OPERATOR ALGEBRAS AND HYPERBOLIC DYNAMICS

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1. Introduction

The interaction between operator algebras and ergodic theory is a well established part of both areas. Similarly, the connections between C^*-algebras and topological dynamics enriches both subjects. In this note we discuss some algebras associated to hyperbolic dynamical systems and describe some of their properties. They are higher dimensional generalizations of Cuntz-Krieger algebras.

2. Hyperbolic dynamical systems

The class of dynamical systems we will consider are expansive homeomorphisms of compact metric spaces, (X, f), which have canonical coordinates. These have been called Smale spaces by Ruelle, [17]. To be more precise, a homeomorphism $f\colon X \to X$ is expansive if there is an expansivity constant $\delta > 0$ so that if $d(f^n(x), f^n(y)) < \delta$ for all n, then $x = y$. The ε-stable set at x is $W^s(x, \varepsilon) = \{y\colon d(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n \geq 0\}$ and the ε-unstable set is $W^u(x, \varepsilon) = \{y\colon d(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n \leq 0\}$. The meaning of canonical coordinates is that there exist an $\varepsilon_0 > 0$ so that for any ε with $\varepsilon_0 > \varepsilon > 0$ there is a homeomorphism of $W^s(x, \varepsilon) \times W^u(x, \varepsilon)$ with a neighborhood of x satisfying certain properties, [17]. Further, there is a constant λ, with $0 < \lambda < 1$ such that the homeomorphism f expands by a factor of λ in the ε-unstable set and contracts by λ on the ε-stable set. We require also that (X, f) is topologically mixing.

It is a consequence of these properties that a Smale space has a dense set of periodic points and also a dense orbit. Moreover, its topological entropy is $\log \lambda > 0$, so it can be considered a "chaotic" dynamical system. Basic sets

The first and third authors were supported by NSF grants. The second author was supported by an NSERC grant.

of Axiom A diffeomorphisms have this structure. If X is a Smale space of dimension zero then (X, f) is conjugate to (Σ_A, σ_A), a subshift of finite type. The 2-adic solenoid with the shift automorphism and the 2-dimensional torus with the automorphism induced by the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ are higher dimensional examples.

There is an invariant measure, μ, on X, the Bowen measure, distinguished by the property of having maximal entropy. It has the additional property that in a canonical coordinate system at x it can be expressed as a product measure,

$$\mu = \mu_s^x \times \mu_u^x,$$

with the measures μ_s^x and μ_u^x no longer invariant, but rather scale by λ and λ^{-1} respectively under the action of f, [19].

In the next section we will associate operator algebras to these dynamical systems.

3. Ruelle algebras

One possibility for a C^*-algebra to associate to (X, f) is the crossed product, $C(X) \rtimes \mathbb{Z}$. While this can be useful, it is possible to make a different construction in the spirit of the work of Cuntz and Krieger, [5]. These algebras, which are called Ruelle algebras, possess several properties which the crossed products do not have. The basic difference in the construction is that we replace $C(X)$ by a noncommutative algebra before taking the crossed product. In many cases this algebra is a deformation of $C(X)$ and it would be interesting to obtain a general result of this sort. While $C(X)$ is unital with many ideals and with an invariant trace given by μ, its replacement will be stable, simple and have a trace which is scaled by the automorphism induced by f.

There are two equivalence relations defined on X called stable and unstable equivalence. Points x and y are stably equivalent if $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$, and they are unstably equivalent if $\lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0$. We denote this by $x \underset{s}{\sim} y$ or $x \underset{u}{\sim} y$, and write $\mathcal{S}, \mathcal{U} \subseteq X \times X$ for the equivalence relations. The algebras we will construct are obtained as the C^*-algebras of these groupoids, [16]. In order to obtain this we must assign locally compact topologies and Haar systems to the relations. We will give the definitions for $\mathcal{S}$, those for $\mathcal{U}$ being strictly analogous.

One can write $\mathcal{S}$ as a union $\mathcal{S} = \bigcup \mathcal{S}^n$, where $\mathcal{S}^n$ is defined inductively as follows. Let $\mathcal{S}^0 = \{(x, y) : y \in W^s(x, \varepsilon_0)\}$ and set $\mathcal{S}^n = (f \times f)^{-n}(\mathcal{S}^0)$. Give $\mathcal{S}^n$ the relative topology as a subset of $X \times X$ and let $\mathcal{S}$ have the direct limit topology. With this topology $\mathcal{S}$ is a locally compact groupoid. We next define a Haar system for $\mathcal{S}$. For $x \in X$ let μ_s^x be the measure coming from decomposing the Bowen measure in a canonical coordinate neighborhood of x. Let $\mu_0^x = \delta_x \times \mu_s^x$. This defines a family of measures parametrized by X on $\mathcal{S}^0$. Extend them to $\mathcal{S}$ by defining $\mu_n^x = \lambda^{-n}(\delta_x \times \mu_s^x) \circ (f \times f)^n$ on $\mathcal{S}^n$, and call the resulting measure $\tilde{\mu}_s^x$.

3.1 Proposition [13]. *The measures $\{\tilde{\mu}_s^x\}$ form a Haar system on the locally compact groupoid $\mathcal{S}$.*

In a similar manner one obtains a Haar system $\{\tilde{\mu}_u^x\}$ on $\mathcal{U}$.

3.2 Definition. *The stable and unstable algebras associated to* (X, f) *are* $C^*(\mathcal{S})$ *and* $C^*(\mathcal{U})$*, the reduced* C^**-algebras of the groupoids* $\mathcal{S}$ *and* $\mathcal{U}$*.*

The homeomorphism f induces automorphisms, f_*^s and f_*^u, of these algebras and we may form the crossed products.

3.3 Definition. *The stable and unstable Ruelle algebras are* $\mathcal{R}^s = \mathcal{S} \rtimes \mathbb{Z}$ *and* $\mathcal{R}^u = \mathcal{U} \rtimes \mathbb{Z}$*.*

Properties of the algebras.

We will now list some basic properties of these algebras, referring to [6], [7], [13], [14], [15] for proofs.

3.4 Theorem. *Let* (X, f) *be a topologically mixing Smale space. Then one has*

(1) *The algebras* $\mathcal{S}$ *and* $\mathcal{U}$ *are separable, nuclear and simple.*
(2) *There are traces,* τ_s *and* τ_u *which are scaled by the automorphisms in the sense that* $\tau_s(f_*^s(a)) = \lambda \tau_s(a)$ *and* $\tau_u(f_*^u(b)) = \lambda^{-1}\tau_u(b)$*.*
(3) *There are faithful representations* $\rho_s,\ \rho_u \colon \mathcal{S},\ \mathcal{U} \to \mathcal{L}(\mathcal{H})$ *which have the property that*
$$\rho_s(a)\rho_u(b) \in \mathcal{K}.$$
(4) *The images of* ρ_s *and* ρ_u *asymptotically commute in the sense that, for* $a \in \mathcal{S}$ *and* $b \in \mathcal{U}$*, one has*
$$\lim_{n\to\infty} \|[\rho_s((f_*^s)^n(a)), \rho_u(b)]\| = 0.$$

In all known examples the algebras $\mathcal{S}$ and $\mathcal{U}$ are stable and we expect that this holds in general. More interestingly, in all known cases they are of the form $\varinjlim (C(X) \otimes F)_i$ where F is a finite dimensional algebra.

For the Ruelle algebras we have equally nice properties.

3.5 Theorem. *Let* (X, f) *be a topologically mixing Smale space. Then the algebras* $\mathcal{R}^s$ *and* $\mathcal{R}^u$ *are separable, nuclear, simple, stable and purely infinite.*

It follows from Theorem 3.6 that Ruelle algebras are very close to belonging to the class of algebras covered by the Kirchberg-Phillips classification theorem, [9], [12]. Indeed, the only missing property is the requirement that the algebras satisfy the Universal Coefficient Theorem of Rosenberg and Schochet. This is an interesting point and we will see in the next section that possessing this property has several strong implications.

For a subshift of finite type, Ruelle algebras are stable Cuntz-Krieger algebras. In higher dimensions this need not be true, but one does have the following result.

3.7 Proposition. *Let (X,f) be a topologically mixing Smale space. Assume that* $\mathcal{R}^s$ *satisfies the Universal Coefficient Theorem. If there is no torsion in* $K_1(\mathcal{R}^s)$*, then* $\mathcal{R}^s$ *is isomorphic to* $O_A \otimes \mathcal{K}$ *for some aperiodic 0-1 matrix A.*

A similar result holds for $\mathcal{R}^u$.

4. Duality

The original motivation for the definition of $\mathcal{R}^s$ and $\mathcal{R}^u$ was to understand the duality inherent in the transversality of the stable and unstable equivalence classes and the expanding and contracting nature of the homeomorphism. The C^*-algebraic version of these properties are (3) and (4) of Theorem 3.5. In this section we will describe how this leads to a K-theoretic version of Spanier-Whitehead duality. We note that Alain Connes has incorporated a notion of noncommutative Poincaré duality into the definition of noncommutative manifold, [4]. It is related to what is discussed here, and we will address this point later.

4.1 Definition. *Let A and B be C^*-algebras. Then A and B are Spanier-Whitehead dual if there are classes $\Delta \in KK^p(A \otimes B, \mathbb{C})$ and $\delta \in KK^p(\mathbb{C}, A \otimes B)$ such that*

$$\otimes \Delta \colon K_i(A) \to K^{i+p}(B)$$

and

$$\delta \otimes \colon K^{i+p}(B) \to K_i(A)$$

are inverse isomorphisms.

We will refer to the classes Δ and δ as duality elements. If A and B are $C(X)$ and $C(Y)$ respectively, with X and Y finite complexes that are Spanier-Whitehead dual in the usual sense, then A and B are also dual in the sense of Definition 4.1.

It is possible to interpret the Baum-Connes conjecture for a countable torsion free hyperbolic group in terms of this type of duality, [4]. Indeed, the Miscenko line bundle determines a class $\delta \in KK(\mathbb{C}, C_0(B\Gamma) \otimes C_r^*(\Gamma))$ and the dual Dirac operator determines a class $\Delta \in KK(C_0(B\Gamma) \otimes C_r^*(\Gamma), \mathbb{C})$. The Baum-Connes conjecture holds for the group Γ if Δ and δ are duality elements in the sense of Definition 4.1.

We will next discuss how the algebras $\mathcal{R}^s$ and $\mathcal{R}^u$ are Spanier-Whitehead dual. This requires the construction of the elements Δ and δ and the proof that they induce isomorphisms via Kasparov product. Since the dynamics plays such a strong role here we shall sketch the constructions. To obtain Δ we use Theorem 3.5, (3) and (4). Those conditions allow us to define a sequence of maps

$$\Delta_n \colon C^*(\mathcal{S}) \otimes C^*(\mathcal{U}) \to \mathcal{K}$$

via

$$\Delta_n(a \otimes b) = \rho_s((f_*^s)^n(a))\rho_u(b).$$

One can obtain from this an asymptotic morphism defined on the tensor product of the mapping cylinders of the automorphisms

$$\Delta_t \colon Cyl(C^*(S), f_*^s) \otimes Cyl(C^*(U), f_*^u) \to C_0(0,1) \otimes \mathcal{K}$$

which determines an element in $KK^1(Cyl(C^*(S), f_*^s) \otimes Cyl(C^*(U), f_*^u), \mathbb{C})$. Since $Cyl(C^*(S), f_*^s)$ and $Cyl(C^*(U), f_*^u)$ are KK-equivalent (in the odd sense via invertible KK-elements in KK^1) to $\mathcal{R}^s$ and $\mathcal{R}^u$, respectively, we obtain the class

$$\Delta \in KK^1(\mathcal{R}^s \otimes \mathcal{R}^u, \mathbb{C}).$$

Note that this made strong use of the hyperbolic nature of the dynamics, just as the construction of dual Dirac requires negative curvature assumptions.

The construction of δ does not require hyperbolicity, but does use the transversality of the stable and unstable equivalence relations. To obtain our element we will construct a unitary $u \in (Cyl(C^*(\mathcal{S}), f^s_*) \otimes Cyl(C^*(\mathcal{U}), f^u_*))^+$ which will determine the class $\delta \in KK^1(\mathbb{C}, \mathcal{R}^s \otimes \mathcal{R}^u)$ as above. Transversality implies that $C^*(\mathcal{S}) \otimes C^*(\mathcal{U})$ is strongly Morita equivalent to the C^*-algebra of the r-discrete groupoid which is obtained as the intersection of the equivalence relations, $C^*(A)$. An explicit Morita equivalence bimodule can be used to associate to the unit, $1 \in C^*(A)$ a projection $p \in C^*(\mathcal{S}) \otimes C^*(\mathcal{U})$. Using a twisted version of Bott periodicity, the projection p yields the required class $[u] \in KK^1(\mathbb{C}, Cyl(C^*(\mathcal{S}), f^s_*) \otimes Cyl(C^*(\mathcal{U}), f^u_*))$. A careful analysis yields that these elements induce inverse isomorphisms.

4.2 Theorem. *Let (X, f) be a topologically mixing Smale space. Then the classes Δ and δ constructed above induce Spanier-Whitehead duality isomorphisms,*

$$\otimes\Delta \colon K_i(\mathcal{R}^s) \rightleftarrows K^{i+1}(\mathcal{R}^u) \colon \delta \otimes .$$

The special case of zero dimensional Smale spaces was worked out in [6] using different definitions of the duality elements. In that case, one can use the construction of Cuntz-Krieger algebras using the full Fock space of a finite dimensional Hilbert space to implement the duality. It would be very interesting to find an analogous construction in the case of higher dimensional Smale spaces. This could lead to new connections between hyperbolic dynamics and physics.

Implications of duality.

Let us assume for this subsection that $\mathcal{R}^s$ and $\mathcal{R}^u$ satisfy the Universal Coefficient Theorem. Then the following Propositions are easy to prove.

4.3 Proposition. *The groups, $K_*(\mathcal{R}^s)$ and $K_*(\mathcal{R}^u)$ are finitely generated.*

One may ask if Proposition 4.3 follows from dynamical properties of Smale spaces without the assumption that the Universal Coefficient Theorem holds. For example, Smale spaces are automatically finite dimensional as a consequence of the dynamics.

4.4 Proposition. *Poincaré duality holds in the sense that $K_i(\mathcal{R}^s) \cong K^{i+1}(\mathcal{R}^u)$.*

Proposition 4.4 follows from the fact that, under our hypothesis, $\mathcal{R}^s \cong \mathcal{R}^u$. Note that, even in the case of a zero dimensional Smale space, where $\mathcal{R}^s = O_A$ and $\mathcal{R}^u = O_{A^t}$, one needs the Kirchberg-Phillips Theorem to verify that they are isomorphic. As mentioned earlier, Connes has introduced the notion of Poincaré duality in his work on the standard model in particle physics, [4]. It is viewed there as a requirement that every non-commutative manifold should satisfy. Proposition 4.4 suggests that Ruelle algebras might be viewed as odd-dimensional non-commutative manifolds.

5. Compact abelian groups with expansive automorphisms

Compact abelian groups which admit expansive automorphisms have many beautiful properties, [1], [3], [10]. For example, if they are connected, (and possibly in general), they admit the structure of Smale space, [3]. The examples of the 2-adic solenoid with the shift automorphism and the 2-dimensional torus with the automorphism induced by $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ are of this type. They have the property of being Smale spaces. In this setting, the study of the Ruelle algebras and their duality, as described in Section 4, lead to some results about the groups themselves which we describe in this section.

Let (G, φ) be a compact abelian group with an expansive automorphism. Let G^s and G^u denote the stable and unstable equivalence classes of the identity element, e. It is easy to see that these are subgroups. Moreover, the stable and unstable equivalence relations can be described in terms of them via

$$\mathcal{S} \cong G \rtimes G^s$$

and

$$\mathcal{U} \cong G \rtimes G^u.$$

One may use these identifications to induce topologies on the groups G^s and G^u which will be different from their relative topology as subgroups of G. Let $G^h = G^s \cap G^u$, (where G^h stands for the homoclinic subgroup to conform with terminology in dynamics). Then G^h, viewed as a subgroup of either G^s or G^u, is discrete.

The main result which comes from the study of this class of examples is the following.

5.1 Theorem [8]. *Let G be a connected compact abelian group with an expansive automorphism. Then*

(1) *the groups G^s and G^u are their own Pontrjagin duals,*

$$\widehat{G^s} \cong G^s$$

and

$$\widehat{G^u} \cong G^u$$

(2) *G^h is the Pontrjagin dual of G,*

$$\widehat{G^h} \cong G.$$

These ideas have proved useful in dynamics, [11]. It is instructive to look at the two examples we have been referring to. For the case of $G = T^2$ and $f = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$), G^s and G^u are both copies of the real line which are densely immersed in the torus with slopes corresponding to the eigenvectors of the matrix. The topology induced on the subgroups from the equivalence relations is the usual topology on $\mathbb{R}$. The homoclinic subgroup G^h is isomorphic to a lattice in $\mathbb{R}^2$ and the duality relations of Theorem 5.1 are apparent.

The 2-adic solenoid (Σ, σ) is a somewhat more interesting example. In this case $\Sigma^s \cong \mathbb{Q}_2$, where $\mathbb{Q}_2$ is the 2-adic numbers with its natural locally compact topology, and $\Sigma^u \cong \mathbb{R}$, where $\mathbb{R}$ has its usual topology. In this case, $\Sigma^h \cong \mathbb{Z}[\frac{1}{2}]$, where $\mathbb{Z}[\frac{1}{2}]$ is the dyadic rationals which is included in the natural way in both

$\mathbb{Q}_2$ and $\mathbb{R}$ as a discrete subgroup. Again, the duality relations are seen to follow from standard facts about these groups. This example has a further property, which need not hold in general. Namely, the stable and unstable algebras are isomorphic. From the computations,

$$C^*(\mathcal{S}) \cong C_0(\mathbb{Q}_2) \rtimes \mathbb{Z}[\frac{1}{2}]$$

and

$$C^*(\mathcal{U}) \cong C_0(\mathbb{R}) \rtimes \mathbb{Z}[\frac{1}{2}]$$

one obtains the isomorphism

$$C_0(\mathbb{Q}_2) \rtimes \mathbb{Z}[\frac{1}{2}] \cong C_0(\mathbb{R}) \rtimes \mathbb{Z}[\frac{1}{2}].$$

There are several ways to verify this isomorphism. One that is being investigated involves wavelets and may provide some insights into connections between dynamics and wavelets.

6. Final remarks

(1) An interesting question which remains is whether the Ruelle algebras satisfy the Universal Coefficient Theorem. If they do, then according to the Kirchberg-Phillips Theorem, they are in the class of groups for which the K-groups are a complete isomorphism invariant. Note that any pair of countable abelian groups can be realized as the K-groups of an algebra in this class. Since the K-groups of the Ruelle algebras will be finitely generated, one could ask if any classifiable algebra is a direct limit of Ruelle algebras. If this holds, then then it would establish an interesting link between classification of nuclear C^*-algebras, in the sense of George Elliott's program, [5], and hyperbolic dynamics.

(2) Consideration of Axiom A diffeomorphisms of compact manifolds leads one to consider a class of algebras with a finite filtrations by ideals

$$A \supseteq A^{(1)} \supseteq A^{(2)} \ldots \supseteq A^{(N)} \supseteq \mathbb{C}$$

, where

$$A^{(i)}/A^{(i+1)}$$

is isomorphic to a Ruelle algebras satisfying the Universal Coefficient Theorem. Since then the subquotients are determined by K-theory, it would be interesting to find additional invariants which would determine the algebras themselves up to isomorphism.

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