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OPTIMALITY CONDITIONS FOR OPTIMIZATION PROBLEMS WITH COMPLEMENTARITY CONSTRAINTS*

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Abstract. Optimization problems with complementarity constraints are closely related to optimization problems with variational inequality constraints and bilevel programming problems. In this paper, under mild constraint qualifications, we derive some necessary and sufficient optimality conditions involving the proximal coderivatives. As an illustration of applications, the result is applied to the bilevel programming problems where the lower level is a parametric linear quadratic problem.

Key words. optimization problems, complementarity constraints, optimality conditions, bilevel programming problems, proximal normal cones

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1. Introduction.

The main purpose of this paper is to derive necessary and sufficient optimality conditions for the *optimization problem with complementarity constraints* (OPCC) defined as follows:

$$\begin{array}{lll} \text{(OPCC)} & \min & f(x,y,u) \\ & \text{s.t.} & \langle u, \psi(x,y,u)\rangle = 0, \quad u \geq 0, \quad \psi(x,y,u) \leq 0 \\ & & L(x,y,u) = 0, \quad g(x,y,u) \leq 0, \quad (x,y,u) \in \Omega, \end{array} \tag{1.1}$$

where $f : R^{n+m+q} \to R$, $\psi : R^{n+m+q} \to R^q$, $L : R^{n+m+q} \to R^l$, $g : R^{n+m+q} \to R^d$, and Ω is a nonempty subset of R^{n+m+q}.

(OPCC) is an optimization problem with equality and inequality constraints. However, due to the complementarity constraint (1.1), the Karush–Kuhn–Tucker (KKT) necessary optimality condition is rarely satisfied by (OPCC) since it can be shown as in [9, Proposition 1.1] that there always exists a nontrivial abnormal multiplier. This is equivalent to saying that the usual constraint qualification conditions, such as the Mangasarian–Fromovitz condition, will never be satisfied (see [8, Proposition 3.1]). The purpose of this paper is to derive necessary and sufficient optimality conditions under mild constraint qualifications that are satisfied by a large class of OPCCs.

To motivate our main results, we formulate problem (OPCC), where $\Omega = R^{n+m+q}$, as the following optimization problem with a generalized equation constraint:

$$\begin{array}{lll} \text{(GP)} & \min & f(x,y,u) \\ & \text{s.t.} & 0 \in -\psi(x,y,u) + N(u, R^q_+), \\ & & L(x,y,u) = 0, \quad g(x,y,u) \leq 0, \end{array} \tag{1.2}$$

where

$$N(u,C) := \begin{cases} \text{the normal cone of } C \text{ at } y & \text{if } u \in C, \\ \emptyset & \text{if } u \notin C \end{cases}$$

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is the normal cone operator in the sense of convex analysis.

Let $(\bar{x}, \bar{y}, \bar{u})$ be a solution of (OPCC), where $\Omega = R^{n+m+q}$. If $N(u, R^q_+)$ were single-valued and smooth, then the generalized equation constraint (1.2) would reduce to an ordinary equation. Using the KKT condition, we could deduce that if a constraint qualification is satisfied for (GP) and the problem data are smooth, then there exist KKT multipliers $\xi \in R^l$, $\zeta \in R^d$, $\eta \in R^q$ such that

$$\begin{aligned} 0 &= \nabla f(\bar{x}, \bar{y}, \bar{u}) + \nabla L(\bar{x}, \bar{y}, \bar{u})^\top \xi + \nabla g(\bar{x}, \bar{y}, \bar{u})^\top \zeta \\ &\quad - \nabla \psi(\bar{x}, \bar{y}, \bar{u})^\top \eta + \{0\} \times \{0\} \times \nabla N_{R^q_+}(\bar{u})^\top \eta, \\ 0 &= \langle \zeta, g(\bar{x}, \bar{y}, \bar{u}) \rangle, \ \zeta \geq 0, \end{aligned}$$

where ∇ denotes the usual gradient, $M^\top$ denotes the transpose of the matrix M, and N_C denotes the map $y \to N(y, C)$. However, $u \rightrightarrows N(u, R^q_+)$ is in general a set-valued map. Naturally, we hope to replace $\nabla N_{R^q_+}(\bar{u})^\top \eta$ by the image of some derivatives of the set-valued map $u \rightrightarrows N(u, R^q_+)$ acting on the vector η. The natural candidate for such a derivative of set-valued maps is the Mordukhovich coderivative (see Definition 2.3) since the Mordukhovich coderivatives have a good calculus, and in the case when the set-valued map is single-valued and smooth, the image of the Mordukhovich coderivative acting on a vector coincides with the usual gradient operator acting on the vector (see [6, Proposition 2.4]). Indeed, as in [7], we can show that if $(\bar{x}, \bar{y}, \bar{u})$ is an optimal solution of (OPCC) and a constraint qualification holds, then there exist $\xi \in R^l$, $\zeta \in R^d$, $\eta \in R^q$ such that

$$\begin{aligned} 0 &\in \nabla f(\bar{x}, \bar{y}, \bar{u}) + \nabla L(\bar{x}, \bar{y}, \bar{u})^\top \xi + \nabla g(\bar{x}, \bar{y}, \bar{u})^\top \zeta \\ &\quad - \nabla \psi(\bar{x}, \bar{y}, \bar{u})^\top \eta + \{0\} \times \{0\} \times D^* N_{R^q_+}(\bar{u}, \psi(\bar{x}, \bar{y}, \bar{u}))(\eta), \\ 0 &= \langle \zeta, g(\bar{x}, \bar{y}, \bar{u}) \rangle, \qquad \zeta \geq 0, \end{aligned}$$

where D^* denotes the Mordukhovich coderivative (see Definition 2.3). Recall from [7, Definition 2.8] that a set-valued map $\Phi : R^n \rightrightarrows R^q$ with a closed graph is said to be pseudo-upper-Lipschitz continuous at $(\bar{z}, \bar{v})$ with $\bar{v} \in \Phi(\bar{z})$ if there exist a neighborhood U of $\bar{z}$, a neighborhood V of $\bar{v}$, and a constant $\mu > 0$ such that

$$\Phi(z) \cap V \subset \Phi(\bar{z}) + \mu \|z - \bar{z}\| B \quad \forall z \in U.$$

The constraint qualification for the above necessary condition involving the Mordukhovich coderivative turns out to be the pseudo-upper-Lipschitz continuity of the set-valued map

$$\Sigma(v_1, v_2, v_3) := \{(x, y, u) : v_1 \in -\psi(x, y, u) + N(u, R^q_+), L(x, y, u) = v_2, g(x, y, u) + v_3 \leq 0\}$$

at $(\bar{x}, \bar{y}, \bar{u}, 0)$. This constraint qualification is very mild since the pseudo-upper-Lipschitz continuity is weaker than both the upper-Lipschitz continuity and the pseudo-Lipschitz continuity (the so-called Aubin property). However, the Mordukhovich normal cone involved in the necessary condition may be too large sometimes. For example, in [7, Example 4.1], both $(0, 0)$ and $(1, 1)$ satisfy the above necessary conditions, but only $(1, 1)$ is the unique optimal solution. Can one replace the Mordukhovich normal cone involved in the necessary condition by the potentially smaller proximal normal cone? The answer is negative in general, since the proximal coderivative as defined in Definition 2.3 usually has only a "fuzzy" calculus. Consider the following optimization problem:

$$\begin{array}{ll} \min & -y \\ \text{s.t.} & y-u=0, \quad yu=0, \quad y\geq 0, \quad u\geq 0. \end{array}$$

The unique optimal solution $(0,0)$ does not satisfy the KKT condition but satisfies the necessary condition involving the Mordukhovich coderivatives. It does not satisfy the necessary condition with the Mordukhovich normal cone replaced by the proximal normal cone. This example shows that some extra assumptions are needed for the necessary condition involving the proximal coderivatives to hold. In this paper such a condition is found. Moreover, we show that the proximal normal cone involved in the necessary condition can be represented by a system of linear and nonlinear equations, and the necessary optimality conditions involving the proximal coderivatives turn out to be sufficient under some convexity assumptions on the problem data.

Although the optimization problems with complementarity constraints are a class of optimization problems with independent interest, the incentive to study (OPCC) mainly comes from the following optimization problem with variational inequality constraints (OPVIC), where the constraint region of the variational inequality is a system of inequalities:

$$\begin{array}{lll} \text{(OPVIC)} & \min & f(x,y) \\ & \text{s.t.} & y\in S(x), \quad g(x,y)\leq 0, \quad (x,y)\in\Omega, \end{array}$$

where $f: R^{n+m}\to R$, Ω is a nonempty subset of R^{m+n} and $S(x)$ is the *solution set of a variational inequality with parameter* x; i.e.,

$$S(x)=\{y\in R^m: \psi(x,y)\leq 0 \text{ and } \langle F(x,y), z-y\rangle \geq 0 \ \ \forall z \text{ s.t. } \psi(x,z)\leq 0\},$$

where $F: R^{n+m}\to R^m$ and $\psi: R^{n+m}\to R^q$. The recent monograph [4] by Luo, Pang, and Ralph has an extensive study for (OPVIC). The reader may find the references for the various optimality conditions for (OPVIC) from [4].

(OPCC) is closely related to OPVICs and bilevel programming problems. Indeed, if ψ is $C^1$and quasi convex in y and a certain constraint qualification condition holds at $\bar{y}$ for the optimization problem

$$\min \quad \langle F(\bar{x},\bar{y}), z\rangle \quad \text{ s.t. } \psi(\bar{x},z)\leq 0,$$

then by the KKT necessary and sufficient optimality condition, $(\bar{x},\bar{y})$ is a solution of (OPVIC) if and only if there exists $\bar{u}\in R^q$ such that $(\bar{x},\bar{y},\bar{u})$ is a solution of the following optimization problem:

$$\begin{array}{lll} \text{(KS)} & \min & f(x,y) \\ & \text{s.t.} & \langle u, \psi(x,y)\rangle = 0, \quad u\geq 0, \quad \psi(x,y)\leq 0, \\ & & F(x,y)+\nabla_y\psi(x,y)^\top u=0, \\ & & g(x,y)\leq 0, \qquad (x,y)\in\Omega, \end{array}$$

which is a special case of (OPCC).

In the case where $F(x,y)=\nabla_y h(x,y)$, where $h: R^{n+m}\to R$ is differentiable and pseudoconvex in y, (KS) is equivalent to the following *bilevel programming problem* (BLPP), or so-called Stackelberg game:

$$\begin{array}{lll} \text{(BLPP)} & \min & f(x,y) \\ & \text{s.t.} & y\in S(x), \quad g(x,y)\leq 0, \quad (x,y)\in\Omega, \end{array}$$

where $S(x)$ is the set of solutions of the problem (P_x):

$$(P_x) \qquad \text{minimize} \qquad h(x,y) \qquad \text{s.t. } \psi(x,y) \le 0.$$

We organize the paper as follows. Section 2 contains background material on nonsmooth analysis and preliminary results. In section 3 we derive the necessary and sufficient optimality conditions for (OPCC). As an illustration of applications, we also apply the result to (BLPP), where the lower level is a linear quadratic programming problem.

2. Preliminaries. This section contains some background material on nonsmooth analysis and preliminary results which will be used later. We give only concise definitions that will be needed in the paper. For more detailed information on the subject, our references are Clarke [1, 2], Loewen [3], and Mordukhovich [6].

First we give some concepts for various normal cones and subgradients.

DEFINITION 2.1. *Let Ω be a nonempty subset of R^n. Given $\bar{z} \in cl\Omega$, the closure of set Ω, the convex cone*

$$N^\pi(\bar{z},\Omega) := \{\xi \in R^n : \exists M > 0 \text{ s.t. } \langle \xi, z-\bar{z}\rangle \le M\|z-\bar{z}\|^2 \ \forall z \in \Omega\}$$

is called the proximal normal cone to set Ω at point $\bar{z}$, and the closed cone

$$\hat{N}(\bar{z},\Omega) := \{\lim_{i\to\infty} \xi_i : \xi_i \in N^\pi(z_i,\Omega), z_i \to \bar{z}\}$$

is called the limiting normal cone to Ω at point $\bar{z}$.

REMARK 2.1. *It is known that if Ω is convex, then the proximal normal cone and the limiting normal cones coincide with the normal cone in the sense of convex analysis.*

DEFINITION 2.2. *Let $f : R^n \to R \cup \{+\infty\}$ be lower semicontinuous and finite at $\bar{z} \in R^n$. The limiting subgradient of f at $\bar{z}$ is defined to be the set*

$$\hat{\partial} f(\bar{z}) := \{\zeta : (\zeta,-1) \in \hat{N}(\bar{z}, epi\ f)\},$$

where $epi\ f := \{(z,v) : v \ge f(z)\}$ denotes the epigragh of f.

REMARK 2.2. *It is known that if f is a convex function, the limiting subgradient coincides with the subgradient in the sense of convex analysis. For a locally Lipschitz function f, $\partial f = co\hat{\partial} f(x)$, where ∂ denotes the Clarke generalized gradient and co denotes the convex hull. Hence the limiting subgradient is in general a smaller set than the Clarke generalized gradient.*

For set-valued maps, the definition for limiting normal cone leads to the definition of coderivative of a set-valued map introduced by Mordukhovich (see, e.g., [6]).

DEFINITION 2.3. *Let $\Phi : R^n \rightrightarrows R^q$ be an arbitrary set-valued map (assigning to each $z \in R^n$ a set $\Phi(z) \subseteq R^q$ which may be empty) and $(\bar{z},\bar{v}) \in cl\ Gr\Phi$, where $Gr\Phi$ denotes the graph of Φ; i.e., $(z,v) \in Gr\Phi$ if and only if $v \in \Phi(z)$. The set-valued maps from R^q into R^n defined by*

$$D^*_\pi\Phi(\bar{z},\bar{v})(\eta) = \{\zeta \in R^n : (\zeta,-\eta) \in N^\pi((\bar{z},\bar{v}), Gr\Phi)\},$$
$$D^*\Phi(\bar{z},\bar{v})(\eta) = \{\zeta \in R^n : (\zeta,-\eta) \in \hat{N}((\bar{z},\bar{v}), Gr\Phi)\}$$

are called the proximal and Mordukhovich coderivatives of Φ at point $(\bar{z},\bar{v})$, respectively.

PROPOSITION 2.4. *Suppose B is closed, $\bar{x} \in A$, $\bar{x} \notin B$. Then*

$$N^\pi(\bar{x}, A \cup B) = N^\pi(\bar{x}, A).$$

Proof. Since $\bar{x} \notin B$ and B is closed, there exists a neighborhood of $\bar{x}$ that is not contained in B. Therefore, from the definition of the proximal normal cone, we have

$$N^\pi(\bar{x}, A \cup B) = N^\pi(\bar{x}, A). \qquad \square$$

In the following proposition we show that the proximal normal cone of a union of a finite number of sets is the intersection of the proximal cones.

PROPOSITION 2.5. *Let $\Omega = \cup_{i=1}^m \Omega_i$ and $\bar{x} \in \cap_{i=1}^m \Omega_i$. Suppose Ω_i $\forall$ $i = 1, 2, \ldots, m$ are closed. Then*

$$N^\pi(\bar{x}, \Omega) = \cap_{i=1}^m N^\pi(\bar{x}, \Omega_i).$$

Proof. Let $\zeta \in N^\pi(\bar{x}, \Omega)$. Then, by definition, there exists a constant $M > 0$ such that

$$\langle \zeta, x - \bar{x} \rangle \leq M \left\| x - \bar{x} \right\|^2 \qquad \forall x \in \Omega = \cup_{i=1}^m \Omega_i.$$

Since $\bar{x} \in \cap_{i=1}^m \Omega_i$, the above inequality implies that $\zeta \in \cap_{i=1}^m N^\pi(\bar{x}, \Omega_i)$.

Conversely, suppose $\zeta \in \cap_{i=1}^m N^\pi(\bar{x}, \Omega_i)$. Then for all $i = 1, 2, \ldots, m$, there exists $M_i > 0$ such that

$$\langle \zeta, x - \bar{x} \rangle \leq M_i \left\| x - \bar{x} \right\|^2 \qquad \forall x \in \Omega_i.$$

That is, there exists $M = \max_{i \in \{1,2,\ldots,m\}} M_i > 0$ such that

$$\langle \zeta, x - \bar{x} \rangle \leq M \left\| x - \bar{x} \right\|^2 \qquad \forall x \in \Omega = \cup_{i=1}^m \Omega_i,$$

which implies that $\zeta \in N^\pi(\bar{x}, \Omega)$. $\square$

The above decomposition formula for calculating the proximal normal cones turns out to be very useful, since when a set can be written as a union of some convex sets, the task of calculating the proximal normal cones is reduced to calculating the normal cone to convex sets which are easier to calculate. The following proposition is a nice application of the decomposition formula and will be used to calculate the proximal normal cone to the graph of the set-valued map $N_{R_+^q}$ for general q in Proposition 2.7.

PROPOSITION 2.6.

$$N^\pi((\bar{x}, \bar{y}), GrN_{R_+}) = \begin{cases} \{0\} \times R & \textit{if } \bar{x} > 0, \bar{y} = 0, \\ R \times \{0\} & \textit{if } \bar{x} = 0, \bar{y} < 0, \\ (-\infty, 0] \times [0, \infty) & \textit{if } \bar{x} = \bar{y} = 0. \end{cases}$$

Proof. It is easy to see that $\mathrm{Gr}N_{R_+} = \Omega_1 \cup \Omega_2$, where $\Omega_1 = [0, \infty) \times \{0\}$ and $\Omega_2 = \{0\} \times (-\infty, 0]$.

We discuss the following three cases.

Case 1. $\bar{x} > 0$, $\bar{y} = 0$.

In this case, $(\bar{x}, \bar{y}) \in \Omega_1$ and $(\bar{x}, \bar{y}) \notin \Omega_2$. Since Ω_2 is closed, by Proposition 2.4 we have in this case

$$N^\pi((\bar{x}, \bar{y}), GrN_{R_+}) = N((\bar{x}, \bar{y}), \Omega_1) = \{0\} \times R.$$

Case 2. $\bar{x} = 0$, $\bar{y} < 0$.

In this case, $(\bar{x}, \bar{y}) \in \Omega_2$ and $(\bar{x}, \bar{y}) \notin \Omega_1$. Since Ω_1 is closed, by Proposition 2.4 we have in this case

$$N^\pi((\bar{x}, \bar{y}), GrN_{R_+}) = N((\bar{x}, \bar{y}), \Omega_2) = R \times \{0\}.$$

Case 3. $\bar{x} = \bar{y} = 0$.

In this case, $(\bar{x}, \bar{y}) \in \Omega_1 \cap \Omega_2$. By Proposition 2.5 we have

$$\begin{aligned} N^\pi((\bar{x}, \bar{y}), GrN_{R_+}) &= N((\bar{x}, \bar{y}), \Omega_1) \cap N((\bar{x}, \bar{y}), \Omega_2) \\ &= ((-\infty, 0] \times R) \cap (R \times [0, \infty)) \\ &= (-\infty, 0] \times [0, \infty). \qquad \square \end{aligned}$$

Now we are in a position to give an expression for the proximal normal cone to the graph of the set-valued map $N_{R^q_+}$ for general q.

PROPOSITION 2.7. *For any* $(\bar{x}, \bar{y}) \in GrN_{R^q_+}$, *define*

$$\begin{aligned} L := L(\bar{x}) &:= \{i \in \{1, 2, \ldots, q\} : \bar{x}_i > 0\}, \\ I_+ := I_+(\bar{x}, \bar{y}) &:= \{i \in \{1, 2, \ldots, q\} : \bar{x}_i = 0, \bar{y}_i < 0\}, \\ I_0 := I_0(\bar{x}, \bar{y}) &:= \{i \in \{1, 2, \ldots, q\} : \bar{x}_i = 0, \bar{y}_i = 0\}. \end{aligned}$$

Then

$$N^\pi((\bar{x}, \bar{y}), GrN_{R^q_+}) = \{(\gamma, -\eta) \in R^{2q} : \eta_{I_0} \leq 0, \eta_{I_+} = 0, \gamma_L = 0, \gamma_{I_0} \leq 0\}.$$

Proof. Since

$$\begin{aligned} GrN_{R^q_+} &= \left\{(x, y) \in R^{2q} : y \in N(x, R^q_+)\right\} \\ &= \left\{(x, y) \in R^{2q} : y \in N(x_1, R_+) \times N(x_2, R_+) \times \cdots \times N(x_q, R_+)\right\} \\ &= \left\{(x, y) \in R^{2q} : (x_i, y_i) \in GrN_{R_+} \forall i = 1, 2, \ldots, q\right\}, \end{aligned}$$

we have

$$(x, y) \in \mathrm{Gr}N_{R^q_+} \quad \text{if and only if} \quad (x_1, y_1, x_2, y_2, \ldots, x_q, y_q) \in \prod_{i=1}^q \mathrm{Gr}N_{R_+}.$$

Hence from the definition, it is clear that

$$(\gamma, -\eta) \in N^\pi((\bar{x}, \bar{y}), GrN_{R^q_+})$$

if and only if

$$\begin{aligned} (\gamma_1, -\eta_1, \gamma_2, -\eta_2, \ldots, \gamma_q, -\eta_q) &\in N^\pi\left((\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2, \ldots, \bar{x}_q, \bar{y}_q), \prod_{i=1}^q GrN_{R_+}\right) \\ &= \prod_{i=1}^q N^\pi((\bar{x}_i, \bar{y}_i), \mathrm{Gr}N_{R_+}). \end{aligned}$$

The rest of the proof follows from Proposition 2.6. $\square$

It turns out that we can express any element of $N^\pi((\bar{x}, \bar{y}), \mathrm{Gr}N_{R^q_+})$ by a system of nonlinear equations as in the following proposition.

PROPOSITION 2.8.

$$(\gamma, -\eta) \in N^\pi((\bar{x}, \bar{y}), GrN_{R^q_+})$$

if and only if there exist $\alpha, \beta \in R^{2q}_+$ *such that*

$$0 = \sum_{i=1}^{q} \bar{x}_i(\alpha_i + \beta_i) - \sum_{i=1}^{q} \bar{y}_i(\alpha_{q+i} + \beta_{q+i}), \tag{2.1}$$

$$\gamma_i = -\alpha_i + \bar{y}_i\beta_i \quad \forall i = 1, 2, \ldots, q, \tag{2.2}$$

$$\eta_i = -\alpha_{q+i} + \bar{x}_i\beta_{q+i} \quad \forall i = 1, 2, \ldots, q. \tag{2.3}$$

Proof. By Proposition 2.7, $(\gamma, -\eta) \in N^\pi((\bar{x}, \bar{y}), GrN_{R^q_+})$ if and only if

$$\eta_{I_0} \leq 0, \quad \eta_{I_+} = 0, \quad \gamma_L = 0, \quad \gamma_{I_0} \leq 0.$$

By the definition for the index sets I_0, I_+, L in Proposition 2.7, we have

$$\begin{aligned} \eta_{I_0} \leq 0, \gamma_{I_0} \leq 0 \quad & \text{if and only if } \bar{x}_i = \bar{y}_i = 0 \Longrightarrow \eta_i \leq 0, \quad \gamma_i \leq 0, \\ \eta_{I_+} = 0 \quad & \text{if and only if } \bar{y}_i < 0 \Longrightarrow \eta_i = 0, \\ \gamma_L = 0 \quad & \text{if and only if } \bar{x}_i > 0 \Longrightarrow \gamma_i = 0. \end{aligned}$$

Since for any $(\bar{x}, \bar{y}) \in GrN_{R^q_+}$, $\bar{x} \geq 0, \bar{y} \leq 0$, for nonnegative vectors α and β, (2.1) is equivalent to

$$\bar{x}_i(\alpha_i + \beta_i) = 0, \quad \bar{y}_i(\alpha_{q+i} + \beta_{q+i}) = 0 \quad \forall i = 1, \ldots, q.$$

Hence the existence of nonnegative vectors α and β satisfying (2.1)–(2.2) is equivalent to the following condition:

$$\begin{aligned} & \bar{x}_i = \bar{y}_i = 0 \Longrightarrow \eta_i \leq 0, \qquad \gamma_i \leq 0, \\ & \bar{y}_i < 0 \Longrightarrow \eta_i = 0, \\ & \bar{x}_i > 0 \Longrightarrow \gamma_i = 0. \end{aligned}$$

Consequently, it is equivalent to

$$\eta_{I_0} \leq 0, \quad \eta_{I_+} = 0, \quad \gamma_L = 0, \quad \gamma_{I_0} \leq 0.$$

The proof of the proposition is therefore complete. □

Finally, we would like to recall the following definition of a very mild constraint qualification called "calmness," introduced by Clarke [1].

DEFINITION 2.9. *Let* $\bar{x}$ *be a local solution to the following mathematical programming problem:*

$$\begin{aligned} \textit{minimize} \quad & f(x) \\ \textit{s.t.} \quad & g(x) \leq 0, \\ & h(x) = 0, \\ & x \in C, \end{aligned}$$

where $f : R^d \to R^n$, $g : R^d \to R^m$, *and* C *is a closed subset of* R^d. *The above mathematical programming problem is said to be calm at* $\bar{x}$ *provided that there exist positive* ϵ *and* M *such that for all* $(p,q) \in \epsilon B$, *for all* x *in* $\bar{x} + \epsilon B$ *satisfying* $g(x) + p \leq 0, h(x) + q = 0, x \in C$, *one has*

$$f(x) - f(\bar{x}) + M\|(p,q)\| \geq 0,$$

where B *is the open unit ball in the appropriate space.*

It is well known that the calmness condition is a constraint qualification for the existence of a KKT multiplier and the sufficient conditions for the calmness condition include the linear independence condition, the Slater condition, and the Mangasarian–Fromowitz condition. Moreover, the calmness condition is satisfied automatically in the case where the feasible region is a polyhedron.

3. Optimality conditions for OPCC. Let $(\bar{x}, \bar{y}, \bar{u}) \in \Omega$ and $g(\bar{x}, \bar{y}, \bar{u}) \leq 0$, $L(\bar{x}, \bar{y}, \bar{u}) = 0$. Let

$$\begin{aligned} L(\bar{u}) &:= \{1 \leq i \leq q : \bar{u}_i > 0\}, \\ I_+(\bar{x}, \bar{y}, \bar{u}) &:= \{1 \leq i \leq q : \bar{u}_i = 0, \psi_i(\bar{x}, \bar{y}, \bar{u}) < 0\}, \\ I_0(\bar{x}, \bar{y}, \bar{u}) &:= \{1 \leq i \leq q : \bar{u}_i = 0, \psi_i(\bar{x}, \bar{y}, \bar{u}) = 0\}. \end{aligned}$$

Where there is no confusion, we simply use L, I_+, I_0 instead of $L(\bar{u}), I_+(\bar{x}, \bar{y}, \bar{u})$, $I_0(\bar{x}, \bar{y}, \bar{u})$, respectively. It is clear that $\{1, 2, \ldots, q\} = L(\bar{u}) \cup I_+(\bar{x}, \bar{y}, \bar{u}) \cup I_0(\bar{x}, \bar{y}, \bar{u})$. Let

$$F = \left\{ (x,y,u) \in \Omega : \begin{array}{l} L(x,y,u) = 0, \ g(x,y,u) \leq 0 \\ \langle u, \psi(x,y,u) \rangle = 0, u \geq 0, \psi(x,y,u) \leq 0 \end{array} \right\}$$

be the feasible region of (OPCC). For any $I \subseteq \{1, 2, \ldots, q\}$, let

$$F_I := \left\{ (x,y,u) \in \Omega : \begin{array}{l} L(x,y,u) = 0, g(x,y,u) \leq 0 \\ u_i \geq 0, \ \psi_i(x,y,u) = 0 \ \forall i \in I \\ u_i = 0, \ \psi_i(x,y,u) \leq 0 \ \forall i \in \{1,2,\ldots,q\} \backslash I \end{array} \right\}$$

denote a piece of the feasible region F.

Taking the "piecewise programming" approach in the terminology of [4], as in Corollary 2 of [5], we observe that the feasible region of the problem (OPCC) can be rewritten as a union of all pieces $F = \cup_{I \subseteq \{1,2,\ldots,q\}} F_I$. Therefore, a local solution $(\bar{x}, \bar{y}, \bar{u})$ for (OPCC) is also a local solution for each subproblem of minimizing the objective function f over a piece which contains the point $(\bar{x}, \bar{y}, \bar{u})$. Moreover, if $(\bar{x}, \bar{y}, \bar{u})$ is contained in all pieces and all subproblems are convex, then it is a global minimum for the original problem (OPCC). Hence the following proposition follows from this observation.

PROPOSITION 3.1. *Let* $(\bar{x}, \bar{y}, \bar{u})$ *be a local optimal solution to* (OPCC). *Suppose that* f, g, ψ, L *are locally Lipschitz near* $(\bar{x}, \bar{y}, \bar{u})$ *and* Ω *is closed. If for any given index set* $\alpha \subseteq I_0$, *the problem of minimizing* f *over* $F_{\alpha \cup L}$ *is calm in the sense of Definition* 2.9 *at* $(\bar{x}, \bar{y}, \bar{u})$, *then there exist* $\xi \in R^l$, $\zeta \in R^d$, $\eta \in R^q, \gamma \in R^q$ *such that*

$$\begin{aligned} 0 \in \hat{\partial} f(\bar{x}, \bar{y}, \bar{u}) &+ \sum_{i=1}^{l} \xi_i \hat{\partial} L_i(\bar{x}, \bar{y}, \bar{u}) + \sum_{i=1}^{d} \zeta_i \hat{\partial} g_i(\bar{x}, \bar{y}, \bar{u}) + \hat{N}((\bar{x}, \bar{y}, \bar{u}), \Omega) \\ &- \sum_{i=1}^{q} \eta_i \hat{\partial} \psi_i(\bar{x}, \bar{y}, \bar{u}) + \{(0, 0, \gamma)\}, \end{aligned} \tag{3.1}$$

$$\zeta \geq 0, \qquad \langle \zeta, g(\bar{x}, \bar{y}, \bar{u}) \rangle = 0, \tag{3.2}$$

$$\eta_{I_0 \backslash \alpha} \leq 0, \quad \eta_{I_+} = 0, \quad \gamma_L = 0, \quad \gamma_\alpha \leq 0. \tag{3.3}$$

Conversely, let $(\bar{x}, \bar{y}, \bar{u})$ *be a feasible solution for* (OPCC), *and for all index sets* $\alpha \subseteq I_0$, *there exist* $\xi \in R^l$, $\zeta \in R^d$, $\eta \in R^q, \gamma \in R^q$ *such that* (3.1)–(3.3) *are satisfied. If* f *is either convex or pseudoconvex,* g *is convex,* ψ, L *are affine, and* Ω *is convex, then* $(\bar{x}, \bar{y}, \bar{u})$ *is a minimum of* f *over all* $(x, y, u) \in \cup_{\alpha \subseteq I_0} F_{\alpha \cup L}$. *If in addition to the above assumptions* $I_0 = \{1, 2, \ldots, q\}$, *then* $(\bar{x}, \bar{y}, \bar{u})$ *is a global solution for* (OPCC).

Proof. It is obvious that the feasible region of (OPCC) can be represented as the union of pieces $F = \cup_{I \subseteq \{1,2,\ldots,q\}} F_I$. Since $\bar{u}_i > 0 \ \forall i \in L(\bar{u})$ and $\psi_i(\bar{x}, \bar{y}, \bar{u}) < 0$ $\forall i \in I_+(\bar{x}, \bar{y}, \bar{u})$, and

$$F_{\alpha \cup L} = \left\{ (x, y, u) \in \Omega : \begin{array}{l} L(x, y, u) = 0, g(x, y, u) \leq 0 \\ u_i \geq 0,\ \psi_i(x, y, u) = 0 \ \forall i \in \alpha \\ u_i \geq 0,\ \psi_i(x, y, u) = 0 \ \forall i \in L \\ u_i = 0,\ \psi_i(x, y, u) \leq 0 \ \forall i \in I_+ \\ u_i = 0,\ \psi_i(x, y, u) \leq 0 \ \forall i \in I_0 \backslash \alpha \end{array} \right\},$$

we have

$$(\bar{x}, \bar{y}, \bar{u}) \in \cap_{\alpha \subseteq I_0} F_{\alpha \cup L}$$

and

$$(\bar{x}, \bar{y}, \bar{u}) \notin F \backslash (\cup_{\alpha \subseteq I_0} F_{\alpha \cup L}).$$

Hence if $(\bar{x}, \bar{y}, \bar{u})$ is optimal for (OPCC), then for any given index set $\alpha \subseteq I_0, (\bar{x}, \bar{y}, \bar{u})$ is also a minimum for f over $F_{\alpha \cup L}$. Since this problem is calm, by the well-known nonsmooth necessary optimality condition (see, e.g., [1, 2, 3]), there exist $\xi \in R^l$, $\zeta \in R^d$, $\eta \in R^q$, $\gamma \in R^q$ such that (3.1)–(3.3) are satisfied. Conversely, suppose that for each $\alpha \subseteq I_0$ there exist $\xi \in R^l$, $\zeta \in R^d$, $\eta \in R^q$, $\gamma \in R^q$ such that (3.1)–(3.3) are satisfied and the problem is convex. By virtue of Remarks 2.1 and 2.2, the limiting subgradients and the limiting normal cones coincide with the subgradients and the normal cone in the sense of convex analysis, respectively. Hence, by the standard first-order sufficient optimality conditions, $(\bar{x}, \bar{y}, \bar{u})$ is a minimum of f over $F_{\alpha \cup L}$ for each $\alpha \subseteq I_0$ and hence is a minimum of f over $\cup_{\alpha \subseteq I_0} F_{\alpha \cup L}$. In the case when $I_0 = \{1, 2, \ldots, q\}$, $L = \emptyset$ and the feasible region $F = \cup_{\alpha \subseteq I_0} F_{\alpha \cup L}$. Hence $(\bar{x}, \bar{y}, \bar{u})$ is a global optimal for (OPCC) in this case. The proof of the proposition is now complete. □

REMARK 3.1. *The necessary part of the above proposition with smooth problem data is given by Luo, Pang, and Ralph in* [4] *under the so-called "basic constraint qualification."*

Note that the multipliers in Proposition 3.1 depend on the index set α through (3.3). However, if for some pair of index sets α $(\subseteq I_0)$ and $I_0 \backslash \alpha$, the components $(\eta_{I_0}, \gamma_{I_0})$ of the multipliers are the same, then we would have a necessary condition that does not depend on the index set α. In this case the necessary condition turns out to be the necessary condition involving the proximal coderivatives as in (b) of the following theorem.

THEOREM 3.2. *Suppose* f, g, L, ψ *are continuously differentiable. Then the following three conditions are equivalent:*

(a) *There exist* $\xi \in R^l$, $\zeta \in R^d$, $\eta, \gamma \in R^q$ *such that*

$$0 = \nabla f(\bar{x}, \bar{y}, \bar{u}) + \sum_{i=1}^{l} \xi_i \nabla L_i(\bar{x}, \bar{y}, \bar{u}) + \sum_{i=1}^{d} \zeta_i \nabla g_i(\bar{x}, \bar{y}, \bar{u})$$

$$- \sum_{i=1}^{q} \eta_i \nabla_i \psi_i(\bar{x}, \bar{y}, \bar{u}) + \{(0, 0, \gamma)\}, \tag{3.4}$$

$$\zeta \geq 0, \qquad \langle \zeta, g(\bar{x}, \bar{y}, \bar{u}) \rangle = 0, \tag{3.5}$$

$$\eta_{I_0} \leq 0, \quad \eta_{I_+} = 0, \quad \gamma_L = 0, \quad \gamma_{_{I_0}} \leq 0. \tag{3.6}$$

(b) *There exist* $\xi \in R^l$, $\zeta \in R^d$, $\eta \in R^q$ *such that*

$$0 = \nabla f(\bar{x}, \bar{y}, \bar{u}) + \sum_{i=1}^{l} \xi_i \nabla L_i(\bar{x}, \bar{y}, \bar{u}) + \sum_{i=1}^{d} \zeta_i \nabla g_i(\bar{x}, \bar{y}, \bar{u})$$

$$- \sum_{i=1}^{q} \eta_i \nabla \psi_i(\bar{x}, \bar{y}, \bar{u}) + \{0\} \times \{0\} \times D^*_\pi N_{R^q_+}(\bar{u}, \psi(\bar{x}, \bar{y}, \bar{u}))(\eta), \tag{3.7}$$

$$\zeta \geq 0, \qquad \langle \zeta, g(\bar{x}, \bar{y}, \bar{u}) \rangle = 0. \tag{3.8}$$

(c) *There exist* $\xi \in R^l$, $\zeta \in R^d$, $\eta, \gamma \in R^q$, $\alpha, \beta \in R^{2q}_+$ *such that* (3.4) *and* (3.5) *are satisfied and*

$$\begin{aligned} 0 &= \sum_{i=1}^{q} \bar{u}_i(\alpha_i + \beta_i) - \sum_{i=1}^{q} \psi_i(\bar{x}, \bar{y}, \bar{u})(\alpha_{q+i} + \beta_{q+i}), \\ \eta_i &= -\alpha_{q+i} + \bar{u}_i \beta_{q+i} \quad \forall i = 1, 2, \ldots, q, \\ \gamma_i &= -\alpha_i + \psi_i(\bar{x}, \bar{y}, \bar{u}) \beta_i \quad \forall i = 1, 2, \ldots, q. \end{aligned}$$

Let $(\bar{x}, \bar{y}, \bar{u})$ *be a local optimal solution to* (OPCC), *where* $\Omega = R^{n+m+q}$. *Suppose that there exists an index set* $\alpha \subseteq I_0$ *such that the problem of minimizing* f *over* $F_{\alpha \cup L}$ *and the problem of minimizing* f *over* $F_{(I_0 \setminus \alpha) \cup L}$ *are calm. Furthermore, suppose that*

$$0 = \sum_{i=1}^{l} \xi_i \nabla L_i(\bar{x}, \bar{y}, \bar{u}) + \sum_{i=1}^{d} \zeta_i \nabla g_i(\bar{x}, \bar{y}, \bar{u}) - \sum_{i=1}^{q} \eta_i \nabla \psi_i(\bar{x}, \bar{y}, \bar{u}) + \{(0, 0, \gamma)\}, \tag{3.9}$$

$$0 = \langle \zeta, g(\bar{x}, \bar{y}, \bar{u}) \rangle, \quad \eta_{I_+} = 0, \quad \gamma_L = 0 \tag{3.10}$$

implies that $\eta_{I_0} = 0, \gamma_{I_0} = 0$. *Then the three equivalent conditions* (a)–(c) *hold. Conversely, let* $(\bar{x}, \bar{y}, \bar{u})$ *be a feasible solution to* (OPCC), *where* $\Omega = R^{n+m+q}$ *and let* f *be pseudoconvex,* g *be convex,* ψ, L *be affine. If one of the equivalent conditions* (a)–(c) *holds, then* $(\bar{x}, \bar{y}, \bar{u})$ *is a minimum of* f *over all* $(x, y, u) \in \cup_{\alpha \subseteq I_0} F_{\alpha \cup L}$. *If in addition to the above assumptions* $I_0 = \{1, 2, \ldots, q\}$, *then* $(\bar{x}, \bar{y}, \bar{u})$ *is a global solution for* (OPCC).

Proof. By the definition of the proximal coderivatives (Definition 2.3),

$$\gamma \in D^*_\pi N_{R^q_+}(\bar{u}, \psi(\bar{x}, \bar{y}, \bar{u}))(\eta)$$

if and only if

$$(\gamma, -\eta) \in N^\pi((\bar{u}, \psi(\bar{x}, \bar{y}), GrN^q_+).$$

Hence the equivalence of condition (a) and condition (b) follows from Proposition 2.7. The equivalence of condition (b) and condition (c) follows from Proposition 2.8.

Let $(\bar{x}, \bar{y}, \bar{u})$ be a local optimal solution to (OPCC), where $\Omega = R^{n+m+q}$. Then it is also a local optimal solution to the problem of minimizing f over $F_{\alpha \cup L}$ and the problem of minimizing f over $F_{(I_0 \backslash \alpha) \cup L}$. By the calmness assumption for these two problems, there exist $\xi^i \in R^l$, $\zeta^i \in R^d$, $\eta^i \in R^q$, $\gamma^i \in R^q, i = 1, 2$, satisfying (3.1)–(3.3), which implies that

$$
\begin{aligned}
0 = &\sum_{i=1}^{l} (\xi_i^1 - \xi_i^2) \nabla L_i(\bar{x}, \bar{y}, \bar{u}) + \sum_{i=1}^{d} (\zeta_i^1 - \zeta_i^2) \nabla g_i(\bar{x}, \bar{y}, \bar{u}) \\
&\qquad - \sum_{i=1}^{q} (\eta_i^1 - \eta_i^2) \nabla \psi_i(\bar{x}, \bar{y}, \bar{u}) + \{(0, 0, \gamma^1 - \gamma^2)\}, \\
0 = &\langle \zeta^1 - \zeta^2, g(\bar{x}, \bar{y}, \bar{u}) \rangle, \quad (\eta^1 - \eta^2)_{I_+} = 0, \quad (\gamma^1 - \gamma^2)_L = 0.
\end{aligned}
$$

By the assumption we arrive at $\eta_{I_0}^1 = \eta_{I_0}^2, \gamma_{I_0}^1 = \gamma_{I_0}^2$. Since by (3.3), $\eta^1_{I_0 \backslash \alpha} \leq 0, \gamma^1_\alpha \leq 0$ and $\eta^2_\alpha \leq 0, \gamma^2_{I_0 \backslash \alpha} \leq 0$, we have

$$
\eta_{I_0}^1 = \eta_{I_0}^2 \leq 0, \qquad \gamma_{I_o}^1 = \gamma_{I_0}^2 \leq 0.
$$

That is, condition (a) holds.

The sufficient part of the theorem follows from the sufficient part of Proposition 3.1. $\square$

As observed in [4, Proposition 4.3.5], the necessary optimality conditions (3.4)–(3.6) happen to be the KKT condition for the relaxed problem

$$
\begin{aligned}
\text{(RP)} \qquad & \min f(x, y, u) \\
& \text{s.t. } u_i \geq 0, \quad \psi_i(x, y, u) = 0 \quad \forall i \in L(\bar{u}), \\
& \qquad u_i = 0, \quad \psi_i(x, y, u) \leq 0 \quad \forall i \in I_+(\bar{x}, \bar{y}, \bar{u}), \\
& \qquad u_i \geq 0, \quad \psi_i(x, y, u) \leq 0 \quad \forall i \in I_0(\bar{x}, \bar{y}, \bar{u}), \\
& \qquad L(x, y, u) = 0, \qquad g(x, y, u) \leq 0,
\end{aligned}
$$

and $(\xi, \zeta, \eta, \gamma)$ satisfies (3.4)–(3.6) if and only if it satisfies the KKT condition for the subproblem of minimizing f over the feasible region $F_{\alpha \cup L}$, i.e., (3.1)–(3.3) with the smooth problem data and $\Omega = R^{n+m+q}$, for all index sets $\alpha \subseteq I_0(\bar{x}, \bar{y}, \bar{u})$. Consequently, if the strict Mangasarian–Fromovitz constraint qualification (SMFCQ) holds for problem (RP) at $(\xi, \zeta, \eta, \gamma)$ which satisfies (3.4)–(3.6), then $(\xi, \zeta, \eta, \gamma)$ is the unique multiplier which satisfies (3.4)–(3.6). Since the index sets α only affect the $(\eta_{I_0}, \gamma_{I_0})$ components of the multiplier $(\xi, \zeta, \eta, \gamma)$, we observe that the existence of multipliers satisfying (3.4)–(3.6) is equivalent to the existence of multipliers satisfying (3.1)–(3.3) for all index sets $\alpha \subseteq I_0(\bar{x}, \bar{y}, \bar{u})$ with the components $(\eta_{I_0}, \gamma_{I_0})$ having the same sign. From the proof of Theorem 3.2, it is easy to see that the condition that no nonzero vectors satisfy (3.9)–(3.10) is a sufficient condition for the existence of common $(\eta_{I_0}, \gamma_{I_0})$ components of the multiplier $(\xi, \zeta, \eta, \gamma)$ for all index sets $\alpha \subseteq I_0(\bar{x}, \bar{y}, \bar{u})$. Hence this condition refines the sufficient condition of a unique multiplier such as the SMFCQ for the relaxed problem proposed in [4, Proposition 4.3.5].

We now give an example which does not have a unique multiplier satisfying (3.4)–(3.6) but does satisfy the condition proposed in Theorem 3.2.

EXAMPLE 3.1 (see [4, Example 4.3.6]). *Consider the following OPCC:*

$$
\begin{aligned}
\textit{minimize} \quad & x_3 + u_1 + u_2 \\
\textit{s.t.} \quad & u \geq 0, \qquad \psi(x,u) := (-x_1 - u_1, -x_2 - u_2) \leq 0, \\
& \langle u, \psi(x,u) \rangle = 0, \\
& x_3 \geq 0, \qquad 2x_3 \geq 0.
\end{aligned}
$$

$(\bar{x}, \bar{u}) = (\bar{x}_1, \bar{x}_2, 0, 0, 0)$, *where* $\bar{x}_1, \bar{x}_2$ *are any real numbers, are obviously solutions to the above problem. As pointed out in* [4, *Example* 4.3.6], *SMFCQ does not hold for this problem. However, we can verify that it satisfies our condition. Indeed, the equation* (3.9) *for this problem is*

$$
\begin{aligned}
0 = {} & \zeta_1(0,0,-1,0,0) + \zeta_2(0,0,-2,0,0) - \eta_1(-1,0,0,-1,0) \\
& -\eta_2(0,-1,0,0,-1) + (0,0,0,\gamma_1,\gamma_2),
\end{aligned}
$$

which implies that $\eta = 0, \gamma = 0$.

Moreover, the calmness condition is satisfied since the constraint region for each subproblem $F_{\alpha \cup L}$ *is a polyhedron due to the fact that* ψ *and* g *are both affine. Hence by Theorem* 3.2, *if* $(\bar{x}, \bar{u})$ *is a local minimum to the above problem, then there exist* ζ, η, γ *such that*

$$
\begin{aligned}
0 = {} & (0,0,1,1,1) + \zeta_1(0,0,-1,0,0) + \zeta_2(0,0,-2,0,0) \\
& -\eta_1(-1,0,0,-1,0) - \eta_2(0,-1,0,0,-1) + (0,0,0,\gamma_1,\gamma_2), \\
& \zeta \geq 0, \quad \zeta_1 \bar{x}_3 = 0, \quad 2\zeta_2 \bar{x}_3 = 0, \\
& \eta_{I_0} \leq 0, \quad \eta_{I_+} = 0, \quad \gamma_L = 0, \quad \gamma_{I_0} \leq 0,
\end{aligned}
$$

which implies $\eta_1 = \eta_2 = 0$, $\gamma_1 = \gamma_2 = 1$, *and* $\bar{x}_3 = 0$. *Since* $I_0(\bar{x}, \bar{u}) = \{1, 2\}$ *for* $(\bar{x}, \bar{u}) = 0$, 0 *is a global optimal solution according to Theorem* 3.2 *and* $(\bar{x}, 0, 0)$ *with* $\bar{x} \neq 0$ *are local optimal solutions.*

To illustrate the application of the result obtained, we now consider the following bilevel programming problem (BLQP), where the lower level problem is linear quadratic:

$$
\begin{aligned}
(\text{BLQP}) \qquad \min \quad & f(x,y) \\
\text{s.t.} \quad & y \in S(x), \\
& Gx + Hy + a \leq 0,
\end{aligned}
$$

where G and H are $l \times n$ and $l \times m$ matrices, respectively, $a \in R^l$, and $S(x)$ is the solution set of the quadratic programming problem with parameter x:

$$
\begin{aligned}
(\text{QP}_x) \qquad \min \quad & \langle y, Px \rangle + \frac{1}{2}\langle y, Qy \rangle + p^t x + q^t y \\
\text{s.t.} \quad & Dx + Ey + b \leq 0,
\end{aligned}
$$

where $Q \in R^{m \times m}$ is a symmetric and positive semidefinite matrix, $p \in R^n$, $q \in R^m$, $P \in R^{m \times n}$, D and E are $q \times n$ and $q \times m$ matrices, respectively, and $b \in R^q$.

Replacing the bilevel constraint by the KKT condition for the lower level problem, it is easy to see that (BLQP) is equivalent to the problem

$$
\begin{aligned}
(\text{KKT}) \quad \min \quad & f(x,y) \\
\text{s.t.} \quad & \langle Dx + Ey + b, u \rangle = 0, \quad u \geq 0, \quad Dx + Ey + b \leq 0, \\
& Qy + Px + q + E^\top u = 0, \\
& Gx + Hy + a \leq 0,
\end{aligned}
$$

which is an OPCC. Let $(\bar{x}, \bar{y})$ be an optimal solution of (BLQP) and $\bar{u}$ a corresponding multiplier; i.e,

$$0 = Q\bar{y} + P\bar{x} + q + E^\top \bar{u}, \tag{3.11}$$

$$\langle D\bar{x} + E\bar{y} + b, \bar{u}\rangle = 0, \qquad u \geq 0. \tag{3.12}$$

Then

$$\begin{aligned} L &= \{1 \leq i \leq q : \bar{u}_i > 0\}, \\ I_+ &= \{1 \leq i \leq q : \bar{u}_i = 0, (D\bar{x} + E\bar{y} + b)_i < 0\}, \\ I_0 &= \{1 \leq i \leq q : \bar{u}_i = 0, (D\bar{x} + E\bar{y} + b)_i = 0\}. \end{aligned}$$

The feasible region of problem (KKT) is

$$F = \left\{ (x, y, u) \in R^{n+m+q} : \begin{array}{l} Qy + Px + q + E^\top u = 0,\ Gx + Hy + a \leq 0 \\ \langle u, Dx + Ey + b\rangle = 0, u \geq 0, Dx + Ey + b \leq 0 \end{array} \right\},$$

and for any $I \subseteq \{1, 2, \ldots, q\}$,

$$F_I = \left\{ (x, y, u) \in R^{n+m+q} : \begin{array}{l} Qy + Px + q + E^\top u = 0,\ Gx + Hy + a \leq 0 \\ u_i \geq 0,\ (Dx + Ey + b)_i = 0\ \forall i \in I \\ u_i = 0,\ (Dx + Ey + b)^i \leq 0\ \forall i \in \{1, 2, \ldots, q\} \backslash I \end{array} \right\}.$$

Since $F_{\alpha \cup L}$ for any index set $\alpha \subseteq I_0$ has linear constraints only, the problem of minimizing f over $F_{\alpha \cup L}$ is calm. Hence the following result follows from Proposition 3.1.

COROLLARY 3.3. *Let $(\bar{x}, \bar{y})$ be an optimal solution of* (BLQP) *and $\bar{u}$ a corresponding multiplier. Suppose that f is locally Lipschitz near $(\bar{x}, \bar{y})$. Then for each $\alpha \subseteq I_0$, there exist $\xi \in R^m$, $\zeta \in R^d$, $\eta \in R^q$ such that*

$$\begin{aligned} &0 \in \hat{\partial} f(\bar{x}, \bar{y}) + \{P^\top \xi\} \times Q^\top \xi + \{G^\top \zeta\} \times \{H^\top \zeta\} - \{D^\top \eta\} \times \{E^\top \eta\}, \\ &\zeta \geq 0, \qquad \langle G\bar{x} + H\bar{y} + a, \zeta\rangle = 0, \\ &\eta_\alpha \leq 0, \quad \eta_{I_+} = 0, \quad (E\xi)_L = 0, \quad (E\xi)_\alpha \geq 0. \end{aligned}$$

If f is either convex or pseudoconvex, then the above necessary condition is also sufficient for a feasible solution $(\bar{x}, \bar{y}, \bar{u})$ of (KKT) *to be a minimum of f over all $(x, y, u) \in \cup_{\alpha \subseteq I_0} F_{\alpha \cup L}$. In particular, if f is either convex or pseudoconvex and $I_0 = \{1, 2, \ldots, q\}$, then the above condition is sufficient for a feasible solution $(\bar{x}, \bar{y})$ to be a global optimum for* (BLQP).

The following result follows from Theorem 3.2.

COROLLARY 3.4. *Let $(\bar{x}, \bar{y})$ be an optimal solution of* (BLQP) *and $\bar{u}$ a corresponding multiplier. Suppose that f is C^1 and*

$$0 = P^\top \xi + G^\top \zeta - D^\top \eta, \tag{3.13}$$

$$0 = Q^\top \xi + H^\top \zeta - E^\top \eta, \tag{3.14}$$

$$0 = \langle G\bar{x} + H\bar{y} + a, \zeta\rangle, \tag{3.15}$$

$$\eta_{I_+} = 0, \qquad (E\xi)_L = 0$$

implies $\eta_{I_0} = (E\xi)_{I_0} = 0$. Then there exist $\xi \in R^m$, $\zeta \in R^d$, $\eta \in R^q$ such that

$$0 = \nabla f(\bar{x}, \bar{y}) + \{P^\top \xi\} \times Q^\top \xi + \{G^\top \zeta\} \times \{H^\top \zeta\} - \{D^\top \eta\} \times \{E^\top \eta\}, \tag{3.16}$$

$$\zeta \geq 0, \qquad \langle G\bar{x} + H\bar{y} + a, \zeta\rangle = 0, \tag{3.17}$$

$$\eta_{I_0} \leq 0, \quad \eta_{I_+} = 0, \quad (E\xi)_L = 0, \quad (E\xi)_{I_0} \geq 0.$$

Equivalently, there exist $\xi \in R^l$, $\zeta \in R^d$, $\eta \in R^q$ *such that* (3.16)–(3.17) *are satisfied and*

$$(-E\xi, -\eta) \in N^\pi((\bar{u}, D\bar{x} + E\bar{u} + b), GrN_{R^q_+}).$$

Equivalently, there exist $\xi \in R^l$, $\zeta \in R^d$, $\eta \in R^q$, $\alpha, \beta \in R^{2q}_+$ *such that* (3.16)–(3.17) *are satisfied and*

$$\begin{aligned} 0 &= \sum_{i=1}^{q} \bar{u}_i(\alpha_i + \beta_i) - \sum_{i=1}^{q}(D\bar{x} + E\bar{y} + b)_i(\alpha_{q+i} + \beta_{q+i}), \\ \eta_i &= -\alpha_{q+i} + \bar{u}_i\beta_{q+i} \ \forall i = 1, 2, \ldots, q, \\ (E\xi)_i &= \alpha_i - (D\bar{x} + E\bar{y} + b)_i\beta_i \ \forall i = 1, 2, \ldots, q. \end{aligned}$$

Conversely, let $(\bar{x}, \bar{y})$ *be any vector in* R^{n+m} *satisfying the constraints* $G\bar{x} + H\bar{y} + a \leq 0$ *and* $D\bar{x} + E\bar{y} + b \leq 0$ *and* f *be pseudoconvex. If there exists* $\bar{u} \in R^q$ *that satisfies* (3.11)–(3.12) *such that one of the above equivalent conditions holds, then* $(\bar{x}, \bar{y}, \bar{u})$ *is a minimum of* f *over all* $(x, y, u) \in \cup_{\alpha \subseteq I_0} F_{\alpha \cup L}$. *In addition to the above assumptions, if* $I_0 = \{1, 2, \ldots, q\}$, *then* $(\bar{x}, \bar{y})$ *is a global minimum for* (BLQP).

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