AN EXCISION THEOREM FOR THE K-THEORY OF C^*-ALGEBRAS

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Abstract. We consider a pair of C^*-algebras $A' \subseteq A$. The K-theory of the mapping cone for this inclusion can be regarded as a relative K-group. We describe a situation where two such pairs have isomorphic relative groups.

§1. Introduction

This paper is concerned with a certain excision result for K-theory of C^*-algebras.

Let us begin by setting out some notation. Let A be any C^*-algebra. We let $A^\sim$ be the C^*-algebra obtained by adjoining a unit to A (even if A is already unital). Let $M_n(A)$ denote the C^*-algebra of $n \times n$ matrices with entries from A. For any a in $A^\sim$ (respectively, $M_n(A^\sim)$), let $\dot{a}$ denote its image in $\mathbb{C}$, the complex numbers, (respectively, $M_n(\mathbb{C})$), under the map moding out by A. We also regard $\mathbb{C}$ and $M_n(\mathbb{C})$ implicitly as subalgebras of $A^\sim$ and $M_n(A^\sim)$, respectively.

Suppose A' is a C^*-subalgebra of A. We regard $A'^\sim \subseteq A^\sim$ as the natural unital inclusion. Recall [Sch, W-O, B1] that the mapping cone for the inclusion $A' \subseteq A$ is

$$C(A'; A) = \Big\{ f : [0,1] \longrightarrow A \mid f \text{ is continuous}, \quad f(0) = 0, \quad f(1) \in A' \Big\}.$$

It is a C^*-algebra with pointwise operations and

$$\|f\| = \sup\{\|f(t)\| \mid 0 \le t \le 1\}$$

for f in $C(A'; A)$. There is a natural short exact sequence

$$0 \longrightarrow C_0(0,1) \otimes A \xrightarrow[i]{} C(A'; A) \xrightarrow[ev]{} A' \longrightarrow 0$$

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where

$$ev(f) = f(1), \qquad f \in C(A'; A)$$

$$i(g \otimes a)(t) = g(t)a, \qquad g \in C_0(0,1), \quad a \in A, \quad 0 \leq t \leq 1.$$

Let $b : K_i(A) \to K_{i+1}(C_0(0,1) \otimes A)$ denote the usual isomorphism [B1]. After using b to replace the terms involving $K_*(C_0(0,1) \otimes A)$, the six-term exact squence for K-groups associated with the sequence above becomes

$$\begin{tikzcd}
K_1(A) \arrow[r, "i_* b"] & K_0(C(A'; A)) \arrow[r, "ev_*"] & K_0(A') \arrow[d, "j_*"] \\
K_1(A') \arrow[u, "j_*"] & K_1(C(A'; A)) \arrow[l, "ev_*"] & K_0(A) \arrow[l, "i_* b"]
\end{tikzcd}$$

where $j : A' \to A$ denotes the inclusion map. We regard $K_*(C(A'; A))$ as a "relative group" for the C^*-algebra inclusion $A' \subseteq A$. Indeed, if A' is actually an ideal in A, then there is a natural isomorphism

$$K_*(C(A'; A)) \cong K_*(A/A').$$

To see this, let

$$J = \{f \in C(A'; A) \mid f(t) \in A' \text{ for all } 0 \leq t \leq 1\},$$

which is an ideal in $C(A'; A)$. Moreover, $J \cong C_0(0,1] \otimes A'$ and so $K_*(J) = 0$, since $C_0(0,1]$ is contractible [W-O, B1]. We also have a short exact sequence

$$0 \to J \to C(A, A') \to C_0(0,1) \otimes (A/A') \to 0.$$

Taking the six-term exact sequence for K-groups and noting $K_*(J) = 0$ yields the result. Thus, if A' is an ideal, $K_*(C(A'; A))$ depends only on A/A'.

Our goal is to describe two pairs of inclusions $A' \subseteq A$ and $B' \subseteq B$ which are related in a specific eay that we may conclude that there is an isomorphism

$$K_*(C(A'; A)) \cong K_*(C(B'; B)),$$

which is natural in some sense. The rôles of A and B here will not be symmetric. In some sense, the inclusion $A' \subseteq A$ will be the more tractible. We suppose that A and B are both

C^*-algebras of operators acting on the Hilbert space $\mathcal{H}$. We suppose that z is a self adjoint unitary on $\mathcal{H}$ and that the following conditions are satisfied. First, B should lie in the multiplier algebra of A. We should have $zAz = A$ and, for all b in B, $zbz - b$ lies in A. One interesting case where this occurs is when $(\mathcal{H}, z)$ is a Fredholm module for B [B1]. Let A be the C^*-algebra of compact operators on $\mathcal{H}$. These conditions are satisfied. Returning to the general situation, we let A' and B' be those operators in A and B, respectively, which commute with z. We require three more technical assumptions on A, B and z (given as 4, 5, 6 in section 3). Under these hypotheses, we construct a homomorphism

$$\alpha : K_*\left(C(B'; B)\right) \longrightarrow K_*\left(C(A'; A)\right)$$

and prove that it is an isomorphism.

The main applications of this result are in various situations arising from dynamical systems where B, B', A and A' can all be described as groupoid C^*-algebras. For example, $B = C(X) \times_\phi \mathbb{Z}$ and $B' = A_Y$ of [Put1], where ϕ is a minimal homeomorphism of a Cantor set X, can be described in this way. Here, A is the compact operator on $\ell^2(\mathbb{Z})$ and A' is the direct sum of compact operators on two orthogonal subspaces. More applications can be found in [Put2]. (Also, see [GPS].)

In Section 2, we provide a description of $K_0\left(C(A'; A)\right)$ which will be useful. In Section 3, we state and prove the main results (3.1 and 3.7).

§2. K-theory of Mapping Cones

Our aim in this section is to provide a natural description of $K_0\left(C(A', A)\right)$.

We begin, as in Section 1, with C^*-algebras $A' \subseteq A$. For each $n = 1, 2, 3, \cdots$, we let $V_n(A'; A)$, or simply V_n, denote the set of elements v in $M_n(A^\sim)$ such that

(i) v is a partial isometry.

(ii) v^*v is in $M_n(\mathbb{C})$.

(iii) vv^* is in $M_n(A'^\sim)$.

(In some ways, it would be more natural to required v^*v to be in $M_n(A'^\sim)$; our definition will be more convenient, however.) We regard $V_n \subseteq V_{n+1}$ by identifying v and $v \oplus 0$, for all v in V_n. We let

$$V(A'; A) = \bigcup_n V_n(A'; A).$$

We will make use of the following two facts:

1. If h is a self-adjoint element of a C^*-algebra and $\|h - h^2\| < \delta < \frac{1}{2}$, then the spectrum of h is contained in $(-2\delta, 2\delta) \cup (1 - 2\delta,\ 1 + 2\delta)$. The proof is an easy application of the spectral theorem.
2. If p_1 and p_2 are projections in a C^*-algebra with $\|p_1 - p_2\| < \delta < \frac{1}{2}$, then there is a unitary u in the C^*-algebra such that $up_1u^* = p_2$ and $\|u - 1\| < \pi\delta$. For a proof, see 4.3.2, 4.6.5 of [B1].

Lemma 2.1. *Suppose* $0 < \epsilon < 100^{-1}$ *and* v *in* $M_n(A^\sim)$ *satisfies* (i) *and* (ii) *above and there exists* q *in* $M_n(A'^\sim)$ *such that* $\|vv^* - q\| < \epsilon$. *Then there exists a unitary* u *in* $M_n(A^\sim)$ *such that* $\|u - 1\| < 30\epsilon$ *and* uv *is in* $V_n(A'; A)$.

Proof. First replace q by $(q + q^*)/2$ so we may assume it is self-adjoint. Since v is a partial isometry, vv^* is a projection and so

$$\|q^2 - q\| < 4\epsilon.$$

Then, using the first fact above, $q_1 = \chi_{(\frac{1}{2}, \infty)}(q)$ is a projection and $\|q_1 - q\| < 8\epsilon$ hence

$$\|q_1 - vv^*\| < 9\epsilon.$$

The second fact above then gives the desired u. ∎

We define a map

$$\kappa : V(A'; A) \longrightarrow K_0\left(C(A'; A)\right).$$

Begin with v in $V_n(A'; A)$. Consider

$$v_1 = \begin{bmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{bmatrix}$$

in $M_{2n}(A^\sim)$. It is easily verified that v_1 is a self-adjoint unitary. We define a path of self-adjoint unitaries in $M_{2n}(A^\sim)$ by

$$v_2(t) = \left[\dot{v}_1 + 1 + e^{i\pi t}(1 - \dot{v}_1)\right]^{-1} \left[v_1 + 1 + e^{i\pi t}(1 - v_1)\right],$$

for $0 \leq t \leq 1$. Notice that v_2 satisfies

(i) $v_2(t)$ is unitary for all t,

(ii) v_2 is in $C[0,1] \otimes M_{2n}(A^\sim)$,

(iii) $\dot{v}_2(t) = 1$, for all t,

(iv) $v_2(0) = 1$,

(v) $v_2(1) = \dot{v}_1^{-1} v_1$.

Together, (ii), (iii) and (iv) imply that v_2 may be regarded as an element of

$$[C_0(0,1] \otimes M_{2n}]^{\sim}.$$

Finally, we define

$$p_v(t) = v_2(t)\, e_{11}\, v_2(t)^*,$$

for $0 \leq t \leq 1$, where e_{11} denotes $1_n \oplus 0$ in $M_{2n}(A^{\sim})$. It is easy to verify that

(i) $p_v(0) = e_{11}$

(ii) $p_v(1) = (1_n - v^*v) \oplus vv^* \in M_{2n}(A'^{\sim})$

(iii) $\dot{p}_v(t) = e_{11}$, for all $0 \leq t \leq 1$.

Thus, p_v is in $M_{2n}\left(C(A';A)^{\sim}\right)$ and $[p_v] - [e_{11}]$ is in $K_0\left(C(A';A)\right)$. We denote this element by $\kappa(v)$. We summarize the properties of κ.

Lemma 2.2.

(i) *For v, w in $V(A';A)$,*

$$\kappa(v \oplus w) = \kappa(v) + \kappa(w).$$

(ii) *If v, w are in $V_n(A';A)$ and $\|v - w\| < 200^{-1}$, then $\kappa(v) = \kappa(w)$.*

(iii) *For v in $V_n(A';A)$, w_1 in $U_n(A'^{\sim})$ and w_2 in $U_n(\mathbb{C})$, then $w_1 v w_2$ is in $V_n(A';A)$ and*

$$\kappa(w_1) = \kappa(w_2) = 0$$

$$\kappa(w_1 v w_2) = \kappa(v).$$

(iv) *For any projection p in $M_n(\mathbb{C})$, $\kappa(p) = 0$.*

(v) *If v is a partial isometry in $M_n(A'^{\sim})$, then $\kappa(v) = 0$.*

Proof. Parts (i) and (iv) are verified by direct computations, which we omit.

In proving (ii), one notes that the construction of p_v depends continuously on v. In fact, $\|v - w\| < 200^{-1}$ implies $\|p_v - p_w\| < \frac{1}{2}$ (we omit the details), which implies $[p_v] = [p_w]$ and the conclusion. As a consequence of (ii), if v and w are homotopic in $V_n(A';A)$ then $\kappa(v) = \kappa(w)$.

In part (iii), we begin by considering $v \oplus 0$, $w_1 \oplus w_1^*$ and $w_2 \oplus w_2^*$. By standard methods (see 4.2.9 of [W-O]), $w_1 \oplus w_1^*$ and $w_2 \oplus w_2^*$ are both homotopic to the identity in

$U_{2n}(A'^{\sim})$ and $U_{2n}(\mathbb{C})$ respectively. Thus, $w_1 v w_2 \oplus 0$ is homotopic to $v \oplus 0$ in $V_{2n}(A';A)$, so $\kappa(v) = \kappa(w_1 v w_2)$ by (ii) and (i). Finally, $\kappa(w_1) = \kappa(w_2) = 0$ both following as special cases ($v = w_2 = 1$, $w_1 = v = 1$) of (iii) and (iv). As for (v), writing

$$v \oplus 0 = \begin{bmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$$

the conclusion follows from (iii) and (iv). ∎

We now want to see how this map κ relates to the six-term exact sequence (1.2).

Lemma 2.3.

(i) *For v in $V_n(A';A)$,*

$$ev_*(\kappa(v)) = [vv^*] - [v^*v].$$

(ii) *For v in $U_n(A^{\sim})$*

$$i_* b[v] = \kappa(v).$$

Proof.

(i) We compute

$$\begin{aligned} ev_*(\kappa(v)) &= [p_v(1)] - [e_{11}] \\ &= [(1_n - v^*v) \oplus vv^*] - [e_{11}] \\ &= [vv^*] - [v^*v]. \end{aligned}$$

(ii) In the construction of $\kappa(v)$, v_2 is a path of unitaries in $M_{2n}(A^{\sim})$ from 1 to $\dot{v}_1^{-1} v_1$. Let $v_3(t)$ be any path of unitaries in $M_{2n}(\mathbb{C})$ from 1 to $\dot{v} \oplus \dot{v}^*$. Then $v_3(t)v_2(t)$ is a path from 1 to $v \oplus v^*$. By the definition of b

$$\begin{aligned} b[v] &= [v_3 v_2 e_{11} v_2 v_3^*] - [e_{11}] \\ &= [v_3 p_v v_3^*] - [e_{11}] \\ &= [p_v] - [e_{11}] \\ &= \kappa(v), \end{aligned}$$

since $v_3(t)$ is in $M_{2n}(\mathbb{C})$. ∎

Lemma 2.4. $\kappa : V(A';A) \to K_0(C(A';A))$ *is onto.*

Proof. Let p, q be projections in $M_m\left(C(A';A)^{\sim}\right)$ with $[\dot{p}] = [\dot{q}]$ in $K_0(\mathbb{C})$; *i.e.* $[p]-[q]$ is in $K_0\left(C(A';A)\right)$. By exactness of (1.2), $j_* ev_*\left([p]-[q]\right) = 0$ in $K_0(A)$. This means $[p(1)] = [q(1)]$ in $K_0(A)$. So there exists positive integers k, $n = 2m+k$ and a partial isometry v in $M_n(A^{\sim})$ such that

$$v^*v = 1_m \oplus 0_m \oplus 1_k$$

$$vv^* = p(1) \oplus (1_m - q(1)) \oplus 1_k.$$

Then v is in $V_n(A';A)$ and by (i) of 2.3, we have

$$ev_*\left([p]-[q]\right) = ev_*\left(\kappa(v)\right).$$

Hence, $\kappa(v) - [p] + [q]$ is in the kernel of ev_* which is the image of i_*. For some unitary w in $M_\ell(A'^{\sim})$, $i_*(w) = \kappa(v) - [p] + [q]$. Using (ii) of 2.3, we have

$$\begin{aligned}\kappa(v \oplus w^*) &= \kappa(v) + \kappa(w^*)\\ &= \kappa(v) - i_*(w)\\ &= [p]-[q]. \qquad \blacksquare\end{aligned}$$

Lemma 2.5. *Let $\approx$ denote the equivalence relation on $V(A';A)$ generated by*

(i) $v \approx v \oplus p$, $v \in V(A';A)$, *p a projection in $M_n(\mathbb{C})$.*

(ii) *If $v(t)$ is a continuous path in $V_n(A';A)$, then $v(0) \approx v(1)$.*

Then $\kappa : V(A';A)/\approx \longrightarrow K_0\left(C(A';A)\right)$ is a well-defined bijection.

Proof. It follows from 2.2 (i), (ii) and (iv) that κ is well-defined. From 2.4, we see that κ is onto. It remains to show that if v_1, v_2 are in $V_n(A';A)$ and $\kappa(v_1) = \kappa(v_2)$, then $v_1 \approx v_2$.

First, note that if v, w_1 and w_2 are as in 2.2(iii), then

$$\begin{aligned}w_1vw_2 &= w_1vw_2 \oplus 0\\ &= (w_1 \oplus w_1^*)(v \oplus 0)(w_2 \oplus w_2^*).\end{aligned}$$

By homotoping the first and third terms of the right hand side, we see that $w_1vw_2 \approx v$.

Returning to v_1 and v_2 with $\kappa(v_1) = \kappa(v_2)$, we may first assume that by taking direct sums with (different) scalar projections that the ranks of $v_1^*v_2$ and $v_2^*v_2$ are equal. We can then right multiply v_1 by a scalar unitary — without changing its $\approx$-equivalence class — to obtain $v_1^* v_1 = v_2^* v_2$.

From $\kappa(v_1) = \kappa(v_2)$, we apply ev_* to both sides, use 2.3(i) and $v_1^* v_1 = v_2^* v_2$ to conclude that $[v_1 v_1^*] = [v_2 v_2^*]$ in $K_0(A'^{\sim})$. Again we may take direct sum with a scalar projection and reduce to the case $v_1 v_1^*$ and $v_2 v_2^*$ are unitarily equivalent. By left multiplying v_1 be a unitary in $M_n(A'^{\sim})$, we obtain $v_1 v_1^* = v_2 v_2^*$, $v_1^* v_1 = v_2^* v_2$, without changing the $\approx$-equivalence class of v_1 or v_2.

Let

$$R_n(t) = \begin{bmatrix} t & -\sqrt{1-t^2} \\ \sqrt{1-t^2} & t \end{bmatrix}, \qquad 0 \le t \le 1$$

be in $M_{2n}(\mathbb{C})$ and define the path in $M_{2n}(A^{\sim})$

$$v(t) = R_n(t)\,[v_1 \oplus v_1^* v_1]\; R_n(t)^{-1}\,[(v_1^* v_2 + 1 - v_1^* v_1) \oplus 1]$$

for $0 \le t \le 1$. Observe that for all t, $v(t)$ is in $V_{2n}(A'; A)$, $v(0) = v_1^* v_2 \oplus v_1$ and $v(1) = v_2 \oplus v_1^* v_1$. We have $v_1^* v_2$ is in $V_n(A'; A)$ and

$$\begin{aligned} \kappa\left(v_1^* v_2\right) &= \kappa\left(v(0)\right) - \kappa(v_1) \\ &= \kappa\left(v(1)\right) - \kappa(v_1) \\ &= \kappa(v_2) - \kappa(v_1) \\ &= 0. \end{aligned}$$

Now, consider the unitary $v = v_1^* v_2 + (1 - v_1^* v_1)$ in $M_n(A^{\sim})$. We have

$$i_* b[v] = \kappa(v) = \kappa(v_1^* v_2) = 0,$$

which implies $[v]$ is in the image of j_*. That is, v is homotopic (after direct summing with the identity) to a unitary in $M_n(A'^{\sim})$. Let $v'(t)$ be any path of unitaries in $M_n(A^{\sim})$ with $v'(0) = v$ and $v'(1) \in M_n(A'^{\sim})$.

Now define a path in $M_{4n}(A^{\sim})$

$$w(t) = \begin{bmatrix} v'(t)v_1 & v'(t)(1 - v_1 v_1^*) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 - v_1^* v_1 & 0 & 0 & 0 \\ 0 & v_1 v_1^* & 0 & 0 \end{bmatrix}.$$

It is straightforward to verify that, for all $0 \leq t \leq 1$,

$$w(t)^* w(t) = 1_n \oplus 1_n \oplus 0_n \oplus 0_n$$

$$w(t) w(t)^* = 1_n \oplus 0 \oplus (1 - v_1^* v_1) \oplus v_1 v_1^*$$

and so $w(t)$ is a path in $V_{4n}(A'; A)$. Evaluating at $t = 0$, we see

$$w(0) = \begin{bmatrix} v_2 & 1 - v_1 v_1^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 - v_1^* v_1 & 0 & 0 & 0 \\ 0 & v_1 v_1^* & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 v_1^* & 1 - v_1 v_1^* & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 - v_1 v_1^* & v_1 v_1^* & 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} v_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 - v_2^* v_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} v_2^* v_2 & 0 & 1 - v_2^* v_2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 - v_2^* v_2 & 0 & v_2^* v_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The first matrix in this product is a unitary in $M_{4n}(A'^{\sim})$, the last in $M_{4n}(\mathbb{C})$ and so

$$w(0) \approx v_2 \oplus 1 \oplus (1 - v_2^* v_2) \oplus 0 \approx v_2.$$

A similar calculation shows $w(1) \approx v_1$ and we are done. ▮

Regarding the relation $\approx$, it is clear that if v_0 and v_1 are homotopic, then for any scalar projection p, $v_0 \oplus p$ and $v_1 \oplus p$ are homotopic. Therefore, if $v_0 \approx v_1$ then there are scalar projections p_0 and p_1 such that $v_0 \oplus p_0$ and $v_1 \oplus p_1$ are homotopic.

A few other remarks are in order. Following exactly as in the beginning of the proof (before $\kappa(v_1) = \kappa(v_2)$ was used), given any v_1 and v_2 in $V(A'; A)$ we may direct sum scalar projections and right multiply by one by a scalar unitary to get $v_1^* v_1 = v_2^* v_2$. Finally, if $v(r)$ is a path in $V_n(A'; A)$, we may right multiply by a path of scalar unitaries so that $v(r)^* v(r) = v(0)^* v(0)$, for all r.

For each $0 < \epsilon < 400^{-1}$, we let $V_n^\epsilon(A'; A)$ denote the set of v in $M_n(A^\sim)$ such that

(i) v is a partial isometry,

(ii) $v^* v$ is in $M_n(\mathbb{C})$,

(iii) $\|vv^* - q\| < \epsilon$, for some q in $M_n(A'^\sim)$.

We let $V^\epsilon(A'; A)$ denote the union of the $V_n^\epsilon(A'; A)$, with the usual inclusion of V_n^ϵ in V_{n+1}^ϵ. For any a in $V^\epsilon(A'; A)$, let v be as in 2.1. We define $\kappa(a) = \kappa(v)$. This is independent of the choice of v by 2.2(ii). It is also easy to see that 2.2 is valid if we replace $V(A'; A)$ by $V^\epsilon(A'; A)$. We extend the definition of $\approx$ to $V^\epsilon(A', A)$ in the obvious way.

Lemma 2.6. *Suppose A has a countable approximate unit $\{e_n\}_1^\infty$ contained in A'. Then for every v in $V_n(A'; A)$ and $0 < \epsilon < 400^{-1}$, $v \approx w$, for some w in $V_{2n}^\epsilon(A'; A)$ such that*

$$w = \begin{bmatrix} w_0 & 0 \\ (p - w_0^* w_0)^{\frac{1}{2}} & 0 \end{bmatrix},$$

where w_0 is in $M_n(A)$, p is a projection in $M_n(\mathbb{C})$ and $0 \le w_0^* w_0 \le p$. Moreover if

$$w = \begin{bmatrix} w_0 & 0 \\ (p - w_0^* w_0)^{\frac{1}{2}} & 0 \end{bmatrix} \qquad w' = \begin{bmatrix} w_0' & 0 \\ (p - w_0'^* w_0')^{\frac{1}{2}} & 0 \end{bmatrix}$$

are homotopic in $V_{2n}^\epsilon(A'; A)$ then there is a path

$$w(t) = \begin{bmatrix} w_0(t) & 0 \\ (p - w_0(t)^* w_0(t))^{\frac{1}{2}} & 0 \end{bmatrix}$$

joining them.

(The point here is that w_0 lies in $M_n(A)$ and not just $M_n(A^\sim)$.)

Proof. Notice that $v \approx \dot{v}^* v$ — see the proof of 2.5 — and $(\dot{v}^* v)^\cdot = \dot{v}^* \dot{v} = p$ is a projection in $M_n(\mathbb{C})$. Thus, we may assume $\dot{v} = p$. Using e_m to denote $1_n \otimes e_m$ in $M_n(A)$, notice that

$$e_m' = \begin{bmatrix} e_m & -(1 - e_m^2)^{\frac{1}{2}} \\ (1 - e_m^2)^{\frac{1}{2}} & e_m \end{bmatrix}$$

is a unitary in $M_{2n}(A'^{\sim})$ so

$$v \approx e_m' \,(v \oplus 0) = \begin{bmatrix} e_m v & 0 \\ (1-e_m^2)^{\frac{1}{2}} v & 0 \end{bmatrix}.$$

We will let $w_0 = e_m v$, for some sufficiently large m, which is in $M_n(A)$. It is clear that $w_0^* w_0 \le p$. Consider

$$\begin{aligned} &\left\| \left(1-e_m^2\right)^{\frac{1}{2}} v - (p - w_0^* w_0)^{\frac{1}{2}} \right\| \\ &\quad \le \left\| \left(1-e_m^2\right)^{\frac{1}{2}} (v-p) \right\| \\ &\qquad + \left\| \left(1-e_m^2\right)^{\frac{1}{2}} p - (p - w_0^* w_0)^{\frac{1}{2}} \right\|. \end{aligned}$$

The first term tends to zero since $v - p$ is in $M_n(A)$ and e_m is an approximate unit. As for the second, since $(1 - e_m^2)$ and p commute, their product is positive and

$$\begin{aligned} &\left\| \left(1-e_m^2\right)^{\frac{1}{2}} p - (p - w_0^* w_0)^{\frac{1}{2}} \right\| \\ &\qquad \le \left\| \left(1-e_m^2\right) p - (p - w_0^* w_0) \right\|^{\frac{1}{2}} \\ &\quad = \left\| (p-v)^* \left(1-e_m^2\right)(p-v) \right\|^{\frac{1}{2}} \end{aligned}$$

which tends to zero as m goes to infinity. Therefore, we may choose m so that $e_m'(v \oplus 0)$ and

$$\begin{bmatrix} w_0 & 0 \\ (p - w_0^* w_0)^{\frac{1}{2}} & 0 \end{bmatrix}$$

are sufficiently close so that the latter is in $V_{2n}^{\epsilon}(A'; A)$ and is $\approx$-equivalent to the former.

For the final part, consider the C^*-algebra $C[0,1] \otimes A$. We omit the details. ∎

§3. The Excision Theorem

Here, we state and prove our main results (Theorems 3.1-3.7). We describe the hypotheses. We suppose that A and B are C^*-algebras acting on the Hilbert space $\mathcal{H}$. We also suppose that z is a self-adjoint unitary operator on $\mathcal{H}$. Note that we regard $M_n(A)$ and $M_n(B)$ as acting on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$, the n-fold direct sum. We also let z denote the operator $z \oplus \cdots \oplus z$ on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. We let $[a, b] = ab - ba$ for any operators a, b on $\mathcal{H}$.

We will assume conditions 1-6 hold.

1. For all a in A, b in B, ab is in A; *i.e.* B acts as multipliers of A.
2. $zAz = A$.
3. For all b in B, $zbz - b$ is in A.
4. There is a continuous path $\{e_t \mid t \geq 0\}$ in A such that
   (i) $0 \leq e_t \leq e_s \leq 1$, for $t \leq s$,
   (ii) $e_s e_t = e_t$ for $s \geq t + 2$,
   (iii) for all a in A,
   $$\lim_{t\to\infty} \|e_t a - a\| = 0 = \lim_{t\to\infty} \|ae_t - a\|.$$
   (iv) $[e_t, z] = 0$, for all t.
   We define C^*-subalgebras
   $$A' = \{a \in A \mid [a, z] = 0\}$$
   $$B' = \{b \in B \mid [b, z] = 0\}.$$
5. For all b in B, there exists b' in B' such that
   $$\|b - b'\| \leq 2\|[b, z]\|.$$
   (In the terminology of M.-D. Choi, almost commuting with z implies nearly commuting with z.)
6. There is a dense $*$-subalgebra $\mathcal{A} \subseteq A$ such that for a in $\mathcal{A}$, there is $t_0 \geq 1$ such that
   (i) $ae_t = e_t a = a$, for all $t \geq t_0$,
   and, for any such t_0 as above, there is b in B such that
   (ii) $be_t = e_t b = a, \quad t_0 \leq t \leq t_0 + 2$.
   (iii) $[b, z] = [a, z]$.
   (iv) $\|b\| \leq \|a\|$.
   (The choice of b will depend on t_0 as well as a.)

Note that the condition on A analogous to 5 is valid; let $a' = (a + zaz)/2$.

Many examples are found in the theory of C^*-algebras associated to dynamical systems via the crossed product or groupoid C^*-algebra constructions. Let us mention one explicit example.

Fix an irrational number θ, $0 < \theta < 1$. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and let u and v denote the unitary operators

$$(u\xi)(n) = \xi(n-1)$$

$$(v\xi)(n) = \exp(2\pi i\theta)\xi(n),$$

for ξ in $\ell^2(\mathbb{Z})$, n in $\mathbb{Z}$. Then u and v satisfy the relation $uv = \exp(2\pi i\theta)\, vu$ and generate a C^*-algebra, B, isomorphic to the irrational rotation C^*-algebra, A_θ. We let $A = K(\mathcal{H})$, the compact operators, and

$$(z\xi)(n) = \begin{cases} \xi(n) & n \geq 1 \\ -\xi(n) & n \leq 0. \end{cases}$$

It is easy to verify 1, 2 and 3. It is also easy to see that

$$A' = K\left(\ell^2\{n \mid n \leq 0\}\right) \oplus K\left(\ell^2\{n \mid n \geq 1\}\right).$$

The proofs that 4, 5 and 6 hold can be found in [Put2]. Also the techniques of [Put2] show that B' is the C^*-subalgebra of B generated by v and $u(v-1)$. (See example 2.6 of [Put2].)

Theorem 3.1. *Let A, B, z satisfy* 1-6 *as above. Then there is an isomorphism*

$$\alpha : K_0\left(C(B'; B)\right) \to K_0\left(C(A', A)\right)$$

which is natural in a sense to be described.

Let us take a moment to try to justify our description of 3.1 as an "excision" theorem. Section 2 describes the K-theory of the mapping cone $C(A'; A)$ as partial isometries in A with initial and final projection in A'. The extent to which an element a lies in A' can be measured by $zaz - a = z[a, z]$. A similar remark applies to B' and B. Conditions 2, 3 and 6(iii) essentially mean that the sets

$$\{zaz - a \mid a \in A\}$$

$$\{zbz - b \mid b \in B\}$$

"agree". The conclusion is then that the corresponding "relative K-groups" are isomorphic.

We begin by describing the map α. We use e_t to also denote the element $1_n \otimes e_t$ in $M_n(A)$, for any $n = 1, 2, 3, \cdots$. We will use the description of $K_0\left(C(B'; B)\right)$ provided by

Lemma 2.5 and the discussion following it. Let v be in $V_n^\epsilon(B';B)$. For all $t \geq 1$, we define $\alpha(v)_t$ by

$$\alpha(v)_t = \begin{bmatrix} ve_t & 0 \\ (v^*v - e_t v^* v e_t))^{\frac{1}{2}} & 0 \end{bmatrix}$$

Since B acts as multipliers of A, ve_t is in $M_n(A)$. Also, v^*v is a projection in $M_n(\mathbb{C})$ and it follows that $\alpha(v)_t$ lies in $M_{2n}(A^\sim)$. It is also worth noting that e_t and v^*v commute so that

$$(v^*v - e_t v^* v e_t)^{\frac{1}{2}} = v^*v\left(1 - e_t^2\right)^{\frac{1}{2}}.$$

It is easy to check that

$$\alpha(v)_t^* \, \alpha(v)_t = v^*v \oplus 0,$$

which is in $M_{2n}(\mathbb{C})$ and is a projection.

Lemma 3.2. *For v in $V_n^\epsilon(B';B)$ and $0 < \epsilon < 400^{-1}$, there is $t \geq 1$ such that $\alpha(v)_s$ is in $V_{2n}^\epsilon(A';A)$ for all $s \geq t$.*

Proof. We claim that

$$\limsup_{t\to\infty} \|[\alpha(v)_t \, \alpha(v)_t^*, \ z]\| \leq \epsilon.$$

To see this,

$$\alpha(v)_t \, \alpha(v)_t^* = \begin{bmatrix} ve_t^2 v^* & ve_t\left(1 - e_t^2\right)^{\frac{1}{2}} \\ \left(1 - e_t^2\right)^{\frac{1}{2}} e_t v^* & v^*v\left(1 - e_t^2\right) \end{bmatrix}$$

and we will check the commutators of the four entries with z separately. The lower right entry actually commutes with z since e_t does and v^*v is in $M_n(\mathbb{C})$. As for the upper right (or lower left)

$$\begin{aligned} \lim_{t\to\infty} \left[ve_t\left(1 - e_t^2\right)^{\frac{1}{2}}, \ z\right] &= \lim_{t\to\infty} [v, z]\, e_t\left(1 - e_t^2\right)^{\frac{1}{2}} \\ &= 0 \end{aligned}$$

since $z[v, z]$ is in $M_n(A)$ and e_t is an approximate unit for A. For the upper left entry, we have

$$\begin{aligned} &\limsup_{t\to\infty} \left\| \left[ve_t^2 v^*, \ z\right] \right\| \\ &\quad = \limsup_{t\to\infty} \left\|[v, z]\, e_t^2 v^* + ve_t^2 [v^*, z]\right\|. \end{aligned}$$

Since $z[v, z]$ and $z[v^*, z]$ are both in A, e_t will asymptotically commute both, so this equals

$$\limsup_{t\to\infty} \left\|e_t^2 [v, z]\, v^* + v[v^*, z]\, e_t^2\right\|.$$

Applying the same argument and noting $[v, z]\, v^*$ is in $M_n(A)$ since v^* is in the multiplier algebra of $M_n(A)$, this equals

$$\begin{aligned}\limsup_{t\to\infty} &\left\| \left([v, z]\, v^* + v[v^*, z] \right)\, e_t^2 \right\| \\ &= \limsup_{t\to\infty} \left\| [vv^*,\, z]\, e_t^2 \right\| \\ &\leq \epsilon\end{aligned}$$

since vv^* is within ϵ of an element of in $M_n(A'^{\sim})$. The claim is established.

To see the conclusion, let

$$q = \frac{z\, \alpha(v)_t\, \alpha(v)_t^* z + \alpha(v)_t\, \alpha(v)_t^*}{2}.$$

Now, (iii) follows from the claim and it is clear that q is in $M_{2n}(A'^{\sim})$. ∎

Notice that

$$\alpha(v \oplus w)_t = \alpha(v)_t \oplus \alpha(w)_t$$

(at least after a change of basis which we will suppress). It follows from 3.2 that letting

$$\alpha\left(\kappa(v)\right) = \kappa\left(\alpha(v)_s\right),)$$

for any sufficiently large s defines an element in $K_0\left(C(A'; A)\right)$. To see that α is well-defined it suffices to apply Lemma 2.5 and observe the following. If p is a projection in $M_n(\mathbb{C})$ then

$$\alpha(p)_t = e_t'(p \oplus 0),$$

where e_t' is as in 2.6. So then $\kappa\left(\alpha(p_t)_t\right) = 0$ by 2.2(ii), (iii).

Also observe that if $v(r)$, $0 \leq r \leq 1$ is a path in $V_n^{\epsilon}(B'; B)$ then the limit in 3.2 can be made uniform over r, and, hence, for s large $\alpha\left(v(r)\right)_s$ will be a homotopy in $V_{2n}^{2\epsilon}(A'; A)$.

The proof of 3.1 will require several technical Lemmas.

Lemma 3.3. *Let w_0 be in $M_n(\mathcal{A})$ and p be a projection in $M_n(\mathbb{C})$ such that $p \geq w_0^* w_0$. Then there is $t_0 \geq 1$ and v_0 in $M_n(B)$ with $v_0^* v_0 \leq p$ such that*

(i) $w_0 e_s = e_s w_0 = w_0$, *for* $s \geq t_0$

(ii) $v_0 e_s = e_s v_0 = w_0$, *for* $t_0 + 2 \geq s \geq t_0$

(iii) $[v_0, z] = [w_0, z]$

(iv) $[v_0^* v_0,\, z] = [w_0^* w_0,\, z]$

(v) $[v_0 v_0^*,\, z] = [w_0 w_0^*,\, z]$

(vi) $\left[(p - v_0^* v_0)^{\frac{1}{2}},\, z\right] = \left[(p - w_0^* w_0)^{\frac{1}{2}}\,,\, z\right]$.

Proof. Choose any t_0 and b as in hypothesis 6 for $a = w_0$. Then let

$$v_0 = bp \text{ so } v_0^* v_0 = p\, b^* b\, p \leq p \|b\|^2\, p \leq p.$$

Conditions (i), (ii) and (iii) follow at once from hypothesis 6.

We have

$$\begin{aligned}
[v_0^* v_0,\, z] &= [v_0^*,\, z]\, v_0 + v_0^*[v_0,\, z] \\
&= [w_0^*,\, z]\, v_0 + v_0^*[w_0,\, z] \\
&= [w_0^* e_t,\, z]\, v_0 + v_0^*[e_t w_0,\, z], \quad \text{for } t_0 \leq t \leq t_0 + 2 \\
&= [w_0^*,\, z]\, e_t v_0 + v_0^* e_t [w_0,\, z] \\
&= [w_0^*,\, z]\, w_0 + w_0^*[w_0,\, z] \quad \text{by (ii)} \\
&= [w_0^* w_0,\, z]
\end{aligned}$$

and so (iv) holds. A similar argument establishes (v). As for (vi), it follows from (iv) that

$$[f(p - v_0^* v_0),\, z] = [f(p - w_0^* w_0),\, z]$$

for any polynomial f. By standard approximation arguments, the same holds for $f(t) = t^{\frac{1}{2}}$. ▮

Lemma 3.4. *Let w_0, p, t_0, v_0 be as in* 3.3. *Define w in $M_{2n}(A^{\sim})$ and v in $M_{2n}(B^{\sim})$ by*

$$w = \begin{bmatrix} w_0 & 0 \\ (p - w_0^* w_0)^{\frac{1}{2}} & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} v_0 & 0 \\ (p - v_0^* v_0)^{\frac{1}{2}} & 0 \end{bmatrix}.$$

Then

(i) $w^* w = v^* v = p \oplus 0$,

(ii) $e_s[v, z] = [v, z]\, e_s = [v, z] = [w, z] \quad \text{for} \quad s \geq t_0$,

(iii) $[ww^*,\, z] = [vv^*,\, z]$.

The proof is an easy consequence of 3.3; we omit the details.

Lemma 3.5. *Let w_0 be in $M_n(\mathcal{A}^\sim)$, p a projection in $M_n(\mathbb{C})$ with $p \geq w_0^* w_0$. Let t_0, v_0 be as in 3.3, w, v as in 3.4 and assume w is in $V_{2n}^{\epsilon}(A'; A)$ for some $0 < \epsilon < 400^{-1}$. Then*

(i) *v is in $V_{2n}^{4\epsilon}(B'; B)$,*

(ii) *$\alpha(v)_s$ is in $V_{4n}^{4\epsilon}(A'; A)$, for all $s \geq t_0$,*

(iii) *$\kappa\left(\alpha(v)_s\right) = \kappa(w)$, for $t_0 \leq s \leq t_0 + 2$.*

Proof.

(i) From 3.4(i), $v^* v = p \oplus 0$ and we must check only that vv^* is close to an element of $M_{2n}(B'^\sim)$. From 3.4(iii)

$$\|[vv^*,\, z]\| = \|[ww^*,\, z]\| \leq 2\epsilon$$

since w is in $V_{2n}^{\epsilon}(A'; A)$. Apply hypothesis 5 to find q in $M_{2n}(\mathcal{B}'^\sim)$ so that $\|q - vv^*\| \leq 4\epsilon$, and (i) is complete.

(ii) As before, we must compute

$$\|[\alpha(v)_s\, \alpha(v)_s^*,\, z]\|\,.$$

Now, for $s \geq t_0$,

$$\alpha(v)_s\, \alpha(v)_s^* = \begin{bmatrix} v e_s^2 v^* & v e_t \left(1 - e_t^2\right)^{\frac{1}{2}}\, v^* v \\ v^* v \left(1 - e_t^2\right)^{\frac{1}{2}}\, e_t v^* & v^* v \left(1 - e_t^2\right) \end{bmatrix}$$

and commutators with z for each of the entries is done separately. The off-diagonal entries commute with z because $v^* v = p$ and by condition (ii) of 3.4, so $(1 - e_t)[v, z] = 0$. The lower right entry also commutes with z while

$$\left[v e_s^2 v^*,\, z\right] = [ww^*,\, z] \quad \text{for} \quad s \geq t_0.$$

This completes the proof of (ii).

(iii) By direct computation

$$\alpha(v)_s = \begin{bmatrix} v_0 e_s & 0 & 0 & 0 \\ (p - v_0^* v_0)^{\frac{1}{2}}\, e_s & 0 & 0 & 0 \\ p(1-e_2^2)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e_s & -\left(1-e_s^2\right)^{\frac{1}{2}} & 0 \\ 0 & \left(1-e_s^2\right)^{\frac{1}{2}} & e_s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} w_0 & 0 & 0 & 0 \\ (p - w_0^* w_0)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for $t_0 \leq s \leq t_0 + 2$, using Lemma 3.2. The first matrix above is in $M_{4n}\ (A'^{\sim})$ and so the result follows from 2.2(iii). ▮

Lemma 3.6 *Suppose v is in $V_n(B'; B)$ and $\|[v, z]\| \leq \epsilon \leq 10^{-6}$. Then $\kappa(v) = 0$.*

Proof. By hypothesis 5, there is a v' in $M_n(B'^{\sim})$ such that $\|v'\| \leq 1$ and $\|v - v'\| \leq 2\epsilon$. Let

$$w = \begin{bmatrix} v'p & 0 \\ (p - pv'^* v' p)^{\frac{1}{2}} & 0 \end{bmatrix},$$

where $p = v^* v$, so w is in $V_{2n}(B'; B)$ and in $M_{2n}(B'^{\sim})$ and

$$\|v \oplus 0 - w\| \leq 4\epsilon^{\frac{1}{2}}.$$

Moreover, $\kappa(w) = 0$ by 2.2(v) and $\kappa(v) = \kappa(w)$ by 2.2(ii). ▮

Let us describe the naturality of the isomorphism described in 3.1. Suppose $\left(A_1, B_1, z_1, \{e_t^{(1)}\}\right)$ and $\left(A_2, B_2, z_2, \{e_t^{(2)}\}\right)$ are two systems satisfying 1-6. Also suppose

$$\sigma : A_1 \longrightarrow A_2$$
$$\pi : B_1 \longrightarrow B_2$$

a $*$-homomorphisms such that

$$\sigma(ab) = \sigma(a)\pi(b), \qquad a \in A_1,\ b \in B_1$$

$$\sigma(z_1 a z_1) = z_2\, \sigma(a)\, z_2, \qquad a \in A_1$$

$$\pi(z_1 b z_1) = z_2\, \pi(b)\, z_2, \qquad b \in B_1$$

$$\sigma(z_1 b z_1 - b) = z_2\, \pi(b)\, z_2 - \pi(b), \qquad b \in B_1$$

$$\sigma\left(e_t^{(1)}\right) = e_t^{(2)}, \qquad \text{for all } t.$$

It is easy to see that σ and π induce $*$-homomorphisms

$$\tilde{\sigma} : C\left(A_1';\, A_1\right) \longrightarrow C\left(A_2';\, A_2\right)$$

$$\tilde{\pi} : C\left(B_1';\, B_1\right) \longrightarrow C\left(B_2';\, B_2\right).$$

The map α is natural in the sense that the following diagram commutes:

$$\begin{array}{ccc} K_0\left(C(B_1';\, B_1)\right) & \xrightarrow{\ \alpha\ } & K_0\left(C(A_1';\, A_1)\right) \\ \downarrow{\scriptstyle \tilde{\pi}_*} & & \downarrow{\scriptstyle \tilde{\sigma}_*} \\ K_0\left(C(B_2';\, B_2)\right) & \xrightarrow{\ \alpha\ } & K_0\left(C(A_2';\, A_2)\right) \end{array}$$

The proof of this is immediate. We omit the details.

As an application, suppose (A, B, z, e_t) satisfies 1-6 and suppose X is a compact second countable Hausdorff space. Fix some regular Borel measure μ on X with full support. Then we can regard $A \otimes C(X)$, $B \otimes C(X)$ and $z \otimes 1$ as operators on $\mathcal{H} \otimes L^2(X, \mu)$. Hypotheses 1-3 are easily checked and $e_t \otimes 1$ satisfies 4. We also have

$$(A \otimes C(X))' = A' \otimes C(X)$$

$$(B \otimes C(X))' = B' \otimes C(X)$$

and 5 follows. The algebraic tensor produce of $\mathcal{A}$ and $C(X)$ can be seen to satisfy 6.

Proof of 3.1. First of all, it is fairly clear that α is additive. The surjectivity of α follows at once from Lemmas 2.6 and 3.5.

Suppose v is in $V_n(B';B)$ and $\alpha(\kappa(v)) = 0$ in $K_0(C(A';A))$. Let $p = v^*v$ which is a projection in $M_n(\mathbb{C})$. Fix $\epsilon = 10^{-7}$. Choose $t_1 \geq 1$ such that

(1) $$\|[v,z]\,e_t - [v,z]\| \leq \epsilon$$

$$\|[v,z] - [v,z]\,e_t\| \leq \epsilon, \quad t \geq t_1$$

and such that

(2) $$\alpha(v)_t \in V_{2n}^{\epsilon}(A';A), \quad t \geq t_1.$$

Since $\kappa(\alpha(v)) = 0$, we may direct sum $\alpha(v)_{t_1}$ with a scalar projection q so that the result is homotopic to a scalar projection in $V^{\epsilon}(A';A)$. By replacing v by $v \oplus q$, we may assume simply that $\alpha(v)_{t_1}$ is homotopic to $\begin{bmatrix} 0 & 0 \\ p & 0 \end{bmatrix}$, which is homotopic to $p \oplus 0$. We apply Lemma 2.6 to obtain a path as described there. We may then approximate the "w_0" part of this path by a path in $M_n(\mathcal{A})$. We right multiply this path by p and we obtain a path $a(s)$, $0 \leq s \leq 1$, such that a is in the algebraic tensor product of $C[0,1]$ and $M_n(\mathcal{A})$,

(3) $$w(s) = \begin{bmatrix} a(s) & 0 \\ (p - a(s)^*a(s))^{\frac{1}{2}} & 0 \end{bmatrix}, \quad 0 \leq s \leq 1,$$

$$\in V_{2n}^{2\epsilon}(A';A)$$

(4) $$a(1) = 0$$

$$\|w(0) - \alpha(v)_{t_1}\| \leq 2\epsilon,$$

hence,

$$\|a(0) - ve_{t_1}\| \leq 2\epsilon,$$

$$\left\|(p - a(0)^*a(0))^{\frac{1}{2}} - p\left(1 - e_{t_1}^2\right)^{\frac{1}{2}}\right\| \leq 2\epsilon.$$

We may apply the sequence of Lemmas 3.3, 3.4 and 3.5 to the element a in $M_n(\mathbb{C}[0,1] \odot \mathcal{A})$ (algebraic tensor product) and p in $M_n(\mathbb{C})$ to obtain a path $b(s)$, $0 \leq s \leq 1$

$$v_1(s) = \begin{bmatrix} b(s) & 0 \\ (p - b(s)^*b(s))^{\frac{1}{2}} & 0 \end{bmatrix}$$

$0 \le s \le 1$ and $t_2 \ge t_1 + 2$ such that

$$[b(s), z] = [a(s), z], \tag{6}$$

$$b(s)\, e_t = e_t\, b(s), \quad t_2 \le t \le t_2 + 2, \tag{7}$$

$$a(s)\, e_t = e_t\, a(s) = a(s), \quad t \ge t_2, \tag{8}$$

$$[b(s)^* b(s),\, z] = [a(s)^* a(s),\, z] \tag{9}$$

$$[b(s) b(s)^*,\, z] = [b(s) b(s)^*,\, z] \tag{10}$$

$$\left[(p - b(s)^* b(s))^{\frac{1}{2}}\,,\, z\right] = \left[(p - a(s)^* a(s))^{\frac{1}{2}}\,,\, z\right], \tag{11}$$

$$v_1(s) \text{ is in } V_{2n}^{4\epsilon}(B'; B)$$

$$\alpha\,(v_1(s))_t \text{ is in } V_{4n}^{4\epsilon}(A'; A), \quad t \ge t_2.$$

Let us evaluate v_1 at $s = 1$. Making use of (4), (6) and (9), we see that

$$[v_1(1),\, z] = 0 \tag{12}$$

and so $v_1(1)$ is in $M_n(B'^{\sim})$. Next, we claim that

$$\|[v\, b(0)^*,\, z]\| \le 3\epsilon, \tag{13}$$

$$\left\| \left[v\, (p - b(0)^* b(0))^{\frac{1}{2}}\,,\, z \right] \right\| \le 3\epsilon. \tag{14}$$

To see the first, we have

$$\|[v\, b(0)^*,\, z]\| = \|[v, z]\, b(0)^* + v\, [b(0)^*,\, z]\|$$

$$\le \|[v, z]\, e_{t_1}\, b(0)^* + v\, [a(0)^*,\, z]\| + \epsilon$$

by (1) and (6),

$$\le \|[v, z]\, e_{t_1} e_{t_2}\, b(0)^* + v\, [e_{t_1}\, v^*,\, z]\| + \epsilon$$

by hypothesis 4(ii) and (5),

$$= \|[v, z]\, e_{t_1}\, a(0)^* + v e_{t_1} [v^*, z]\| + \epsilon$$

by (7)

$$\leq \left\|[v, z]\, e_{t_1}^2\, v^* + v e_{t_1}^2 [v^*, z]\right\| + 2\epsilon$$

by (5) and (1)

$$= \left\| \left[v e_{t_1}^2\, v^*,\; z\right] \right\| + 2\epsilon$$

$$\leq 3\epsilon$$

because of (2). To see the second, there is a similar computation which we omit.

Now consider

$$v_2(s) = (v \oplus 0)\, v_1(s)^*, \qquad 0 \leq s \leq 1.$$

This is a path of partial isometries in $M_{2n}(B^{\sim})$. For each s, its range projection is the range projection of v which is in $M_{2n}(B'^{\sim})$. Its initial projection is the range projection of $v_1(s)$ which is in $M_{2n}(B'^{\sim})$, for all s. As noted in (12), when $s = 1$, this projection is actually Murray-von Neumann equivalent to $p \oplus 0$ in $M_{2n}(B'^{\sim})$. So we may find a path of unitaries $u(s)$, $0 \leq s \leq 1$ in $M_{2n}(B'^{\sim})$ (actually, it may be necessary to pass to $M_{4n}(B'^{\sim})$) such that

$$v_1(1)^*\, u(1) = p \oplus 0$$

$$v_1(s)^*\, u(s) \text{ has initial projection } p \oplus 0,$$

$$0 \leq s \leq 1.$$

Now, consider the path

$$v_3(s) = (v \oplus 0)\, v_1(s)^*\, u(s), \qquad 0 \leq s \leq 1.$$

It is a path in $V_{2n}(B'; B)$. Moreover, for $s = 1$,

$$v_3(1) = v \oplus 0$$

while for $s = 0$,

$$v_3(0) = \begin{bmatrix} v\, b(0)^* & v\, (p - b(0)^*\, b(0))^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix} u(0)$$

which commutes with z, to within 3ϵ, by (13) and (14). By Lemma 2.2(v) and the homotopy invariance of κ,

$$\kappa(v) = \kappa\left(v_3(1)\right) = \kappa\left(v_3(0)\right) = 0.$$

This proves that α is injective and we are done. ∎

Theorem 3.7. *Let A, B, z satisfy* 1-6 *as before. Then there are isomorphisms*

$$\alpha : K_i\left(C(B'; B)\right) \longrightarrow K_i\left(C(A'; A)\right),$$

which are natural, for $i = 0, 1$.

Proof. The case $i = 0$ is done. For the other case, let $B_1 = C(S^1) \otimes B$, $A_1 = C(S^1) \otimes A$, $z_1 = 1 \otimes z$ and $\sigma : A_1 \to A$, $\pi : B_1 \to B$ be given by evaluation at some fixed point of the circle, S^1. There is a split exact sequence

$$0 \to C_0(0,1) \otimes C(B'; B) \to C(B'_1; B_1) \xrightarrow{\pi} C(B'; B) \to 0$$

and a corresponding one for A and A_1. Using the naturality of α on K_0 and the usual isomorphism

$$K_1\left(C(B'; B)\right) \cong K_0\left(C_0(0,1) \otimes C(B'; B)\right)$$

and the usual techniques, one obtains the result for K_1 groups as well. ∎

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