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ERRATUM: SENSITIVITY ANALYSIS OF THE VALUE FUNCTION FOR OPTIMIZATION PROBLEMS WITH VARIATIONAL INEQUALITY CONSTRAINTS*

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Abstract. In our paper [*SIAM J. Control Optim.*, 40 (2001), pp. 699–723], due to an error in the proof, an additional assumption is needed for the conclusion of Theorem 3.6 to hold. In this erratum, we restate and prove Theorem 3.6 and correct other related mistakes accordingly.

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In our paper [1], due to an error in the proof, an additional assumption is needed for the conclusion of Theorem 3.6 to hold. As a consequence, Theorem 4.2 does not hold, each of Theorems 4.4, 4.8, 4.11, and 4.13 requires an additional assumption, and the last two lines on page 701 and the first two lines on page 702 should be changed to

$$M^1 = M^1_{CD}(\Sigma), M^1_C(\Sigma), M^1_S(\Sigma),$$
$$M^0 = M^0_{CD}(\Sigma), M^0_C(\Sigma), M^0_S(\Sigma).$$

We first correct Theorem 3.6 by adding the additional assumption (0.1) as follows.

THEOREM 3.6. *In addition to the basic assumption* (BH), *assume that there exists* $\delta > 0$ *such that the set*

$$\{(x,y) \in C \ : \ \Psi(x,y,\bar\alpha) \le p, H(x,y,\bar\alpha) = q, r \in F(x,y,\bar\alpha) + N_\Omega(y), f(x,y,\bar\alpha) \le M, \\ (p,q,r) \in B(0;\delta)\}$$

is bounded for each M *and the following assumption holds:*

$$(\gamma,\beta,\eta,0) \in M^0(\bar x,\bar y,\bar\alpha) \ \textit{implies}\ \gamma = 0,\ \beta = 0,\ \eta = 0. \tag{0.1}$$

Then the value function $V(\alpha)$ *is lower semicontinuous near* $\bar\alpha$, *and*

$$\partial V(\bar\alpha) \subseteq \bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)} \{-\zeta : (\gamma,\beta,\eta,\zeta) \in M^1(\bar x,\bar y,\bar\alpha)\},$$
$$\partial^\infty V(\bar\alpha) \subseteq \bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)} \{-\zeta : (\gamma,\beta,\eta,\zeta) \in M^0(\bar x,\bar y,\bar\alpha)\},$$

where $M^\lambda(\bar x,\bar y,\bar\alpha)$ *is the set of index* λ *multipliers for problem* GP(p,q,r,α) *at* $(0,0,0,\bar\alpha)$, *i.e., vectors* $(\gamma,\beta,\eta,\zeta)$ *in* $R^d \times R^l \times R^m \times R$ *satisfying*

$$\begin{cases} 0 \in \lambda\partial f(\bar x,\bar y,\bar\alpha) + \partial\langle\Psi,\gamma\rangle(\bar x,\bar y,\bar\alpha) + \partial\langle H,\beta\rangle(\bar x,\bar y,\bar\alpha) + \partial\langle F,\eta\rangle(\bar x,\bar y,\bar\alpha) \\ +\{0\} \times D^*N_\Omega(\bar y, -F(\bar x,\bar y,\bar\alpha))(\eta) \times \{0\} + \{(0,0,\zeta)\} + N_C(\bar x,\bar y) \times \{0\}, \\ \gamma \ge 0 \ \textit{and}\ \langle\Psi(\bar x,\bar y,\bar\alpha),\gamma\rangle = 0, \end{cases}$$

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and $\Sigma(\bar\alpha)$ is the set of solutions of problem $GP(\bar\alpha)$.

We now make the correct statements for Theorems 4.4, 4.8, 4.11, and 4.13 by translating assumption (0.1) to the case of CD, C, P, and S multipliers, respectively. Unless otherwise indicated, we denote by $\nabla f(x,y,\alpha)$ the gradient of function f with respect to (x,y,α) and not the gradient of f with respect to (x,y) as in section 4 of [1].

THEOREM 4.4. *Assume that there exists $\delta > 0$ such that the set*

$$\begin{aligned}\{(x,y)\in C :&(p,q,r)\in B(0;\delta), \Psi(x,y,\bar\alpha)\le p, H(x,y,\bar\alpha)=q,\\ &y\ge 0,\ F(x,y,\bar\alpha)\ge r, \langle y, F(x,y,\bar\alpha)-r\rangle = 0, f(x,y,\bar\alpha)\le M\}\end{aligned}$$

is bounded for each M. Assume also that

$$\begin{aligned}&0\in \nabla\Psi(\bar x,\bar y,\bar\alpha)^\top\gamma+\nabla H(\bar x,\bar y,\bar\alpha)^\top\beta+\nabla F(\bar x,\bar y,\bar\alpha)^\top\eta+(0,\xi,0)+N_C(\bar x,\bar y)\times\{0\},\\ &\gamma\ge 0 \text{ and } \langle \Psi(\bar x,\bar y,\bar\alpha),\gamma\rangle = 0,\\ &\xi_i = 0 \qquad \text{if } \bar y_i>0 \text{ and } F_i(\bar x,\bar y,\bar\alpha)=0,\\ &\eta_i = 0 \qquad \text{if } \bar y_i=0 \text{ and } F_i(\bar x,\bar y,\bar\alpha)>0,\\ &\ \text{either } \xi_i<0,\eta_i<0 \text{ or } \xi_i\eta_i=0 \qquad \text{if } \bar y_i = 0 \text{ and } F_i(\bar x,\bar y)=0\end{aligned}$$

implies that $\gamma=0$, $\beta=0$, $\eta=0$. Then the value function V is lower semicontinuous near $\bar\alpha$, and

$$\begin{aligned}\partial V(\bar\alpha)\subseteq \bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)}&\{\nabla_\alpha f(\bar x,\bar y,\bar\alpha)+\nabla_\alpha\Psi(\bar x,\bar y,\bar\alpha)^\top\gamma+\nabla_\alpha H(\bar x,\bar y,\bar\alpha)^\top\beta\\ &+\nabla_\alpha F(\bar x,\bar y,\bar\alpha)^\top\eta : (\gamma,\beta,\eta)\in M^1_{CD}(\bar x,\bar y)\},\end{aligned}\tag{0.2}$$

$$\begin{aligned}\partial^\infty V(\bar\alpha)\subseteq \bigcup_{(\bar x,\bar y)\in\Sigma(\bar\alpha)}&\{\nabla_\alpha\Psi(\bar x,\bar y,\bar\alpha)^\top\gamma+\nabla_\alpha H(\bar x,\bar y,\bar\alpha)^\top\beta\\ &+\nabla_\alpha F(\bar x,\bar y,\bar\alpha)^\top\eta : (\gamma,\beta,\eta)\in M^0_{CD}(\bar x,\bar y)\}.\end{aligned}\tag{0.3}$$

If the set in the right-hand side of inclusion (0.3) *contains only the zero vector, then the value function V is Lipschitz near $\bar\alpha$. If the set in the right-hand side of inclusion* (0.3) *contains only the zero vector and the set in the right-hand side of inclusion* (0.2) *is a singleton, then the value function is strictly differentiable at $\bar\alpha$.*

THEOREM 4.8. *Assume that there exists $\delta>0$ such that the set*

$$\begin{aligned}\{(x,y)\in C :&(p,q,q^m)\in B(0;\delta), \Psi(x,y,\bar\alpha)\le p, H(x,y,\bar\alpha)=q,\\ &\min\{y_i, F_i(x,y,\bar\alpha)\} = q_i^m, i=1,\dots,m, f(x,y,\bar\alpha)\le M\}\end{aligned}$$

is bounded for each M. Assume also that

$$\begin{aligned}&0\in \nabla\Psi(\bar x,\bar y,\bar\alpha)^\top\gamma+\nabla H(\bar x,\bar y,\bar\alpha)^\top\beta+\nabla F(\bar x,\bar y,\bar\alpha)^\top\eta+(0,\xi,0)+N_C(\bar x,\bar y)\times\{0\},\\ &\gamma\ge 0, \langle\Psi,\gamma\rangle(\bar x,\bar y,\bar\alpha)=0,\end{aligned}$$

where

$$\begin{aligned}&\eta_i = 0 \quad \forall i\in I_+,\\ &\xi_i=0 \quad \forall i\in L,\\ &\eta_i = r_i(1-\bar t_i), \xi_i = r_i\bar t_i \text{ for some } \bar t_i\in[0,1], \quad \forall i\in I_0\end{aligned}$$

implies that $\gamma = 0$, $\beta = 0$, $\eta = 0$, $r_i = 0$, $i = 1, \ldots, m$. *Then the value function* V *is lower semicontinuous near* $\bar{\alpha}$, *and*

$$\partial V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{\nabla_\alpha f(\bar{x},\bar{y},\bar{\alpha}) + \nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta$$
$$+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in M_C^1(\bar{x},\bar{y})\}, \tag{0.4}$$
$$\partial^\infty V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{\nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta$$
$$+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in M_C^0(\bar{x},\bar{y})\}. \tag{0.5}$$

If the set in the right-hand side of inclusion (0.5) *contains only the zero vector, then the value function* V *is Lipschitz near* $\bar{\alpha}$. *If the set in the right-hand side of inclusion* (0.5) *contains only the zero vector and the set in the right-hand side of inclusion* (0.4) *is a singleton, then the value function is strictly differentiable at* $\bar{\alpha}$.

THEOREM 4.11. *Assume that there exists* $\delta > 0$ *such that, for* $(\bar{x},\bar{y}) \in \Sigma(\bar{\alpha})$ *and each index set* $\sigma \subseteq I_0(\bar{x},\bar{y})$, *the set in Proposition* 4.10 *is bounded for each* M *and*

$$\begin{cases} 0 = \nabla\Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla H(\bar{x},\bar{y},\bar{\alpha})^\top \beta + \nabla F(\bar{x},\bar{y},\bar{\alpha})^\top \eta + (0,\xi,0) + N_C(\bar{x},\bar{y}) \times \{0\}, \\ \gamma_{J(\Psi)} = 0, \eta_{I_+} = 0, \xi_L = 0, \xi_\sigma \leq 0, \eta_{I_0\setminus\sigma} \leq 0, \end{cases}$$

implies that $\gamma = 0$, $\beta = 0$, $\eta = 0$. *Then the value function* V *is lower semicontinuous near* $\bar{\alpha}$, *and*

$$\partial V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{\nabla_\alpha f(\bar{x},\bar{y},\bar{\alpha}) + \nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta$$
$$+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in \cup_{\sigma\subseteq I_0} M_\sigma^1(\bar{x},\bar{y})\}, \tag{0.6}$$
$$\partial^\infty V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{\nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta$$
$$+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in \cup_{\sigma\subseteq I_0} M_\sigma^0(\bar{x},\bar{y})\}. \tag{0.7}$$

If the set in the right-hand side of inclusion (0.7) *contains only the zero vector, then the value function* V *is Lipschitz near* $\bar{\alpha}$. *If the set in the right-hand side of inclusion* (0.7) *contains only the zero vector and the set in the right-hand side of inclusion* (0.6) *is a singleton, then the value function is strictly differentiable at* $\bar{\alpha}$.

THEOREM 4.13. *In addition to the assumptions of Theorem* 4.11, *assume that* $C = R^n \times R^a \times R^b$ *and, for all* $(\bar{x},\bar{z},\bar{u}) \in \Sigma(\bar{\alpha})$, *the partial MPEC linear independence constraint qualification is satisfied; i.e.,*

$$\begin{cases} 0 = \nabla_{x,y}\Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_{x,y} H(\bar{x},\bar{y},\bar{\alpha})^\top \beta + \nabla_{x,y} F(\bar{x},\bar{y},\bar{\alpha})^\top \eta + (0,\xi), \\ \gamma_{J(\Psi)} = 0, \eta_{I_+} = 0, \xi_L = 0, \end{cases}$$

implies that $\eta_{I_0} = 0, \xi_{I_0} = 0$, *where* $J(\Psi) := \{i : \Psi_i(\bar{x},\bar{y},\bar{\alpha}) < 0\}$. *Further assume that*

$$\begin{cases} 0 = \nabla\Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla H(\bar{x},\bar{y},\bar{\alpha})^\top \beta + \nabla F(\bar{x},\bar{y},\bar{\alpha})^\top \eta + (0,\xi,0), \\ \gamma_{J(\Psi)} = 0, \eta_{I_+} = 0, \xi_L = 0, \eta_{I_0} \leq 0, \xi_{I_0} \leq 0, \end{cases}$$

implies that $\gamma = 0$, $\beta = 0$, $\eta = 0$. *Then the value function* V *is lower semicontinuous near* $\bar{\alpha}$, *and*

$$\partial V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{\nabla_\alpha f(\bar{x},\bar{y},\bar{\alpha}) + \nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta$$
$$+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in M_S^1(\bar{x},\bar{y})\},$$
$$\partial^\infty V(\bar{\alpha}) \subseteq \bigcup_{(\bar{x},\bar{y})\in\Sigma(\bar{\alpha})} \{\nabla_\alpha \Psi(\bar{x},\bar{y},\bar{\alpha})^\top \gamma + \nabla_\alpha H(\bar{x},\bar{y},\bar{\alpha})^\top \beta$$
$$+\nabla_\alpha F(\bar{x},\bar{y},\bar{\alpha})^\top \eta : (\gamma,\beta,\eta) \in M_S^0(\bar{x},\bar{y})\}.$$

Note that the additional assumption (0.1) and its corresponding assumptions in Theorems 4.4., 4.8, 4.11, and 4.13 are automatically satisfied in the case in which the perturbation is additive. In the case of nonadditive perturbations, they are needed even in the case of nonlinear programming, i.e., when $\Omega = R^m$ in Theorem 3.6.

The main error occurs in the proof of Theorem 3.6 when we applied [1, Proposition 2.6] to obtain the partial subdifferentials from the subdifferentials of the fully perturbed value function. The positions of vectors ζ and 0 were switched by mistake. Instead of proving that $(\zeta, 0) \in \partial^\infty \widetilde{V}(0,\bar{\alpha})$ implies $\zeta = 0$, we proved that $(0,\zeta) \in \partial^\infty \widetilde{V}(0,\bar{\alpha})$ implies $\zeta = 0$. Hence, on page 709 in lines 13–18, "For any $(0,0,0,\zeta) \in \partial^\infty \widetilde{V}(0,0,0,\bar{\alpha})$, we have $(0,0,0,\zeta) \in -M^0(\bar{x},\bar{y},\bar{\alpha})$ for some point $(\bar{x},\bar{y},\bar{\alpha}) \in \Sigma(0,0,0,\bar{\alpha})$. Therefore,

$$(0,0,\zeta) \in N_C(\bar{x},\bar{y}) \times \{0\},$$

which implies that $\zeta = 0$" should be changed to "For any $(-\gamma,-\beta,-\eta,0) \in \partial^\infty \widetilde{V}(0,0,0,\bar{\alpha})$, we have $(-\gamma,-\beta,-\eta,0) \in -M^0(\bar{x},\bar{y},\bar{\alpha})$ for some point $(\bar{x},\bar{y},\bar{\alpha}) \in \Sigma(0,0,0,\bar{\alpha})$. Hence $(\gamma,\beta,\eta,0) \in M^0(\bar{x},\bar{y},\bar{\alpha})$, which implies $\gamma = 0$, $\beta = 0$, $\eta = 0$ by assumption (0.1)."

Consider the nonlinear programming formulation of (OPCC) in [1, section 4.1]. Assumption (0.1) amounts to the nonexistence of a nonzero vector $(\gamma,\beta,r^F,r^y,\mu)$ such that

$$\begin{aligned}
&0 \in \nabla\Psi(\bar{x},\bar{y},\bar{\alpha})^\top\gamma + \nabla H(\bar{x},\bar{y},\bar{\alpha})^\top\beta \\
&-\nabla F(\bar{x},\bar{y},\bar{\alpha})^\top r^F - \{(0,r^y,0)\} + \mu\nabla\langle y,F\rangle(\bar{x},\bar{y},\bar{\alpha}) + N_C(\bar{x},\bar{y}) \times \{(0)\}, \\
&\gamma \geq 0, \langle\gamma,\Psi(\bar{x},\bar{y},\bar{\alpha})\rangle = 0, \\
&r^F \geq 0, r^y \geq 0, \langle r^F, F(\bar{x},\bar{y},\bar{\alpha})\rangle = 0, \langle r^y,\bar{y}\rangle = 0.
\end{aligned}$$

However, using [1, Proposition 4.16] with x replaced by (x,α), the above assumption will never be satisfied, and hence [1, Theorem 4.2] does not hold. Consider the following example, which is the example in [1] with the extra constraint $(x,y) \in [-1,1] \times [-1,1]$:

$$\begin{aligned}
\text{minimize}\quad & -y \\
\text{subject to}\quad & x - y = 0, \\
& x \geq 0, y \geq 0, xy = 0, (x,y) \in [-1,1] \times [-1,1].
\end{aligned}$$

Note that the growth hypothesis holds since the set $[-1,1] \times [-1,1]$ is compact. The normal multiplier set $M^1_{NLP}(0,0) = \emptyset$. So [1, Theorem 4.2] is not true for this example.

That is, the nonlinear programming multipliers may not be useful in the sensitivity analysis.

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REFERENCE

[1] Y. Lucet and J. J. Ye, *Sensitivity analysis of the value function for optimization problems with variational inequality constraints*, SIAM J. Control Optim., 40 (2001), pp. 699–723.