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DISCONTINUOUS SOLUTIONS OF THE HAMILTON–JACOBI EQUATION FOR EXIT TIME PROBLEMS*

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Abstract. In general, the value function associated with an exit time problem is a discontinuous function. We prove that the lower (upper) semicontinuous envelope of the value function is a supersolution (subsolution) of the Hamilton–Jacobi equation involving the proximal subdifferentials (superdifferentials) with subdifferential-type (superdifferential-type) mixed boundary condition. We also show that if the value function is upper semicontinuous, then it is the maximum subsolution of the Hamilton–Jacobi equation involving the proximal superdifferentials with the natural boundary condition, and if the value function is lower semicontinuous, then it is the minimum solution of the Hamilton–Jacobi equation involving the proximal subdifferentials with a natural boundary condition. Futhermore, if a compatibility condition is satisfied, then the value function is the unique lower semicontinuous solution of the Hamilton–Jacobi equation with a natural boundary condition and a subdifferential type boundary condition. Some conditions ensuring lower semicontinuity of the value functions are also given.

Key words. Hamilton–Jacobi equation, dynamic programming principle, exit time problems, proximal subdifferentials

AMS subject classifications. 49L05, 49L20

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1. Introduction. In this paper we study the exit time problem (also called the control problem with a boundary condition as in [10]). In its simplest form, the exit time problem involves a given open set E in R^n, and asks for choices for the time $t^* \geq 0$ and the measurable function u on $[0, t^*)$ which will

$$\begin{aligned} \text{minimize} \quad & J(x,u) := \int_0^{t^*} e^{-\lambda s} f(y(s),u(s))ds + e^{-\lambda t^*} h(y(t^*)) \\ \text{subject to (s.t.)} \quad & \dot{y}(t) = g(y(t),u(t)) \quad \text{a.e. } \ t \in [0,t^*], \\ & u(t) \in U \quad \text{a.e. } \ t \in [0,t^*), \\ & y(0) = x, y(t) \in E, 0 \leq t < t^*, y(t^*) \notin E. \end{aligned}$$

By the classical Hamilton–Jacobi (H–J) theory (or the so-called dynamic programming theory), if the value function V is continuously differentiable, then it is the unique solution of the following H–J equation:

$$\lambda V(x) + H(x, -\nabla V(x)) = 0 \quad \forall x \in E, \tag{1}$$

where the Hamiltonian $H(x,p) := \max\{p \cdot g(x,u) - f(x,u) : u \in U\}$, with the natural boundary condition

$$V(x) = h(x) \quad \forall x \in \partial E.$$

Due to the complicated behavior of the trajectories at the boundary of the state space, the value function for the exit time problem is in general discontinuous, even

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if all the problem data are Lipschitz continuous, unless some nontangency condition is imposed on the boundary (see, e.g., [12, 23, 10] for the Lipschitz continuity of the value function). Solving the H–J equation (1) with appropriate boundary conditions in some nonclassical sense has become an active research area. Gonzalez and Rofman [13] proved that the value function is an upper bound of a suitable set of subsolutions of the H–J equation. Dempster and Ye [10] characterized the Lipschitz value function as a solution of the H–J equation involving the Clarke generalized gradient. Bardi and Soravia [2], Barles and Perthame [4, 5], Blanc [6], Ishii [14], and Soravia [17, 18] have studied the solution of the H–J equation (1) with various boundary conditions in the framework of the viscosity solutions first introduced by Crandall and Lions [9] for continuous functions and later defined for discontinuous functions by Ishii [14, 15] and modified by Barron and Jenson [3] for the case of convex Hamiltonians. The reader is also referred to the recent monograph of Bardi and Capuzzo-Dolcetta [1] for the history and the recent development of the H–J equation using the viscosity approach.

Under assumptions that reduce the exit time problem to a generalized optimal stopping time problem, Ye and Zhu [24] showed that the value function of the exit time problem with relaxed controls is the unique lower semicontinuous solution of the H–J equation with the usual gradient replaced by the proximal subdifferential $\partial_p V(x)$ (see Definition 2.1) with the natural boundary condition

$$V(x) = h(x) \ \forall x \in E^c,$$

where E^c denotes the complement of the state space E and the subdifferential type boundary condition, i.e.,

$$\lambda V(x) + H(x, -\partial_p V(x)) \leq 0 \quad \forall x \in \partial E.$$

The purpose of this paper is to extend the H–J theory using the equivalence between the invariance and the H–J equation to treat exit time problems under assumptions that are much more general than those in [24]. In particular, we allow the discount rate λ to be zero and the exit cost h to be unbounded. In Theorem 2.2 we show that the lower (upper) semicontinuous envelope of the value function is a supersolution (subsolution) of the H–J equation involving the proximal subdifferentials (superdifferentials) with subdifferential-type (superdifferential-type) mixed boundary condition. In Theorems 2.3 and 2.4 we show that if the value function is upper semicontinuous, then it is the maximum subsolution of the H–J equation involving the proximal superdifferentials with the natural boundary condition, and if the value function is lower semicontinuous, then it is the minimum solution of the H–J equation involving the proximal subdifferentials with a natural boundary condition. Some conditions ensuring lower semicontinuity of the value functions are given in Proposition 2.5.

The technique of treating semicontinuous solutions to the H–J equation by using equivalence between the invariance property and the H–J equation was first introduced by Subbotin [19] for differential games (see also Subbotin [20]) and has been used in [8, 11] for finite horizon problems and in [22] for minimal time problems. The equivalence of the various concepts of the solution to the H–J equation in an open set was also given in [8].

We arrange the paper as follows: In the next section we state the problem formulation for the exit time problem and our main results. In section 3 we establish the equivalence among the optimality principle, the invariance property, and the H–J equations. The proofs of the main results are contained in section 4.

2. The exit time problems and the H–J equation. Let U be a compact subset of R^m and $\mathrm{Prob}(U)$ the set of all Borel probability measures on U. Consider $\mathrm{Prob}(U)$ as a subset of the dual of $C(U)$ endowed with the weak star topology, where $C(U)$ is the Banach space of continuous functions on U with the supremum norm. For any $\phi \in C(U)$ and $u \in \mathrm{Prob}(U)$, we denote the pairing of ϕ and u by $\phi(u) := \int_U \phi(r)u(dr)$. Let $\mathcal{U}$ be the set of all Lebesgue measurable mappings from R to $\mathrm{Prob}(U)$. For finite real numbers $a < b$, define $\mathcal{U}_{[a,b]} := \{u|_{[a,b]} : u \in \mathcal{U}\}$. Then $\mathcal{U}_{[a,b]}$ is a weak star compact subset of $L^1([a,b]; C(U))^*$. We endow $\mathcal{U}$ with the following topology: u^n converges to u in $\mathcal{U}$ provided that $u^n|_{[a,b]}$ converges to $u|_{[a,b]}$ in $\mathcal{U}_{[a,b]}$ for any finite real numbers $a < b$. The set $\mathcal{U}_{[a,b]}$ is the collection of relaxed control functions defined in Warga [21]. It is the compactification of the set of usual control functions in the weak star topology of $L^1([a,b]; C(U))^*$. Elements of $\mathcal{U}_{[a,b]}$ are called relaxed controls. Using the set of relaxed controls ensures the existence of the optimal solution and also ensures the convexity of the velocity set so that the invariance theorems can be used. Any relaxed control can be approximated by usual controls. We refer to [21] for more details.

Let the state space E be an open subset of R^d, $\bar{E}$ be the closure of E, and O be an open set containing $\bar{E}$. Assume that $g : O \times U \to R^d$ satisfies the following condition.

(H1) $g(x,u)$ is continuous, bounded, and Lipschitz in x uniformly in $u \in U$.

Under such a condition, for each $x \in O$ and $u \in \mathcal{U}$, the differential equation

$$\dot{y}(s) = g(y(s), u(s)) \quad \text{a.e.}$$

has a unique solution defined on R that satisfies the side condition $y(0) = x$. We denote this solution by $y[x,u](\cdot)$ to indicate its dependence on x and u.

For each initial state $x \in E$ and control function u, define the exit time $t^*[x,u]$ to be the first time the trajectory starting from $x \in E$ corresponding to the control u exits from the state space E, or infinity if it never exits the state space; i.e.,

$$t^*[x,u] := \inf\{t > 0 : y[x,u](t) \notin E\},$$

where $\inf \emptyset = \infty$ by convention. For any $x \in E^c$, we define $t^*[x,u] := 0$. Where there is no confusion, we will simply use t^* instead of $t^*[x,u]$.

Let $\lambda \geq 0$ be the discount rate. Consider the following exit time problem:

$$\begin{aligned} P_x \quad & \text{minimize} \quad J(x,u) := \int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda t^*} h(y[x,u](t^*)) \\ & \text{s.t.} \quad u \in \mathcal{U}. \end{aligned}$$

We state some further basic assumptions:

(H2) The running cost $f(x,u) : O \times U \to R^d$ is continuous, bounded, and Lipschitz in x uniformly in $u \in U$. The exit cost $h(x) : O \to R$ is lower semicontinuous. Furthermore, when $t^*[x,u] = \infty$ the integral $\int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s))ds$ converges and the limit $e^{-\lambda\infty} h(y[x,u](\infty)) := \lim_{r\to\infty} e^{-\lambda r} h(y[x,u](r))$ exists and is finite.

REMARK 1. *The exit time problem we consider in this paper is more general than that usually considered in the literature* (*see, e.g.,* [1, 6]) *in that we allow the discount rate* λ *to be zero and the exit cost* h *to be unbounded. Notice that under the assumption that* f *is bounded, the integral* $\int_0^\infty e^{-\lambda s} f(y[x,u](s), u(s))ds$ *converges* *automatically for the case* $\lambda > 0$ *so the assumption* (H2) *is mainly for the case when* $\lambda = 0$.

Under our assumptions, it is known that there exists an optimal control for the exit time problem for each $x \in E$. Define the value function of the family of problems P_x as

$$V(x) := \min \left\{ \int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s)) ds + e^{-\lambda t^*} h(y[x,u](t^*)) : u \in \mathcal{U} \right\}.$$

Unlike a standard free end point optimal control problem whose value function is continuous if the terminal cost is continuous, the value function for the exit time problem is in general discontinuous even in the case where the terminal cost h is smooth. To see this we examine two simple examples.

EXAMPLE 1. *Let* $E = (0,1)$ *be the state space and the control set* $U = \{-1\}$. *Consider the following exit time problem where* $f(x,u) \equiv 0, g(x,u) = u, h(x) = x$, $\lambda = 0$:

$$\begin{aligned} \min \quad & y(t^*) \\ s.t. \quad & \dot{y} = u, u(t) = -1, \\ & y(0) = x. \end{aligned}$$

It is easy to see that the value function

$$V(x) = \begin{cases} 0 & \textit{if } x \in [0,1), \\ x & \textit{if } x \notin [0,1) \end{cases}$$

is upper semicontinuous with discontinuity at $x = 1$.

EXAMPLE 2. *In Example* 1, *change the control set to* $U = \{1\}$. *Then the value function becomes*

$$V(x) = \begin{cases} 1 & \textit{if } x \in (0,1], \\ x & \textit{if } x \notin (0,1], \end{cases}$$

which is lower semicontinuous with discontinuity at $x = 0$.

In order to see the connections between the value function and the H–J equations we define the lower and upper semicontinuous envelopes of a function $W : O \to R$ as

$$W_*(x) := \liminf_{y \to x} W(y)$$

and

$$W^*(x) := \limsup_{y \to x} W(y),$$

respectively. Then it is easy to see that W_* is lower semicontinuous and W^* is upper semicontinuous.

We will use the concept of proximal subdifferentials (superdifferentials) for any lower (upper) semicontinuous functions defined as follows.

DEFINITION 2.1 (see, e.g., Clarke [7] and Loewen [16]). *Let* $\phi : R^d \to (-\infty, \infty]$ *be an extended-valued lower semicontinuous function. The proximal subdifferential of* ϕ *at* $x \in R^d$ *where* $\phi(x) \neq \infty$ *is a set-valued map defined by*

$$\begin{aligned} \partial_p \phi(x) := \{ \xi \in R^d : & \exists \sigma > 0, \delta > 0 \\ & s.t.\ \phi(y) \geq \phi(x) - \sigma \|y - x\|^2 + \langle \xi, y - x \rangle \quad \forall y \in x + \delta B \}, \end{aligned}$$

where $\langle a, b\rangle$ denotes the inner product of the vectors a and b and B denotes the open unit ball. Let $\phi : R^d \to [-\infty, \infty)$ be an extended-valued upper semicontinuous function. The proximal superdifferential of ϕ at x where $\phi(x) \neq \infty$ is defined by

$$\partial^p\phi(x) := -\partial_p(-\phi)(x),$$

i.e.,

$$\partial^p\phi(x) := \{\xi \in R^d : \exists\sigma > 0, \delta > 0 \quad s.t.\ \phi(y) \leq \phi(x) + \sigma\|y - x\|^2 + \langle\xi, y - x\rangle \quad \forall x + \delta B\}.$$

REMARK 2. *Since the function $y \to \phi(x) - \sigma\|y-x\|^2 + \langle\xi, y-x\rangle$ in the right-hand side of the inequality in the definition of the proximal subdifferential is a quadratic, it is easy to see that $\xi \in \partial_p\phi(x)$ if and only if there is a parabola fitting under the epigraph of ϕ at $(x, \phi(x))$ with ξ as the slope of ϕ at x. Hence, in the case where there does not exist a parabola fitting under the epigraph of ϕ at $(x, \phi(x))$, the proximal subdifferential of ϕ at x may be empty* (*e.g., $\phi(x) = -|x|$ has an empty proximal subdifferential at* 0). *Similarly the proximal superdifferential may be empty. However, we shall see later that the emptiness of the proximal subdifferential* (*superdifferential*) *is actually an advantage.*

We now state our main results. The first result gives the connection between the semicontinuous envelopes of the value function and the H–J inequalities.

THEOREM 2.2. *Under assumptions* (H1)–(H2) *the lower semicontinuous envelope of the value function $V_*(x)$ is a supersolution of the H–J equation involving the proximal subdifferentials* (*in E*), *i.e.,*

$$\lambda V_*(x) + H(x, -\partial_p V_*(x)) \geq 0 \quad \forall x \in E \tag{2}$$

with the subdifferential-type mixed boundary condition

$$\max\{V_*(x) - h(x), \lambda V_*(x) + H(x, -\partial_p V_*(x))\} \geq 0 \quad \forall x \in \partial E, \tag{3}$$

and the upper semicontinuous envelope of the value function $V^*(x)$ is a subsolution of the H–J equation involving the proximal superdifferentials (*in E*), *i.e.,*

$$\lambda V^*(x) + H(x, -\partial^p V^*(x)) \leq 0 \quad \forall x \in E, \tag{4}$$

with the superdifferential-type mixed boundary condition

$$\min\{V^*(x) - h^*(x), \lambda V^*(x) + H(x, -\partial^p V^*(x))\} \leq 0 \quad \forall x \in \partial E, \tag{5}$$

where ∂E denotes the boundary of E.

REMARK 3. *Equation* (2) *should be understood in the following sense: At any point $x \in E$ where $\partial_p V_*(x) \neq \emptyset$,*

$$\lambda V_*(x) + H(x, -\xi) \geq 0 \quad \forall \xi \in \partial_p V_*(x).$$

Hence the points x where $\partial_p V_*(x) = \emptyset$ can be neglected. Equation (4) *is understood in a similar way. Equation* (3) *means that if $x \in \partial E$ is a point where $V_*(x) < h(x)$ and $\partial_p V_*(x) \neq \emptyset$, then*

$$\lambda V_*(x) + H(x, -\xi) \geq 0 \quad \forall \xi \in \partial_p V_*(x).$$

Similarly, (5) *means that if $x \in \partial E$ is a point where $V^*(x) > h^*(x)$ and $\partial^p V^*(x) \neq \emptyset$, then*

$$\lambda V^*(x) + H(x, -\xi) \le 0 \quad \forall \xi \in \partial^p V^*(x).$$

Note that a similar result was given in Theorem 2.9 *of Blanc* [6] *in the viscosity solution sense for the case $\lambda > 0$ and bounded exit cost h. In general, as in Remark* 2.7 *of Blanc* [6], *we do not expect to have a unique function that satisfies* (2)–(5).

When the value function has a semicontinuity property, the following two theorems give connections between the value function (instead of its semicontinuous envelopes) and the H–J equation with natural boundary condition (instead of the mixed boundary condition).

THEOREM 2.3. *In additions to assumptions* (H1)–(H2), *assume that the value function is upper semicontinuous. Then it is the maximum upper semicontinuous function that is a subsolution of the H–J equation involving the proximal superdifferentials* (*in E*), *i.e.,*

$$\lambda W(x) + H(x, -\partial^p W(x)) \le 0 \quad \forall x \in E,$$

with the natural boundary condition

$$W(x) = h(x) \quad \forall x \in \partial E.$$

THEOREM 2.4. *In additions to assumptions* (H1)–(H2), *assume that the value function is lower semicontinuous. Then it is the minimum lower semicontinuous solution of the H–J equation involving the proximal subdifferentials* (*in E*), *i.e.,*

$$\lambda W(x) + H(x, -\partial_p W(x)) = 0 \quad \forall x \in E,$$

with the natural boundary condition

$$W(x) = h(x) \quad \forall x \in \partial E.$$

We now give some conditions which ensure lower semicontinuity of the value function. First we state the required assumptions.

(A) $\forall\ x \in \partial E$ and $u \in \mathcal{U}$ such that $y[x,u](t) \in \bar{E} \quad \forall t \in [0,\infty)$, the limit $\lim_{r\to\infty} e^{-\lambda r} h(y[x,u](r))$ exists. Also, $\forall x \in \partial E$, all controls $u \in \mathcal{U}$ and $r \ge 0$ such that $y[x,u](t) \in \bar{E} \forall t \in [0,r], y[x,u](r) \in \partial E$, or $r = \infty$,

$$h(x) \le \int_0^r e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda r} h(y[x,u](r)).$$

PROPOSITION 2.5. *In addition to assumptions* (H1)–(H2), *if* (A) *is satisfied, then the value function $V(x)$ is lower semicontinuous on $\bar{E}$.*

Proof. Let $x \in \bar{E}$ and $x_n \in \bar{E}, x_n \to x$. By definition of the value function for each n, there exists $u^n \in \mathcal{U}$ and $t^*[x_n, u_n] := t_n^*$ such that

$$V(x_n) = \int_0^{t_n^*} e^{-\lambda s} f(y[x_n,u_n](s), u_n(s))ds + e^{-\lambda t_n^*} h(y[x_n,u_n](t_n^*)). \tag{6}$$

We now consider two cases.

Case 1. The sequence $\{t_n^*\}$ is bounded. Without loss of generality we may assume that t_n^* converges to r, $t_n^* \in [0, r+1]$ and $u_n|_{[0,r+1]}$ converges to $u \in \mathcal{U}_{[0,r+1]}$ in the topology of $\mathcal{U}_{[0,r+1]}$. Then $y[x_n, u_n](s)$ uniformly converges to $y[x,u](s)$ in $[0, r+1]$. Since $\bar{E}$ and ∂E are closed, $y[x,u](t) \in \bar{E}\ \forall t \in [0,r]$ and $y[x,u](r) \in \partial E$. Taking lim inf in (6) when $n \to \infty$ yields by virtue of lower semicontinuity of h

$$\begin{aligned}
\liminf_{n\to\infty} V(x_n) &\geq \int_0^r e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda r} h(y[x,u](r)) \\
&= \int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s))ds \\
&\quad + \int_{t^*}^r e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda r} h(y[x,u](r)) \\
&\geq \int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda t^*} h(y[x,u](t^*)) \\
&\qquad\qquad \text{(by assumption (A))} \\
&\geq V(x).
\end{aligned}$$

Case 2. The sequence $\{t_n^*\}$ is unbounded. Without loss of generality we may assume that $t_n^* \to \infty$. For each integer $m > 0$, consider the restriction of u_n to $[0, m]$. Since $\mathcal{U}_{[0,m]}$ is compact we can extract a convergent subsequence from $\{u_n|_{[0,m]}\}$. Using the diagonal method we can choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ and an element $u \in \mathcal{U}$ such that $u_{n_i}|_{[0,m]}$ converges to $u|_{[0,m]}$ in $\mathcal{U}|_{[0,m]}$ for any m. We may assume that

$$\lim_{i\to\infty} V(x_{n_i}) = \liminf_{n\to\infty} V(x_n).$$

Taking lim inf in (6) when $n_i \to \infty$ yields in the case $t^*[x,u] = \infty$ that

$$\begin{aligned}
\liminf_{i\to\infty} V(x_{n_i}) &\geq \int_0^\infty e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda \infty} h(y[x,u](\infty)) \\
&\geq V(x)
\end{aligned}$$

and in the case $t^*[x,u] < \infty$ that

$$\begin{aligned}
\liminf_{i\to\infty} V(x_{n_i}) &\geq \int_0^\infty e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda \infty} h(y[x,u](\infty)) \\
&= \int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s))ds \\
&\quad + \int_{t^*}^\infty e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{\lambda \infty} h(y[x,u](\infty)) \\
&\geq \int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda t^*} h(y[x,u](t^*)) \\
&\qquad\qquad \text{(by assumption (A))} \\
&\geq V(x).
\end{aligned}$$

The proof of the proposition is complete. □

REMARK 4. *In its essentials the above result says that if we allow the trajectory to continue after the first exit time but assume that it is cheaper to stop at the first exit time than to continue until the trajectory reaches the boundary again, then the value function is lower semicontinuous. The assumption* (A) *is not satisfied at the boundary* *point $x = 1$ for Example 1 since the only r in this case is $r = 1$ and the trajectory starting at $x = 1$ using the only control $u = -1$ reaches the boundary state $x = 0$ at time $r = 1$, but $1 = h(1) \not\leq h(y[1, u](1)) = 0$. On the other hand, the assumption* (A) *is satisfied by Example* 2 *and hence the value function is lower semicontinuous.*

Combining Theorem 2.4 and Proposition 2.5, we have the following corollary.

COROLLARY 2.6. *Under assumptions* (H1), (H2), *and* (A), *the value function is the minimum lower semicontinuous solution of the H–J equation involving the proximal subdifferentials* (*in E*), *i.e.,*

$$\lambda W(x) + H(x, -\partial_p W(x)) = 0 \quad \forall x \in E, \tag{7}$$

with the natural boundary condition

$$W(x) = h(x) \quad \forall x \in \partial E. \tag{8}$$

One may wonder whether the natural boundary condition (8) is enough for the uniqueness of the solution to (7). The following example gives a negative answer.

EXAMPLE 3. *Let $E = R^2 \backslash \{0\}$ be the state space and the control set $U = [-1, 1]$. Consider the exit time problem where $f(x, u) \equiv 1, g(x, u) = (u, 0), h(x) = 0, \lambda = 1$. It is easy to see that*

$$V(x) = \begin{cases} \int_0^{|x_1|} e^{-s} ds = 1 - e^{-|x_1|} & \text{if } x_2 = 0, \\ \int_0^{\infty} e^{-s} ds = 1 & \text{if } x_2 \neq 0, \end{cases}$$

$$H(x, p) = \max\{p_1 u : u \in [-1, 1]\} - 1 = |p_1| - 1,$$

and

$$\partial_p V(x_1, 0) = \begin{cases} (e^{-x_1}, 0) & \text{if } x_1 > 0, \\ (-e^{x_1}, 0) & \text{if } x_1 < 0, \\ [-1, 1] \times \{0\} & \text{if } x_1 = 0. \end{cases}$$

Hence the value function is a lower semicontinuous solution of the H–J equation (7) *with the natural boundary condition* (8). *However, the function $W(x) = 1$ if $x \neq 0$ and $W(0) = 0$ is also a lower semicontinuous solution of* (7), (8). *Indeed, by Corollary* 2.6, *the value function is the minimum solution of the H–J equation* (7) *with the natural boundary condition $V(0) = 0$.*

The above example shows that the natural boundary condition (8) may not be enough to ensure the uniqueness of the solution to the H–J equation involving the proximal subdifferentials (7). However, V satisfies the subdifferential-type boundary condition

$$\lambda V(0) + H(x, -\partial_p V(0)) \leq 0$$

while $W(x)$ does not satisfy the above boundary condition. (Note $\partial_p W(0) = R^2$.) We now give a compatibility condition stronger than assumption (A) under which the value function is not only lower semicontinuous but also a unique lower semicontinuous solution to the H–J equation involving the proximal subdifferentials with the natural boundary condition and the subdifferential-type boundary condition.

(H3) $\forall\ x \in O \setminus E$, all controls $u \in \mathcal{U}$

$$h(x) \leq \int_0^r e^{-\lambda s} f(y[x, u](s), u(s)) ds + e^{-\lambda r} h(y[x, u](r)) \quad \forall 0 \leq r < \tau^*, \tag{9}$$

where

$$\tau^* := \inf\{t > 0 : y[x,u](t) \notin O \setminus E\}$$

and when

$$\tau^* = \infty, \int_0^{\tau^*} e^{-\lambda s} f(y[x,u](s), u(s))ds$$

converges and the limit $\lim_{r\to\infty} e^{-\lambda r} h(y[x,u])(r))$ exists and is finite.

REMARK 5. *When $E = R^d$ and $h = 0$ the exit time problem becomes an infinite horizon problem and assumption* (H3) *is satisfied vacuously. If $f \geq 0$ and $h = 0$, then assumption* (H3) *is also satisfied. This includes the time optimal control problem since solving a time optimal control problem is equivalent to solving an exit time problem with $f = 1$ and $h = 0$. Note that* (H3) *is a local version of the assumption* (5.8) *in Bardi and Capuzzo-Dolcetta* [1] *and* (H3′) *in Ye and Zhu* [24].

The statement of the following theorem is known from Corollary 4.5 of Ye and Zhu [24] for the case where $\lambda > 0$ and h is bounded under the assumption that (9) is satisfied globally $\forall x \in R^d$. However, the proof we give here is independent and different.

THEOREM 2.7. *Under assumptions* (H1)–(H3)*, the value function $V(x)$ is a unique lower semicontinuous solution of the H–J equation involving the proximal subdifferentials* (*in E*)*, i.e.,*

$$\lambda V(x) + H(x, -\partial_p V(x)) = 0 \quad \forall x \in E$$

with the natural boundary condition

$$V(x) = h(x) \qquad \forall x \in O\backslash E$$

and the subdifferential-type boundary condition, i.e.,

$$\lambda V(x) + H(x, -\partial_p V(x)) \leq 0 \forall x \in \partial E.$$

REMARK 6. *Note that a similar result was proved in Theorem* 5.5 *of Bardi and Capuzzo-Dolcetta* [1] *for the lower semicontinuous envelope of the value function in the viscosity solution sense in the case where $\lambda > 0$ and h is bounded under the assumption that* (9) *is satisfied globally $\forall x \in R^d$.*

3. Optimality principle, invariance, and the H–J equation.

DEFINITION 3.1. *Let $W(x) : G :\to R$ where $G \subseteq R^d$ is an open set. We say that $W(x)$ satisfies*

(a) *the* superoptimality principle *in G if and only if $\forall x \in G$ there exists a control $u \in \mathcal{U}$ such that*

$$W(x) \geq \int_0^\tau e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda\tau} W(y[x,u](\tau)) \quad \forall 0 \leq \tau < \tau^*;$$

(b) *the* suboptimality principle *in G if and only if $\forall x \in G$ and $\forall u \in \mathcal{U}$,*

$$W(x) \leq \int_0^\tau e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda\tau} W(y[x,u](\tau)) \qquad \forall 0 \leq \tau < \tau^*.$$

Here $\tau^* := \inf\{t > 0 : y[x,u](t) \notin G\}$ *is the exit time from set* G.

The following facts are well known from the Bellman optimality principle.

PROPOSITION 3.2. *The value function* $V(x)$ *satisfies the superoptimality principle in* E *and suboptimality principle in* E.

Furthermore, the following results indicate that not only the value function but also its lower (upper) semicontinuous envelope satisfies the superoptimality (suboptimality) principle in E.

PROPOSITION 3.3. *The lower semicontinuous envelope of the value function* $V_*(x)$ *satisfies the superoptimality principle in* E *and the upper semicontinuous envelope of the value function* $V^*(x)$ *satisfies the suboptimality principle in* E.

Proof. Fix $x \in E$, and suppose $V_*(x) = \lim_{n\to\infty} V(x_n)$ where $\lim_{n\to\infty} x_n = x, x_n \in E$. Then by Proposition 3.2, there exists $u_n \in \mathcal{U}$ and $t^*[x_n, u_n] = t_n^*$ such that

$$V(x_n) \geq \int_0^\tau e^{-\lambda s} f(y[x_n,u_n](s), u_n(s))ds + e^{-\lambda\tau} V(y[x_n,u_n](\tau)) \quad \forall 0 \leq \tau < t_n^*.$$

Without loss of generality assume that $t_n^* \to t^*$ where t^* may be finite or infinity. Let $0 \leq \tau < t^*$. Hence, for n large enough $\tau \leq t_n^*$. Taking limits and using the compactness of relaxed controls, we find a control $u \in \mathcal{U}$ such that

$$V_*(x) \geq \int_0^\tau e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda\tau} V_*(y[x,u](\tau)) \quad \forall 0 \leq \tau < t^*.$$

Similarly we can prove that $V^*(x)$ satisfies the suboptimality principle. ☐

In the following proposition, we show that either semicontinuity on $\bar{E}$ and the optimality principle in E or the suboptimality principle in an open set containing $\bar{E}$ gives the comparison results.

PROPOSITION 3.4.

(a) *Suppose that* W *satisfies the superoptimality principle in* E. *If* W *is lower semicontinuous on* $\bar{E}$ *and*

$$W(x) \geq h(x) \quad \forall x \in \partial E,$$

then $W(x) \geq V(x) \quad \forall x \in \bar{E}$.

(b) *Suppose that* W *satisfies the suboptimality principle in* E. *If* W *is upper semicontinuous on* $\bar{E}$ *and*

$$W(x) \leq h(x) \quad \forall x \in \partial E,$$

then $W(x) \leq V(x) \quad \forall x \in \bar{E}$.

(b′) *Suppose that* W *satisfies the suboptimality principle in the open set* O *containing* $\bar{E}$ *and*

$$W(x) \leq h(x) \quad \forall x \in \partial E.$$

Then $W(x) \leq V(x) \quad \forall x \in \bar{E}$.

Proof. (a) Suppose that W satisfies the superoptimality in E and $W(x) \geq h(x)$ $\forall x \in \partial E$. Then $\forall x \in E$ there exists $u \in \mathcal{U}$ such that $\forall \tau_n \in [0, t^*)$.

$$W(x) \geq \int_0^{\tau_n} e^{-\lambda s} f(y[x,u](s), u(s))ds + e^{-\lambda\tau_n} W(y[x,u](\tau)).$$

Without loss of generality, assume that $\tau_n \to t^*$. Taking limits in the above inequality, we have by the compactness of relaxed controls and the lower semicontinuity of the function W that

$$\begin{aligned} W(x) &\geq \int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s)) ds + e^{-\lambda t^*} W(y[x,u](t^*)) \\ &\geq \int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s)) ds + e^{-\lambda t^*} h(y[x,u](t^*)) \\ &\geq V(x). \end{aligned}$$

Similarly we can prove (b).

(b′) Now suppose W satisfies the suboptimality in O and $W(x) \leq h(x)\ \forall x \in \partial E$. If $x \in \partial E$, then $W(x) \leq h(x) = V(x)$. If $x \in E$, then $\forall u \in \mathcal{U}$ we have

$$W(x) \leq \int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s)) ds + e^{-\lambda t^*} W(y[x,u](t^*)) \tag{10}$$

$$\leq \int_0^{t^*} e^{-\lambda s} f(y[x,u](s), u(s)) ds + e^{-\lambda t^*} h(y[x,u](t^*)). \tag{11}$$

Hence $W(x) \leq V(x)$. □

Definition 3.5 (see [22, Definition 3.1]). *Suppose $\Omega \subset R^n$ is nonempty, $\Theta \subset R^n$ is open, and $\Gamma : R^n \Longrightarrow R^n$ is a set-valued map.*

(a) *Then (Γ, Ω) is weakly invariant in Θ provided that $\forall x \in \Omega \cap \Theta$, there exists an absolutely continuous function $y(\cdot)$ that satisfies $\dot{y}(s) \in \Gamma(y(s))$ a.e., $y(0) = x$, and*

$$y(s) \in \Omega \quad \forall s \in [0, \tau^*).$$

(b) *Then (Γ, Ω) is strongly invariant in Θ provided that $\forall x \in \Omega \cap \Theta$ and for any absolutely continuous function $y(\cdot)$ that satisfies $\dot{y}(s) \in \Gamma(y(s))$ a.e., $y(0) = x$ one has*

$$y(s) \in \Omega \quad \forall s \in [0, \tau^*)$$

where $\tau^* := \inf\{t > 0 : y(t) \notin \Theta\}$.

Define

$$\begin{aligned} F(s,x,r) &:= \{(g(x,u), -e^{-\lambda s} f(x,u)) : u \in U\}, \\ \widetilde{F}(s,x,r) &:= \{(g(x,u), e^{-\lambda s} f(x,u)) : u \in U\}. \end{aligned}$$

We write $\{1\} \times F$ for the set-valued map defined as

$$(\{1\} \times F)(s,x,r) := \{(1, g(x,u), -e^{-\lambda s} f(x,u)) : u \in U\}.$$

Similarly, we define $\{1\} \times \widetilde{F}$ and $\{-1\} \times \{-F\}$. Let $W : G \to R$. We denote the epigraph of the function $e^{-\lambda t} W(x)$ by X_W, i.e.,

$$X_W := \{(t,x,r) : r \geq e^{-\lambda t} W(x)\}.$$

The following results show that the optimality principles are equivalent to the invariance properties.

Proposition 3.6 (equivalence of optimality principles and invariances). *Let G be an open set in R^d.*

(a) *A function W satisfies the superoptimality principle in G if and only if* $(\{1\} \times F, X_W)$ *is weakly invariant in* $R \times G \times R$*;*

(b) *A function W satisfies the suboptimality principle in G if and only if either* $(\{1\} \times \{\widetilde{F}\}, X_{-W})$ *is strongly invariant in* $R \times G \times R$ *or* $(\{-1\} \times \{-F\}, X_W)$ *is strongly invariant in* $R \times G \times R$.

Proof. Since the proof is straightforward by using definitions, we prove only the second part of (b). Let $(t, x, r) \in X_W \cap R \times G \times R$. Then $x \in G$ and $r \geq e^{-\lambda t} W(x)$. By suboptimality principle, we have $\forall u \in \mathcal{U}$,

$$e^{\lambda \tau} W(y[x,u](-\tau)) \leq \int_{-\tau}^{0} e^{-\lambda s} f(y[x,u](s), u(s)) ds + W(x) \quad \forall 0 \leq \tau \leq \tau^*,$$

where $\tau^* := \inf\{t > 0 : y[x,u](-\tau) \notin G\}$. Let $z_0(\tau) = -\tau + t$, $z(\tau) = y[x,u](-\tau)$, $z_{d+1}(\tau) = r - \int_0^{-\tau} e^{-\lambda(t+s)} f(y[x,u](s), u(s)) ds$. Then $z_0(0) = t, z(0) = y[x,u](0) = x, z_{d+1}(0) = r$, $(\dot{z_0}, \dot{z}, \dot{z_{d+1}})(s) \in (\{-1\} \times \{-F\})(z_0(s), z(s))$, and

$$\begin{aligned}
z_{d+1}(\tau) &= r - \int_0^{-\tau} e^{-\lambda(t+s)} f(y[x,u](s), u(s)) ds \\
&\geq e^{-\lambda t} W(x) - \int_0^{-\tau} e^{-\lambda(t+s)} f(y[x,u](s), u(s)) ds \\
&= e^{-\lambda t} (W(x) - \int_0^{-\tau} e^{-\lambda s} f(y[x,u](s), u(s)) ds) \\
&\geq e^{-\lambda(t-\tau)} W(y[x,u](-\tau)) \\
&= e^{-\lambda z_0(\tau)} W(z(\tau)).
\end{aligned}$$

That is, $(z_0, z, z_{d+1})(\tau) \in X_W$ $\forall 0 \leq \tau < \tau^*$. So $(\{-1\} \times \{-F\}, X_W)$ is strongly invariant in $R \times G \times R$. Conversely, we can show that if $(\{-1\} \times \{-F\}, X_W)$ is strongly invariant in $R \times G \times R$, then W satisfies the suboptimality principle in G. ∎

In the case when the function satisfying the optimality principles has semicontinuity properties, the invariances can be described by the H–J equations in the following way.

PROPOSITION 3.7 (equivalence of invariances and the H–J equations). *Let G be an open subset in* R^d.

(a) *Let $W : G \to R$ be a lower semicontinuous function. Then* $(\{1\} \times F, X_W)$ *is weakly invariant in* $R \times G \times R$ *if and only if*

$$\lambda W(x) + H(x, -\partial_p W(x)) \geq 0 \quad \forall x \in G.$$

(b) *Let $W : G \to R$ be an upper semicontinuous function. Then* $(\{1\} \times \widetilde{F}, X_{-W})$ *is strongly invariant in* $R \times G \times R$ *if and only if*

$$\lambda W(x) + H(x, -\partial^p W(x)) \leq 0 \quad \forall x \in G.$$

(b′) *Let $W : G \to R$ be a lower semicontinuous function. Then* $(\{-1\} \times \{-F\}, X_W)$ *is strongly invariant in* $R \times G \times R$ *if and only if*

$$\lambda W(x) + H(x, -\partial_p W(x)) \leq 0 \quad \forall x \in G.$$

The proof is based on the following lemmas. We denote $N^p_\Omega(x) = \partial_p \delta_\Omega(x)$, where δ_Ω is the indicator function of a set Ω defined by

$$\delta_\Omega(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ +\infty & \text{if } x \notin \Omega. \end{cases}$$

LEMMA 3.8 (see, e.g., [22, Theorem 3.1]). *Suppose that for each $x \in R^n$, $\Gamma(x)$ is not empty, convex, and compact, and the graph $gph\Gamma := \{(x, v) : v \in \Gamma(x)\}$ is closed in R^{2n}. Let $\Omega \subseteq R^n$ be closed and $\Theta \subseteq R^n$ be open.*

(a) *Then (Γ, Ω) is weakly invariant in Θ if and only if*

$$\min\{\langle v, \xi\rangle : v \in \Gamma(x)\} \le 0 \quad \forall x \in \Omega \cap \Theta, \xi \in N^p_\Omega(x),$$

where $N^p_\Omega(x)$ is the proximal normal cone to Ω at $x \in \Omega$ defined by

$$N^p_\Omega(x) := \{\xi \in R^n : \exists M > 0 \text{ s.t. } \langle \xi, x' - x\rangle \le M\|x' - x\|^2 \quad \forall x' \in \Omega\}.$$

(b) *In addition, assume that Γ is Lipschitz continuous; i.e., for each compact subset $C \subset R^n$, there exists $K > 0$ so that*

$$\Gamma(x) \subset \Gamma(y) + K\|x - y\|B \quad \forall x, y \in C.$$

Then (Γ, Ω) is strongly invariant in Θ if and only if

$$\max\{\langle v, \xi\rangle : v \in \Gamma(x)\} \le 0 \quad \forall x \in \Omega \cap \Theta, \xi \in N^p_\Omega(x).$$

LEMMA 3.9. *Suppose that for each $x \in R^n$, $\Gamma(x)$ is not empty, convex, and compact, and the graph $gph\Gamma := \{(x, v) : v \in \Gamma(x)\}$ is closed in R^{2n}. Let θ be a lower semicontinuous function and Θ be an open subset of R^n. Then*

(a)

$$\begin{aligned} &\min\{\langle v_1, \eta\rangle + v_2\rho : (v_1, v_2) \in \Gamma(z, r)\} \le 0 \quad \forall (z, r) \in epi\theta \cap \Theta, \\ &\qquad\qquad (\eta, \rho) \in N^p_{epi\theta}(z, r) \end{aligned} \tag{12}$$

if and only if

$$\min\{\langle v_1, \eta\rangle - v_2 : (v_1, v_2) \in \Gamma(z, \theta(z))\} \le 0 \quad \forall z \in \Theta, \eta \in \partial_p \theta(z). \tag{13}$$

(b) *In addition, assume that Γ is Lipschitz continuous. Then*

$$\begin{aligned} &\max\{\langle v_1, \eta\rangle + v_2\rho : (v_1, v_2) \in \Gamma(z, \theta(z))\} \le 0 \quad \forall (z, r) \in epi\theta \cap \Theta, \\ &\qquad\qquad (\eta, \rho) \in N^p_{epi\theta}(z, r) \end{aligned} \tag{14}$$

if and only if

$$\max\{\langle v_1, \eta\rangle - v_2 : (v_1, v_2) \in \Gamma(z, \theta(z)))\} \le 0 \quad \forall z \in \Theta, \eta \in \partial_p \theta(z). \tag{15}$$

Proof. Since an equivalent definition of the proximal subdifferential of ϕ at z is that

$$\eta \in \partial_p \theta(z) \text{ if and only if } (\eta, -1) \in N^p_{epi\theta}(z, \theta(z)),$$

(13) and (15) are (12) and (14) with $r = \theta(z)$ and $\rho = -1$, respectively. So it suffices to prove that (13) and (15) imply (12) and (14), respectively.

We first suppose that (13) holds. Let $(z, r) \in epi\theta \cap \Theta, (\eta, \rho) \in N^p_{epi\theta}(z, r)$. Then by the nature of epigraphs, we have $\rho \leq 0$. Let us assume first that $\rho < 0$ from which it follows that $r = \theta(z)$. Since $N^p_{epi\theta}(z, \theta(z))$ is a cone, we have $(-\frac{\eta}{\rho}, -1) \in N^p_{epi\theta}(z, \theta(z))$ and consequently $-\frac{\eta}{\rho} \in \partial_p \theta(z)$. By (13), we have

$$\min\left\{-\left\langle v_1, \frac{\eta}{\rho}\right\rangle - v_2 : (v_1, v_2) \in \Gamma(z, \theta(z))\right\} \leq 0.$$

Since $\rho < 0$, we have

$$\min\{\langle v_1, \eta\rangle + v_2 \rho : (v_1, v_2) \in \Gamma(z, \theta(z))\} \leq 0.$$

That is, (12) holds $\forall\ \rho < 0$.

We see that $(\eta, \rho) = 0$ trivially satisfies (12). Now suppose $\rho = 0$ and $\eta \neq 0$, from which it follows that $(\eta, 0) \in N^p_{epi\theta}(z, \theta(z))$. By definition (cf. [16]), η is in the singular limiting subdifferential of θ at z. So there exists $\{z_i\}, \{\eta_i\}$, and $\{\rho_i\}$ so that $z_i \to z, \theta(z_i) \to \theta(z), \eta_i \to \eta, \rho_i < 0, \rho_i \uparrow 0$, and $-\frac{\eta_i}{\rho_i} \in \partial_p \theta(z_i)$. By (13), we have

$$\min\left\{-\left\langle v_1, \frac{\eta_i}{\rho_i}\right\rangle - v_2 : (v_1, v_2) \in \Gamma(z_i, \theta(z_i))\right\} \leq 0.$$

So there exist $(v_1^i, v_2^i) \in \Gamma(z_i, \theta(z_i))$ such that

$$\langle v_1^i, \eta_i\rangle + v_2^i \rho_i \leq 0.$$

Without loss of generality, assume that $v_1^i \to v_1, v_2^i \to v_2$. Then $(v_1, v_2) \in \Gamma(z, \theta(z))$ and

$$\langle v_1, \eta\rangle + v_2 \cdot 0 \leq 0 \quad \forall (\eta, 0) \in N_{epi\theta}(z, \theta(z)),$$

which is (12) when $\rho = 0$.

Now suppose (15) holds. Let $(\eta, \rho) \in N^p_{epi\theta}(z, r), (z, r) \in epi\theta \cap \Theta$. Then $\rho \leq 0$. If $\rho < 0$, then the proof is similar to that in (a). If $\rho = 0$, then $r = \theta(z), (\eta, 0) \in N^p_{epi\theta}(z, \theta(z))$. So there exist $\{z_i\}, \{\eta_i\}$, and $\{\rho_i\}$ such that $z_i \to z, \theta(z_i) \to \theta(z), \eta_i \to \eta, \rho_i < 0, \rho_i \uparrow 0$, and $-\frac{\eta_i}{\rho_i} \in \partial_p \theta(z_i)$. By (15), we have

$$\max\left\{-\left\langle v_1, \frac{\eta_i}{\rho_i}\right\rangle - v_2 : (v_1, v_2) \in \Gamma(z_i, \theta(z_i))\right\} \leq 0.$$

That is,

$$\max\{\langle v_1, \eta_i\rangle + v_2 \rho_i : (v_1, v_2) \in \Gamma(z_i, \theta(z_i))\} \leq 0.$$

Since Γ is Lipschitz continuous, letting $(v_1, v_2) \in \Gamma(z_i, \theta(z_i))$, we have

$$\Gamma(z_i, \theta(z_i)) \subset \Gamma(z, \theta(z)) + K(\|z - z_i\|^2 + |\theta(z) - \theta(z_i)|^2)^{1/2} B.$$

Therefore there exists $(v_1^i, v_2^i) \in \Gamma(z, \theta(z))$ such that

$$(v_1, v_2) = (v_1^i, v_2^i) + K(\|x_1 - z_i\|^2 + |\theta(x_1) - \theta(z_i)|^2)^{1/2} e,$$

where $\|e\| \leq 1$. Hence

$$\begin{aligned}\langle v_1, \eta_i\rangle + v_2 \rho_i &= \langle v_1^i, \eta_i\rangle + v_2^i \rho_i + \langle \lambda_i e_1, \eta_i\rangle + \langle \lambda_i e_2, \rho_i\rangle \\ &\leq \langle \lambda_i e_1, \eta_i\rangle + \langle \lambda_i e_2, \rho_i\rangle,\end{aligned}$$

where $\lambda_i = K(\|x_1 - z_i\|^2 + |\theta(x_1) - \theta(z_i)|^2)^{1/2} \to 0$ as $i \to \infty$. Taking limits, we have

$$\langle v_1, \eta \rangle + v_2 \cdot 0 \leq 0 \quad \forall (\eta, 0) \in N_{epi\theta}(x_1, \theta(x_1)).$$

That is, (14) when $\rho = 0$. □

LEMMA 3.10 (see [24, Lemma 4.1]). *Let W be an extended-valued lower semicontinuous function. Then*

$$\partial_p(e^{-\lambda t} W(x)) = \{-\lambda e^{-\lambda t} W(x)\} \times \{e^{-\lambda t} \partial_p W(x)\}.$$

Proof of Proposition 3.7. By virtue of (a) in Lemmas 3.8 and 3.9, $(\{1\} \times F, X_W)$ is weakly invariant in $R \times G \times R$ if and only if

$$\min\{\xi_1 + \xi_2 \cdot g(x,u) + e^{-\lambda t} f(x,u) : u \in U\} \leq 0 \quad \forall x \in G, \xi \in \partial_p(e^{-\lambda t} W(x)).$$

By Lemma 3.10, that is,

$$\min\{-\lambda W(x) + \eta \cdot g(x,u) + f(x,u) : u \in U\} \leq 0 \quad \forall x \in G, \eta \in \partial_p W(x).$$

Hence

$$\lambda W(x) + H(x, -\eta) \geq 0 \quad \forall x \in G, \eta \in \partial_p W(x).$$

By virtue of (b) in Lemmas 3.8 and 3.9, $(\{1\} \times \widetilde{F}, X_{-W})$ is strongly invariant in $R \times G \times R$ if and only if

$$\max\{\xi_1 + \xi_2 \cdot g(x,u) - e^{-\lambda t} f(x,u) : u \in U\} \leq 0 \quad \forall x \in G, \xi \in \partial_p(-e^{-\lambda t} W(x)).$$

By Lemma 3.10, that is,

$$\max\{\lambda W(x) - \eta \cdot g(x,u) - f(x,u) : u \in U\} \leq 0 \quad \forall x \in G, \eta \in \partial_p(-W(x)).$$

Hence

$$\lambda W(x) + H(x, -\eta) \leq 0 \quad \forall x \in G, \eta \in \partial^p W(x).$$

By virtue of (b) in Lemmas 3.8 and 3.9, $(\{-1\} \times \{-F\}, X_W)$ is strongly invariant in $R \times G \times R$ if and only if

$$\max\{-\xi_1 - \xi_2 \cdot g(x,u) - e^{-\lambda t} f(x,u) : u \in U\} \leq 0 \quad \forall x \in G, \xi \in \partial_p(e^{-\lambda t} W(x)).$$

By Lemma 3.10, that is,

$$\max\{\lambda W(x) - \eta \cdot g(x,u) - f(x,u) : u \in U\} \leq 0 \quad \forall x \in R^d, \eta \in \partial_p W(x).$$

Hence

$$\lambda W(x) + H(x, -\eta) \leq 0 \quad \forall x \in G, \eta \in \partial_p W(x). \qquad \square$$

We now derive from Propositions 3.6 and 3.7 the equivalence between the optimality principles and the H–J equations.

PROPOSITION 3.11 (equivalence of optimality principles and the H–J equations). *Let G be an open subset of R^d.*

(a) *Let $W : G \to R$ be a lower semicontinuous function. Then it satisfies the superoptimality principle in G if and only if it is a supersolution of the H–J equation involving the proximal subdifferentials in G; i.e.,*

$$\lambda W(x) + H(x, -\partial_p W(x)) \geq 0 \quad \forall x \in G.$$

(b) *Let $W : G \to R$ be an upper semicontinuous function. Then it satisfies the suboptimality principle in G if and only if it is a subsolution of the H–J equation involving the proximal superdifferentials in G; i.e.,*

$$\lambda W(x) + H(x, -\partial^p W(x)) \leq 0 \quad \forall x \in G.$$

(b′) *Let $W : G \to R$ be a lower semicontinuous function. Then it satisfies the suboptimality principle in G if and only if it is a subsolution of the H–J equation involving the proximal subdifferentials in G; i.e.,*

$$\lambda W(x) + H(x, -\partial_p W(x)) \leq 0 \quad \forall x \in G.$$

4. Proof of main results.

Proof of Theorem 2.2. By Proposition 3.3, $V_*(x)$ satisfies the superoptimality principle in E. So by (a) of Proposition 3.11, it is a supersolution of the H–J equation involving the proximal subdifferentials in E.

We now prove that V_* satisfies the boundary condition

$$\max\{V_*(x) - h(x), \lambda V_*(x) + H(x, -\partial_p V_*(x))\} \geq 0 \quad \forall x \in \partial E. \tag{16}$$

If $V_*(x) - h(x) \geq 0$ $\forall x \in \partial E$, then the boundary condition (16) holds. Otherwise suppose that there exists $x \in \partial E$ such that $V_*(x) < h(x)$.

Let $x_n \to x, V(x_n) \to V_*(x)$. We may assume without loss of generality that $x_n \in E$ $\forall n$. Indeed, if there exists a subsequence $\{x_p\}$ of $\{x_n\}$ such that $x_p \in \partial E$ $\forall p$, then, by definition of the value function on the boundary of E and the lower semicontinuity of the exit cost h, we have

$$V_*(x) = \lim_{n\to\infty} V(x_n) = \lim_{p\to\infty} V(x_p) = \lim_{p\to\infty} h(x_p) \geq h(x),$$

which contradicts the assumption that $V_*(x) < h(x)$.

Now by the Bellman optimality principle, there exists a control $u_n \in \mathcal{U}$, $t_n^* := t^*[x_n, u_n] > 0$ such that

$$V(x_n) \geq \int_0^r e^{-\lambda s} f(y[x_n, u_n](s), u_n(s))ds + e^{-\lambda \tau_n} V(y[x_n, u_n](r)) \quad \forall 0 \leq r \leq t_n^*.$$

Now let $\bar{r} = \liminf t_n^*$. We must have $\bar{r} > 0$, since otherwise we can find a subsequence of $\{t_n^*\}$ such that $t_n^* \to 0$ so that

$$\begin{aligned} V_*(x) &= \lim_{n\to\infty} V(x_n) \\ &\geq \liminf_{n\to\infty} \int_0^{t_n^*} e^{-\lambda s} f(y[x_n, u_n](s), u_n(s))ds + \liminf_{n\to\infty} e^{-\lambda t_n^*} h(y[x_n, u_n](t_n^*)) \\ &\geq h(x) \quad \text{ since } h \text{ is lower semicontinuous,} \end{aligned}$$

which is a contradiction. Now by the compactness of relaxed controls on $[0, \bar{r}]$, there exists $u = \lim_{n\to\infty} u_n$ such that

$$V_*(x) \geq \int_0^r e^{-\lambda s} f(y[x, u](s), u(s))ds + e^{-\lambda r} V_*(y[x, u](r)) \quad \forall r \in (0, \bar{r}].$$

Let $\xi \in \partial_p V_*(x)$. Then there exist $\sigma > 0, \delta > 0$ such that

$$V_*(x') - V_*(x) + \sigma\|x' - x\|^2 \geq \langle \xi, x' - x\rangle \quad \forall x' \in x + \delta B.$$

Let $x' = y[x,u](r)$ where $r \in [0,\bar{r}]$ is fixed. Then

$$\begin{aligned}\langle \xi, y[x,u](r) - x\rangle &\leq \sigma\|y[x,u](r) - x\|^2 + V_*(y[x,u](r)) - V_*(x)\\ &\leq \sigma\|y[x,u](r) - x\|^2 - e^{\lambda r}\int_0^r e^{-\lambda s} f(y[x,u](s), u(s))ds\\ &\quad + e^{\lambda r}V_*(x) - V_*(x).\end{aligned}$$

Since $y[x,u](r) - x = \int_0^r g(y[x,u](s),u(s))ds$, one has

$$\begin{aligned}&\int_0^r [\langle -\xi, g(y[x,u](s),u(s))\rangle - f(y[x,u](s),u(s))]ds\\ &+\int_0^r [(1 - e^{\lambda(r-s)})f(y[x,u](s),u(s))]ds + (e^{\lambda r} - 1)V_*(x) \geq -\sigma\|y[x,u](r) - x\|^2.\end{aligned}$$

By virtue of the boundedness of g and the Lipschitz continuity of g, f uniformly in $u \in U$, one has

$$\begin{aligned}&\|y[x,u](r) - x\| \leq M_g r\\ &(\|\xi\|L_g + L_f)M_g s + \langle \xi, g(x,u(s))\rangle - f(x,u(s))\\ &\quad \geq \langle \xi, g(y[x,u](s),u(s))\rangle - f(y[x,u](s),u(s)),\end{aligned}$$

where M_g, L_g, L_f denote the bound of g and the Lipschitz constants of g, f, respectively. Therefore, one has

$$\begin{aligned}&\int_0^r [\langle -\xi, g(x,u(s))\rangle - f(x,u(s))]ds + (e^{\lambda r} - 1)V_*(x)\\ &\quad \geq o(r) - \int_0^r [(1 - e^{\lambda(r-s)})f(y[x,u](s),u(s))]ds\\ &\quad \geq o(r) - \int_0^r (1 - e^{\lambda(r-s)})M_f ds,\end{aligned}$$

where $o(r)$ indicates a function $g(r)$ such that $\lim_{t\to 0^+} |g(r)|/r = 0$ and M_f is the bound of f. Since the term in the square bracket in the first integral is bounded from above by

$$H(x,-\xi) = \max\{\langle -\xi, g(x,u)\rangle - f(x,u) : u \in U\},$$

(17) implies that

$$H(x,-\xi)r + (e^{\lambda r} - 1)V_*(x) \geq o(r) - \int_0^r (1 - e^{\lambda(r-s)})M_f ds.$$

Dividing the above inequality by r and letting $r \to 0$, we have

$$\lambda V_*(x) + H(x, -\partial_p V_*(x)) \geq 0.$$

Similarly by Proposition 3.3, $V^*(x)$ satisfies the suboptimality principle. So by Proposition 3.6, $(\{-1\} \times \{-F\}, X_{V^*})$ is strongly invariant in $R \times E \times R$. Hence, by

Proposition 3.7, V^* is a proximal subsolution of the H–J equation. The boundary condition can be proved similarly. □

Proof of Theorem 2.3. By Proposition 3.2, the value function $V(x)$ satisfies the suboptimality principle in E. Since $V(x)$ is upper semicontinuous, by (b) of Proposition 3.11, it is a subsolution of the H–J equation involving the proximal superdifferentials, i.e.,

$$\lambda V(x) + H(x, -\partial^p V(x)) \leq 0 \quad \forall x \in E.$$

Conversely, let $W(x)$ be an upper semicontinuous function such that

$$\begin{aligned} \lambda W(x) + H(x, -\partial^p W(x)) \leq 0 \quad & \forall x \in E \\ W(x) \leq h(x) \quad & \forall x \in \partial E. \end{aligned}$$

Then by (b) of Proposition 3.11, W satisfies the suboptimality principle in E. By (b) of Proposition 3.4, $W(x) \leq V(x)\ \forall x \in \bar{E}$. □

Proof of Theorem 2.4. By Proposition 3.2, the value function V satisfies both the superoptimality principle in E and the suboptimality principle in E. Since the value function is assumed to be lower semicontinuous, by the equivalence of the optimality principles and the H–J equations ((a) and (b′) of Proposition 3.11), the value function is both a supersolution and subsolution (hence a solution) of the H–J equation involving the proximal subdifferentials. Now if $W(x)$ is a lower semicontinuous solution of the H–J equation involving the proximal subdifferentials in E with the natural boundary condition $W(x) = h(x)\ \forall x \in \partial E$, then by (a) of Proposition 3.4, $W(x) \geq V(x)\ \forall x \in E$. □

Proof of Theorem 2.7. By Proposition 3.2, the value function V satisfies both the superoptimality principle in E and the suboptimality principle in E. Observing that $V(x) = h(x)\ \forall x \in E^c$ we have by assumption (H3) that the value function also satisfies the suboptimality principle in O which contains $\bar{E}$. Since by Proposition 2.5 the value function is lower semicontinuous, by (a) and (b′) of Proposition 3.11,

$$\lambda V(x) + H(x, -\partial_p V(x)) \geq 0 \quad \forall x \in E, \tag{17}$$

$$\lambda V(x) + H(x, -\partial_p V(x)) \leq 0 \quad \forall x \in O. \tag{18}$$

Now suppose W is a lower semicontinuous function that satisfies (17), (18), and the natural boundary condition $W(x) = h(x)\ \forall x \in O \backslash E$. Then by Proposition 3.11, W satisfies both the superoptimality principle in E and the suboptimality principle in O. Hence by (a) and (b′) of Proposition 3.4, $W(x) = V(x)\ \forall x \in \bar{E}$. □

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