mathcal{K} \otimes O_A & \longrightarrow & \mathcal{E}'' & \longrightarrow & O_A \otimes C(S^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \bar{\alpha}} & & \\
0 & \longrightarrow & \mathcal{K} \otimes O_A & \longrightarrow & \mathcal{E} \otimes O_A & \longrightarrow & O_A \otimes O_{A^t} \otimes O_A & \longrightarrow & 0
\end{array}$$

where $\alpha = (\Theta_{O_A})^{-1} \otimes \tau_{O_A}(\delta) = \bar{\alpha} \circ (1_{O_A} \otimes i)$, and $\alpha \otimes 1_{\mathcal{S}} = \tau_{\mathcal{S}}((\Theta_{O_A})^{-1}) \otimes \sigma_{12}\tau_{\mathcal{S}}\tau_{O_A}(\delta)$.

The crucial step is to untwist the middle row by finding an isomorphism $\mathcal{E}'' \cong \mathcal{T} \otimes O_A$ so that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} \otimes O_A & \longrightarrow & \mathcal{E}'' & \longrightarrow & O_A \otimes C(S^1) & \longrightarrow & 0 \\
 & & {\scriptstyle =}\downarrow & & {\scriptstyle \cong}\downarrow & & {\scriptstyle \sigma_{12}}\downarrow & & \\
0 & \longrightarrow & \mathcal{K} \otimes O_A & \longrightarrow & \mathcal{T} \otimes O_A & \xrightarrow[\pi_{\mathcal{T}} \otimes 1_{O_A}]{} & C(S^1) \otimes O_A & \longrightarrow & 0,
\end{array}$$

where $\mathcal{T}$ is the Toeplitz extension.

Assuming this, the proof can be completed as follows. We have

$$\tau_{O_A}(\mathcal{T}) = \sigma_{12}\bar{\alpha}^*(\tau^{O_A}(\sigma_{12}(\Delta))).$$

Hence, one has

$$\begin{aligned}\tau_{O_A}(\mathcal{T}_0) &= (i \otimes 1_{O_A})^*(\tau^{O_A}(\mathcal{T})) \\ &= (i \otimes 1_{O_A})^*\sigma_{12}\bar{\alpha}^*(\tau^{O_A}(\sigma_{12}(\Delta))) \\ &= \sigma_{12}(1_{O_A} \otimes i)^*\bar{\alpha}^*(\tau^{O_A}(\sigma_{12}(\Delta))) \\ &= \sigma_{12}\alpha^*(\tau^{O_A}(\sigma_{12}(\Delta))).\end{aligned}$$

Thus, substituting in for α, we obtain

$$\tau_{O_A}(\mathcal{T}_0) = \tau_{\mathcal{S}}((\Theta_{O_A})^{-1}) \otimes \sigma_{12}\tau_{\mathcal{S}}\tau^{O_A}(\delta) \otimes \tau^{O_A}(\sigma_{12}(\Delta)),$$

which is the desired formula.

We now turn to the issue of obtaining the explicit isomorphism between $\mathcal{E}''$ and $\mathcal{T} \otimes O_A$. For convenience, we will suppress the A in our notation from the elements such as ${R_i}^A$, and ${L_i}^A$. Define W in $\mathcal{E} \otimes O_A$ by

$$W = \sum_{i=1}^{n} R_i \otimes {s_i}^*.$$

We will need two technical lemmas.

Lemma 5.2. *One has*

i) $\pi \otimes 1_{O_A}(W) = \bar{\alpha}(1 \otimes z)$.

ii) $W^*W = \sum_{i,j} A_{ji} R_j {R_j}^* \otimes s_i {s_i}^* + P_\Omega \otimes 1$.

iii) $[W^*, W] = P_\Omega \otimes 1$.

iv) $(P_\Omega \otimes 1)W = 0$.

v) $[W, L_k \otimes 1] = 0$ *for* $k = 1, \ldots, n$.

vi) $[W^*, L_k \otimes 1] = P_\Omega \otimes s_k$ *for* $k = 1, \ldots, n$.

Proof. For (**??**), one proceeds as follows. Note first that

$$\begin{aligned}\bar{\alpha}(1 \otimes z) &= \sigma^{23}(\bar{\Theta}_{O_A}^{-1})^*(1 \otimes \bar{\omega}(z)) \\ &= \sigma^{23}(1 \otimes \bar{\omega}(z)) \\ &= \sigma^{23}(\sum_i 1 \otimes s_i^* \otimes t_i) \\ &= \sum_i 1 \otimes t_i \otimes s_i^*.\end{aligned}$$

Moreover,

$$\begin{aligned}(\pi \otimes 1_{O_A})(W) &= (\pi \otimes 1_{O_A})(\sum_i R_i \otimes s_i^*) \\ &= \sum_i \pi(R_i) \otimes s_i^* \\ &= \sum_i 1 \otimes t_i \otimes s_i^*.\end{aligned}$$

The remaining parts of lemma can be verified in a routine manner. □

The remaining facts we need are incorporated into the following.

Lemma 5.3. *Let $V_k = W^*(L_k \otimes 1)$, for $k = 1 \ldots n$. Then we have, for each k,*

i) $\pi \otimes 1_{O_A}(V_k) = \bar{\alpha}(s_k \otimes 1)$,

ii) $\sum_j V_j V_j^* = W^*W$,

iii) $V_k^* V_k = \sum_j A_{kj} V_j V_j^*$,

iv) $[W, V_k] = 0$,

v) $[W^*, V_k] = 0$.

Proof. As in the previous lemma, we will verify (??) and leave the remaining parts of the proof to the reader, since they are essentially routine. For ??, we check

$$\begin{aligned}(\pi \otimes 1_{O_A})(V_k) &= (\pi \otimes 1_{O_A})(W^*)(\pi \otimes 1_{O_A})(L_k \otimes 1) \\ &= \bar{\alpha}(1 \otimes z)(s_k \otimes 1 \otimes 1),\end{aligned}$$

and

$$\begin{aligned}\bar{\alpha}(s_k \otimes 1) &= \sigma^{23}(1_{O_A} \otimes \bar{w})(\bar{\Theta}_A^{-1})^*(s_k \otimes 1) \\ &= \sigma^{23}(1_{O_A} \otimes \bar{w})(s_k \otimes 1) \\ &= \sigma^{23}(1_{O_A} \otimes w)(s_k \otimes 1 \otimes 1) \\ &= \bar{\alpha}(1 \otimes z)(s_k \otimes 1 \otimes 1).\end{aligned}$$

□

Now we may define the isomorphism from $\mathcal{T} \otimes O_A$ to $\mathcal{E}''$. Let S denote the unilateral shift. The required map is defined by sending $S \otimes 1$ to W, and $1 \otimes S_k$ to V_k. Note that the unit of $\mathcal{T} \otimes O_A$ is mapped to W^*W in $\mathcal{E}''$. The fact that this assignment extends to a *-homomorphism follows from the universal properties of $\mathcal{T}$ and O_A. The fact that it is onto follows from observing that $\mathcal{E}''$ is generated by $\{W, V_1, \ldots, V_k\}$ which is straightforward. Finally, the fact that the appropriate diagram commutes follows from ?? and ??.

6. Final comments

1. As mentioned earlier, the duality theorem holds for the stable and unstable Ruelle algebras associated to a Smale space, [?]. However, the duality classes, Δ and δ, must be constructed in a different way. This is done using asymptotic morphisms and uses the fact that locally the Smale space decomposes into a product of expanding and

contracting sets. It would be very interesting to have a Fock space construction of the more general classes as well.

2. The duality result for Cuntz-Krieger algebras sheds some light on the computations of the K-theory of O_A's as in [**?**]. Recall that if A is an $n \times n$ aperiodic matrix of 0's and 1's , then there are *canonical* isomorphisms

$$K_0(O_A) \cong \mathbb{Z}^n/(1-A^T)\mathbb{Z}^n$$

$$K_1(O_A) \cong \ker(1-A^T)$$

$$K^0(O_A) \cong \ker(1-A)$$

$$K^1(O_A) \cong \mathbb{Z}^n/(1-A)\mathbb{Z}^n.$$

Note that $\mathbb{Z}^n/(1-A)\mathbb{Z}^n \cong \mathbb{Z}^n/(1-A^T)\mathbb{Z}^n$ by the structure theorem for finitely generated abelian groups, but the isomorphism is not natural. The explanation for why one has A^T in the formulas now comes from duality, since one has the diagram

$$\begin{array}{ccc} K_0(O_A) & \xrightarrow{\cong} & K^1(O_{A^t}) \\ \downarrow & & \downarrow \\ \mathbb{Z}^n/(1-A^T)\mathbb{Z}^n & \xrightarrow{=} & \mathbb{Z}^n/(1-A^T)\mathbb{Z}^n. \end{array}$$

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