SIAM J. CONTROL OPTIM.
Vol. 40, No. 3, pp. 699–723

SENSITIVITY ANALYSIS OF THE VALUE FUNCTION FOR OPTIMIZATION PROBLEMS WITH VARIATIONAL INEQUALITY CONSTRAINTS*

YVES LUCET† AND JANE J. YE‡

Abstract. In this paper we perform sensitivity analysis for optimization problems with variational inequality constraints (OPVICs). We provide upper estimates for the limiting subdifferential (singular limiting subdifferential) of the value function in terms of the set of normal (abnormal) coderivative (CD) multipliers for OPVICs. For the case of optimization problems with complementarity constraints (OPCCs), we provide upper estimates for the limiting subdifferentials in terms of various multipliers. An example shows that the other multipliers may not provide useful information on the subdifferentials of the value function, while the CD multipliers may provide tighter bounds. Applications to sensitivity analysis of bilevel programming problems are also given.

Key words. sensitivity analysis, optimization problems, variational inequality constraints, complementarity constraints, limiting subdifferentials, value functions, bilevel programming problems

AMS subject classification. 49K40

PII. S0363012999361718

1. Introduction.

In this paper, we consider the sensitivity analysis for the following optimization problem with variational inequality constraints (OPVIC):

$$\begin{aligned} \text{(OPVIC)} \qquad & \text{minimize} \quad f(x,y) \\ & \text{subject to} \quad \Psi(x,y) \le 0, H(x,y) = 0, (x,y) \in C, \\ (1) \qquad & \qquad y \in \Omega, \langle F(x,y), y-z \rangle \le 0 \quad \forall z \in \Omega, \end{aligned}$$

where the following basic assumptions are satisfied:

(BA) The functions $f : R^{n+m} \to R$, $\Psi : R^{n+m} \to R^d$, $H : R^{n+m} \to R^l$, and $F : R^{n+m} \to R^m$ are Lipschitz near any given point of C; C is a closed subset of R^{n+m}, and Ω is a closed convex subset of R^m. Note that the OPVIC is also called the mathematical program with equilibrium constraints (MPEC).

By definition of a normal cone in the sense of convex analysis, the variational inequality (1) is equivalent to saying that $y \in \Omega$ and the vector $-F(x,y)$ is in the normal cone of the convex set Ω at y. Hence the OPVIC can be rewritten as an optimization problem with a generalized equation constraint:

$$\begin{aligned} \text{(GP)} \qquad & \text{minimize} \ f(x,y) \\ & \text{subject to} \ \Psi(x,y) \le 0, H(x,y) = 0, (x,y) \in C, \\ (2) \qquad & \qquad 0 \in F(x,y) + N_\Omega(y), \end{aligned}$$

where

$$N_\Omega(y) := \begin{cases} \text{the normal cone of } \Omega \text{ if } y \in \Omega, \\ \emptyset \text{ otherwise} \end{cases}$$

*Received by the editors September 15, 1999; accepted for publication (in revised form) April 2, 2001; published electronically September 7, 2001. This work was partly supported by the Pacific Institute for the Mathematical Sciences and by an NSERC research grant.

http://www.siam.org/journals/sicon/40-3/36171.html

†Centre for Experimental and Constructive Mathematics, Simon Fraser University, 8888 University Dr., Burnaby, British Columbia, Canada V5A 1S6 (lucet@cecm.sfu.ca).

‡Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada V8W 3P4 (janeye@Math.UVic.CA).

is the normal cone operator.

Let $(\bar{x}, \bar{y})$ be an optimal solution of the OPVIC. If $N_\Omega(y)$ is single-valued and smooth, then the generalized equation constraint (2) would reduce to an ordinary equation $0 = F(x, y) + N_\Omega(y)$. Moreover, if all problem data are smooth and there is no abstract constraint, then the Fritz John necessary optimality condition can be stated as follows. There exist scalar $\lambda \geq 0$ and the vectors (γ, β, η) not all zero such that

$$\begin{cases} 0 = \lambda \nabla f(\bar{x}, \bar{y}) + \nabla \Psi(\bar{x}, \bar{y})^\top \gamma + \nabla H(\bar{x}, \bar{y})^\top \beta + \nabla F(\bar{x}, \bar{y})^\top \eta + \{0\} \times \nabla N_\Omega(\bar{y})^\top \eta, \\ \gamma \geq 0, \text{ and } \langle \Psi(\bar{x}, \bar{y}), \gamma \rangle = 0, \end{cases}$$

where ∇ denotes the usual gradient and $A^\top$ denotes the transpose of a matrix A. In general, however, the map $y \rightrightarrows N_\Omega(y)$ is a set-valued map. Naturally, the usual gradient $\nabla N_\Omega(\bar{y})$ has to be replaced by some kinds of derivatives of set-valued maps.

The Kuhn–Tucker-type necessary conditions with the transpose of the usual gradient ∇N_Ω replaced by the Mordukhovich coderivative $D^* N_\Omega$ were first derived in Ye and Ye [24] under the so-called pseudo-upper-Lipschitz condition for the case of no inequality, no equality constraints, and an abstract constraint in x only. They were further studied under the strong regularity condition in the sense of Robinson and the generalized Mangasarian–Fromovitz constraint qualifications by Outrata in [14] in the case of complementarity constraints and constraints in x only. The first order theory including the necessary optimality conditions involving the Mordukhovich coderivative, various constraint qualifications and their relationships for the general setting of this paper was given in Ye [23]. (Although the equality constraint $H(x, y) = 0$ was not considered explicitly there, the general results under the presence of an equality constraint still hold without any difficulty.) In Ye [22] the Kuhn–Tucker-type necessary conditions with the proximal coderivative for the case of optimization problems with complementarity constraints (OPCCs) were also studied . For recent developments and references on other optimality conditions and computational algorithms, the reader is referred to recent monographs of Luo, Pang, and Ralph [8] and Outrata, Kočvara, and Zowe [15].

In this paper we continue the study by considering the value function $V(p, q, r)$ associated with the right-hand side perturbations

$$\begin{array}{lll} \mathrm{GP}(p,q,r) & \text{minimize} & f(x,y) \\ & \text{subject to} & \Psi(x,y) \leq p, H(x,y) = q, (x,y) \in C, \\ & & r \in F(x,y) + N_\Omega(y), \end{array} \tag{3}$$

i.e.,

$$\begin{aligned} V(p,q,r) := \inf\{ f(x,y) : \quad & \Psi(x,y) \leq p, H(x,y) = q, (x,y) \in C \\ & r \in F(x,y) + N_\Omega(y)\}, \end{aligned}$$

where by convention $\inf \emptyset := +\infty$.

Our main result shows that as in sensitivity analysis for ordinary nonlinear programming (NLP) problems, under certain growth hypotheses, the value function V is lower semicontinuous near 0, and the limiting subdifferentials of the value functions are contained in the negative of the multiplier sets, i.e.,

$$\partial V(0) \subseteq -M^1(\Sigma), \tag{4}$$

$$\partial^\infty V(0) \subseteq -M^0(\Sigma), \tag{5}$$

where Σ is the set of solutions of GP and $M^\lambda(\Sigma)$ is the set of index λ CD multipliers for problem GP, which is the set of vectors (γ, β, η) satisfying the Fritz John necessary condition stated above with the transpose of the usual gradient ∇N_Ω replaced by the Mordukhovich coderivative $D^* N_\Omega$ in the case of smooth problem data and no abstract constraints.

In the case of $M^0(\Sigma) = \{0\}$, (5) implies that the singular limiting subgradient $\partial^\infty V(0)$ contains only the zero vector, and hence the value function is Lipschitz continuous near 0. Moreover, if the optimal solution is unique, if the set of abnormal multipliers $M^0(\Sigma)$ contains only the zero vector, and if the set of Kuhn–Tucker multipliers $M^1(\Sigma)$ is a singleton ζ, then inclusion (4) implies that the value function is smooth and $\nabla V(0) = -\zeta$.

In the case where $\Omega = R^m_+, C = R^{n+m}$, OPVIC reduces to the following OPCC.

$$\text{(OPCC)} \qquad \begin{array}{rl} \text{minimize} & f(x,y), \\ \text{subject to} & \Psi(x,y) \le 0, H(x,y) = 0, \\ & y \ge 0,\ F(x,y) \ge 0, \langle y, F(x,y)\rangle = 0. \end{array}$$

In this case (when all functions involved are smooth), an index λ CD multiplier set corresponding to a feasible solution $(\bar{x}, \bar{y})$ denoted by $M^\lambda_{CD}(\bar{x}, \bar{y})$ consists of $(\gamma, \beta, \eta) \in R^d \times R^l \times R^m$ such that

$$0 = \lambda \nabla f(\bar{x}, \bar{y}) + \nabla \Psi(\bar{x}, \bar{y})^\top \gamma + \nabla H(\bar{x}, \bar{y})^\top \beta + \nabla F(\bar{x}, \bar{y})^\top \eta + (0, \xi), \tag{6}$$

$$\gamma \ge 0 \text{ and } \langle \Psi(\bar{x}, \bar{y}), \gamma \rangle = 0, \tag{7}$$

$$\xi_i = 0 \qquad \text{if } \bar{y}_i > 0 \text{ and } F_i(\bar{x}, \bar{y}) = 0, \tag{8}$$

$$\eta_i = 0 \qquad \text{if } \bar{y}_i = 0 \text{ and } F_i(\bar{x}, \bar{y}) > 0, \tag{9}$$

and

$$\text{either } \xi_i < 0,\ \eta_i < 0, \text{or } \xi_i \eta_i = 0 \qquad \text{if } \bar{y}_i = 0 \text{ and } F_i(\bar{x}, \bar{y}) = 0.$$

We call vectors $(\gamma, \beta, \eta) \in R^d \times R^l \times R^m$ satisfying (6)–(9) and

$$\xi_i \eta_i \ge 0 \qquad \text{if } \bar{y}_i = 0 \text{ and } F_i(\bar{x}, \bar{y}) = 0$$

an index λ C-multiplier set and denote it by $M^\lambda_C(\bar{x}, \bar{y})$, and we call those satisfying (6)–(9) and

$$\xi_i \le 0, \eta_i \le 0 \qquad \text{if } \bar{y}_i = 0 \text{ and } F_i(\bar{x}, \bar{y}) = 0$$

an index λ S-multiplier set and denote it by $M^\lambda_S(\bar{x}, \bar{y})$.

Under certain growth hypotheses, we show that the value function

$$\begin{aligned} V(p,q,r) := \{f(x,y) : &\Psi(x,y) \le p, H(x,y) = q, \\ & y \ge 0, F(x,y) - r \ge 0, \langle y, F(x,y) - r\rangle = 0\} \end{aligned}$$

is lower semicontinuous near 0 and

$$\partial V(0) \subseteq -M^1 \quad \partial^\infty V(0) \subseteq -M^0,$$

where

$$M^1 = M^1_{CD}(\Sigma), M^1_C(\Sigma), M^1_S(\Sigma), \text{ or } \{(\gamma, \beta, \mu\bar{y} - r^F) : (\gamma, \beta, r^F, r^y, \mu) \in M^1_{NLP}(\Sigma)\},$$
$$M^0 = M^0_{CD}(\Sigma), M^0_C(\Sigma), M^0_S(\Sigma), \text{ or } \{(\gamma, \beta, \mu\bar{y} - r^F) : (\gamma, \beta, r^F, r^y, \mu) \in M^0_{NLP}(\Sigma)\},$$

where $M^\lambda_{NLP}(\bar{x},\bar{y})$ is the set of index λ ordinary NLP multipliers when the OPCC is treated as an ordinary NLP problem.

Moreover, we show that the above multiplier sets can be ordered as follows:

$$\{(\gamma,\beta,\mu\bar{y}-r^F):(\gamma,\beta,r^F,r^y,\mu)\in M^\lambda_{NLP}(\Sigma)\}\subseteq M^\lambda_S(\Sigma)\subseteq M^\lambda_{CD}(\Sigma)\subseteq M^\lambda_C(\Sigma).$$

It is obvious that one should use the smallest multiplier sets as possible. However, the smaller multiplier sets may be empty and hence may not provide any information on the properties of the value function. We show that under reasonable constraint qualifications such as the generalized Mangasarian–Fromovitz constraint qualification and the strongly regular constraint qualification, the abnormal CD multiplier set contains only the zero vector, and the set of normal CD multipliers is nonempty. An example is given to show that in sensitivity analysis the CD multipliers may provide more useful information than the other multipliers. In this example, the value function is Lipschitz, and the limiting subdifferentials of the value function coincide with the set of negative CD multipliers, while the limiting subdifferentials are contained strictly in the set of negative C multipliers and the set of P multipliers, NLP multipliers, and S multipliers are empty. Applications to the bilevel programming problem are also given.

In this paper we deal only with the sensitivity analysis of the optimal values. For the sensitivity analysis of the optimal solutions, the reader is referred to Scheel and Scholtes [19].

The following notations are used throughout the paper: B denotes the open unit ball; $B(\bar{z};\delta)$ denotes the open ball centered at $\bar{z}$ with radius $\delta>0$. For a set E, co$E$ denotes the convex hull of E, and int$E$ and cl$E$ denote the interior and the closure of E, respectively. The notation $\langle a,b\rangle$ denotes the inner products of vectors a and b. For a differentiable function f, $\nabla f(\bar{x})$ denotes the gradient of f at $\bar{x}$. For a vector $a\in R^n$, a_i denotes the ith component of a. For an m by n matrix A and index sets $I\subseteq\{1,2,\ldots,m\}$, $J\subseteq\{1,2,\ldots,n\}$, A_I and $A_{I,J}$ denote the submatrix of A with rows specified by I and the submatrix of A with rows and columns specified by I and J, respectively. $A^\top$ denotes the transpose of a matrix A. For a vector $d\in R^m$, d_I is the subvector composed from the components $d_i, i\in I$.

2. Preliminaries. The purpose of this section is to provide the background material on nonsmooth analysis which will be used later. We give only concise definitions and facts that will be needed in the paper. For more detailed information on the subject, our references are Clarke [3], Loewen [7], Rockafellar and Wets [18], and Mordukhovich [10, 12, 13].

First we give some definitions for various subdifferentials and normal cones.

DEFINITION 2.1. *Let $f:R^n\to R\cup\{+\infty\}$ be lower semicontinuous and finite at $\bar{x}\in R^n$. The proximal subdifferential of f at $\bar{x}$ is the set defined by*

$$\partial^\pi f(\bar{x})=\{v\in R^n:\exists M>0,\delta>0 \text{ s.t.}\quad f(x)\geq f(\bar{x})+\langle v,x-\bar{x}\rangle+M\|x-\bar{x}\|^2\ \ \forall x\in\bar{x}+\delta B\},$$

the limiting subdifferential of f at $\bar{x}$ is the set defined by

$$\partial f(\bar{x}):=\left\{v\in R^n:v=\lim_{\nu\to\infty}v^\nu \text{ with } v^\nu\in\partial^\pi f(x^\nu) \text{ and } x^\nu\to\bar{x}\right\},$$

the singular limiting subdifferential of f at $\bar{x}$ is the set defined by

$$\partial^\infty f(\bar{x}):=\left\{v\in R^n:v=\lim_{\nu\to\infty}\lambda^\nu v^\nu \text{ with } v^\nu\in\partial^\pi f(x^\nu) \text{ and } \lambda^\nu\downarrow 0,x^\nu\to\bar{x}\right\}.$$

Let $f : R^n \to R$ be Lipschitz near $\bar{x} \in R^n$. The Clarke generalized gradient of f at $\bar{x}$ is the set

$$\partial_C f(\bar{x}) := clco\partial f(\bar{x}).$$

For set-valued maps, the definition for a limiting normal cone leads to the definition of the coderivative of a set-valued map introduced by Mordukhovich in [9].

DEFINITION 2.2. *For a closed set $C \subset R^n$ and $\bar{x} \in C$, the proximal normal cone to C at $\bar{x}$ is defined by*

$$N_C^\pi(\bar{x}) := \{v \in R^n : \exists M > 0 \ s.t. \ \langle v, x - \bar{x}\rangle \leq M\|x - \bar{x}\|^2 \quad \forall x \in C\},$$

and the limiting normal cone to C at $\bar{x}$ is defined by

$$N_C(\bar{x}) := \left\{ \lim_{\nu\to\infty} v^\nu : v^\nu \in N_C^\pi(x^\nu), x^\nu \to \bar{x} \right\}.$$

DEFINITION 2.3. *Let $\Phi : R^n \rightrightarrows R^q$ be a set-valued map. Let $(\bar{x}, \bar{p}) \in clGph\Phi$, where $Gph\Phi := \{(x, p) \ : \ p \in \Phi(x)\}$ is the graph of the set-valued map Φ. The set-valued map $D^*\Phi(\bar{x}, \bar{p})$ from R^q into R^n, defined by*

$$D^*\Phi(\bar{x}, \bar{p})(\eta) := \{\xi \in R^n \ : \ (\xi, -\eta) \in N_{Gph\Phi}(\bar{x}, \bar{p})\},$$

is called the Mordukhovich coderivative of Φ at $(\bar{x}, \bar{p})$.

In general, we have the following inclusions, which may be strict:

$$\partial^\pi f(\bar{x}) \subseteq \partial f(\bar{x}) \subseteq \partial_C f(\bar{x}).$$

In the case where f is a convex function, all subdifferentials coincide with the subdifferentials in the sense of convex analysis, i.e.,

$$\partial^\pi f(\bar{x}) = \partial f(\bar{x}) = \partial_C f(\bar{x}) = \{\zeta : f(x) - f(\bar{x}) \geq \langle\ \zeta, x - \bar{x}\rangle \quad \forall x\}.$$

In the case where f is strictly differentiable (see the definition, e.g., in Clarke [2]), we have

$$\partial f(\bar{x}) = \partial_C f(\bar{x}) = \{\nabla f(\bar{x})\}.$$

The following facts about the subdifferentials are well known.

PROPOSITION 2.4.

(i) *A function $f : R^n \to R$ is Lipschitz near $\bar{x}$ and $\partial f(\bar{x}) = \{\zeta\}$ if and only if f is strictly differentiable at $\bar{x}$ and the gradient of f at $\bar{x}$ equals ζ.*
(ii) *A function $f : R^n \to R$ is Lipschitz near $\bar{x}$ if and only if $\partial^\infty f(\bar{x}) = \{0\}$.*
(iii) *If a function $f : R^n \to R$ is Lipschitz near $\bar{x}$ with positive constant L_f, then $\partial f(\bar{x}) \subseteq L_f clB$.*

The following calculus rules will be useful and can be found in the references given in the beginning of this section.

PROPOSITION 2.5 (see, e.g., [7, Proposition 5A.4]). *Let $f : R^n \to R$ be Lipschitz near $\bar{x}$, and let $g : R^n \to R \cup \{+\infty\}$ be lower semicontinuous and finite at $\bar{x}$. Then*

$$\partial(f + g)(\bar{x}) \subseteq \partial f(\bar{x}) + \partial g(\bar{x}),$$
$$\partial^\infty(f + g)(\bar{x}) \subseteq \partial^\infty g(\bar{x}).$$

PROPOSITION 2.6 (see, e.g., [7, Lemma 5A.3]). *Let $f : R^n \times R^m \to R \cup \{+\infty\}$ be lower semicontinuous and finite at $(\bar{x}, \bar{y})$. If $(\zeta, 0) \in \partial^\infty f(\bar{x}, \bar{y})$ implies that $\zeta = 0$, then*

$$\begin{aligned} \partial_y f(\bar{x}, \bar{y}) &\subseteq \{\eta : (\zeta, \eta) \in \partial f(\bar{x}, \bar{y}) \quad \text{for some } \zeta\}, \\ \partial_y^\infty f(\bar{x}, \bar{y}) &\subseteq \{\eta : (\zeta, \eta) \in \partial^\infty f(\bar{x}, \bar{y}) \quad \text{for some } \zeta\}. \end{aligned}$$

PROPOSITION 2.7 (see, e.g., [13, Theorem 7.6]). *Let the minimum function be*

$$(\wedge f_j)(x) := \min\{f_j(x) | j = 1, 2, \ldots, m\},$$

where $f_j : R^n \to R \cup \{+\infty\}$. Assume that f_j are lower semicontinuous around $\bar{x}$ for $j \in J(\bar{x})$ and lower semicontinuous at $\bar{x}$ for $j \notin J(\bar{x})$, where

$$J(x) := \{j | f_j(x) = \wedge f_j(x)\}.$$

Then the minimum function $\wedge f_j(x)$ is lower semicontinuous around $\bar{x}$ and

$$\begin{aligned} \partial(\wedge f_j)(\bar{x}) &\subseteq \bigcup \{\partial f_j(\bar{x}) | j \in J(\bar{x})\}, \\ \partial^\infty(\wedge f_j)(\bar{x}) &\subseteq \bigcup \{\partial^\infty f_j(\bar{x}) | j \in J(\bar{x})\}. \end{aligned}$$

Classical results on the value function can be found in [2, 4, 7, 11, 18], while the results we quote are from [7].

PROPOSITION 2.8 (see [7, (b) and (d) of Theorem 5A.2]). *Let $g : R^n \times R^m \to R \cup \{+\infty\}$ be lower semicontinuous everywhere and finite at $(\bar{z}, \bar{\alpha})$. Suppose g is bounded below on some set $E \times O$, where E is a compact neighborhood of $\bar{z}$ and O is an open set containing $\bar{\alpha}$. Define the value function $V : R^m \to R \cup \{+\infty\}$ and the set of minimizers Σ as follows:*

$$\begin{aligned} V(\alpha) &:= \inf\{g(z, \alpha) : z \in E\}, \\ \Sigma(\alpha) &:= \{z \in E : g(z, \alpha) = V(\alpha)\}. \end{aligned}$$

If $\Sigma(\bar{\alpha}) \subseteq \operatorname{int} E$, then the value function V is lower semicontinuous on O, and the subdifferentials of V satisfy these estimates:

$$\begin{aligned} \partial V(\bar{\alpha}) &\subseteq \{\eta \in R^m : (0, \eta) \in \partial g(z, \bar{\alpha}) \quad \text{for some } z \in \Sigma(\bar{\alpha})\}, \\ \partial^\infty V(\bar{\alpha}) &\subseteq \{\eta \in R^m : (0, \eta) \in \partial^\infty g(z, \bar{\alpha}) \quad \text{for some } z \in \Sigma(\bar{\alpha})\}. \end{aligned}$$

Our results are stated using the limiting subdifferentials. Alternatively, they could be derived by using the Fréchet subdifferentials instead of the proximal subdifferentials. (Both lead to the same limiting subdifferentials in finite dimensional spaces.) In [18] arguments are given in favor of the former (called there the regular subdifferentials). In the present paper we use the proximal subdifferentials to provide the same framework as in [23].

3. Main results. Let $(\bar{x}, \bar{y})$ be a feasible solution of the OPVIC and let λ be a nonnegative number. We define $M^\lambda(\bar{x}, \bar{y})$, the index λ CD multiplier set corresponding to $(\bar{x}, \bar{y})$, to be the set of vectors (γ, β, η) in $R^d \times R^l \times R^m$ satisfying the Fritz John-type necessary optimality condition involving the Mordukhovich coderivatives for GP, that is, the vectors (γ, β, η) such that

$$\begin{cases} 0 \in \lambda \partial f(\bar{x}, \bar{y}) + \partial\langle \Psi, \gamma \rangle(\bar{x}, \bar{y}) + \partial\langle H, \beta \rangle(\bar{x}, \bar{y}) + \partial\langle F, \eta \rangle(\bar{x}, \bar{y}) \\ \qquad + \{0\} \times D^* N_\Omega(\bar{y}, -F(\bar{x}, \bar{y}))(\eta) + N_C(\bar{x}, \bar{y}), \\ \gamma \geq 0, \text{ and } \langle \Psi(\bar{x}, \bar{y}), \gamma \rangle = 0. \end{cases}$$

Then by Ye [23, Theorem 3.1], the Fritz John-type necessary optimality condition involving the Mordukhovich coderivatives can be rephrased as follows.

PROPOSITION 3.1. *Under the basic assumption* (BA)*, if* $(\bar{x}, \bar{y})$ *is a local solution of* OPVIC*, then either the set of normal* CD *multipliers is nonempty or there is a nonzero abnormal* CD *multiplier, i.e.,*

$$M^1(\bar{x}, \bar{y}) \cup (M^0(\bar{x}, \bar{y}) \setminus \{0\}) \neq \emptyset.$$

Note that by the definition of the Mordukhovich coderivative,

$$\xi \in D^* N_\Omega(\bar{y}, -F(\bar{x}, \bar{y}))(\eta) \text{ if and only if } (\xi, -\eta) \in N_{GphN_\Omega}(\bar{y}, -F(\bar{x}, \bar{y})).$$

In the case where $\Omega = \{0\}$, OPVIC reduces to an ordinary mathematical programming problem with equality, inequality, and abstract constraints. The term $D^* N_\Omega(\bar{y}, -F(\bar{x}, \bar{y}))(\eta)$ vanishes, and the above Fritz John condition can be considered as a limiting subdifferential version of the generalized Lagrange multiplier rule as found in Clarke [2, Theorem 6.1.1] and was obtained by Mordukhovich [9, Theorem 1(b)].

In the case where $\Omega = R^m_+$, (1) reduces to a complementarity constraint,

$$y \geq 0, \ F(x, y) \geq 0, \ \langle F(x, y), y \rangle = 0,$$

and the coderivative of the normal cone to the set R^m_+ can be calculated using the following lemma whose proof follows from [22, Proposition 2.7] and the definition of the limiting normal cones.

LEMMA 3.2. *For any* $(\bar{u}, -\bar{v}) \in GphN_{R^m_+}$*,*

$$\begin{aligned} N_{GphN_{R^m_+}}(\bar{u}, -\bar{v}) = \{(\xi, -\eta) \in R^{2m} : & \xi_i = 0 \ \text{if } \bar{u}_i > 0, \bar{v}_i = 0, \\ & \eta_i = 0 \ \text{if } \bar{u}_i = 0, \bar{v}_i > 0, \\ & \text{either } \xi_i \eta_i = 0 \ \text{or } \xi_i < 0 \ \text{and } \eta_i < 0 \ \text{if } \bar{u}_i = 0, \bar{v}_i = 0\}. \end{aligned}$$

In the case where Ω is a polyhedral convex set, one can calculate the Mordukhovich coderivative of the normal cone to the set Ω by using the formula of the limiting normal cone to the graph of the normal cone to the set Ω, which was first given in the proof of Dontchev and Rockafellar [5, Theorem 2] and stated in Poliquin and Rockafellar [16, Proposition 4.4].

We first consider the following additively (right-hand side) perturbed GP:

$$\begin{array}{lll} \text{GP}(p, q, r) & \text{minimize} & f(x, y) \\ & \text{subject to} & \Psi(x, y) \leq p, H(x, y) = q, (x, y) \in C, \\ & & r \in F(x, y) + N_\Omega(y), \end{array}$$

with the solution set denoted by $\Sigma(p, q, r)$.

In order to obtain useful information on the subdifferentials of the value function at $(\bar{p}, \bar{q}, \bar{r})$, some hypotheses are usually made for $\text{GP}(p, q, r)$, where (p, q, r) are sufficiently close to the point of interest $(\bar{p}, \bar{q}, \bar{r})$ (see, for example, [4, Growth Hypothesis 3.1.1], [2, Hypothesis 6.5.1], [18, Definition 1.8]). In this paper, we make the following growth hypothesis [7, Theorem 5A.2]:

(GH) at $(\bar{p}, \bar{q}, \bar{r})$: There exists $\delta > 0$ such that the set

$$\begin{aligned} \{(x, y) \in C \ : \ & \Psi(x, y) \leq p, H(x, y) = q, r \in F(x, y) + N_\Omega(y), f(x, y) \leq M, \\ & (p, q, r) \in B(\bar{p}, \bar{q}, \bar{r}; \delta)\} \end{aligned}$$

is bounded for each M.

In order to apply Proposition 2.8, we rewrite the value function in the following form:

$$V(p,q,r) = \inf g(x,y,p,q,r),$$

where g is the extended-value function defined by

$$g(x,y,p,q,r) := f(x,y) + I_{(Gph\Phi)\cap(C\times R^{d+l+m})}(x,y,p,q,r)$$

with I_E being the indicator function of a set E defined by

$$I_E(x) := \begin{cases} 0 \text{ if } x \in E, \\ \infty \text{ if } x \notin E \end{cases}$$

and Φ being the set-valued map defined by

$$\Phi(x,y) = (\Psi(x,y), H(x,y), F(x,y)) + R^d_+ \times \{0\} \times N_\Omega(y).$$

The growth hypothesis (GH) amounts to saying the function g is level-bounded in (x,y) uniformly for any $(p,q,r) \in B(\bar{p},\bar{q},\bar{r};\delta)$. Hence by virtue of [18, Theorem 1.9], $\bigcup_{(p,q,r)\in B(\bar{p},\bar{q},\bar{r};\delta)} \Sigma(p,q,r)$ is a compact set and for all $(p,q,r) \in B(\bar{p},\bar{q},\bar{r};\delta)$,

$$V(p,q,r) = \inf\{g(x,y,p,q,r) : (x,y) \in E\},$$

where E is a compact set with interior containing $\bigcup_{(p,q,r)\in B(\bar{p},\bar{q},\bar{r};\delta)} \Sigma(p,q,r)$. It is clear that g is lower semicontinuous everywhere and finite at any $(x,y,p,q,r) \in (Gph\Phi) \cap (C \times R^{d+l+m})$. Since f is Lipschitz on E, g is bounded below on $E \times B(\bar{x},\bar{y};\epsilon)$. The following result then follows immediately by applying Proposition 2.8.

PROPOSITION 3.3. *Under the basic assumption* (BA) *and the growth hypothesis* (GH) *at* $(\bar{p},\bar{q},\bar{r})$ *the value function* V *is lower semicontinuous on* $B(\bar{p},\bar{q},\bar{r};\delta)$ *and*

$$\partial V(\bar{p},\bar{q},\bar{r}) \subseteq \bigcup_{(\bar{x},\bar{y})\ \in\Sigma(\bar{p},\bar{q},\bar{r})} \{(u,v,w)\ :\ (0,0,u,v,w) \in \partial g(\bar{x},\bar{y},\bar{p},\bar{q},\bar{r})\}, \tag{10}$$

$$\partial^\infty V(\bar{p},\bar{q},\bar{r}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{p},\bar{q},\bar{r})} \{(u,v,w)\ :\ (0,0,u,v,w) \in \partial^\infty g(\bar{x},\bar{y},\bar{p},\bar{q},\bar{r})\}. \tag{11}$$

We now prove that the set in the right-hand side of (10) (respectively, (11)) is included in the normal multiplier set M^1 (respectively, the abnormal multiplier set M^0).

By the sum rule (see Proposition 2.5) and the fact that for any closed set E with $\bar{z} \in E$

$$\partial I_E(\bar{z}) = \partial^\infty I_E(\bar{z}) = N_E(\bar{z}),$$

we have

$$\begin{aligned} \partial g(\bar{x},\bar{y},\bar{p},\bar{q},\bar{r}) &\subset \partial f(\bar{x},\bar{y}) \times \{(0,0)\} + N_{(Gph\Phi)\cap(C\times R^{d+l+m})}(\bar{x},\bar{y},\bar{p},\bar{q},\bar{r}), \\ \partial^\infty g(\bar{x},\bar{y},\bar{p},\bar{q},\bar{r}) &\subset N_{(Gph\Phi)\cap(C\times R^{d+l+m})}(\bar{x},\bar{y},\bar{p},\bar{q},\bar{r}). \end{aligned}$$

Hence we need only to compute the normal cone.

LEMMA 3.4. *If* $(s_x,s_y,s_p,s_q,s_r) \in N_{(Gph\Phi)\cap(C\times R^{d+l+m})}(\bar{x},\bar{y},\bar{p},\bar{q},\bar{r})$, *then*

$$\begin{cases} (s_x,s_y) \in \partial\langle\Psi,-s_p\rangle(\bar{x},\bar{y}) + \partial\langle H,-s_q\rangle(\bar{x},\bar{y}) + \partial\langle F,-s_r\rangle(\bar{x},\bar{y}) + N_C(\bar{x},\bar{y}) \\ \qquad\qquad +\{0\} \times D^*N_\Omega(\bar{y},\bar{r}-F(\bar{x},\bar{y}))(-s_r), \\ s_p \geq 0,\ \textit{and}\ \langle\Psi(\bar{x},\bar{y})-\bar{p},s_p\rangle = 0. \end{cases}$$

Proof. Step 1. Let $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}, \tilde{r})$ be any point in a neighborhood of $(\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{r})$ on which Ψ, H, and F are Lipschitz continuous and

$$(s_x, s_y, s_p, s_q, s_r) \in N^{\pi}_{(Gph\Phi)\cap(C\times R^{d+l+m})}(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}, \tilde{r}).$$

By definition of the proximal normal cone, there is $M > 0$ such that for all $(x, y, p, q, r) \in (Gph\Phi) \cap (C \times R^{d+l+m})$,

$$\langle (s_x, s_y, s_p, s_q, s_r), (x, y, p, q, r) - (\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}, \tilde{r}) \rangle \le M\|(x, y, p, q, r) - (\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}, \tilde{r})\|^2.$$

In other words, $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}, \tilde{r})$ is a solution to the optimization problem

$$\begin{array}{rl} \text{minimize} & \langle -(s_x, s_y, s_p, s_q, s_r), (x, y, p, q, r) \rangle + M\|(x, y, p, q, r) - (\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}, \tilde{r})\|^2 \\ \text{subject to} & \Psi(x, y) \le p, H(x, y) = q, (x, y) \in C, \\ & r \in F(x, y) + N_\Omega(y). \end{array}$$

We now prove that the only abnormal CD multiplier for the above problem is the zero vector. Indeed, the set of abnormal CD multipliers at $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}, \tilde{r})$ for the above problem are the vectors (γ, β, η) satisfying

$$\begin{cases} 0 \in \partial\langle \Psi, \gamma\rangle(\tilde{x}, \tilde{y}) \times \{(-\gamma, 0, 0)\} + \partial\langle H, \beta\rangle(\tilde{x}, \tilde{y}) \times \{(0, -\beta, 0)\} + \partial\langle F, \eta\rangle(\tilde{x}, \tilde{y}) \\ \times\{(0, 0, -\eta)\} + \{0\} \times D^*N_\Omega(\tilde{y}, \tilde{r} - F(\tilde{x}, \tilde{y}))(\eta) \times \{(0, 0, 0)\} + N_C(\tilde{x}, \tilde{y}) \\ \times\{(0, 0, 0)\},\ \gamma \ge 0,\ \text{and } \langle \Psi(\tilde{x}, \tilde{y}) - \tilde{p}, \gamma\rangle = 0, \end{cases}$$

which obviously coincides with the set $\{(0, 0, 0)\}$. Applying Proposition 3.1, we conclude that the set of normal CD multipliers for the above problem must be nonempty. That is, there are vectors $\eta \in R^m, \beta \in R^l$, and $\gamma \in R^d$ such that

$$\begin{cases} 0 \in -\{(s_x, s_y, s_p, s_q, s_r)\} + \partial\langle \Psi, \gamma\rangle(\tilde{x}, \tilde{y}) \times \{(-\gamma, 0, 0)\} + \partial\langle H, \beta\rangle(\tilde{x}, \tilde{y}) \times \{(0, -\beta, 0)\} \\ \qquad +\partial\langle F, \eta\rangle(\tilde{x}, \tilde{y}) \times \{(0, 0, -\eta)\} + \{0\} \times D^*N_\Omega(\tilde{y}, \tilde{r} - F(\tilde{x}, \tilde{y}))(\eta) \times \{(0, 0, 0)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +N_C(\tilde{x}, \tilde{y}) \times \{(0, 0)\}, \\ \gamma \ge 0,\ \text{and } \langle \Psi(\tilde{x}, \tilde{y}) - \tilde{p}, \gamma\rangle = 0. \end{cases}$$

That is,

$$\begin{cases} (s_x, s_y) \in \partial\langle \Psi, -s_p\rangle(\tilde{x}, \tilde{y}) + \partial\langle H, -s_q\rangle(\tilde{x}, \tilde{y}) \\ \qquad\qquad +\partial\langle F, -s_r\rangle(\tilde{x}, \tilde{y}) + \{0\} \times D^*N_\Omega(\tilde{y}, \tilde{r} - F(\tilde{x}, \tilde{y}))(-s_q) + N_C(\tilde{x}, \tilde{y}), \\ s_p \ge 0,\ \text{and } \langle \Psi(\tilde{x}, \tilde{y}) - \tilde{p}, s_p\rangle = 0. \end{cases}$$

Step 2. Now take any $(s_x, s_y, s_p, s_q, s_r) \in N_{(Gph\Phi)\cap(C\times R^{d+l+m})}(\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{r})$. Then by definition of limiting normal cones, there are sequences $(x^\nu, y^\nu, p^\nu, q^\nu, r^\nu) \to (\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{r})$ and $(s_x^\nu, s_y^\nu, s_p^\nu, s_q^\nu, s_r^\nu) \to (s_x, s_y, s_p, s_q, s_r)$ with

$$(s_x^\nu, s_y^\nu, s_p^\nu, s_q^\nu, s_r^\nu) \in N^{\pi}_{(Gph\Phi)\cap(C\times R^{d+l+m})}(x^\nu, y^\nu, p^\nu, q^\nu, r^\nu).$$

By virtue of step 1,

$$\begin{cases} (s_x^\nu, s_y^\nu) \in \partial\langle \Psi, -s_p^\nu\rangle(x^\nu, y^\nu) + \partial\langle H, -s_q^\nu\rangle(x^\nu, y^\nu) + \partial\langle F, -s_r^\nu\rangle(x^\nu, y^\nu) \\ \qquad\qquad + \{0\} \times D^*N_\Omega(y^\nu, r^\nu - F(x^\nu, y^\nu))(-s_r^\nu) + N_C(x^\nu, y^\nu), \\ s_p^\nu \ge 0,\ \text{and } \langle \Psi(x^\nu, y^\nu) - p^\nu, s_p^\nu\rangle = 0. \end{cases}$$

Since Ψ is Lipschitz near $(\bar{x},\bar{y})$, we have

$$\begin{aligned}\partial\langle\Psi,-s_p^\nu\rangle(x^\nu,y^\nu) &\subseteq \partial\langle\Psi,-s_p\rangle(x^\nu,y^\nu)+\partial\langle\Psi,s_p-s_p^\nu\rangle(x^\nu,y^\nu) \text{ by Proposition 2.5}\\ &\subseteq \partial\langle\Psi,-s_p\rangle(x^\nu,y^\nu)+\|s_p^\nu-s_p\|L_\Psi clB \text{ by Proposition 2.4,}\end{aligned}$$

where L_Ψ is the Lipschitz constant of Ψ. Similarly,

$$\begin{aligned}\partial\langle H,-s_q^\nu\rangle(x^\nu,y^\nu) &\subseteq \partial\langle H,-s_q\rangle(x^\nu,y^\nu)+\|s_q^\nu-s_q\|L_H clB,\\ \partial\langle F,-s_r^\nu\rangle(x^\nu,y^\nu) &\subseteq \partial\langle F,-s_r\rangle(x^\nu,y^\nu)+\|s_r^\nu-s_r\|L_F clB,\end{aligned}$$

where L_H, L_F are the Lipschitz constants of F and H. Hence we have

$$\begin{cases}(s_x^\nu,s_y^\nu)\in\partial\langle\Psi,-s_p\rangle(x^\nu,y^\nu)+\partial\langle H,-s_q\rangle(x^\nu,y^\nu)+\partial\langle F,-s_r\rangle(x^\nu,y^\nu)\\ \qquad\qquad +(\|s_p^\nu-s_p\|+\|s_q^\nu-s_q\|+\|s_r^\nu-s_r\|)(L_\Psi+L_H+L_F)clB\\ \qquad\qquad +\{0\}\times D^*N_\Omega(y^\nu,r^\nu-F(x^\nu,y^\nu))(-s_r^\nu)+N_C(x^\nu,y^\nu),\\ s_p^\nu\ge 0, \text{ and } \langle\Psi(x^\nu,y^\nu)-p^\nu,s_p^\nu\rangle=0.\end{cases}$$

Taking limits as $\nu\to\infty$ and using the definitions of the limiting normal cone and the limiting subdifferentials completes the proof. ∎

Remark. As is pointed out by referee 1, alternatively, Lemma 3.4 can also be proved by formulating the constraints in the form of [12, equation (6.19)] and applying [12, Theorem 6.10].

All in all, we proved the following result.

THEOREM 3.5. *Assume* (GH) *and* (BA) *hold. Then the value function* V *is lower semicontinuous on* $B(\bar{p},\bar{q},\bar{r};\delta)$ *and*

$$\partial V(\bar{p},\bar{q},\bar{r})\subset\bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{p},\bar{q},\bar{r})}-M^1(\bar{x},\bar{y}) \text{ and } \partial^\infty V(\bar{p},\bar{q},\bar{r})\subset\bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{p},\bar{q},\bar{r})}-M^0(\bar{x},\bar{y}).$$

We now consider the value function $V(\alpha)$ associated with the following perturbed GP:

$$\begin{array}{lll}\mathrm{GP}(\alpha) & \text{minimize} & f(x,y,\alpha)\\ & \text{subject to} & \Psi(x,y,\alpha)\le 0, H(x,y,\alpha)=0,(x,y)\in C,\\ & & 0\in F(x,y,\alpha)+N_\Omega(y),\end{array}$$

i.e.,

$$\begin{aligned}V(\alpha):=\inf\{f(x,y,\alpha):\quad &\Psi(x,y,\alpha)\le 0, H(x,y,\alpha)=0,(x,y)\in C,\\ &0\in F(x,y,\alpha)+N_\Omega(y)\},\end{aligned}$$

where the following basic assumptions are satisfied:

(BH) The functions $f:R^{n+m+c}\to R,\Psi:R^{n+m+c}\to R^d, H:R^{n+m+c}\to R^l$, and $F:R^{n+m+c}\to R^m$ are locally Lipschitz near any points in $C\times R^c$; C is a closed subset of R^{n+m}; and Ω is a closed convex subset of R^m.

It is easy to see that we can turn the nonadditive perturbations into additive perturbations by adding an auxiliary variable:

$$\begin{array}{lll}\mathrm{GP}(\alpha) & \text{minimize} & f(x,y,z)\\ & \text{subject to} & \Psi(x,y,z)\le 0, H(x,y,z)=0,(x,y,z)\in C\times R^c,\\ & & 0\in F(x,y,z)+N_\Omega(y),\\ & & z=\alpha,\end{array}$$

which is the partially perturbed problem of the fully perturbed problem

$$\begin{array}{llll} \mathrm{GP}(p,q,r,\alpha) & \text{minimize} & f(x,y,z) \\ & \text{subject to} & \Psi(x,y,z) \le p, H(x,y,z) = q, (x,y,z) \in C \times R^c, \\ & & r \in F(x,y,z) + N_\Omega(y), \\ & & z = \alpha. \end{array}$$

By Theorem 3.5, if the fully perturbed problem $\mathrm{GP}(p,q,r,\alpha)$ satisfies the growth hypothesis (GH) at $(0,0,0,\bar{\alpha})$, then the value function $\widetilde{V}(p,q,r,\alpha)$ defined by

$$\widetilde{V}(p,q,r,\alpha) := \inf\{f(x,y,z) : \Psi(x,y,z) \le p, H(x,y,z) = q, (x,y,z) \in C \times R^c, \\ r \in F(x,y,z) + N_\Omega(y), z = \alpha\}$$

is lower semicontinuous on $B(0,0,0,\bar{\alpha};\delta)$ and

$$\partial \widetilde{V}(0,0,0,\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y},\bar{\alpha})\in\Sigma(0,0,0,\bar{\alpha})} -M^1(\bar{x},\bar{y},\bar{\alpha}),$$
$$\partial^\infty \widetilde{V}(0,0,0,\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y},\bar{\alpha})\in\Sigma(0,0,0,\bar{\alpha})} -M^0(\bar{x},\bar{y},\bar{\alpha}).$$

For any $(0,0,0,\zeta) \in \partial^\infty \widetilde{V}(0,0,0,\bar{\alpha})$, we have $(0,0,0,\zeta) \in -M^0(\bar{x},\bar{y},\bar{\alpha})$ for some point $(\bar{x},\bar{y},\bar{\alpha}) \in \Sigma(0,0,0,\bar{\alpha})$. Therefore,

$$(0,0,\zeta) \in N_C(\bar{x},\bar{y}) \times \{0\},$$

which implies that $\zeta = 0$. By Proposition 2.6, we have

$$\partial_\alpha \widetilde{V}(0,0,0,\bar{\alpha}) \subseteq \{-\zeta : -(\gamma,\beta,\eta,\zeta) \in \partial \widetilde{V}(0,0,0,\bar{\alpha}) \text{ for some } (\gamma,\beta,\eta)\},$$
$$\partial_\alpha^\infty \widetilde{V}(0,0,0,\bar{\alpha}) \subseteq \{-\zeta : -(\gamma,\beta,\eta,\zeta) \in \partial^\infty \widetilde{V}(0,0,0,\bar{\alpha}) \text{ for some } (\gamma,\beta,\eta)\}.$$

Moreover, since all functions involved are continuous, it suffices to fix α at $\bar{\alpha}$ in the growth hypothesis (GH) at $(0,0,0,\bar{\alpha})$ for the fully perturbed problem $\mathrm{GP}(p,q,r,\alpha)$. Consequently, noticing that $V(\alpha) = \widetilde{V}(0,0,0,\alpha)$, we have proved the following theorem.

THEOREM 3.6. *In addition to the basic assumption* (BH), *assume that there exists $\delta > 0$ such that the set*

$$\{(x,y) \in C \ : \ \Psi(x,y,\bar{\alpha}) \le p, H(x,y,\bar{\alpha}) = q, r \in F(x,y,\bar{\alpha}) + N_\Omega(y), f(x,y,\bar{\alpha}) \le M, \\ (p,q,r) \in B(0;\delta)\}$$

is bounded for each M. Then the value function $V(\alpha)$ is lower semicontinuous near $\bar{\alpha}$ and

$$\partial V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{-\zeta : (\gamma,\beta,\eta,\zeta) \in M^1(\bar{x},\bar{y},\bar{\alpha})\},$$
$$\partial^\infty V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{-\zeta : (\gamma,\beta,\eta,\zeta) \in M^0(\bar{x},\bar{y},\bar{\alpha})\},$$

where $M^\lambda(\bar{x},\bar{y},\bar{\alpha})$ is the set of index λ multipliers for problem $\mathrm{GP}(p,q,r,\alpha)$ *at* $(0,0,0,\bar{\alpha})$, *i.e., vectors $(\gamma,\beta,\eta,\zeta)$ in $R^d \times R^l \times R^m \times R$ satisfying*

$$\begin{cases} 0 \in \lambda\partial f(\bar{x},\bar{y},\bar{\alpha}) + \partial\langle\Psi,\gamma\rangle(\bar{x},\bar{y},\bar{\alpha}) + \partial\langle H,\beta\rangle(\bar{x},\bar{y},\bar{\alpha}) + \partial\langle F,\eta\rangle(\bar{x},\bar{y},\bar{\alpha}) \\ \qquad +\{0\} \times D^*N_\Omega(\bar{y}, -F(\bar{x},\bar{y},\bar{\alpha}))(\eta) \times \{0\} + \{(0,0,\zeta)\} + N_C(\bar{x},\bar{y}) \times \{0\}, \\ \gamma \ge 0, \ \textit{and } \langle\Psi(\bar{x},\bar{y},\bar{\alpha}),\gamma\rangle = 0, \end{cases}$$

and $\Sigma(\bar\alpha)$ is the set of solutions of problem $GP(\bar\alpha)$.

The above estimates may not be useful in the case where $\partial V(\bar\alpha)$ is empty. The following consequence of Theorem 3.6 and Proposition 2.4 provides conditions which rule out this possibility.

COROLLARY 3.7. *Under the assumption of Theorem 3.6, if the set of ζ components of the abnormal CD multiplier set contains only the zero vector, i.e.,*

$$\bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)} \{-\zeta : (\gamma,\beta,\eta,\zeta)\in M^0(\bar x,\bar y,\bar\alpha)\} = \{0\},$$

then $V(\bar\alpha)$ is finite and Lipschitz near $\bar\alpha$ with

$$\emptyset \neq \partial V(\bar\alpha) \subset \bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)} \{-\zeta : (\gamma,\beta,\eta,\zeta)\in M^1(\bar x,\bar y,\bar\alpha)\}.$$

In addition to the above assumptions, if the ζ components of the normal CD multiplier set are unique, i.e.,

$$\bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)} \{-\zeta : (\gamma,\beta,\eta,\zeta)\in M^1(\bar x,\bar y,\bar\alpha)\} = \{-\zeta\},$$

then V is strictly differentiable at $\bar\alpha$ and $\nabla V(\bar\alpha) = -\zeta$.

In the case where all functions are smooth, the estimates have the following simple expression.

COROLLARY 3.8. *In addition to the assumptions in Theorem 3.6, assume that f,Ψ,H,F are C^1 at each $(\bar x,\bar y,\bar\alpha)$, where $(\bar x,\bar y)\in\Sigma(\bar\alpha)$; then the value function V is lower semicontinuous near $\bar\alpha$, and*

$$\begin{aligned}
\partial V(\bar\alpha) \subseteq \bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)} &\{\nabla_\alpha f(\bar x,\bar y,\bar\alpha) + \nabla_\alpha\Psi(\bar x,\bar y,\bar\alpha)^\top\gamma + \nabla_\alpha H(\bar x,\bar y,\bar\alpha)^\top\beta\\
&+\nabla_\alpha F(\bar x,\bar y,\bar\alpha)^\top\eta : \qquad (\gamma,\beta,\eta)\in M^1(\bar x,\bar y)\},\\
\partial^\infty V(\bar\alpha) \subseteq \bigcup_{(\bar x,\bar y,\bar z)\in\Sigma(\bar\alpha)} &\{\nabla_\alpha\Psi(\bar x,\bar y,\bar\alpha)^\top\gamma + \nabla_\alpha H(\bar x,\bar y,\bar\alpha)^\top\beta\\
&+\nabla_\alpha F(\bar x,\bar y,\bar\alpha)^\top\eta : \qquad (\gamma,\beta,\eta)\in M^0(\bar x,\bar y)\},
\end{aligned}$$

where $M^\lambda(\bar x,\bar y)$ is the set of index λ CD multipliers for problem GP$(\bar\alpha)$.

Note that in the case where there are no variational inequality constraints, the CD multipliers are the ordinary NLP multipliers, and the above results recover the well-known results in the sensitivity analysis of NLP.

4. Applications to OPCCs. In this section, we apply our main results to the following perturbed OPCC:

$$\begin{array}{lll}
(\mathrm{OPCC})(\alpha) & \text{minimize} & f(x,y,\alpha),\\
& \text{subject to} & \Psi(x,y,\alpha)\le 0, H(x,y,\alpha)=0, (x,y)\in C,\\
& & y\ge 0,\ F(x,y,\alpha)\ge 0,\\
& & \langle y, F(x,y,\alpha)\rangle = 0,
\end{array}$$

which is GP(α) with $\Omega = R^m_+$.

For easier exposition, we assume in this section that all problem data f,Ψ,H,F are C^1. We denote by $\nabla f(x,y,\alpha)$ the gradient of function f with respect to (x,y).

For $(\bar{x}, \bar{y})$, a feasible solution of (OPCC)$(\bar{\alpha})$, we define the index sets

$$\begin{aligned} L &:= L(\bar{x}, \bar{y}) := \{1 \le i \le m : \bar{y}_i > 0, F_i(\bar{x}, \bar{y}, \bar{\alpha}) = 0\}, \\ I_+ &:= I_+(\bar{x}, \bar{y}) := \{1 \le i \le m : \bar{y}_i = 0, F_i(\bar{x}, \bar{y}, \bar{\alpha}) > 0\}, \\ I_0 &:= I_0(\bar{x}, \bar{y}) := \{1 \le i \le m : \bar{y}_i = 0, F_i(\bar{x}, \bar{y}, \bar{\alpha}) = 0\}. \end{aligned}$$

4.1. Sensitivity analysis of the value function via NLP multipliers. Let $(\bar{x}, \bar{y})$ be a local optimal solution for (OPCC)$(\bar{\alpha})$. Treating (OPCC)$(\bar{\alpha})$ as an ordinary NLP problem with inequality constraints

$$\Psi(x, y, \bar{\alpha}) \le 0, y \ge 0, F(x, y, \bar{\alpha}) \ge 0,$$

equality constraints

$$H(x, y, \bar{\alpha}) = 0, \langle y, F(x, y, \bar{\alpha}\rangle = 0,$$

and the abstract constraint $(x, y) \in C$, it is easy to see that the Fritz John optimality condition implies the existence of $\lambda \ge 0, \gamma \in R^d, \beta \in R^l, r^F \in R^m, r^y \in R^m, \mu \in R$, not all zero, such that

$$\begin{aligned} &0 \in \lambda \nabla f(\bar{x}, \bar{y}, \bar{\alpha}) + \nabla \Psi(\bar{x}, \bar{y}, \bar{\alpha})^\top \gamma + \nabla H(\bar{x}, \bar{y}, \bar{\alpha})^\top \beta \\ &\quad -\nabla F(\bar{x}, \bar{y}, \bar{\alpha})^\top r^F - \{(0, r^y)\} + \mu \nabla \langle y, F\rangle(\bar{x}, \bar{y}, \bar{\alpha}) + N_C(\bar{x}, \bar{y}), \\ &\gamma \ge 0, \langle \gamma, \Psi(\bar{x}, \bar{y}, \bar{\alpha})\rangle = 0, \\ &r^F \ge 0, r^y \ge 0, \langle r^F, F(\bar{x}, \bar{y}, \bar{\alpha})\rangle = 0, \langle r^y, \bar{y}\rangle = 0. \end{aligned}$$

Using the sum and product rules, we have

$$\nabla \langle y, F\rangle(\bar{x}, \bar{y}, \bar{\alpha}) = \{(0, F(\bar{x}, \bar{y}, \bar{\alpha}))\} + \nabla F(\bar{x}, \bar{y}, \bar{\alpha})^\top \bar{y}.$$

Therefore, the Fritz John necessary condition becomes

$$\begin{aligned} &0 \in \lambda \nabla f(\bar{x}, \bar{y}, \bar{\alpha}) + \nabla \Psi(\bar{x}, \bar{y}, \bar{\alpha})^\top \gamma + \nabla H(\bar{x}, \bar{y}, \bar{\alpha})^\top \beta \\ &\quad +\nabla F(\bar{x}, \bar{y}, \bar{\alpha})^\top (\mu \bar{y} - r^F) + \{(0, \mu F(\bar{x}, \bar{y}, \bar{\alpha}) - r^y)\} + N_C(\bar{x}, \bar{y}), \\ &\gamma \ge 0, \langle \gamma, \Psi(\bar{x}, \bar{y}, \bar{\alpha})\rangle = 0, \\ &r^F \ge 0, r^y \ge 0, \text{ and } \langle r^F, F(\bar{x}, \bar{y}, \bar{\alpha})\rangle = 0, \langle r^y, \bar{y}\rangle = 0. \end{aligned}$$

DEFINITION 4.1 (NLP multipliers). *We call all vectors $(\gamma, \beta, r^F, r^y, \mu) \in R^d \times R^l \times R^m \times R^m \times R$ satisfying the above Fritz John necessary condition for any $\lambda \ge 0$ the index λ NLP multipliers for OPCC$(\bar{\alpha})$ and denote the set by $M^{\lambda}_{NLP}(\bar{x}, \bar{y})$.*

Since we treat OPCC(α) as an ordinary NLP problem, $\Omega = \{0\}$ in the corresponding problem GP(α). Hence the CD multipliers for the corresponding $GP(\alpha)$ are the NLP multipliers defined above. Applying Corollary 3.8 and Proposition 2.4, we derive the following upper estimates of the limiting subdifferentials of the value function in terms of the NLP multipliers.

THEOREM 4.2. *Assume that there exists $\delta > 0$ such that the set*

$$\begin{aligned} \{(x, y) \in C :&(p, q, p_y, p_F, q_\mu) \in B(0; \delta), \Psi(x, y, \bar{\alpha}) \le p, H(x, y, \bar{\alpha}) = q, \\ &y \ge p_y, \ F(x, y, \bar{\alpha}) \le p_F, \langle y, F(x, y, \bar{\alpha})\rangle = q_\mu, f(x, y, \bar{\alpha}) \le M\} \end{aligned}$$

is bounded for each M. Then the value function V is lower semicontinuous near $\bar\alpha$ and

$$\partial V(\bar\alpha) \subseteq \bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)} \{\nabla_\alpha f(\bar x,\bar y,\bar\alpha) + \nabla_\alpha \Psi(\bar x,\bar y,\bar\alpha)^\top \gamma + \nabla_\alpha H(\bar x,\bar y,\bar\alpha)^\top \beta$$
$$+\nabla_\alpha F(\bar x,\bar y,\bar\alpha)^\top(\mu\bar y - r^F) : (\gamma,\beta,r^F,r^y,\mu) \in M^1_{NLP}(\bar x,\bar y)\}, \tag{12}$$
$$\partial^\infty V(\bar\alpha) \subseteq \bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)} \{\nabla_\alpha \Psi(\bar x,\bar y,\bar\alpha)^\top \gamma + \nabla_\alpha H(\bar x,\bar y,\bar\alpha)^\top \beta$$
$$+\nabla_\alpha F(\bar x,\bar y,\bar\alpha)^\top(\mu\bar y - r^F) : (\gamma,\beta,r^F,r^y,\mu) \in M^0_{NLP}(\bar x,\bar y)\}. \tag{13}$$

If the set in the right-hand side of inclusion (13) *contains only the zero vector, then the value function V is Lipschitz near $\bar\alpha$. If the set in the right-hand side of inclusion* (13) *contains only the zero vector and the set in the right-hand side of inclusion* (12) *is a singleton, then the value function is strictly differentiable at $\bar\alpha$.*

4.2. Sensitivity analysis of the value function via CD multipliers. Since OPCC($\bar\alpha$) is OPVIC($\bar\alpha$) with $\Omega = R^m_+$, the following expression of CD multiplers follows immediately from Lemma 3.2.

PROPOSITION 4.3. *For* OPCC($\bar\alpha$)*, an index λ* CD *multiplier corresponding to a feasible solution $(\bar x,\bar y)$ is a vector $(\gamma,\beta,\eta) \in R^d \times R^l \times R^m$ such that*

$$0 \in \lambda\nabla f(\bar x,\bar y,\bar\alpha) + \nabla\Psi(\bar x,\bar y,\bar\alpha)^\top\gamma + \nabla H(\bar x,\bar y,\bar\alpha)^\top\beta$$
$$+\nabla F(\bar x,\bar y,\bar\alpha)^\top\eta + (0,0,\xi) + N_C(\bar x,\bar y), \tag{14}$$
$$\gamma \ge 0 \text{ and } \langle \Psi(\bar x,\bar y,\bar\alpha),\gamma\rangle = 0, \tag{15}$$
$$\xi_i = 0 \qquad \text{if } \bar y_i > 0 \text{ and } F_i(\bar x,\bar y,\bar\alpha) = 0, \tag{16}$$
$$\eta_i = 0 \qquad \text{if } \bar y_i = 0 \text{ and } F_i(\bar x,\bar y,\bar\alpha) > 0, \tag{17}$$
$$\text{either } \xi_i < 0,\ \eta_i < 0,\ \text{ or } \xi_i\eta_i = 0 \qquad \text{if } \bar y_i = 0 \text{ and } F_i(\bar x,\bar y) = 0. \tag{18}$$

Corollary 3.8 and Proposition 2.4 now lead to the following result.

THEOREM 4.4. *Assume that there exists $\delta > 0$ such that the set*

$$\{(x,y) \in C : (p,q,r) \in B(0;\delta), \Psi(x,y,\bar\alpha) \le p, H(x,y,\bar\alpha) = q,$$
$$y \ge 0,\ F(x,y,\bar\alpha) \ge r, \langle y, F(x,y,\bar\alpha) - r\rangle = 0, f(x,y,\bar\alpha) \le M\}$$

is bounded for each M. Then the value function V is lower semicontinuous near $\bar\alpha$ and

$$\partial V(\bar\alpha) \subseteq \bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)} \{\nabla_\alpha f(\bar x,\bar y,\bar\alpha) + \nabla_\alpha \Psi(\bar x,\bar y,\bar\alpha)^\top \gamma + \nabla_\alpha H(\bar x,\bar y,\bar\alpha)^\top \beta$$
$$+\nabla_\alpha F(\bar x,\bar y,\bar\alpha)^\top\eta : (\gamma,\beta,\eta) \in M^1_{CD}(\bar x,\bar y)\}, \tag{19}$$
$$\partial^\infty V(\bar\alpha) \subseteq \bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)} \{\nabla_\alpha \Psi(\bar x,\bar y,\bar\alpha)^\top \gamma + \nabla_\alpha H(\bar x,\bar y,\bar\alpha)^\top \beta$$
$$+\nabla F(\bar x,\bar y,\bar\alpha)^\top\eta : (\gamma,\beta,\eta) \in M^0_{CD}(\bar x,\bar y)\}. \tag{20}$$

If the set in the right-hand side of inclusion (20) *contains only the zero vector, then the value function V is Lipschitz near $\bar\alpha$. If the set in the right-hand side of inclusion* (20) *contains only the zero vector and the set in the right-hand side of inclusion* (19) *is a singleton, then the value function is strictly differentiable at $\bar\alpha$.*

We say that the generalized Mangasarian–Fromovitz constraint qualification for OPCC($\bar\alpha$) is satisfied at $(\bar x,\bar y)$ if $C = D \times R^m$ and

(i) for every partition of I_0 into sets P, Q, R with $R \neq \emptyset$, there exist vectors $k \in \mathrm{int} T_C(\bar{x}, D), h \in R^m$ such that $h_{I_+} = 0, h_Q = 0, h_R \geq 0$,

$$\begin{aligned}
&\nabla_x \Psi_{I(\Psi)}(\bar{x}, \bar{y}, \bar{\alpha})k + \nabla_y \Psi_{I(\Psi)}(\bar{x}, \bar{y}, \bar{\alpha})h \leq 0,\\
&\nabla_x H(\bar{x}, \bar{y}, \bar{\alpha})k + \nabla_y H(\bar{x}, \bar{y}, \bar{\alpha})h = 0,\\
&\nabla_x F_{L\cup P}(\bar{x}, \bar{y}, \bar{\alpha})k + \nabla_y F_{L\cup P}(\bar{x}, \bar{y}, \bar{\alpha})h = 0,\\
&\nabla_x F_R(\bar{x}, \bar{y}, \bar{\alpha})k + \nabla_y F_R(\bar{x}, \bar{y}, \bar{\alpha})h \geq 0,
\end{aligned}$$

and either $h_i > 0$ or

$$\nabla_x F_i(\bar{x}, \bar{y}, \bar{\alpha})k + \nabla_y F_i(\bar{x}, \bar{y}, \bar{\alpha})h > 0 \text{ for some } i \in R;$$

(ii) for every partition of I_0 into the sets P, Q, the matrix

$$\begin{bmatrix} \nabla_x H(\bar{x}, \bar{y}, \bar{\alpha}) & \nabla_y H_{A,L\cup P}(\bar{x}, \bar{y}, \bar{\alpha}) \\ \nabla_x F_{L\cup P}(\bar{x}, \bar{y}, \bar{\alpha}) & \nabla_y F_{L\cup P, L\cup P}(\bar{x}, \bar{y}, \bar{\alpha}) \end{bmatrix}$$

has full row rank and there exist vectors $k \in \mathrm{int} T_C(\bar{x}, D), h \in R^m$ such that

$$\begin{aligned}
&h_{I_+} = 0, h_Q = 0,\\
&\nabla_x \Psi_{I(\Psi)}(\bar{x}, \bar{y}, \bar{\alpha})k + \nabla_y \Psi_{I(\Psi)}(\bar{x}, \bar{y}, \bar{\alpha})h < 0,\\
&\nabla_x H(\bar{x}, \bar{y}, \bar{\alpha})k + \nabla_y H(\bar{x}, \bar{y}, \bar{\alpha})h = 0,\\
&\nabla_x F_{L\cup P}(\bar{x}, \bar{y}, \bar{\alpha})k + \nabla_y F_{L\cup P}(\bar{x}, \bar{y}, \bar{\alpha})h = 0,
\end{aligned}$$

where $A := \{1, \ldots, l\}$, $T_C(\bar{x}, D)$ denotes the Clarke tangent cone of D at $\bar{x}$, and $I(\Psi) := \{i : \Psi_i(\bar{x}, \bar{y}) = 0\}$ is the index set of the binding inequality constraints.

In [23, Proposition 4.5] it was proved that the generalized Mangasarian–Fromovitz constraint qualification implies that the only abnormal CD multiplier is the zero vector. Hence Theorem 4.4 has the following consequence.

COROLLARY 4.5. *In addition to the assumptions of Theorem* 4.4, *if the generalized Mangasarian–Fromovitz constraint qualification as defined above is satisfied for* OPCC($\bar{\alpha}$), *then* $V(\alpha)$ *is finite and Lipschitz near* $\bar{\alpha}$.

Another sufficient condition for $M^0_{CD}(\Sigma(\bar{\alpha})) = \{0\}$ is the strong regularity condition in the sense of Robinson [17]. For OPCC ($\bar{\alpha}$), the strong regularity condition has the following form according to [17, Theorem 3.1].

COROLLARY 4.6. *In addition to the assumptions of Theorem* 4.4, *assume that* $C = D \times R^m$ *for some* $D \subseteq R^n$, *that there are no inequality constraints, and that the following conditions are satisfied:*

(i) *the matrix*

$$\begin{bmatrix} \nabla_y H_{A,L}(\bar{x}, \bar{y}, \bar{\alpha}) \\ \nabla_y F_{L,L}(\bar{x}, \bar{y}, \bar{\alpha}) \end{bmatrix}$$

is nonsingular, where $A := \{1, \ldots, l\}$;

(ii) *the Schur complement of the above matrix in the matrix*

$$\begin{bmatrix} \nabla_y H_{A,L}(\bar{x}, \bar{y}, \bar{\alpha}) & \nabla_y H_{A,I_0}(\bar{x}, \bar{y}, \bar{\alpha}) \\ \nabla_y F_{L,L}(\bar{x}, \bar{y}, \bar{\alpha}) & \nabla_y F_{L,I_0}(\bar{x}, \bar{y}, \bar{\alpha}) \\ \nabla_y F_{I_0,L}(\bar{x}, \bar{y}, \bar{\alpha}) & \nabla_y F_{I_0,I_0}(\bar{x}, \bar{y}, \bar{\alpha}) \end{bmatrix}$$

has positive principle minors;

then $V(\alpha)$ *is finite and Lipschitz near* $\bar{\alpha}$.

4.3. Sensitivity analysis of the value function via C multipliers. It is easy to see that OPCC ($\bar{\alpha}$) can be formulated as the following optimization problem with a nonsmooth equation:

$$
\begin{array}{llll}
& \mathrm{OPCC}(\bar{\alpha}) & \text{minimize} & f(x,y,\alpha) \\
(21) & & \text{subject to} & \Psi(x,y,\alpha) \leq 0, H(x,y,\alpha) = 0, (x,y) \in C, \\
& & & \min\{y_i, F_i\}(x,y,\alpha) = 0, \qquad i = 1,2,\ldots,m.
\end{array}
$$

It can be shown as in Scheel and Scholtes [19, Lemma 1] that a solution of the OPCC is C stationary defined as follows.

DEFINITION 4.7 (C multipliers). *Let $(\bar{x},\bar{y})$ be a feasible point of the* OPCC. *The point $(\bar{x},\bar{y})$ is C stationary if there exist vectors $(\gamma,\beta,\eta,\xi) \in R^d \times R^l \times R^m \times R^m$ satisfying* (14)–(17) *and*

$$\xi_i \eta_i \geq 0 \qquad \text{if } \bar{y}_i = 0 \text{ and } F_i(\bar{x},\bar{y},\bar{\alpha}) = 0.$$

The set of vectors (γ,β,η) satisfying the above condition for some ξ is called the index λ C multiplier set and is denoted by $M_C^\lambda(\bar{x},\bar{y})$.

THEOREM 4.8. *Assume that there exists $\delta > 0$ such that the set*

$$
\begin{aligned}
\{(x,y) \in C : &(p,q,q^m) \in B(0;\delta), \Psi(x,y,\bar{\alpha}) \leq p, H(x,y,\bar{\alpha}) = q, \\
&\min\{y_i, F_i(x,y,\bar{\alpha})\} = q_i^m, i = 1,\ldots,m, f(x,y,\bar{\alpha}) \leq M\}
\end{aligned}
$$

is bounded for each M. Then the value function V is lower semicontinuous near $\bar{\alpha}$ and

$$
\begin{aligned}
\partial V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} &\{\nabla_\alpha f(\bar{x},\bar{y},\bar{\alpha}) + \nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta \\
&+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in M_C^1(\bar{x},\bar{y})\}, \qquad (22)\\
\partial^\infty V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} &\{\nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta \\
&+F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in M_C^0(\bar{x},\bar{y})\}. \qquad (23)
\end{aligned}
$$

If the set in the right-hand side of inclusion (23) *contains only the zero vector, then the value function V is Lipschitz near $\bar{\alpha}$. If the set in the right-hand side of inclusion* (23) *contains only the zero vector and the set in the right-hand side of inclusion* (22) *is a singleton, then the value function is strictly differentiable at $\bar{\alpha}$.*

Proof. By Theorem 3.6, since the growth assumption is satisfied, the value function is lower semicontinuous near $\bar{\alpha}$ and

$$
\begin{aligned}
\partial V(\bar{\alpha}) &\subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{-\zeta : (\gamma,\eta,\zeta) \in M^1(\bar{x},\bar{y},\bar{\alpha})\}, \\
\partial^\infty V(\bar{\alpha}) &\subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{-\zeta : (\gamma,\eta,\zeta) \in M^0(\bar{x},\bar{y},\bar{\alpha})\},
\end{aligned}
$$

where $M^\lambda(\bar{x},\bar{y},\bar{\alpha})$ is the set of vectors $(\gamma,\beta,r,\zeta) \in R^{d+l+m+c}$ such that

$$
\begin{aligned}
&0 \in \lambda \nabla f(\bar{x},\bar{y},\bar{\alpha}) + \nabla \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla H(\bar{x},\bar{y},\bar{\alpha})^\top \beta \\
&\qquad +\partial \sum_{i=1}^m r_i \min\{y_i, F_i\}(\bar{x},\bar{y},\bar{\alpha}) + \{(0,0,\zeta)\} + N_C(\bar{x},\bar{y}) \times \{0\}, \\
&\gamma \geq 0, \langle \gamma, \Psi \rangle(\bar{x},\bar{y},\bar{\alpha}) = 0.
\end{aligned}
$$

Note that, in the above, ∇f denotes the gradient of a function f with respect to (x, y, α). Since

$$\partial \sum_{i=1}^{m} r_i \min\{y_i, F_i\}(\bar{x}, \bar{y}, \bar{\alpha}) \subseteq \sum_{i=1}^{m} r_i \partial_C \min\{y_i, F_i\}(\bar{x}, \bar{y}, \bar{\alpha})$$

and

$$\partial_C \min\{y_i, F_i\}(\bar{x}, \bar{y}, \bar{\alpha}) = \begin{cases} (0, e_i, 0) & \forall i \in I_+, \\ \nabla F_i(\bar{x}, \bar{y}, \bar{\alpha}) & \forall i \in L, \\ \{t(0, , e_i, 0) + (1-t)\nabla F_i(\bar{x}, \bar{y}, \bar{\alpha}) : t \in [0,1]\} & \forall i \in I_0, \end{cases}$$

where e_i is the unit vector whose ith component is 1 and those other components are zero, there exist γ, β, η such that

$$\zeta = \lambda \nabla_\alpha f(\bar{x}, \bar{y}, \bar{\alpha}) + \nabla_\alpha \Psi(\bar{x}, \bar{y}, \bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x}, \bar{y}, \bar{\alpha})^\top \beta + \nabla_\alpha F(\bar{x}, \bar{y}, \bar{\alpha})^\top \eta$$

and

$$\begin{aligned} &0 \in \lambda \nabla f(\bar{x}, \bar{y}, \bar{\alpha}) + \nabla \Psi(\bar{x}, \bar{y}, \bar{\alpha})^\top \gamma + \nabla H(\bar{x}, \bar{y}, \bar{\alpha})^\top \beta + \nabla F(\bar{x}, \bar{y}, \bar{\alpha})^\top \eta \\ &\qquad + (0, \xi) + N_C(\bar{x}, \bar{y}), \\ &\gamma \geq 0, \langle \Psi, \gamma \rangle(\bar{x}, \bar{y}, \bar{\alpha}) = 0, \end{aligned}$$

where

$$\begin{array}{ll} \eta_i = 0 & \forall i \in I_+, \\ \xi_i = 0 & \forall i \in L, \\ \eta_i = r_i(1 - \bar{t}_i), \xi_i = r_i \bar{t}_i \text{ for some } \bar{t}_i \in [0,1], & \forall i \in I_0. \end{array}$$

It is then easy to see that

$$\forall i \in I_0, \eta_i \xi_i \geq 0.$$

Hence (γ, β, η) is a C multiplier, and the proof of the theorem is complete. ∎

4.4. Sensitivity analysis via P multipliers and S multipliers. Taking the "piecewise programming" approach, for any given index set $\nu \subseteq I := \{1, \ldots, m\}$, we consider the subproblem associated with ν:

$$\begin{array}{lll} \text{OPCC}(\alpha)_\nu & \text{minimize} & f(x, y, \alpha) \\ & \text{subject to} & \Psi(x, y, \alpha) \leq 0, H(x, y, \alpha) = 0, (x, y) \in C, \\ & & y_i \geq 0,\ F_i(x, y, \alpha) = 0 \quad \forall i \in \nu \\ & & y_i = 0,\ F_i(x, y, \alpha) \geq 0 \quad \forall i \in I \backslash \nu. \end{array}$$

As suggested by referee 2, since the value function is the minimum of the value functions for the subproblems, i.e.,

$$V(\alpha) = \min_{\nu \subset I} V_\nu(\alpha)$$

and

$$V(\bar{\alpha}) = V_\nu(\bar{\alpha}) \ \forall \nu = L(\bar{x}, \bar{y}) \cup \sigma, \sigma \subseteq I_0(\bar{x}, \bar{y}), (\bar{x}, \bar{y}) \in \Sigma(\bar{\alpha}),$$

applying the calculus for the minimum functions in Proposition 2.7, we conclude that the value function V is lower semicontinuous if each $V_\nu(\alpha), \nu = L(\bar{x},\bar{y}) \cup \sigma, \sigma \subseteq I_0(\bar{x},\bar{y}), (\bar{x},\bar{y}) \in \Sigma(\bar{\alpha})$, is lower semicontinuous and the following inclusion holds:

$$\partial^\infty V(\bar{\alpha}) \subseteq \{\partial^\infty V_\nu(\bar{\alpha}) : \nu = L(\bar{x},\bar{y}) \cup \sigma, \sigma \subseteq I_0(\bar{x},\bar{y}), (\bar{x},\bar{y}) \in \Sigma(\bar{\alpha})\}, \tag{24}$$

$$\partial V(\bar{\alpha}) \subseteq \{\partial V_\nu(\bar{\alpha}) : \nu = L(\bar{x},\bar{y}) \cup \sigma, \sigma \subseteq I_0(\bar{x},\bar{y}), (\bar{x},\bar{y}) \in \Sigma(\bar{\alpha})\}. \tag{25}$$

The Fritz John condition for the subproblem OPCC$(\bar{\alpha})_\nu$ with

$$\nu = L(\bar{x},\bar{y}) \cup \sigma, \sigma \subseteq I_0(\bar{x},\bar{y}), (\bar{x},\bar{y}) \in \Sigma(\bar{\alpha})$$

implies the existence of vectors $(\gamma,\beta,\eta,\xi) \in R^d \times R^a \times R^b \times R^b$ satisfying (14)–(17) and

$$\xi_\sigma \leq 0, \eta_{I_0 \backslash \sigma} \leq 0. \tag{26}$$

DEFINITION 4.9 (P multipliers). *The set of all vectors (γ,β,η) satisfying the above Fritz John condition at $(\bar{x},\bar{y})$ is denoted by $M_\sigma^\lambda(\bar{x},\bar{y})$, and $\bigcup_{\sigma \subseteq I_0} M_\sigma^\lambda(\bar{x},\bar{y})$ is called the set of P multipliers.*

Applying Corollary 3.8, we have the following result.

PROPOSITION 4.10. *For any $(\bar{x},\bar{y}) \in \Sigma(\bar{\alpha})$ and any given index set $\sigma \subseteq I_0(\bar{x},\bar{y})$, assume that there exists $\delta > 0$ such that the set*

$$\begin{aligned}\{(x,y) \in C :&(p,q,q^y,q^F,) \in B(0;\delta), \Psi(x,y,\bar{\alpha}) \leq p, H(x,y,\bar{\alpha}) = q,\\ &y_i \geq q_i^y,\ F_i(x,y,\bar{\alpha}) = q_i^F \quad \forall i \in \nu := \sigma \cup L(\bar{x},\bar{y}),\\ &y_i = q_i^y,\ F_i(x,y,\bar{\alpha}) \geq q_i^F \quad \forall i \in I \backslash \nu, f(x,y,\bar{\alpha}) \leq M\}\end{aligned}$$

is bounded for each M. Then the value function for subproblem OPCC$(\bar{\alpha})_\nu$ *with $\nu = L(\bar{x},\bar{y}) \cup \sigma$ is lower semicontinuous near $\bar{\alpha}$ and*

$$\begin{aligned}\partial V_\nu(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma_\nu(\bar{\alpha})} &\{\nabla_\alpha f(\bar{x},\bar{y},\bar{\alpha}) + \nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta\\ &+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in M_\sigma^1(\bar{x},\bar{y})\},\\ \partial^\infty V_\nu(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma_\nu(\bar{\alpha})} &\{\nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta\\ &+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in M_\sigma^0(\bar{x},\bar{y})\},\end{aligned}$$

where $\Sigma_\nu(\bar{\alpha})$ denotes the set of solutions for the subproblem OPCC $(\alpha)_\nu$.

We have the following estimates for the value function in terms of P multipliers.

THEOREM 4.11. *Assume that there exists $\delta > 0$ such that for $(\bar{x},\bar{y}) \in \Sigma(\bar{\alpha})$ and each index set $\sigma \subseteq I_0(\bar{x},\bar{y})$, the set in Proposition 4.10 is bounded for each M. Then the value function V is lower semicontinuous near $\bar{\alpha}$ and*

$$\begin{aligned}\partial V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} &\{\nabla_\alpha f(\bar{x},\bar{y},\bar{\alpha}) + \nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta\\ &+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in \cup_{\sigma\subseteq I_0} M_\sigma^1(\bar{x},\bar{y})\},\end{aligned} \tag{27}$$

$$\begin{aligned}\partial^\infty V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} &\{\nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta\\ &+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in \cup_{\sigma\subseteq I_0} M_\sigma^0(\bar{x},\bar{y})\}.\end{aligned} \tag{28}$$

If the set in the right-hand side of inclusion (28) *contains only the zero vector, then the value function V is Lipschitz near $\bar{\alpha}$. If the set in the right-hand side of inclusion* (28) *contains only the zero vector and the set in the right-hand side of inclusion* (27) *is a singleton, then the value function is strictly differentiable at $\bar{\alpha}$.*

DEFINITION 4.12 (S multipliers). *The set of index λ S multipliers, denoted by $M_S^\lambda(\bar{x},\bar{y})$, is the set of all vectors $(\gamma,\beta,\eta) \in R^d \times R^a \times R^b$ satisfying* (14)–(17) *and*

$$\xi_i \le 0, \eta_i \le 0 \qquad \text{if } \bar{y}_i = 0 \text{ and } F_i(\bar{x},\bar{y},\bar{\alpha}) = 0.$$

In the following theorem, we give a condition under which the set of P multipliers and S multipliers coincide, and so we have the estimates in terms of the S multipliers.

THEOREM 4.13. *In addition to the assumptions of Theorem* 4.11*, assume that $C = R^n \times R^a \times R^b$ and for all $(\bar{x},\bar{z},\bar{u}) \in \Sigma(\bar{\alpha})$, the partial MPEC linear independence constraint qualification is satisfied, i.e.,*

$$\begin{cases} 0 = \nabla\Psi(\bar{x},\bar{y},\bar{\alpha})^\top\gamma + \nabla H(\bar{x},\bar{y},\bar{\alpha})^\top\beta + \nabla F(\bar{x},\bar{y},\bar{\alpha})^\top\eta + (0,0,\xi), \\ \gamma_{J(\Psi)} = 0,\ \eta_{I_+} = 0,\ \xi_L = 0, \end{cases}$$

implies that $\eta_{I_0} = 0, \xi_{I_0} = 0$, where $J(\Psi) := \{i : \Psi_i(\bar{x},\bar{y},\bar{\alpha}) < 0\}$. Then the value function V is lower semicontinuous near $\bar{\alpha}$ and

$$\partial V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{\nabla_\alpha f(\bar{x},\bar{y},\bar{\alpha}) + \nabla_\alpha\Psi(\bar{x},\bar{y},\bar{\alpha})^\top\gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top\beta$$
$$+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top\eta : (\gamma,\beta,\eta) \in M_S^1(\bar{x},\bar{y})\},$$
$$\partial^\infty V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{\nabla_\alpha\Psi(\bar{x},\bar{y},\bar{\alpha})^\top\gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top\beta$$
$$+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top\eta : (\gamma,\beta,\eta) \in M_S^0(\bar{x},\bar{y})\}.$$

Remark. As in the proof of [22, Theorem 3.2], it is easy to see that under the partial MPEC linear independence constraint qualification, all multipliers including the S multiplier, the CD multiplier, the C multiplier, and the P multiplier coincide.

Recently, the MPEC linear independence constraint qualifications have received a lot of attention. It is known that under the MPEC linear independence constraint qualification, the computation of the OPCC is much easier and more efficient (see, e.g., Scholtes [20]). Furthermore, it was shown in Scholtes [21] that the MPEC linear independence constraint qualification is a generic condition for the OPCC. Here we prove the importance of the MPEC linearly independence constraint qualification from the aspect of the sensitivity analysis: the value function is Lipschitz continuous, and it is even strictly differentiable in the case where the optimal solution set is unique. Note that the MPEC linear independence constraint qualification is stronger than the partial MPEC linear independence constraint qualification.

COROLLARY 4.14. *In addition to the assumptions of Theorem* 4.11*, assume that the MPEC linear independence constraint qualifications are satisfied at all $(\bar{x},\bar{y}) \in \Sigma(\bar{\alpha})$, i.e.,*

$$\begin{cases} 0 = \nabla\Psi(\bar{x},\bar{y},\bar{\alpha})^\top\gamma + \nabla H(\bar{x},\bar{y},\bar{\alpha})^\top\beta + \nabla F(\bar{x},\bar{y},\bar{\alpha})^\top\eta + (0,0,\xi), \\ \gamma_{J(\Psi)} = 0, \eta_{I_+} = 0, \xi_L = 0, \end{cases}$$

implies that $\gamma = 0, \beta = 0, \eta = 0, \xi = 0$. Then the value function is Lipschitz continuous near $\bar{\alpha}$. Furthermore, if the set of optimal solutions $\Sigma(\bar{\alpha})$ is a singleton, then the value function V is strictly differentiable at $\bar{\alpha}$.

Proof. The MPEC linear independence constraint qualification obviously implies that $M_S^0(\bar{x},\bar{y}) = \{0\}$ and $M_S^1(\bar{x},\bar{y})$ is a singleton. Hence the conclusion follows from Theorem 4.13 and Proposition 2.4. ∎

4.5. Relationships between the multipliers for the OPCC. Applying the definitions, it is clear that

$$M_S^\lambda(\bar{x},\bar{y}) \subseteq M_{CD}^\lambda(\bar{x},\bar{y}) \subseteq M_C^\lambda(\bar{x},\bar{y}), \qquad M_S^\lambda(\bar{x},\bar{y}) \subseteq M_P^\lambda(\bar{x},\bar{y}). \tag{29}$$

It is not possible to compare the set of NLP multipliers directly with the other multipliers since the spaces they belong to have different dimensions. However, the following interesting relationships can be obtained.

PROPOSITION 4.15 (relationship between an NLP multiplier and an S multiplier).

$$\{(\gamma,\beta,\mu\bar{y}-r^F) : (\gamma,\beta,r^F,r^y,\mu) \in M_{NLP}^\lambda(\bar{x},\bar{y})\} \subseteq M_S^\lambda(\bar{x},\bar{y})$$

for all $\lambda \geq 0$.

Proof. Let $(\gamma,\beta,r^F,r^y,\mu) \in M_{NLP}^\lambda(\bar{x},\bar{y})$. We consider the following cases.

Case $\bar{y}_i > 0, F_i(\bar{x},\bar{y}) = 0$. Then $r_i^y = 0$. So $\xi_i := \mu F_i - r_i^y = 0$.

Case $\bar{y}_i = 0, F_i(\bar{x},\bar{y}) > 0$. Then $r_i^F = 0$. So $\eta_i = \mu\bar{y}_i - r_i^F = 0$.

Case $\bar{y}_i = 0, F_i(\bar{x},\bar{y}) = 0]$. Then $\xi_i = \mu F_i(\bar{x},\bar{y}) - r_i^y = -r_i^y$ and $\eta_i = \mu\bar{y}_i - r_i^F = -r_i^F$. So $\xi_i = -r_i^y \leq 0$ and $\eta_i = -r_i^F \leq 0$.

Hence (γ,β,η), where $\eta := \mu\bar{y}-r^F$, is an S multiplier, and the proof of the proposition is complete. ∎

The above relationship indicates that one can arrange the upper estimates of the limiting subdifferentials in Theorems 4.2, 4.13, 4.4, and 4.8 from the smallest to the largest in the order of NLP multipliers, S multipliers, CD multipliers, and C multipliers.

One may try to use the smallest multiplier set in sensitivity analysis. However, the smaller multipler sets tend to require stronger constraint qualifications and hence may be empty. In such a case, where the smaller multipler set is empty, one may have to use the larger multiplier set.

We now use the following example to show that in some cases the smaller multiplier sets such as the NLP and the S multiplier sets may be empty while the CD multiplier provides the tightest bound.

Example. Consider the OPCC

$$(P) \qquad \begin{array}{rl} \text{minimize} & -y \\ \text{subject to} & x - y = 0, \\ & x \geq 0, y \geq 0, xy = 0, \end{array}$$

where $x \in R$ and $y \in R$, and its perturbed problem

$$P(q,r) \qquad \begin{array}{rl} \text{minimize} & -y \\ \text{subject to} & x - y = q, \\ & x - r \geq 0, y \geq 0, (x-r)y = 0, \end{array}$$

which is OPCC (α) with $\alpha = (q,r), f = -y, H = x-y-q, F = x-r$. Let $\bar{\alpha} = (0,0)$. It is clear that the only feasible solution for problem $(P) = P(0,0)$ is $(0,0)$. Hence the only optimal solution for *(P)* is $(0,0)$. The set of index λ NLP multipliers (β,r^y,r^F,μ)

at $(0, 0)$ satisfy

$$\begin{cases} 0 = \lambda(0, -1) + \beta(1, -1) - (r^F, 0) - (0, r^y) + \mu(0, 0), \\ r^F, r^y \geq 0. \end{cases}$$

It is clear that any $(\beta, r^y, r^F, \mu) = (0, 0, 0, \mu)$ with $\mu \neq 0$ is a nonzero NLP abnormal multiplier and there is no NLP normal multiplier. Hence $M^0_{NLP}(0, 0) = \{(0, 0, 0)\} \times (-\infty, +\infty) \neq \{(0, 0, 0, 0)\}$ and $M^1_{NLP}(0, 0) = \emptyset$.

Since $\bar{y} = 0$ and $F(\bar{x}, \bar{y}, 0) = \bar{x} = 0$, the index λ CD multipliers (β, η) at $(0, 0)$ satisfy

$$0 = \lambda(0, -1) + \beta(1, -1) + \eta(1, 0) + (0, \xi),$$
$$\text{either } \xi < 0, \eta < 0, \text{ or } \xi\eta = 0.$$

When $\lambda = 0$, the above condition implies that $\beta = \eta = \xi = 0$, while when $\lambda = 1$, either $\eta = 1, \beta = -1, \xi = 0$, or $\beta = \eta = 0, \xi = 1$. So $M^1_{CD}(0, 0) = \{(0, 0)\} \cup \{(-1, 1)\}$ and $M^0_{CD}(0, 0) = \{(0, 0)\}$.

The set of index λ C multipliers (β, η) at $(0, 0)$ satisfy

$$0 = \lambda(0, -1) + \beta(1, -1) + \eta(1, 0) + (0, \xi),$$
$$\xi\eta \geq 0.$$

When $\lambda = 0$, the above condition implies that $\beta = \eta = \xi = 0$, while for $\lambda = 1$, $-\beta = \eta \in [0, 1]$. So $M^1_C(0, 0) = \{(\beta, \eta) : \eta = -\beta \in [0, 1]\}$ and $M^0_C(0, 0) = \{(0, 0)\}$.

Since the optimal solution for (P) is $(\bar{x}, \bar{y}) = (0, 0)$, $(0, 0)$ is also optimal for the subproblem associated with $\nu = \{1\}$,

$$(P_1) \qquad \begin{array}{rl} \text{minimize} & -y \\ \text{subject to} & x - y = 0, \\ & y \geq 0, x = 0, \end{array}$$

and the subproblem associated with $\nu = \emptyset$,

$$(P_2) \qquad \begin{array}{rl} \text{minimize} & -y \\ \text{subject to} & x - y = 0, \\ & y = 0, x \geq 0. \end{array}$$

The index λ multiplier set for (P_1) consists of vectors (β, η) satisfying

$$\begin{cases} 0 = \lambda(0, -1) + \beta(1, -1) + \eta(1, 0) + (0, \xi), \\ \xi \leq 0, \end{cases}$$

and the index λ multiplier set for (P_2) consist of vectors (β, η) satisfying

$$\begin{cases} 0 = \lambda(0, -1) + \beta(1, -1) + \eta(1, 0) + (0, \xi), \\ \eta \leq 0. \end{cases}$$

Therefore, the abnormal P multiplier set is

$$\begin{aligned} M^0_P(0, 0) = M^0_1(0, 0) \cup M^0_2(0, 0) &= \{(\beta, \eta) : \beta = -\eta \leq 0\} \cup \{(\beta, \eta) : \beta = -\eta \geq 0\} \\ &= \{(\beta, \eta) : \beta = -\eta\}, \end{aligned}$$

and the normal P multiplier set is

$$
\begin{aligned}
M_P^1(0,0) = M_1^1(0,0) \cup M_2^1(0,0) &= \{(\beta,\eta) : \beta = -\eta \le -1\} \cup \{(\beta,\eta) : \beta = -\eta \ge 0\} \\
&= \{(\beta,\eta) : \beta = -\eta \in (-\infty,-1] \cup [0,\infty)\}.
\end{aligned}
$$

The index λ S multiplier set consists of vectors (β,η) satisfying

$$
\begin{cases}
0 = \lambda(0,-1) + \beta(1,-1) + \eta(1,0) + (0,\xi), \\
\xi \le 0, \eta \le 0,
\end{cases}
$$

i.e.,

$$
\begin{aligned}
&\beta = -\eta, \ \beta = -\lambda + \xi, \\
&\xi \le 0, \ \eta \le 0.
\end{aligned}
$$

That is, $M_S^0(0,0) = \{0\}$, and $M_S^1(0,0) = \emptyset$.

Consider the value function

$$
V(q,r) := \inf\{-y : x - r \ge 0, y \ge 0, (x-r)y = 0, x - y = q\}.
$$

Then by Theorem 4.4, since the only abnormal CD multiplier is the zero vector, we conclude that the value function is Lipschitz near $(0,0)$, and

$$
\begin{aligned}
\emptyset \ne \partial V(0,0) &\subseteq \{\beta(-1,0) + \eta(0,-1) : (\beta,\eta) \in M_{CD}^1(0,0)\} \\
&= -M_{CD}^1(0,0) = \{(0,0)\} \cup \{(1,-1)\}.
\end{aligned}
$$

In fact, we can easily find the expression for the value function for this simple example since the feasible set of the perturbed problem $P(q,r)$ still reduces to one point. Indeed, we have

$$
\begin{cases}
\Sigma(q,r) = \{(r, r-q)\} \text{ and } V(q,r) = q - r & \text{if } q < r, \\
\Sigma(q,r) = \{(q,0)\} \text{ and } V(q,r) = 0 & \text{if } q \ge r.
\end{cases}
$$

So $V(q,r) = \min(0, q-r)$, which is Lipschitz continuous everywhere. By definition of the limiting subdifferentials, it is easy to see that

$$
\begin{aligned}
\partial V(0,0) &= \{(0,0)\} \cup \{(1,-1)\}, \\
\partial^\infty V(0,0) &= \{(0,0)\}.
\end{aligned}
$$

Therefore, the inclusions in Theorem 4.4 are actually equalities here, i.e.,

$$
\begin{aligned}
\partial V(0,0) &= \{(0,0)\} \cup \{(1,-1)\} = -M_{CD}^1(0,0), \\
\partial^\infty V(0,0) &= \{(0,0)\} = -M_{CD}^0(0,0).
\end{aligned}
$$

Using Theorem 4.8, since the only abnormal C multiplier is the zero vector, one also concludes that the value function is Lipschitz. However, the upper estimate for the limiting subdifferentials of the value function in terms of the C multiplier set is a strict inclusion here:

$$
\begin{aligned}
\partial V(0,0) &= \{(0,0)\} \cup \{(1,-1)\} \subset \{(\beta,\eta) : \beta = -\eta \in [0,1]\} = -M_C^1(0,0), \\
\partial^\infty V(0,0) &= \{(0,0)\} = -M_{CD}^0(0,0).
\end{aligned}
$$

The upper estimate for both the limiting and the singular limiting subdifferentials of the value function in Theorem 4.11 are both strict:

$$\begin{aligned}\partial V(0,0) &= \{(0,0)\} \cup \{(1,-1)\}\\ &\subset \{(\beta,\eta) : \beta = -\eta \in (-\infty, 0] \cup (1,\infty)\} = -M_P^1(0,0),\\ \partial^\infty V(0,0) &= \{(0,0)\}\\ &\subset \{(\beta,\eta) : \beta = -\eta\} = -M_P^0(0,0).\end{aligned}$$

These inclusions are not very helpful since the Lipschitz continuity of the value function cannot be detected and the upper estimate is unbounded.

Since there is no S multiplier for this problem, the limiting subdifferential of the value function cannot be estimated in terms of the S multiplier. In fact, the assumptions in Theorem 4.13 are not satisfied for this problem. Indeed,

$$(0,0) = \beta(1,-1) + \eta(1,0) + (0,\xi)$$

does not imply that $\eta = 0, \xi = 0$.

Note that by Theorem 4.2, if the growth hypotheses were satisfied, then

$$\begin{aligned}\partial V(0,0) &\subseteq \{\beta(-1,0) - r^F(0,-1) : (\beta, r^F, r^y, \mu) \in M_{NLP}^1(\Sigma)\},\\ \partial^\infty V(0,0) &\subseteq \{\beta(-1,0) - r^F(0,-1) : (\beta, r^F, r^y, \mu) \in M_{NLP}^0(\Sigma)\}.\end{aligned}$$

But this is not possible since $M_{NLP}^1(\Sigma) = \emptyset$. Indeed, (GH) is not satisfied for this example.

In the above example, $M_{NLP}^0(\Sigma) \neq \{0\}$, while $M_{CD}^0(\Sigma) = \{0\}$. In fact, it is not just a coincidence that $M_{NLP}^0(\Sigma) \neq \{0\}$. In general, the Mangasarian–Fromovitz constraint qualification satisfying at a feasible solution $(\bar{x},\bar{z},\bar{u})$ implies that $M_{NLP}^0((\bar{x},\bar{z},\bar{u})) = \{0\}$, and in the case of no abstract constraint, the two conditions are equivalent (see, e.g., [6] and [25, Proposition 4.5] for details). It is well known that in the case of no abstract constraint, the Mangasarian–Fromovitz constraint qualification fails to hold at every feasible point of the OPCCs. (The proof for the case where the complementarity constraint comes from the KKT condition of a lower level quadratic programming problem was given in Chen and Florian [1, Lemma 3.1], and the proof for the general case was given in [25, Proposition 1.1].) We now prove that even for the case when the abstract constraint set C is present, there always exist nonzero abnormal NLP multipliers for the OPCC.

PROPOSITION 4.16. *Let $(\bar{x},\bar{y}) \in R^{n+m}$ be any feasible solution of the* OPCC. *Then $M_{NLP}^0(\bar{x},\bar{y})\backslash\{0\} \neq \emptyset$.*

Proof. The point $(\bar{x},\bar{y})$ is obviously a solution to the following optimization problem:

$$\begin{aligned}&\text{minimize} && \langle y, F(x,y)\rangle\\ &\text{subject to} && y \geq 0, F(x,y) \geq 0.\end{aligned}$$

By the multiplier rule, there exists $\mu \geq 0, r^y \in R_+^m, r^F \in R_+^m$ not all zero such that

$$\begin{aligned}&0 = \mu\nabla\langle y, F\rangle(\bar{x},\bar{y}) - (0, r^y) - \nabla F(\bar{x},\bar{y})^\top r^F,\\ &\langle \bar{y}, r^y\rangle = 0, \langle r^F, F(\bar{x},\bar{y}) = 0.\end{aligned}$$

Therefore, taking $\gamma = 0, \beta = 0$ $(\gamma = 0, \beta = 0, r^F, r^y, \mu)$ is a nonzero NLP abnormal multiplier of the OPCC. $\square$

4.6. Applications to the bilevel programming problem. One of the motivations to consider OPCCs is to solve the following bilevel programming problem:

$$\text{(BLPP)} \qquad \begin{array}{cc} \text{minimize} & f(x,z) \\ \text{subject to} & z \in S(x), \Psi(x,z) \le 0, (x,z) \in C, \end{array} \tag{30}$$

where $S(x)$ is the solution of the lower-level problem

$$P_x \qquad \begin{array}{cc} \text{minimize} & g(x,z) \\ \text{subject to} & \psi(x,z) \le 0, \end{array} \tag{31}$$

where $f : R^{n+a} \to R, g : R^{n+a} \to R, \psi : R^{n+a} \to R^b, \Psi : R^{n+a} \to R^d$. Under suitable convexity assumptions, we can replace the lower problem by its KKT conditions. As in [24], we find that any (x,z) is solution of (BLPP) if and only if there is u such that (x,z,u) is solution of the problem

$$\begin{array}{cc} \text{minimize} & f(x,z) \\ \text{subject to} & \psi(x,z) \le 0 \text{ and } u \ge 0, \\ & \langle \psi(x,z), u \rangle = 0, \\ & \nabla_z g(x,z) + \nabla_z \psi(x,z)^\top u = 0, \\ & \Psi(x,z) \le 0, (x,z) \in C, \end{array} \tag{32}$$

which is an OPCC.

Consider the perturbed bilevel programming problem

$$\text{BLPP}(\alpha) \quad \begin{array}{ll} \text{minimize} & f(x,z,\alpha) \\ \text{subject to} & z \in S(x,\alpha), \Psi(x,z,\alpha) \le 0, (x,z) \in C, \end{array} \tag{33}$$

where $S(x,\alpha)$ is the solution of the lower-level problem

$$\begin{array}{cc} \text{minimize} & g(x,z,\alpha) \\ \text{subject to} & \psi(x,z,\alpha) \le 0. \end{array} \tag{34}$$

Under suitable assumptions, BLPP(α) is equivalent to

$$\begin{array}{cc} \text{minimize} & f(x,z,\alpha) \\ \text{subject to} & \psi(x,z,\alpha) \le 0 \text{ and } u \ge 0, \\ & \langle \psi(x,z,\alpha), u \rangle = 0, \\ & \nabla_z g(x,z,\alpha) + \nabla_z \psi(x,z,\alpha)^T u = 0, \\ & \Psi(x,z,\alpha) \le 0, (x,z) \in C. \end{array} \tag{35}$$

Hence the results in this section allow us to derive the properties of the value function and compute the upper estimates of the limiting subdifferentials of V by the various kinds of multipliers for the above problem. For example, we can conclude that the value function is Lipschitz continuous when the strong second order sufficient condition and the linear independence of the binding constraints hold for the lower level problem. Indeed, in this case the corresponding generalized equation is strongly regular; hence the set of abnormal CD multipliers contains only the zero vector (see Ye [23, Theorem 5.1]).

Acknowledgments. The authors would like to thank the anonymous referees whose constructive suggestions led to the better presentation of the results and the consideration of the upper estimates in terms of various multipliers other than the CD multipliers in the last section.

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