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Constraint Qualifications and KKT Conditions for Bilevel Programming Problems

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In this paper we consider the bilevel programming problem (BLPP), which is a sequence of two optimization problems where the constraint region of the upper-level problem is determined implicitly by the solution set to the lower-level problem. We extend well-known constraint qualifications for nonlinear programming problems such as the Abadie constraint qualification, the Kuhn-Tucker constraint qualification, the Zangwill constraint qualification, the Arrow-Hurwicz-Uzawa constraint qualification, and the weak reverse convex constraint qualification to BLPPs and derive a Karash-Kuhn-Tucker (KKT)-type necessary optimality condition under these constraint qualifications without assuming the lower-level problem satisfying the Mangasarian Fromovitz constraint qualification. Relationships among various constraint qualifications are also given.

Key words: necessary optimality conditions; constraint qualifications; nonsmooth analysis; value function; bilevel programming problems
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1. Introduction. In this paper we consider the following bilevel programming problem (BLPP):

$$\begin{aligned}(\text{BLPP})\quad \min_{x,\,y}\ & F(x,y)\\ \text{s.t.}\ & y\in S(x),\\ & G_k(x,y)\le 0\quad k\in K,\end{aligned}$$

where $S(x)$ denotes the set of solutions of the lower-level problem:

$$\begin{aligned}(\text{P}_x)\quad \min_{y}\ & f(x,y)\\ \text{s.t.}\ & g_i(x,y)\le 0\quad i\in I,\end{aligned}$$

and F, G_k, f, g_i are functions on $R^n\times R^m$ with finite index sets

$$I=\{1,2,\ldots,p\},\qquad K=\{1,2,\ldots,q\}.$$

We allow p or q to be zero to signify the case in which there are no explicit inequality constraints. In these cases, it becomes clear below that certain references to such constraints are simply to be deleted.

To simplify the exposition and to concentrate on the main ideas, we assume that all defining functions F, G_k, f, g_i are continuously differentiable, and we do not include equality constraints. The results can be easily generalized to the case of the presence of equality constraints in a straightforward manner.

Although the bilevel programming problem was only introduced to the optimization community in the seventies of the 20th century by Bracken and McGill [7], the first formulation of a simpler case was introduced and used on market economy by Stackelberg [29] in 1934, and hence is also known as a Stackelberg game in economic game theory. Bilevel programming problems can be used to model a two-level hierarchical system where the higher level (the leader) and the lower level (the follower) must find vectors $x\in R^n$ and $y\in R^m$, respectively, to minimize their individual objective functions $F(x,y)$ and $f(x,y)$ subject to certain constraints. The leader is assumed to select his decision vector first and the follower after that. Under these assumptions on the order of the play, the game will proceed as follows: For any possible decision vector $x\in R^n$ chosen by the leader, the follower will react optimally by choosing his decision vector $y\in R^m$ to minimize the objective function $f(x,y)$ subject to constraints $g_i(x,y)\le 0$ $i\in I$. Assume also that if the solution set $S(x)$ of the lower-level problem is not a singleton, the follower allows the leader to choose which of them is actually used. Hence, now the leader chooses his optimal decision vector $x\in R^n$ and $y\in S(x)$ to minimize his objective function $F(x,y)$ subject to the constraints $G_k(x,y)\le 0$ $k\in K$. The BLPP has been a hot research area over the last 20 years, and many researchers have made contributions to the area. The reader is referred to monographs Bard [4], Dempe [10], and Shimizu et al. [27] for applications of bilevel programming and recent developments on the subject; and to Dempe [11] and Vincente and Calamai [28] for a bibliography review.

The classical approach to derive necessary optimality conditions for BLPP (see, e.g., Bard and Falk [5]) was to replace the lower-level problem by its Karush-Kuhn-Tucker (KKT) conditions, and the problem of constraint qualification is usually neglected. This approach, however, is only applicable to the case where the lower-level problem is convex, i.e., $f(x,\cdot)$, $g_i(x,\cdot)$ $(i \in I)$ are convex functions, and a certain constraint qualification is satisfied for the lower level. Moreover, the resulting single-level problem belongs to the class of mathematical programs with equilibrium constraints, or MPECs (Luo et al. [18], Outrata et al. [21]), and it is known that the usual constraint qualifications such as the Mangasarian Fromovitz constraint qualification (MFCQ) will never hold (see Ye et al. [35, Proposition 1.1]). Recently, various optimality conditions for MPECs such as the B-stationary condition, the S-stationary condition, the M-stationary condition, and the C-stationary condition, which are weaker than the classical KKT condition and the corresponding constraint qualifications, are developed (see Pang and Fukushima [22], Scheel and Scholtes [26], Ye [30, 32] for detailed discussions).

Dempe [9] and Outrata [20] derived necessary conditions for the case where the solution set $S(x) = \{y(x)\}$ is a singleton by minimizing the objective function $F(x, y(x))$ over all x satisfying constraints $G_k(x, y(x)) \le 0$ $k \in K$. This approach, however, requires that the solution set $S(x)$ is a singleton, and the map $y(x)$ has certain differentiability properties.

In Ye and Zhu [33, 34], the following approach is taken to reformulate the BLPP. Define *the value function of the lower-level problem* as an extended value function $V\colon R^n \to \bar{R}$ by

$$V(x) := \inf_y \{f(x,y)\colon g_i(x,y) \le 0,\ i \in I\}$$

where $\bar{R} := R \cup \{-\infty\} \cup \{+\infty\}$ is the extended real line and $\inf\{\varnothing\} = +\infty$ by convention. Then it is obvious that the BLPP can be reformulated as the following single-level optimization problem involving the value function:

$$\begin{aligned} (\mathrm{SP})_V \quad \min\ & F(x,y) \\ \text{s.t.}\ & f(x,y) - V(x) \le 0, \\ & g_i(x,y) \le 0 \quad i \in I, \\ & G_k(x,y) \le 0 \quad k \in K. \end{aligned}$$

The above single-level problem is completely equivalent to the BLPP without any convexity assumption on the lower-level problem. However, there are two issues that need to be addressed when using this approach. First, it is well known that $V(x)$ may not be differentiable in general, even in the case where all defining functions f, g_i are continuously differentiable, and hence the problem $(\mathrm{SP})_V$ is in general a nonsmooth problem. To use the generalized Lagrange multiplier rule of Clarke [8], $V(x)$ is required to be Lipschitz continuous. For this to be true, the lower-level problem is assumed to satisfy the MFCQ at the optimal solution. Secondly, due to the bilevel structure, the nonsmooth MFCQ for the single-level problem $(\mathrm{SP})_V$ will never be satisfied, and hence weaker constraint qualifications such as the partial calmness condition was suggested by Ye and Zhu [33, 34] as an applicable constraint qualification.

The purpose of this paper is to derive KKT conditions for general bilevel programming problems without convexity assumptions on the lower-level problem, without the assumption that the solution set of the lower-level problem $S(x)$ is a singleton, without the assumption that the lower-level problem satisfies the MFCQ, and without the partial calmness condition. Our approach is to use a new function $\psi(x,y)$ to replace the function $f(x,y) - V(x)$. The resulting single-level problem may be locally equivalent to the BLPP and the function $\psi(x,y)$ is Lipschitz near the optimal solution without any requirements on the MFCQ of the lower-level problem.

2. A new equivalent single-level problem. In order to derive KKT conditions without the assumption of the MFCQ for the lower-level problem, we consider a new single-level problem that is locally equivalent to the BLPP at the optimal solution in this section.

Denote by

$$Y(x) := \{y \in R^m\colon g_i(x,y) \le 0,\ i \in I\}$$

the feasible region of the lower-level problem P_x. Let $(\bar{x}, \bar{y})$ be a local optimal solution of BLPP. Recall that the set-valued map Y is called uniformly bounded around $\bar{x}$ if there exists a neighborhood U of $\bar{x}$ such that the set $\bigcup_{x \in U} Y(x)$ is bounded. Throughout this paper, unless otherwise specified, we assume that the set-valued map Y is uniformly bounded around $\bar{x}$.

Assume that $U(\bar{x}, \bar{y})$ is a bounded open neighborhood of a local optimal solution $(\bar{x}, \bar{y})$, $U := \{x \in R^n : \exists y \text{ s.t. } (x, y) \in U(\bar{x}, \bar{y})\}$ and the set $\bigcup_{x\in U} Y(x)$ is bounded. Let V be some nonempty and compact set that contains an open neighborhood of $\text{cl} \bigcup_{x\in U} Y(x)$ where $\text{cl}\, A$ denotes the closure of set A. Our uniform boundedness assumption on $\bar{x}$ ensures the existence of V.

Let

$$\psi(x, y) := \max_{y'\in V} \sigma(x, y, y')$$

where $\sigma(x, y, y') := \min\{f(x, y) - f(x, y'), -\max_{i\in I} g_i(x, y')\}$.

Lemma 2.1. (i) $\{x \in U,\ y \in Y(x)\colon f(x, y) - V(x) < 0\} = \{x \in U,\ y \in Y(x)\colon \psi(x, y) < 0\} = \varnothing$.

(ii) $\{x \in U,\ y \in Y(x)\colon f(x, y) - V(x) = 0\} \subseteq \{x \in U,\ y \in Y(x)\colon \psi(x, y) = 0\}$.

(iii) *Let $\bar{y} \in S(\bar{x})$. Then the solution set of the problem* $\max_{y'\in V} \sigma(\bar{x}, \bar{y}, y')$ *is given by* $S(\bar{x})$.

Proof. To see part (i), let $x \in U$, $y \in Y(x)$. Due to the compactness of V, we have $\psi(x, y) < 0$ if and only if $\sigma(x, y, y') < 0$ for all $y' \in V$. The latter holds if and only if for all $y' \in V$ $\max_{i\in I} g_i(x, y') \le 0$ implies $f(x, y) < f(x, y')$. Because for $x \in U$ the set $Y(x)$ is a subset of V, we have, equivalently, that for all $y' \in V \cap Y(x) = Y(x)$ it holds that $f(x, y) < f(x, y')$. This is true if and only if $f(x, y) - V(x) < 0$. By definition of the value function $V(x)$, it is obvious that $f(x, y) \ge V(x)$ for all $y \in Y(x)$ always, and hence the set $\{x \in U, y \in Y(x)\colon f(x, y) - V(x) < 0\}$ is empty.

For the proof of (ii), let $x \in U$, $y \in Y(x)$ so that $f(x, y) - V(x) = 0$. This means that y is a global minimizer of the lower-level problem (P_x). If $\psi(x, y) > 0$, then by definition of ψ, there is $y' \in V$ such that $\sigma(x, y, y') > 0$. That is,

$$f(x, y) - f(x, y') > 0 \qquad -\max_{i\in I} g_i(x, y') > 0,$$

which implies that $y' \in Y(x)$, but $f(x, y') < f(x, y)$. However, this contradicts the fact that y is a global minimizer of (P_x), and hence $\psi(x, y)$ cannot be positive. By part (i), $\psi(x, y)$ cannot be negative either; hence $\psi(x, y) = 0$.

Finally, let us prove part (iii). Because $\bar{y} \in S(\bar{x})$, it is easy to see that $\sigma(\bar{x}, \bar{y}, \bar{y}') = 0$ for any $\bar{y}' \in S(\bar{x})$. Hence, to prove that $S(\bar{x})$ is the set of solutions, it suffices to prove that $\sigma(\bar{x}, \bar{y}, y') \le 0$ for any $y' \in V$. On the contrary, suppose that $\sigma(\bar{x}, \bar{y}, y') > 0$ for some $y' \in V$; then

$$-\max_{i\in I} g_i(\bar{x}, y') > 0, \qquad f(\bar{x}, \bar{y}) > f(\bar{x}, y'),$$

which contradicts the fact that $\bar{y} \in S(\bar{x})$, and hence the solution set of the problem is given by $S(\bar{x})$. □

By virtue of Lemma 2.1, the local optimal solution $(\bar{x}, \bar{y})$ of the BLPP may become a local optimal solution of the following problem

$$\begin{aligned} (\text{SP})_\psi \quad \min\ & F(x, y) \\ \text{s.t.}\ & \psi(x, y) \le 0, \\ & g_i(x, y) \le 0 \quad i \in I, \\ & G_k(x, y) \le 0 \quad k \in K. \end{aligned}$$

Again, as for problem $(\text{SP})_V$, there are two issues to be addressed. The first one concerns the Lipschitz continuity of the function $\psi(x, y)$, and the second one involves KKT necessary optimality conditions. We discuss the first one in the remaining part of this section and leave the second issue for the next two sections.

Because $-\psi(x, y)$ can be considered as a value function for a minimization problem $\text{P}(x, y)$ (to be defined in the proof of Proposition 2.2), we need to recall the sensitivity analysis of the value function for the following parametric mathematical program:

$$\begin{aligned} \text{P}(x) \quad \min_y\ & h(x, y) \\ \text{s.t.}\ & \Psi(x, y) \le 0, \\ & y \in C, \end{aligned}$$

where the defining functions $h(x, y)\colon R^n \times R^m \to R$, $\Psi(x, y)\colon R^n \times R^m \to R^s$ are continuously differentiable functions, and C is a closed subset of R^m. We denote

$$w(x) := \inf_y \{h(x, y)\colon \Psi(x, y) \le 0,\ y \in C\}$$

the associated value function. Let $\bar{x} \in R^n$. We denote by $\Sigma(\bar{x})$ the solution set of problem $\mathrm{P}(\bar{x})$. For any $\bar{y} \in \Sigma(\bar{x}) \cap \operatorname{int} C$ where $\operatorname{int} C$ denotes the interior of set C, define the set of abnormal multipliers and the set of normal (i.e., KKT) multipliers for the problem $\mathrm{P}(\bar{x})$ at $\bar{y}$, respectively, as follows:

$$M^0(\bar{y}) := \left\{\eta \in R^s\colon 0 = \sum_{i=1}^{s} \eta_i \nabla_y \Psi_i(\bar{x}, \bar{y}),\ \eta \ge 0,\ \sum_{i=1}^{s} \eta_i \Psi_i(\bar{x}, \bar{y}) = 0\right\};$$

$$M^1(\bar{y}) := \left\{\eta \in R^s\colon 0 = \nabla_y h(\bar{x}, \bar{y}) + \sum_{i=1}^{s} \eta_i \nabla_y \Psi_i(\bar{x}, \bar{y}),\ \eta \ge 0,\ \sum_{i=1}^{s} \eta_i \Psi_i(\bar{x}, \bar{y}) = 0\right\}.$$

The following result, which generalizes the result of Gauvin and Dubeau [13] to the problem involving abstract constraints, can be derived by using Lucet and Ye [16, 17, Theorem 4.4]. Alternatively, the result can also be obtained by using either Clarke [8, Corollary 1 of Theorem 6.5.2] or Rockafellar and Wets [25, Theorem 10.13] with some calculus. Note that the condition $\bigcup_{\bar{y}\in\Sigma(\bar{x})} M^0(\bar{y}) = \{0\}$ holds if and only if the MFCQ holds for $\mathrm{P}(\bar{x})$ at each $\bar{y} \in \Sigma(\bar{x})$.

Proposition 2.1. *Assume that there exists $\delta > 0$ such that the set*

$$\{y \in C\colon p \in B(0, \delta),\ \Psi(\bar{x}, y) \le p,\ h(\bar{x}, y) \le \alpha\}$$

is bounded for each scalar α. Assume also that $\bigcup_{\bar{y}\in\Sigma(\bar{x})} M^0(\bar{y}) = \{0\}$ and $\Sigma(\bar{x}) \subseteq \operatorname{int} C$. Then the value function $w(x)$ is Lipschitz near $\bar{x}$ and

$$\partial^\circ w(\bar{x}) \subseteq \operatorname{co} \bigcup_{\bar{y}\in\Sigma(\bar{x})} \left\{\nabla_x h(\bar{x}, \bar{y}) + \sum_{i=1}^{s} \eta_i \nabla_x \Psi_i(\bar{x}, \bar{y})\colon \eta \in M^1(\bar{y})\right\},$$

where $\operatorname{co} A$ denotes the convex hull of set A, and $\partial^\circ w$ denotes the Clarke generalized gradient of w (see Clarke [8] for definition).

Define the Fritz John-type Lagrangian of the lower-level problem by

$$\mathscr{L}(x, y, \alpha, \gamma) = \alpha f(x, y) + \sum_{i\in I} \gamma_i g_i(x, y),$$

and the set of Fritz John multipliers of the lower-level problem $(\mathrm{P}_{\bar{x}})$ at $\bar{y}$ by

$$FJ(\bar{x}, \bar{y}) = \left\{(\alpha, \gamma) \in R \times R^p\colon (\alpha, \gamma) \ge 0,\ \|(\alpha, \gamma)\|_1 = 1,\ \nabla_y \mathscr{L}(\bar{x}, \bar{y}, \alpha, \gamma) = 0,\ \sum_{i\in I} \gamma_i g_i(\bar{x}, \bar{y}) = 0\right\},$$

where $\|x\|_1 := \sum_{i=1}^{n} |x_i|$ denotes the one norm for a vector x in R^n. It is easy to see that the set of Fritz John multipliers $FJ(\bar{x}, \bar{y})$ is a nonempty convex polytope.

Proposition 2.2. *Let $(\bar{x}, \bar{y})$ be a local solution of BLPP. Then the function $\psi(x, y)$ is Lipschitz continuous near $(\bar{x}, \bar{y})$, and the Clarke generalized gradient at $(\bar{x}, \bar{y})$ has the following upper approximation:*

$$\partial^\circ \psi(\bar{x}, \bar{y}) \subseteq \operatorname{co} W(\bar{x}, \bar{y}),$$

where

$$W(\bar{x}, \bar{y}) := \bigcup_{\bar{y}'\in S(\bar{x})} \{\alpha \nabla f(\bar{x}, \bar{y}) - (\nabla_x \mathscr{L}(\bar{x}, \bar{y}', \alpha, \gamma), 0)\colon (\alpha, \gamma) \in FJ(\bar{x}, \bar{y}')\}.$$

Proof. It is easy to see that $-\psi(x, y)$ is the optimal value function for the following parametric mathematical programming problem:

$$\begin{aligned} \mathrm{P}(x, y) \quad \min_{y', z}\ & -z \\ \text{s.t.}\ & -f(x, y) + f(x, y') + z \le 0, \\ & g_i(x, y') + z \le 0 \quad i \in I, \\ & y' \in V, \quad z \in R. \end{aligned}$$

By Lemma 2.1, if $\bar{y} \in S(\bar{x})$, then the solution set of the problem $\mathrm{P}(\bar{x}, \bar{y})$ is $S(\bar{x}) \times \{0\}$. Let $\bar{y}$ be any element in $S(\bar{x})$ and $(\alpha, \gamma) \in M^0(\bar{y}, 0)$ be any abnormal multiplier of the problem $\mathrm{P}(\bar{x}, \bar{y})$. Then, because the restriction $y' \in V$ is not active at $\bar{y}$,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} \nabla_y f(\bar{x}, \bar{y}) \\ 1 \end{pmatrix} + \sum_{i \in I} \gamma_i \begin{pmatrix} \nabla_y g_i(\bar{x}, \bar{y}) \\ 1 \end{pmatrix}$$

and

$$(\alpha, \gamma) \geq 0, \qquad \sum_{i \in I} \gamma_i g_i(\bar{x}, \bar{y}) = 0,$$

which are only possible if (α, γ) is a zero vector. Hence, $\bigcup_{\bar{y} \in S(\bar{x})} M^0(\bar{y}, 0) = \{0\}$.

Let $(\alpha, \gamma) \in M^1(\bar{y}, 0)$ be any KKT multiplier of the problem $\mathrm{P}(\bar{x}, \bar{y})$. Then

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} \nabla_y f(\bar{x}, \bar{y}) \\ 1 \end{pmatrix} + \sum_{i \in I} \gamma_i \begin{pmatrix} \nabla_y g_i(\bar{x}, \bar{y}) \\ 1 \end{pmatrix}$$

and

$$(\alpha, \gamma) \geq 0, \qquad \sum_{i \in I} \gamma_i g_i(\bar{x}, \bar{y}) = 0,$$

which implies that (α, γ) lies in the set $FJ(\bar{x}, \bar{y})$.

Hence, by Proposition 2.1, the function ψ is Lipschitz near $(\bar{x}, \bar{y})$ and

$$\begin{aligned} \partial^\circ \psi(\bar{x}, \bar{y}) &= -\partial^\circ(-\psi(\bar{x}, \bar{y})) \\ &\subseteq -\mathrm{co} \bigcup_{\bar{y}' \in S(\bar{x})} \{-\alpha \nabla f(\bar{x}, \bar{y}) + (\nabla_x \mathcal{L}(\bar{x}, \bar{y}', \alpha, \gamma), 0) \colon (\alpha, \gamma) \in FJ(\bar{x}, \bar{y}')\} \\ &= \mathrm{co}\, W(\bar{x}, \bar{y}). \quad \square \end{aligned}$$

Now, because all defining functions of problem $(\mathrm{SP})_\psi$ are Lipschitz near the optimal solution, one can apply the generalized Lagrange multiplier rule of Clarke [8, Proposition 6.4.4] and Proposition 2.2 to derive the following KKT condition under the calmness condition. The equivalent form follows from the Carathéodory's theorem, which says that a convex set in R^{n+m} can be represented by not more than $n + m + 1$ elements at a time.

Proposition 2.3. *Let $(\bar{x}, \bar{y})$ be a local optimal solution of* (BLPP) *with $S(\bar{x}) \neq \varnothing$. Suppose the single-level problem* $(\mathrm{SP})_\psi$ *is calm at $(\bar{x}, \bar{y})$ in the sense of Clarke* [8, *Definition* 6.4.1]. *Then there exist multipliers $\mu \geq 0$, $\eta \in R^p_+$, $\beta \in R^q_+$ such that*

$$0 \in \nabla F(\bar{x}, \bar{y}) + \mu\, \mathrm{co}\, W(\bar{x}, \bar{y}) + \sum_{i \in I(\bar{x}, \bar{y})} \eta_i \nabla g_i(\bar{x}, \bar{y}) + \sum_{k \in K(\bar{x}, \bar{y})} \beta_k \nabla G_k(\bar{x}, \bar{y}),$$

where $I(\bar{x}, \bar{y}) := \{i \in I \colon g_i(\bar{x}, \bar{y}) = 0\}$ and $K(\bar{x}, \bar{y}) := \{k \in K \colon G_k(\bar{x}, \bar{y}) = 0\}$. Equivalently, there exist $\lambda^i \geq 0$, $y^i \in S(\bar{x})$, $(\alpha^i, \gamma^i) \in FJ(\bar{x}, y^i)$, $i = 1, 2, \ldots, n + m + 1$, and $\eta \in R^p_+$, $\beta \in R^q_+$ such that

$$\begin{aligned} 0 = \nabla F(\bar{x}, \bar{y}) &+ \sum_{i \in I(\bar{x}, \bar{y})} \eta_i \nabla g_i(\bar{x}, \bar{y}) + \sum_{k \in K(\bar{x}, \bar{y})} \beta_k \nabla G_k(\bar{x}, \bar{y}) \\ &+ \sum_{i=1}^{n+m+1} \lambda^i [\alpha^i \nabla f(\bar{x}, \bar{y}) - (\nabla_x \mathcal{L}(\bar{x}, y^i, \alpha^i, \gamma^i), 0)]. \end{aligned}$$

By virtue of Clarke [8, Corollary 5 of Theorem 6.5.2], if $M^0(\bar{x}, \bar{y})$, the abnormal multiplier set for problem $(\mathrm{SP})_\psi$ at $(\bar{x}, \bar{y})$, contains only the zero vector, then problem $(\mathrm{SP})_\psi$ is calm at $(\bar{x}, \bar{y})$. It is known (see Jourani [14]) that for the nonsmooth problem $(\mathrm{SP})_\psi$, $M^0(\bar{x}, \bar{y}) = \{0\}$ if and only if the generalized MFCQ holds, i.e., there exists $v \in R^{n+m}$ such that

$$\begin{aligned} \psi^\circ((\bar{x}, \bar{y}); v) &< 0, \\ g_i^\circ((\bar{x}, \bar{y}); v) &< 0 \quad i \in I(\bar{x}, \bar{y}), \\ G_k^\circ((\bar{x}, \bar{y}); v) &< 0 \quad k \in K(\bar{x}, \bar{y}), \end{aligned}$$

where $\psi^\circ((\bar{x},\bar{y});v)$ denotes the Clarke generalized directional derivative of a function ψ at $(\bar{x},\bar{y})$ in direction v (see Clarke [8] for definition). In Ye and Zhu [33, Proposition 3.2], under certain conditions it was shown that the generalized MFCQ never holds for the single-level problem $(\text{SP})_V$. We now show that *without any conditions*, the generalized MFCQ never holds for both $(\text{SP})_\psi$ and $(\text{SP})_V$.

Proposition 2.4. *Let* $(\bar{x},\bar{y})$ *be a local solution for* (BLPP). *Then the generalized MFCQ will never hold for* $(\text{SP})_\psi$ *and* $(\text{SP})_V$.

Proof. On the contrary, assume that the generalized MFCQ holds at $(\bar{x},\bar{y})$. Then it is easy to show that there exists a point $(\hat{x},\hat{y})$ such that $\psi(\hat{x},\hat{y})<0$, $g_i(\hat{x},\hat{y})<0$, $\hat{x}\in U$. This contradicts the fact that the set $\{x\in U,\ y\in Y(x)\colon \psi(x,y)<0\}=\varnothing$ (by Lemma 2.1(i)). The proof for $(\text{SP})_V$ is exactly similar. □

Using the new equivalent problem $(\text{SP})_\psi$ instead of $(\text{SP})_V$, similar to that in Ye and Zhu [33, 34], one can also derive the KKT-type optimality condition under the partial calmness condition. The resulting KKT condition will be the one in Proposition 2.3, and no lower-level MFCQ is required to hold.

3. KKT conditions under the extended Abadie CQ. In this section we derive KKT conditions for the cases where the calmness or the partial calmness condition may not hold. For a nonlinear programming problem with smooth defining functions, other than the calmness condition, another well-known constraint qualification that is weaker than most of the other constraint qualifications is the Abadie constraint qualification introduced by Abadie [1]. Our single-level problem, however, is nonsmooth, and hence we need to extend the Abadie CQ to allow the nonsmoothness.

We first recall notions of various tangent cones.

Definition 3.1. Let M be a closed subset in R^n and $\bar{x}\in M$. The contingent cone of M at $\bar{x}$ is the closed cone defined by

$$T(\bar{x},M):=\{v\in X\colon \exists t_n\downarrow 0,\ v_n\to v \text{ s.t. } \bar{x}+t_nv_n\in M\ \forall n\}.$$

The cone of attainable directions of M at $\bar{x}$ is the closed cone defined by

$$A(\bar{x},M)=\Big\{v\colon \text{There exist some } \delta>0 \text{ and a mapping } \alpha\colon R\to X \text{ such that } \alpha(\tau)\in M \\ \text{for all } \tau\in(0,\delta),\ \alpha(0)=\bar{x} \text{ and } \lim_{\tau\downarrow 0}\frac{\alpha(\tau)-\alpha(0)}{\tau}=v\Big\}.$$

The cone of feasible directions of M at $\bar{x}$ is the cone defined by

$$D(\bar{x},M):=\{v\in X\colon \exists\delta>0 \text{ s.t. } \bar{x}+tv\in M\ \forall t\in(0,\delta)\}.$$

The cone of attainable directions is also known as the adjacent cone (see, e.g., Aubin and Frankowska [3]), or the incident cone. In fact,

$$A(\bar{x},M)=\liminf_{\tau\downarrow 0}\frac{M-\bar{x}}{\tau},$$

and hence is a closed cone.

Definition 3.2 (Nonsmooth Linearization Cone). We define the nonsmooth linearization cone of the feasible set $\mathcal{F}$ of the BLPP at the local optimal solution $(\bar{x},\bar{y})$ as the set:

$$L((\bar{x},\bar{y}),\mathcal{F}):=\{v\colon w^\top v\le 0\ w\in\partial^\circ\psi(\bar{x},\bar{y}),\ \nabla g_i(\bar{x},\bar{y})^\top v\le 0\ i\in I(\bar{x},\bar{y}),\ \nabla G_k(\bar{x},\bar{y})^\top v\le 0\ k\in K(\bar{x},\bar{y})\}$$

where $a^\top$ denotes the transpose of vector a.

Note that Lemma 2.1 justifies the use of function ψ in place of the function $f(x,y)-V(x)$ in the definition of the nonsmooth linearization cone of the feasible region of the BLPP at $(\bar{x},\bar{y})$.

We now extend the well-known Abadie CQ [1] to the BLPP at $(\bar{x},\bar{y})$.

Definition 3.3 (N. Abadie CQ). Let $(\bar{x},\bar{y})\in\mathcal{F}$. We say that the nonsmooth Abadie constraint qualification holds at $(\bar{x},\bar{y})\in\mathcal{F}$ if

$$L((\bar{x},\bar{y}),\mathcal{F})\subseteq T((\bar{x},\bar{y}),\mathcal{F}).$$

It is easy to show that in the case when the function ψ is Clarke regular at $(\bar{x},\bar{y})$, i.e., when the usual directional derivative exists at every direction and is equal to the Clarke generalized directional derivative (for example when ψ is convex or smooth), the reverse inclusion

$$L((\bar{x},\bar{y}),\mathcal{F})\supseteq T((\bar{x},\bar{y}),\mathcal{F})$$

always holds. Otherwise the strict inclusion

$$L((\bar{x}, \bar{y}), \mathcal{F}) \subset T((\bar{x}, \bar{y}), \mathcal{F})$$

may be possible. For example, let $\mathcal{F} = \{x \in R\colon -|x| \leq 0\}$. Then it is easy to see that $T(0, \mathcal{F}) = R$ but $L(0, \mathcal{F}) = \{0\}$, and hence $L(0, \mathcal{F}) \subset T(0, \mathcal{F})$.

PROPOSITION 3.1. *Let $(\bar{x}, \bar{y})$ be a local solution of the BLPP. If the nonsmooth Abadie CQ holds at $(\bar{x}, \bar{y})$, then*

$$0 \in \nabla F(\bar{x}, \bar{y}) + \operatorname{cl}\operatorname{cone}\left[\partial^\circ \psi(\bar{x}, \bar{y}) \cup \bigcup_{i \in I(\bar{x}, \bar{y})} \{\nabla g_i(\bar{x}, \bar{y})\} \cup \bigcup_{k \in K(\bar{x}, \bar{y})} \{\nabla G_k(\bar{x}, \bar{y})\}\right],$$

where cone A *denotes the convex cone generated by set A.*

PROOF. It is well known that because $(\bar{x}, \bar{y})$ is a local minimizer for BLPP,

$$\nabla F(\bar{x}, \bar{y})^\top v \geq 0 \quad \text{for all } v \in T((\bar{x}, \bar{y}), \mathcal{F}).$$

Now suppose that the nonsmooth Abadie CQ holds at $(\bar{x}, \bar{y})$. Then

$$\nabla F(\bar{x}, \bar{y})^\top v \geq 0 \quad \text{for all } v \in L((\bar{x}, \bar{y}), \mathcal{F}).$$

Consequently,

$$\nabla F(\bar{x}, \bar{y})^\top v \geq 0 \quad \text{whenever } \max_{a \in C} a^\top v \leq 0,$$

where C denotes the convex cone generated by

$$\partial^\circ \psi(\bar{x}, \bar{y}) \cup \bigcup_{i \in I(\bar{x}, \bar{y})} \{\nabla g_i(\bar{x}, \bar{y})\} \cup \bigcup_{k \in K(\bar{x}, \bar{y})} \{\nabla G_k(\bar{x}, \bar{y})\}.$$

Thus, the function

$$v \to \nabla F(\bar{x}, \bar{y})^\top v + \delta_{C^0}(v)$$

attains its minimum at 0, where $C^0 := \{v \in R^{n+m}\colon v^\top c \leq 0 \text{ for all } v \in C\}$ is the polar cone of C and δ_{C^0} is the indicator function of set C^0. By the sum rule, one has

$$0 \in \nabla F(\bar{x}, \bar{y}) + \partial \delta_{C^0}(0).$$

Because $\partial \delta_{C^0}(0) = C^{00} = \operatorname{cl} C$, the above inclusion is the same as

$$0 \in \nabla F(\bar{x}, \bar{y}) + \operatorname{cl} C. \quad \square$$

REMARK 3.1. Note that the KKT condition under the nonsmooth Abadie CQ differs from the one under the calmness condition in that a closure operation is required. In fact, in Ye [31], the Abadie CQ was also extended to allow the nondifferentiability. However, in Ye [31], the KKT condition was derived under the assumption that the set

$$\operatorname{cone}\left[\partial^\circ \psi(\bar{x}, \bar{y}) \cup \bigcup_{i \in I(\bar{x}, \bar{y})} \{\nabla g_i(\bar{x}, \bar{y})\} \cup \bigcup_{k \in K(\bar{x}, \bar{y})} \{\nabla G_k(\bar{x}, \bar{y})\}\right]$$

is closed.

We now extend the well-known Kuhn-Tucker CQ and Zangwill CQ (Kuhn and Tucker [15], Zangwill [36]) to the BLPP at $(\bar{x}, \bar{y})$.

DEFINITION 3.4 (N. KUHN-TUCKER CQ AND N. ZANGWILL CQ). Let $(\bar{x}, \bar{y}) \in \mathcal{F}$. We say that the nonsmooth Kuhn-Tucker and nonsmooth Zangwill constraint qualification hold at $(\bar{x}, \bar{y}) \in \mathcal{F}$ if

$$L((\bar{x}, \bar{y}), \mathcal{F}) \subseteq A((\bar{x}, \bar{y}), \mathcal{F}), \qquad L((\bar{x}, \bar{y}), \mathcal{F}) \subseteq \operatorname{cl} D((\bar{x}, \bar{y}), \mathcal{F}),$$

respectively.

Because $\operatorname{cl} D(\bar{x}, M) \subseteq A(\bar{x}, M) \subseteq T(\bar{x}, M)$, it is easy to see that N. Zangwill CQ implies N. Kuhn-Tucker CQ, which in turn implies N. Abadie CQ.

Although the Abadie constraint qualification is a weak constraint qualification, it is not very easy to verify because it is not defined in terms of constraint functions. In the rest of this section we will provide some sufficient conditions for the nonsmooth Abadie CQ to hold.

The following definition extends the concept of pseudoconcavity in nonlinear programming (see, e.g., Bazaraa et al. [6] and Mangasarian [19]) to allow the nonsmoothness. The definition depends on the kind of subdifferential used. The reader is referred to the definition for the kind of generalization to a class of subdifferentials in Penot [23].

Definition 3.5. Let φ be a function on R^n, which is Lipschitz near $\bar{x}$. φ is said to be ∂°-pseudoconcave at $\bar{x}$ if for all $x \in R^n$,

$$\max_{w \in \partial^\circ \varphi(\bar{x})} w^\top (x - \bar{x}) \le 0 \quad \Rightarrow \quad \varphi(x) \le \varphi(\bar{x}).$$

It is easy to see that a Lipschitz-continuous concave function on R^n (which may or may not be differentiable) must be ∂°-pseudoconcave at $\bar{x}$.

We now extend the Arrow-Hurwicz-Uzawa constraint qualification introduced by Arrow et al. in [2] to the BLPP.

Definition 3.6 (N. Arrow-Hurwicz-Uzawa CQ). Let $(\bar{x}, \bar{y})$ be a feasible solution of $(\mathrm{SP})_\psi$. We say that the nonsmooth Arrow-Hurwicz-Uzawa CQ holds at $(\bar{x}, \bar{y})$ if $h(x, y) := f(x, y) - V(x)$ is ∂°-pseudoconcave at $(\bar{x}, \bar{y})$, $\partial^\circ h(\bar{x}, \bar{y}) \subseteq \partial^\circ \psi(\bar{x}, \bar{y})$ and there exists v such that

$$w^\top v \le 0 \quad \forall\, w \in \partial^\circ \psi(\bar{x}, \bar{y}), \tag{1}$$

$$\nabla g_i(\bar{x}, \bar{y})^\top v < 0 \quad \forall\, i \in I(\bar{x}, \bar{y}) \backslash \Lambda, \tag{2}$$

$$\nabla g_i(\bar{x}, \bar{y})^\top v \le 0 \quad \forall\, i \in \Lambda, \tag{3}$$

$$\nabla G_k(\bar{x}, \bar{y})^\top v < 0 \quad \forall\, k \in K(\bar{x}, \bar{y}) \backslash \Gamma, \tag{4}$$

$$\nabla G_k(\bar{x}, \bar{y})^\top v \le 0 \quad \forall\, k \in \Gamma, \tag{5}$$

where

$$\Lambda := \{i \in I(\bar{x}, \bar{y}) \colon g_i \text{ is pseudoconcave at } (\bar{x}, \bar{y})\},$$

$$\Gamma := \{k \in K(\bar{x}, \bar{y}) \colon G_k \text{ is pseudoconcave at } (\bar{x}, \bar{y})\}.$$

Proposition 3.2. *The N. Arrow-Hurwicz-Uzawa CQ implies the N. Zangwill CQ.*

Proof. Suppose that the N. Arrow-Hurwicz-Uzawa CQ holds at $(\bar{x}, \bar{y})$. Then there exists v, which satisfies (1)–(5). Then

$$w^\top v \le 0 \quad \text{for all } w \in \partial^\circ \psi(\bar{x}, \bar{y}),$$

and hence for any $t \ge 0$,

$$w^\top ((\bar{x}, \bar{y}) + tv - (\bar{x}, \bar{y})) \le 0 \quad \text{for all } w \in \partial^\circ h(\bar{x}, \bar{y}),$$

which implies by the ∂°-pseudoconcavity of h at $(\bar{x}, \bar{y})$ that $h((\bar{x}, \bar{y}) + tv) \le h(\bar{x}, \bar{y})$ for all $t \ge 0$. Therefore, $h((\bar{x}, \bar{y}) + tv) \le 0$ for all $t \ge 0$. Similarly, one can prove $g_i((\bar{x}, \bar{y}) + tv) \le 0$ $(i \in I(\bar{x}, \bar{y}))$ and $G_k((\bar{x}, \bar{y}) + tv) \le 0$ $(i \in K(\bar{x}, \bar{y}))$ for all $t \ge 0$. By virtue of Lemma 2.1, there exists a small-enough $\delta > 0$ such that $(\bar{x}, \bar{y}) + tv \in \mathcal{F}$. Therefore, $v \in D((\bar{x}, \bar{y}), \mathcal{F})$ and the nonsmooth Zangwill CQ holds at $(\bar{x}, \bar{y})$. □

The following is a nonsmooth extension of the weak reverse convex constraint qualification (see, e.g., Mangasarian [19]). It is easy to see that it is a sufficient condition for the nonsmooth Arrow-Hurwicz-Uzawa CQ to hold because $v = 0$ is always a solution to the system (1), (3), and (5).

Definition 3.7 (N. Weak Reverse Convex CQ). Let $(\bar{x}, \bar{y})$ be a feasible solution of $(\mathrm{SP})_\psi$. We say that the nonsmooth weak reverse convex CQ is satisfied at $(\bar{x}, \bar{y})$ if $h(x, y) = f(x, y) - V(x)$ is ∂°-pseudoconcave at $(\bar{x}, \bar{y})$, $\partial^\circ h(\bar{x}, \bar{y}) \subseteq \partial^\circ \psi(\bar{x}, \bar{y})$, and the functions $g_i(x, y)$ $(i \in I(\bar{x}, \bar{y}))$ and $G_k(x, y)$ $(k \in K(\bar{x}, \bar{y}))$ are pseudoconcave at $(\bar{x}, \bar{y})$.

Because the linearization cone of the feasible set $\mathcal{F}$ involves the function ψ, which is an implicit function of the defining functions, we extended the definition to one that is defined by the defining functions of the problem.

Definition 3.8. We define the extended linearization cone of the feasible set $\mathcal{F}$ as the set:

$$L'((\bar{x}, \bar{y}), \mathcal{F}) := \left\{ v\colon \begin{array}{l} w^\top v \le 0 \quad w \in W(\bar{x}, \bar{y}), \\ \nabla g_i(\bar{x}, \bar{y})^\top v \le 0 \quad i \in I(\bar{x}, \bar{y}), \\ \nabla G_k(\bar{x}, \bar{y})^\top v \le 0 \quad k \in K(\bar{x}, \bar{y}) \end{array} \right\}.$$

Definition 3.9 (E. Abadie CQ, E. Kuhn-Tucker CQ, and E. Zangwill CQ). Let $(\bar{x}, \bar{y}) \in \mathcal{F}$. We say that the extended Abadie constraint qualification, the extended Kuhn-Tucker constraint qualification, and the extended Zangwill constraint qualification holds at $(\bar{x}, \bar{y})$, respectively, if

$$L'((\bar{x}, \bar{y}), \mathcal{F}) \subseteq T((\bar{x}, \bar{y}), \mathcal{F}),$$
$$L'((\bar{x}, \bar{y}), \mathcal{F}) \subseteq A((\bar{x}, \bar{y}), \mathcal{F}),$$
$$L'((\bar{x}, \bar{y}), \mathcal{F}) \subseteq \operatorname{cl} D((\bar{x}, \bar{y}), \mathcal{F}),$$

respectively.

DEFINITION 3.10 (E. ARROW-HURWICZ-UZAWA CQ). Let $(\bar{x}, \bar{y}) \in \mathcal{F}$. We say that the extended Arrow-Hurwicz-Uzawa CQ holds at $(\bar{x}, \bar{y})$ if $h(x, y) = f(x, y) - V(x)$ is ∂°-pseudoconcave at $(\bar{x}, \bar{y})$, $\partial^\circ h(x, y) \subseteq \partial^\circ \psi(x, y)$, and there exists v such that (1)–(5) with $\partial^\circ \psi(\bar{x}, \bar{y})$ replaced by $W(\bar{x}, \bar{y})$ are satisfied.

We are now ready to state the KKT condition for BLPPs under the extended Abadie CQ in the following theorem. Note that the necessary condition under the calmness condition in Proposition 2.3 can be equivalently rewritten as

$$0 \in \nabla F(\bar{x}, \bar{y}) + \operatorname{cone}\left[W(\bar{x}, \bar{y}) \cup \bigcup_{i \in I(\bar{x}, \bar{y})} \{\nabla g_i(\bar{x}, \bar{y})\} \cup \bigcup_{k \in K(\bar{x}, \bar{y})} \{\nabla G_k(\bar{x}, \bar{y})\}\right].$$

Hence, the necessary optimality condition under the extended Abadie constraint qualification differs from the one under the calmness condition in that an extra closure operation is needed.

THEOREM 3.1. *Let $(\bar{x}, \bar{y})$ be a local optimal solution of BLPP with $S(\bar{x}) \neq \varnothing$. Then, under one of the following CQs stated in this section, $(\bar{x}, \bar{y})$ satisfies the KKT condition, i.e.,*

$$0 \in \nabla F(\bar{x}, \bar{y}) + \operatorname{cl}\operatorname{cone}\left[W(\bar{x}, \bar{y}) \cup \bigcup_{i \in I(\bar{x}, \bar{y})} \{\nabla g_i(\bar{x}, \bar{y})\} \cup \bigcup_{k \in K(\bar{x}, \bar{y})} \{\nabla G_k(\bar{x}, \bar{y})\}\right].$$

Moreover, the relationships of the CQs are summarized as follows:

$$\begin{array}{ccc} & & \textit{N. Weak Reverse Convex CQ} \\ & & \Downarrow \\ \textit{E. AHU CQ} & \Leftarrow & \textit{N. AHU CQ} \\ \Downarrow & & \Downarrow \\ \textit{E. Zangwill} & \Leftarrow & \textit{N. Zangwill} \\ \Downarrow & & \Downarrow \\ \textit{E. Kuhn-Tucker} & \Leftarrow & \textit{N. Kuhn-Tucker} \\ \Downarrow & & \Downarrow \\ \textit{E. Abadie} & \Leftarrow & \textit{N. Abadie.} \end{array}$$

PROOF. We first prove that the KKT condition holds under the extended Abadie CQ. By virtue of Lemma 2.1, the local optimal solution $(\bar{x}, \bar{y})$ of BLPP is a feasible solution of the problem $(\mathrm{SP})_\psi$. The desired result follows from replacing the linearization cone $L((\bar{x}, \bar{y}), \mathcal{F})$ by the extended linearization cone $L'((\bar{x}, \bar{y}), \mathcal{F})$ in the proof of Proposition 3.1.

Now Proposition 3.2 shows that the nonsmooth weak reverse convex CQ implies the nonsmooth Zangwill CQ. The relationship between the extended Zangwill, the extended Kuhn-Tucker, the extended Abadie, and the nonsmooth Zangwill, the nonsmooth Kuhn-Tucker, and the nonsmooth Abadie follows by the fact $L((\bar{x}, \bar{y}), \mathcal{F}) \supseteq L'((\bar{x}, \bar{y}), \mathcal{F})$ by virtue of Proposition 2.2. □

4. BLPP where $f(x, y) - V(x)$ is concave. In this section we consider the BLPP where the function $f(x, y) - V(x)$ is concave. This happens, for example, when the lower-level problem is linear convex, i.e., the function $f(x, y)$ is jointly linear and $g_i(x, y)$ $(i \in I(\bar{x}, \bar{y}))$ are jointly convex (see, e.g., Fiacco and Kyparisis [12, Corollary 2.1.9]). In this case the KKT condition takes a simpler form. First we show that the Clarke generalized gradient of the function ψ has the following simpler upper approximation.

PROPOSITION 4.1. *Let $(\bar{x}, \bar{y})$ be a local solution of BLPP where $S(\bar{x}) \neq \varnothing$ and the function ψ is concave. Then for any $\bar{y} \in S(\bar{x})$, the Clarke generalized gradient of ψ at $(\bar{x}, \bar{y})$ has the following upper approximation*:

$$\partial^\circ \psi(\bar{x}, \bar{y}) \subseteq \left\{\left(-\sum_{i \in I(\bar{x}, \bar{y})} \gamma_i \nabla_x g_i(\bar{x}, \bar{y}), \alpha \nabla_y f(\bar{x}, \bar{y})\right) : (\alpha, \gamma) \in FJ(\bar{x}, \bar{y})\right\}.$$

Proof. Note that

$$-\psi(x, y) = \min_{y', z}\{-z\colon -f(x, y) + f(x, y') + z \leq 0,\ g_i(x, y') + z \leq 0\ i \in I,\ y' \in V\},$$

and for $(\bar{x}, \bar{y})$ the solution of the above optimization problem is $S(\bar{x}) \times \{0\}$. because $-\psi(x, y)$ is convex and bounded above on a neighborhood of $(\bar{x}, \bar{y})$, $-\psi$ is Lipschitz near $(\bar{x}, \bar{y})$ (see, e.g., Clarke [8, Proposition 2.2.6]), and the Clarke generalized gradient of $-\psi(x, y)$ coincides with the subgradient in the sense of convex analysis. Let $\xi \in \partial^\circ(-\psi)(\bar{x}, \bar{y})$. By definition of the subdifferential in the sense of convex analysis,

$$-\psi(x, y) - (-\psi(\bar{x}, \bar{y})) \geq \langle \xi, (x, y) - (\bar{x}, \bar{y})\rangle \quad \forall\, (x, y) \in R^n \times R^m,$$

which implies by the definition of the value function that for all (x, y, y', z) satisfying the constraints

$$\begin{aligned} -f(x, y) + f(x, y') + z &\leq 0,\\ g_i(x, y') + z &\leq 0 \quad i \in I,\\ y' &\in V, \end{aligned}$$

one has

$$-z \geq \langle \xi, (x, y) - (\bar{x}, \bar{y})\rangle.$$

That is, $(x, y, y', z) = (\bar{x}, \bar{y}, \bar{y}, 0)$ is a solution to the following optimization problem:

$$\begin{cases} \min\limits_{x, y, y', z} & -z - \langle \xi, (x, y)\rangle\\ \text{s.t.} & -f(x, y) + f(x, y') + z \leq 0,\\ & g_i(x, y') + z \leq 0 \quad i \in I,\\ & y' \in V. \end{cases}$$

Let $\bar{y} \in S(\bar{x})$. It is easy to verify that the MFCQ holds for the above optimization problem at the solution $(\bar{x}, \bar{y}, \bar{y}, 0)$. By the KKT condition, because the constraint $y' \in V$ is not active at $\bar{y}$, there exists $(\alpha, \gamma) \in R \times R^s$, $(\alpha, \gamma) \geq 0$, $\|(\alpha, \gamma)\|_1 = 1$ such that

$$\begin{aligned} \xi &= \alpha[-\nabla f(\bar{x}, \bar{y}) + \nabla_x f(\bar{x}, \bar{y}) \times \{0\}] + \sum_{i\in I} \gamma_i \nabla_x g_i(\bar{x}, \bar{y}) \times \{0\},\\ 0 &= \alpha \nabla_y f(\bar{x}, \bar{y}) + \sum_{i\in I} \gamma_i \nabla_y g_i(\bar{x}, \bar{y}),\\ 0 &= \sum_{i\in I} \gamma_i g_i(\bar{x}, \bar{y}), \end{aligned}$$

which implies that

$$\xi \in \left\{\left(\sum_{i\in I(\bar{x}, \bar{y})} \gamma_i \nabla_x g_i(\bar{x}, \bar{y}), -\alpha \nabla_y f(\bar{x}, \bar{y})\right)\colon (\alpha, \gamma) \in FJ(\bar{x}, \bar{y})\right\}.$$

Consequently,

$$\begin{aligned} \partial^\circ \psi(\bar{x}, \bar{y}) &= -\partial^\circ(-\psi)(\bar{x}, \bar{y})\\ &\subseteq \left\{\left(-\sum_{i\in I(\bar{x}, \bar{y})} \gamma_i \nabla_x g_i(\bar{x}, \bar{y}), \alpha \nabla_y f(\bar{x}, \bar{y})\right)\colon (\alpha, \gamma) \in FJ(\bar{x}, \bar{y})\right\}. \quad \square \end{aligned}$$

Because the upper approximation of the Clarke generalized gradient of ψ has a simpler form when the function is concave, we revise the extended linearization cone as follows:

Definition 4.1. Let $(\bar{x}, \bar{y}) \in \mathcal{F}$. We define the extended linearization cone of the feasible set $\mathcal{F}$ for the concave case as the set:

$$\tilde{L}((\bar{x}, \bar{y}), \mathcal{F}) := \left\{ v\colon \begin{array}{l} \left(-\sum\limits_{i\in I(\bar{x}, \bar{y})} \gamma_i \nabla_x g_i(\bar{x}, \bar{y}), \alpha \nabla_y f(\bar{x}, \bar{y})\right)^\top v \leq 0,\ (\alpha, \gamma) \in FJ(\bar{x}, \bar{y}),\\ \nabla g_i(\bar{x}, \bar{y})^\top v \leq 0 \quad i \in I(\bar{x}, \bar{y}),\\ \nabla G_k(\bar{x}, \bar{y})^\top v \leq 0 \quad k \in K(\bar{x}, \bar{y}) \end{array} \right\}.$$

Definition 4.2. We say that the extended Abadie constraint qualification, the extended Kuhn-Tucker constraint qualification, and the extended Zangwill constraint qualification for the concave case hold at $(\bar{x}, \bar{y})$ if

$$\tilde{L}((\bar{x}, \bar{y}), \mathscr{F}) \subseteq T((\bar{x}, \bar{y}), \mathscr{F}),$$
$$\tilde{L}((\bar{x}, \bar{y}), \mathscr{F}) \subseteq A((\bar{x}, \bar{y}), \mathscr{F}),$$
$$\tilde{L}((\bar{x}, \bar{y}), \mathscr{F}) \subseteq \operatorname{cl} D((\bar{x}, \bar{y}), \mathscr{F}),$$

respectively.

Definition 4.3 (E. Arrow-Hurwicz-Uzawa CQ for the Concave Case). Let $(\bar{x}, \bar{y}) \in \mathscr{F}$. We say that the extended Arrow-Hurwicz-Uzawa CQ for the concave case holds at $(\bar{x}, \bar{y})$ if ψ is concave and there exists v such that

$$\Bigg(-\sum_{i\in I(\bar{x},\bar{y})} \gamma_i \nabla_x g_i(\bar{x}, \bar{y}), \alpha \nabla_y f(\bar{x}, \bar{y})\Bigg)^{\top} v \le 0 \quad \forall (\alpha, \gamma) \in FJ(\bar{x}, \bar{y}),$$
$$\nabla g_i(\bar{x}, \bar{y})^{\top} v < 0 \quad \forall i \in I(\bar{x}, \bar{y}) \backslash \Lambda,$$
$$\nabla g_i(\bar{x}, \bar{y})^{\top} v \le 0 \quad \forall i \in \Lambda,$$
$$\nabla G_k(\bar{x}, \bar{y})^{\top} v < 0 \quad \forall k \in K(\bar{x}, \bar{y}) \backslash \Gamma,$$
$$\nabla G_k(\bar{x}, \bar{y})^{\top} v \le 0 \quad \forall k \in \Gamma,$$

where

$$\Lambda := \{i \in I(\bar{x}, \bar{y}) \colon g_i \text{ is pseudoconcave at } (\bar{x}, \bar{y})\},$$
$$\Gamma := \{k \in K(\bar{x}, \bar{y}) \colon G_k \text{ is pseudoconcave at } (\bar{x}, \bar{y})\}.$$

Theorem 4.1. *Let $(\bar{x}, \bar{y})$ be a local solution of* (BLPP) *where $S(\bar{x}) \neq \varnothing$. Suppose that $h(x, y) = f(x, y) - V(x)$ is concave and one of the constraint qualifications, such as the N. Abadie CQ, N. Kuhn-Tucker CQ, N. Zangwill CQ, N. Arrow-Hurwicz-Uzawa CQ, N. Weak Reverse Convex CQ, E. Abadie CQ, E. Kuhn-Tucker CQ, E. Zangwill CQ, or E. Arrow-Hurwicz-Uzawa CQ for the concave case is satisfied, then there exist $(\alpha, \gamma) \in FJ(\bar{x}, \bar{y})$ and $\lambda \ge 0$, $\eta \in R^p_+$, $\beta \in R^q_+$ such that*

$$0 = \nabla F(\bar{x}, \bar{y}) + \lambda\Bigg(-\sum_{i\in I(\bar{x},\bar{y})} \gamma_i \nabla_x g_i(\bar{x}, \bar{y}), \alpha \nabla_y f(\bar{x}, \bar{y})\Bigg) + \sum_{i\in I(\bar{x},\bar{y})} \eta_i \nabla g_i(\bar{x}, \bar{y}) + \sum_{k\in K(\bar{x},\bar{y})} \beta_k \nabla G_k(\bar{x}, \bar{y}). \qquad (6)$$

Equivalently, there exist $\lambda \ge 0$, $\alpha \ge 0$, $\gamma \in R^p_+$, $\eta \in R^p_+$, $\beta \in R^q_+$ such that $\|(\alpha, \gamma)\|_1 = 1$ and

$$0 = \nabla F(\bar{x}, \bar{y}) + \sum_{i\in I} (\eta_i - \lambda\gamma_i) \nabla g_i(\bar{x}, \bar{y}) + \sum_{k\in K(\bar{x},\bar{y})} \beta_k \nabla G_k(\bar{x}, \bar{y}),$$
$$0 = \alpha \nabla_y f(\bar{x}, \bar{y}) + \sum_{i\in I(\bar{x},\bar{y})} \gamma_i \nabla_y g_i(\bar{x}, \bar{y}),$$
$$\gamma_i = 0, \qquad \eta_i = 0 \quad \forall i \notin I(\bar{x}, \bar{y}).$$

Equivalently, there exist $\alpha \ge 0$, $\gamma \in R^p_+$, $\eta^g \in R^p$, $\beta \in R^q_+$ such that $\|(\alpha, \gamma)\|_1 = 1$ and

$$0 = \nabla F(\bar{x}, \bar{y}) + \sum_{i\in I(\bar{x},\bar{y})} \eta^g_i \nabla g_i(\bar{x}, \bar{y}) + \sum_{k\in K(\bar{x},\bar{y})} \beta_k \nabla G_k(\bar{x}, \bar{y}), \qquad (7)$$
$$0 = \alpha \nabla_y f(\bar{x}, \bar{y}) + \sum_{i\in I(\bar{x},\bar{y})} \gamma_i \nabla_y g_i(\bar{x}, \bar{y}), \qquad (8)$$
$$\eta^g_i \ge 0 \quad i \in I_0(\bar{x}, \bar{y}), \qquad (9)$$

where $I_0(\bar{x}, \bar{y}) := \{i \in I \colon g_i(\bar{x}, \bar{y}) = 0$ and $\gamma_i = 0\}$.

Proof. It is obvious that all constraint qualifications stated in the theorem imply the extended Abadie CQ for the concave case. Now assume that the extended Abadie CQ for the concave case holds. Replacing the linearization cone $L((\bar{x},\bar{y}),\mathcal{F})$ by the extended linearization cone for the concave case $\tilde{L}((\bar{x},\bar{y}),\mathcal{F})$ in the proof of Proposition 3.1, we have

$$0 \in \nabla F(\bar{x},\bar{y}) + \operatorname{cl}\operatorname{cone}\left[A \cup \bigcup_{i\in I(\bar{x},\bar{y})} \{\nabla g_i(\bar{x},\bar{y})\} \cup \bigcup_{k\in K(\bar{x},\bar{y})} \{\nabla G_k(\bar{x},\bar{y})\}\right], \tag{10}$$

where

$$A := \left\{\left(-\sum_{i\in I(\bar{x},\bar{y})} \gamma_i \nabla_x g_i(\bar{x},\bar{y}),\, \alpha\nabla_y f(\bar{x},\bar{y})\right) \colon (\alpha,\gamma)\in FJ(\bar{x},\bar{y})\right\}.$$

Because the set $FJ(\bar{x},\bar{y})$ is a convex polytope and

$$\left(-\sum_{i\in I(\bar{x},\bar{y})} \gamma_i \nabla_x g_i(\bar{x},\bar{y}),\, \alpha\nabla_y f(\bar{x},\bar{y})\right)$$

is a linear mapping of (α,γ), the set A is also a convex polytope by virtue of Rockafellar [24, Theorem 19.3]. By definition of a convex polytope, the above set A is a convex hull of a finite set of points. Consequently, the convex hull of the set

$$B := A \cup \bigcup_{i\in I(\bar{x},\bar{y})} \{\nabla g_i(\bar{x},\bar{y})\} \cup \bigcup_{k\in K(\bar{x},\bar{y})} \{\nabla G_k(\bar{x},\bar{y})\} \cup \{0\}$$

is a polyhedral convex set containing the origin. By Rockafellar [24, Corollary 19.7.1], the convex cone generated by set $\operatorname{co} B$ is polyhedral. However, the convex cone generated by set

$$C := A \cup \bigcup_{i\in I(\bar{x},\bar{y})} \{\nabla g_i(\bar{x},\bar{y})\} \cup \bigcup_{k\in K(\bar{x},\bar{y})} \{\nabla G_k(\bar{x},\bar{y})\}$$

is the same as the convex cone generated by $\operatorname{co} B$, so it is also polyhedral, and hence closed. Therefore, the closure operation in (10) is superfluous. Because the set A is convex, (10) implies the existence of $(\alpha,\gamma)\in F(\bar{x},\bar{y})$ and $\lambda\ge 0$, $\eta\ge 0$, $\beta\ge 0$ such that (6) holds. The equivalence of the first two conditions is obvious. It is also obvious that the second condition implies the third one. We now suppose that there exist $\alpha\ge 0$, $\gamma\in R^p_+$, $\eta^g\in R^p_+$, $\beta\in R^q_+$ such that $\|(\alpha,\gamma)\|_1 = 1$ such that (7)–(9) hold. Let λ be the smallest positive number such that

$$\eta_i^g + \lambda\gamma_i \ge 0 \quad \forall\, i\in I(\bar{x},\bar{y})$$

and $\eta := \eta^g + \lambda\gamma$; then the second condition holds with multipliers $(\lambda,\alpha,\gamma,\eta,\beta)$. □

In the case where the set of Fritz John multipliers $FJ(\bar{x},\bar{y})$ coincides with the set of KKT multipliers for the lower-level problem $P_{\bar{x}}$ at $\bar{y}$, one can take $\alpha = 1$, and hence the necessary condition derived in Theorem 4.1 reduces to the one given in Ye [31, Theorem 4.2]. Hence, Theorem 4.1 extends the result of Ye [31, Theorem 4.2] in that the lower-level problem $P_{\bar{x}}$ is not required to satisfy any constraint qualification at $\bar{y}$.

In the following result we derive the KKT condition for a class of BLPP where no constraint qualifications are required for the KKT condition to hold.

Corollary 4.1. *Let $(\bar{x},\bar{y})$ be a local solution of BLPP where $S(\bar{x})\neq\emptyset$, and suppose that $f(x,y)$ is jointly linear. $g_i(x,y)$ $(i\in I(\bar{x},\bar{y}))$ are jointly convex and pseudoconcave at $(\bar{x},\bar{y})$ and $G_k(x,y)$ $(k\in K(\bar{x},\bar{y}))$ are pseudoconcave at $(\bar{x},\bar{y})$. Then the KKT condition in Theorem 4.1 holds at $(\bar{x},\bar{y})$.*

Proof. By virtue of Fiacco and Kyparisis [12] under the assumptions of the corollary, the function $-\psi$ is convex, and hence ψ is concave. Moreover, $f(x,y)-V(x)$ is also concave, and it is easy to show $\partial^\circ h(\bar{x},\bar{y})\subseteq \partial^\circ\psi(\bar{x},\bar{y})$ in this case. Because all binding constraints of problem $(\mathrm{SP})_\psi$ are pseudoconcave at the optimal solution, the nonsmooth weak reverse convex constraint qualification holds. The result then follows from Theorem 4.1. □

It is interesting to compare our approach with the classical approach, in which the lower-level problem is replaced by the KKT condition of the lower-level problem. First of all, even when the lower-level problem is linear convex, it may happen that the lower-level problem does not satisfy the KKT condition, and hence the classical approach is not applicable to this case. We now consider the case where the lower-level problem is linear convex and the KKT condition is necessary and sufficient for the lower-level problem for each $y\in S(x)$.

Then, by the classical approach, the following single-level problem is considered:

$$\begin{aligned}
\text{(KP)}\quad \min_{x,y,\gamma}\ & F(x,y)\\
\text{s.t.}\ & 0=\nabla_y f(x,y)+\sum_{i\in I}\gamma_i\nabla_y g_i(x,y),\\
& \gamma\ge 0,\qquad g_i(x,y)\le 0,\quad i\in I,\\
& \sum_{i\in I}\gamma_i g_i(x,y)\ge 0,\\
& G_k(x,y)\le 0\quad k\in K.
\end{aligned}\tag{11}$$

It is obvious that if $(\bar{x},\bar{y})$ is a local solution of the BLPP, and the KKT condition is necessary and sufficient for the lower-level problem for each $y\in S(x)$, then there exists γ such that $(\bar{x},\bar{y},\gamma)$ is a local solution to the single-level problem (KP). The converse implication, however, is not true in general unless the KKT multiplier for the lower-level problem is unique.

Problem (KP) belongs to the class of MPECs and it is known (see Ye et al. [35, Proposition 1.1]) that the usual CQs such as the MFCQ do not hold for problem (KP). However, if the defining functions f, g_i $(i\in I)$ are second-order continuously differentiable and the MPEC LICQ holds at $(\bar{x},\bar{y},\gamma)$, a local optimal solution of (KP), then $(\bar{x},\bar{y},\gamma)$ is an S-stationary point: That is, there exist $\mu\in R^m$, $\eta^g\in R^p$, $\beta\in R^q_+$, $\tilde{\eta}\in R^p$ such that

$$\begin{gathered}
0=\nabla F(\bar{x},\bar{y})+\sum_{i\in I(\bar{x},\bar{y})}\eta_i^g\nabla g_i(\bar{x},\bar{y})+\sum_{k\in K(\bar{x},\bar{y})}\beta_k\nabla G_k(\bar{x},\bar{y})+\nabla\Big[\nabla_y f+\sum_{i\in I}\gamma_i\nabla_y g_i\Big](\bar{x},\bar{y})^\top\mu,\\
0=\nabla_y f(\bar{x},\bar{y})+\sum_{i\in I(\bar{x},\bar{y})}\gamma_i\nabla_y g_i(\bar{x},\bar{y}),\\
\tilde{\eta}=\nabla_y g(\bar{x},\bar{y})^\top\mu,\qquad \gamma_i\tilde{\eta}_i=0\quad i\in I,\\
\eta_i^g\ge 0,\qquad \tilde{\eta}_i\ge 0\quad \forall\, i\in I_0(\bar{x},\bar{y}).
\end{gathered}$$

Comparing the above S-stationary condition for (KP) with the KKT condition (7)–(9) in Theorem 4.1, it is easy to see that in the case where $\alpha\neq 0$, $(\bar{x},\bar{y})$ satisfying the KKT condition in Theorem 4.1 implies that there exists γ such that $(\bar{x},\bar{y},\gamma)$ is a S-stationary point of (KP) with the multiplier for the constraint (11) $\mu=0$. Theorem 4.1 therefore provides some sufficient conditions for $(\bar{x},\bar{y},\gamma)$ to be an S-stationary point for (KP) without the MPEC LICQ.

Finally, we would like to comment on the uniform boundedness assumption of the set-valued map Y. Due to the use of function ψ, it is clear that in general Y is required to be uniformly bounded around $\bar{x}$. However, for the case of the "generalized linear" BLPP where $f(x,y)$, $g_i(x,y)$ $(i\in I)$ are jointly linear and $G_k(x,y)$ $(k\in K(\bar{x},\bar{y}))$ are pseudoconcave at $(\bar{x},\bar{y})$, the function $f(x,y)-V(x)$ is concave, and no constraint qualification is required for the KKT condition of any optimization problem with constraints $g_i(x,y)\le 0$ $(i\in I)$ to hold. In this case there is no need to use the function ψ. Indeed, by using the problem $(\text{SP})_V$, Ye [31, Corollary 4.1] has shown that a local optimal solution $(\bar{x},\bar{y})$ satisfies the KKT condition in Theorem 4.1 without the uniform boundedness assumption on the set-valued map Y.

Note that, similar to the proof of the equivalence of the second and the third KKT conditions in Theorem 4.1, it is easy to prove that $(\bar{x},\bar{y},\gamma)$ is an S-stationary point if only it satisfies the classical KKT condition for problem KP (treating it as a nonlinear programming problem with equality and inequality constraints instead of a mathematical program with complementarity constraints). Hence, our approach has the advantage over the classical approach in that the resulting KKT condition is sharper, no second-order differentiability of the functions f, g_i are required and no constraint qualification is needed for the class of BLPP given in Corollary 4.1.

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