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CONSTRAINT QUALIFICATIONS AND NECESSARY OPTIMALITY CONDITIONS FOR OPTIMIZATION PROBLEMS WITH VARIATIONAL INEQUALITY CONSTRAINTS*

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Abstract. A very general optimization problem with a variational inequality constraint, inequality constraints, and an abstract constraint are studied. Fritz John type and Kuhn–Tucker type necessary optimality conditions involving Mordukhovich coderivatives are derived. Several constraint qualifications for the Kuhn–Tucker type necessary optimality conditions involving Mordukhovich coderivatives are introduced and their relationships are studied. Applications to bilevel programming problems are also given.

Key words. optimization problems, variational inequality constraints, necessary optimality conditions, constraint qualifications, coderivatives of set-valued maps, nonsmooth analysis

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1. Introduction.

An optimization problem with variational inequality constraints (OPVIC) is a special class of an optimization problem over variables x and y in which some or all of its constraints are defined by a parametric variational inequality with y as its primary variable and x as the parameter. In this paper we consider a very general optimization problem with variational inequality constraints in finite dimensional spaces defined as follows:

$$\begin{aligned} \text{(OPVIC)} \quad & \text{minimize } f(x,y) \\ & \text{subject to (s.t.) } \psi(x,y) \le 0, (x,y) \in C, \\ & \qquad y \in \Omega, \langle F(x,y), y-z \rangle \le 0 \quad \forall z \in \Omega, \end{aligned}$$

where $f : R^{n+m} \to R$, $\psi : R^{n+m} \to R^d$, $F : R^{n+m} \to R^m$ are Lipschitz near all optimal solutions of (OPVIC), C is a nonempty closed subset of R^{n+m}, and Ω is a closed convex subset of R^m. The above problem is also called a generalized bilevel programming problem (see, e.g., Ye, Zhu, and Zhu [27]) or a mathematical program with equilibrium constraints (see, e.g., Luo, Pang, and Ralph [10]). The reader is referred to [10] for recent developments on the subject and references for other types of optimality conditions.

Although under certain constraint qualifications one can reduce (OPVIC) to an ordinary nonlinear programming problem, it is known that the usual constraint qualification such as the Mangasarian–Fromovitz constraint qualification cannot in general be satisfied for the equivalent nonlinear programming problem (see [27, Proposition 1.1]). In Ye and Ye [26], under the pseudoupper-Lipschitz continuity, a Kuhn–Tucker type necessary optimality condition involving Mordukhovich coderivatives was derived for (OPVIC). In Ye [25], it was shown that a Kuhn–Tucker type necessary optimality condition involving the proximal coderivatives (which are in general smaller than

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Mordukhovich coderivatives) holds under a stronger constraint qualification in the case where the variational inequality is a complementarity system, i.e., $\Omega = R^a \times R^b_+$ with $a + b = m$.

The purpose of this paper is to study (OPVIC) under much weaker assumptions and derive more powerful results than those in [26]. In particular, we incorporate inequality constraints and an abstract constraint in our problems and we do not assume the smoothness of the mapping F as in [26].

As in [26], we formulate (OPVIC) as the following optimization problem with a generalized equation constraint:

$$\begin{aligned} \text{(GP)} \quad & \min f(x,y) \\ & \text{s.t. } \psi(x,y) \le 0, (x,y) \in C, \\ & \qquad 0 \in F(x,y) + N(y,\Omega), \end{aligned}$$

where

$$N(y,\Omega) := \begin{cases} \text{the normal cone of } \Omega \text{ at } y & \text{if } y \in \Omega, \\ \emptyset & \text{if } y \notin \Omega \end{cases}$$

is the normal cone operator.

We show that if $(\bar{x},\bar{y})$ is a local solution of (OPVIC), then there exist $\lambda \ge 0, \eta \in R^m$, and $\gamma \in R^d_+$ not all zero such that

$$\begin{aligned} & 0 \in \lambda \partial f(\bar{x},\bar{y}) + \partial\langle \psi,\gamma\rangle(\bar{x},\bar{y}) + \partial\langle F,\eta\rangle(\bar{x},\bar{y}) \\ & \qquad +\{0\} \times D^* N_\Omega(\bar{y}, -F(\bar{x},\bar{y}))(\eta) + N((\bar{x},\bar{y}),C), \\ & \langle \psi(\bar{x},\bar{y}),\gamma\rangle = 0, \end{aligned}$$

where ∂ denotes the limiting subgradient (see Definition 2.2), N_Ω denotes the set-valued map $y \rightrightarrows N(y,\Omega)$, and D^* denotes the coderivative of a set-valued map (see Definition 2.4). Moreover we introduce the concept of calmness for (OPVIC) and show that under the calmness condition λ can be taken as 1. Several constraint qualifications that are stronger than the calmness condition but easier to verify are introduced and their relationships are indicated.

Note that in the case where $\Omega = R^m$, (OPVIC) is reduced to an ordinary nonlinear programming problem with equality, inequality, and abstract constraints. Hence our results are applicable even for an ordinary nonlinear programming problem.

We organize the paper as follows. Section 2 contains background material on nonsmooth analysis. In section 3, we derive the Fritz John type necessary optimality condition involving Mordukhovich coderivatives and the Kuhn–Tucker type necessary optimality conditions involving Mordukhovich coderivatives under the calmness condition. In section 4 we introduce several constraint qualifications for the Kuhn–Tucker necessary optimality conditions involving the Mordukhovich coderivatives and study the relationships between these constraint qualifications. Applications to bilevel programming problems are discussed in section 5.

The following notations are used throughout the paper. For an m-by-n matrix A and index sets $I \subseteq \{1,2,\ldots,m\}$, $J \subseteq \{1,2,\ldots,n\}$, A_I and $A_{I,J}$ denote the submatrix of A with rows specified by I and the submatrix of A with rows and columns specified by I and J, respectively. For a mapping $\psi : R^n \to R^d$ and a vector $\gamma \in R^d$, $\langle \psi,\gamma\rangle(x)$ is the function defined by $\langle \psi,\gamma\rangle(x) := \langle \psi(x),\gamma\rangle$. For a vector $d \in R^n$, d_i is the ith component of d and d_I is the subvector composed from the components $d_i, i \in I$.

$\langle a, b \rangle$ is the inner product of vectors a and b. $gph\Phi$ is the graph of a set-valued map Φ and $epif$ is the epigraph of a function f. intΩ, clΩ, and coΩ denote the interior, the closure, and the convex hull of a set Ω. We denote by $B_\delta(x_0)$ and B the open ball centered at x_0 with radius $\delta > 0$ and the open unit ball, respectively.

2. Preliminaries. This section contains some background material on nonsmooth analysis which will be used later. We give only concise definitions that will be needed in the paper. For more detailed information on the subject, our references are Clarke [3], Clarke et al. [4], Rockafellar and Wets [19], Loewen [9], and Mordukhovich [13, 14].

First we give some concepts for various normal cones.

DEFINITION 2.1. *Let Ω be a nonempty subset of R^n. Given $\bar{z} \in \mathrm{cl}\Omega$, the convex cone*

$$N^\pi(\bar{z}, \Omega) := \{\xi \in R^n : \exists M > 0 \ s.t. \ \langle \xi, z - \bar{z} \rangle \leq M\|z - \bar{z}\|^2 \ \forall z \in \Omega\}$$

is called the proximal normal cone to set Ω at point $\bar{z}$, the closed cone

$$N(\bar{z}, \Omega) := \left\{ \lim_{k\to\infty} \xi^k : \xi^k \in N^\pi(z^k, \Omega), z^k \to \bar{z} \right\}$$

is called the limiting normal cone to Ω at point $\bar{z}$, and the closed convex hull of the limiting normal cone

$$N_C(\bar{z}, \Omega) := \mathrm{clco}N(\bar{z}, \Omega)$$

is called the Clarke normal cone to set Ω at point $\bar{z}$.

Using the definitions for normal cones, we now give definitions for subgradients of a single-valued map.

DEFINITION 2.2. *Let $f : R^n \to R \cup \{+\infty\}$ be lower semicontinuous and finite at $\bar{z} \in R^n$. The proximal subgradient of f at $\bar{z}$ is defined by*

$$\partial^\pi f(\bar{z}) := \{\xi : (\xi, -1) \in N^\pi((\bar{z}, f(\bar{z})), epif)\},$$

the limiting subgradient of f at $\bar{z}$ is defined by

$$\partial f(\bar{z}) := \{\xi : (\xi, -1) \in N((\bar{z}, f(\bar{z})), epif)\},$$

and the Clarke generalized gradient of f at $\bar{z}$ is defined by

$$\partial_C f(\bar{z}) := \{\xi : (\xi, -1) \in N_C((\bar{z}, f(\bar{z})), epif)\},$$

where $epif := \{(x, r) \in R^n \times R : f(x) \leq r\}$ is the epigraph of f.

The following calculus rules for subgradients are well known and can be found in the references given in the beginning of this section (see, e.g., [9, Proposition 5A.4, Theorem 5A.8], proof of [5, Lemma 2.2]).

PROPOSITION 2.3. *Let functions $f : R^n \to R \cup \{+\infty\}$ be lower semicontinuous and finite at $\bar{z} \in R^n$, $g : R^n \to R$ be Lipschitz near $\bar{z}$, and $h : R^n \to R$ is C^{1+} at $\bar{z}$ (i.e., the gradient of h is Lipschitz near $\bar{z}$). Then the nonnegative scalar multiplication rule is*

$$\partial(\lambda f)(\bar{z}) = \lambda \partial f(\bar{z}) \qquad \forall \lambda \geq 0$$

and the sum rules are

$$\partial(f+g)(\bar{z}) \subseteq \partial f(\bar{z}) + \partial g(\bar{z}),$$

$$\partial^\pi(f+h)(\bar{z}) = \partial^\pi f(\bar{z}) + \nabla h(\bar{z}).$$

Let $\varphi(x) := f(F(x))$, where $F : R^m \to R^n$ is Lipschitz near $\bar{x}$ and $f : R^n \to R$ is Lipschitz near $F(\bar{x})$. Then the chain rule is

$$\partial\varphi(\bar{x}) \subseteq \bigcup\{\partial\langle\eta, F\rangle(\bar{x}) : \eta \in \partial f(F(\bar{x}))\}.$$

For set-valued maps, the definition for a limiting normal cone leads to the definition of coderivative of a set-valued map introduced by Mordukhovich (see, e.g., [14]).

DEFINITION 2.4. *Let $\Phi : R^n \rightrightarrows R^q$ be an arbitrary set-valued map (assigning to each $z \in R^n$ a set $\Phi(z) \subset R^q$ which may be empty) and $(\bar{x}, \bar{y}) \in$ cl$gph\Phi$, where $gph\Phi := \{(z,v) : v \in \Phi(z)\}$ denotes the graph of the set-valued map Φ. The set-valued map $D^*\Phi(\bar{z}, \bar{v})$ from R^q into R^n defined by*

$$D^*\Phi(\bar{z}, \bar{v})(\eta) = \{\xi \in R^n : (\xi, -\eta) \in N((\bar{z}, \bar{v}), gph\Phi)\}$$

is called the coderivative of Φ at the point $(\bar{z}, \bar{v})$. By convention for $(\bar{z}, \bar{v}) \notin$ cl$gph\Phi$ we define $D^*\Phi(\bar{z}, \bar{v})(\eta) = \emptyset$. The symbol $D^*\Phi(\bar{z})$ is used when Φ is single-valued at $\bar{z}$ and $\bar{v} = \Phi(\bar{z})$.

In the special case when a set-valued map is single-valued, the coderivative is related to the limiting subgradient in the following way.

PROPOSITION 2.5 (see [14, Proposition 2.11]). *Let $\Phi : R^n \to R^q$ be single-valued and Lipschitz near $\bar{z}$. Then*

$$D^*\Phi(\bar{z})(\eta) = \partial\langle\Phi, \eta\rangle(\bar{z}) \quad \forall\eta \in R^q.$$

We now give some concepts for Lipschitz behavior of a set-valued map. The following concept for Lipschitz behavior was introduced by Aubin [1].

DEFINITION 2.6. *A set-valued map $\Phi : R^n \rightrightarrows R^q$ is said to be pseudo-Lipschitz continuous around $(\bar{z}, \bar{v}) \in gph\Phi$ if there exist a neighborhood U of $\bar{z}$, a neighborhood V of $\bar{v}$, and $\mu \geq 0$ such that*

$$\Phi(z) \cap V \subset \Phi(z') + \mu\|z' - z\| \text{cl} B \quad \forall z', z \in U.$$

On the other hand, the following upper-Lipschitz behavior was studied by Robinson [21].

DEFINITION 2.7. *A set-valued map $\Phi : R^n \rightrightarrows R^q$ is said to be upper-Lipschitz continuous at $\bar{z} \in R^n$ if there exist a neighborhood U of $\bar{z}$ and $\mu \geq 0$ such that*

$$\Phi(z) \subset \Phi(\bar{z}) + \mu\|z - \bar{z}\| \text{cl} B \quad \forall z \in U.$$

The following proposition is a sum rule for coderivatives.

PROPOSITION 2.8 (see [14, Corollary 4.2]). *Let Φ_1 and Φ_2 be closed-graph set-valued maps from R^n into R^q and let $\bar{v} \in \Phi_1(\bar{z}) + \Phi_2(\bar{z})$. Assume that the multifunction $S : R^{n+q} \rightrightarrows R^{2q}$ defined by*

$$S(z,v) := \{(v_1, v_2) \in R^{2q} | v_1 \in \Phi_1(z), v_2 \in \Phi_2(z), v_1 + v_2 = v\}$$

is locally bounded around $(\bar{z}, \bar{v})$ *and either* Φ_1 *is pseudo-Lipschitz around* $(\bar{z}, v_1)$ *or* Φ_2 *is pseudo-Lipschitz around* $(\bar{z}, v_2)$ *for each* $(v_1, v_2) \in S(\bar{z}, \bar{v})$*. Then for any* $\eta \in R^q$

$$D^*(\Phi_1 + \Phi_2)(\bar{z}, \bar{v})(\eta) \subseteq \cup_{(v_1,v_2)\in S(\bar{z},\bar{v})}[D^*\Phi_1(\bar{z}, v_1)(\eta) + D^*\Phi_2(\bar{z}, v_2)(\eta)].$$

The following sum rule for the case where one of the set-valued maps is single-valued follows from Propositions 2.5 and 2.8.

COROLLARY 2.9. *Let* $\Phi_1 : R^n \to R^q$ *be single-valued and Lipschitz near* $\bar{z}$ *and* $\Phi_2 : R^n \rightrightarrows R^q$ *be an arbitrary closed set-valued map. Then for any* $\bar{v} \in \Phi_1(\bar{z}) + \Phi_2(\bar{z})$ *and* $\eta \in R^q$

$$D^*(\Phi_1 + \Phi_2)(\bar{z}, \bar{v})(\eta) \subseteq \partial\langle\Phi_1, \eta\rangle(\bar{z}) + D^*\Phi_2(\bar{z}, \bar{v} - \Phi_1(\bar{z}))(\eta).$$

3. Necessary optimality conditions. The purpose of this section is to derive the necessary optimality conditions involving Mordukhovich coderivatives for (OPVIC).

The following fundamental results obtained by Mordukhovich will be useful in proving the Fritz John type necessary optimality condition involving Mordukhovich coderivatives.

LEMMA 3.1 (extremal principle). *Let* $\Omega_1, \ldots, \Omega_n$ *be closed sets in* R^m *and let* $\bar{x} \in \cap_{i=1}^n \Omega_i$*. Suppose that there exist a neighborhood* U *of* $\bar{x}$ *and sequences* $\{a_{ik}\} \subseteq R^m$*,* $i = 1, 2, \ldots, n$ *such that* $a_{ik} \to 0$ *as* $k \to \infty$ *for* $i = 1, 2, \ldots, n$ *and*

$$\cap_{i=1}^n (\Omega_i - a_{ik}) \cap U = \emptyset \quad \forall k = 1, 2, \ldots.$$

Then there exists $\xi_i \in N(\bar{x}, \Omega_i), i = 1, \ldots, n$ *such that*

$$\xi_1 + \xi_2 + \cdots + \xi_n = 0, \|\xi_1\| + \|\xi_2\| + \cdots + \|\xi_n\| = 1.$$

Although the terminology of the extremal principle was first given by Mordukhovich [14], the essence of the results can be traced back to Mordukhovich [11]. We may usefully view it as an extension of the Hahn–Banach separation theorem to nonconvex sets. The proof for the case when $n = 2$ can be found in [14, Theorem 3.2]. For the case when $n > 2$, the result can be proved in exactly the same way as the proof of [14, Theorem 3.2] or mathematical induction on n can be used as in the proof of Mordukhovich and Shao [17, Theorem 3.2].

The extremal principle turns out to be very useful in deriving the Fritz John type necessary optimality condition as shown in the following theorem.

THEOREM 3.2. *Let* $(\bar{x}, \bar{y})$ *be a local solution of* (OPVIC)*. Then there exist* $\lambda \geq 0$*,* $\eta \in R^m, \gamma \in R^d_+$ *not all zero such that*

$$\begin{aligned}
&0 \in \lambda\partial f(\bar{x}, \bar{y}) + \partial\langle\psi, \gamma\rangle(\bar{x}, \bar{y}) + \partial\langle F, \eta\rangle(\bar{x}, \bar{y}) \\
&\qquad +\{0\} \times D^*N_\Omega(\bar{y}, -F(\bar{x}, \bar{y}))(\eta) + N((\bar{x}, \bar{y}), C), \\
&\langle\psi(\bar{x}, \bar{y}), \gamma\rangle = 0.
\end{aligned}$$

Proof. Define

$$\begin{aligned}
\Omega_1 &:= \{(x, y, u_0, u, v) : f(x, y) \leq u_0\}, \\
\Omega_2 &:= \{(x, y, f(\bar{x}, \bar{y}), u, 0) : \psi(x, y) \leq u\}, \\
\Omega_3 &:= \{(x, y, f(\bar{x}, \bar{y}), 0, 0) : (x, y) \in C\}, \\
\Omega_4 &:= \{(x, y, f(\bar{x}, \bar{y}), 0, v) : v \in F(x, y) + N(y, \Omega)\}.
\end{aligned}$$

Then $(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}), 0, 0) \in \cap_{i=1}^4 \Omega_i$. By taking $a_{1k} = (0, 0, \nu_k, 0, 0)$ with $\nu_k < 0, \nu_k \to 0$, $a_{ik} = 0 \ \ \forall i = 2, 3, 4$, and $U = V \times R^{1+d+m}$, where V is a neighborhood of the local minimizer $(\bar{x}, \bar{y})$, it is easy to verify that

$$\cap_{i=1}^4 (\Omega_i - a_{ik}) \cap U = \emptyset \quad \forall k = 1, 2, \ldots.$$

By Lemma 3.1, there exist ξ_i, not all zero such that $\xi_i \in N((\bar{x}, \bar{y}, f(\bar{x}, \bar{y}), 0, 0), \Omega_i)$, $i = 1, 2, 3, 4$, and

$$0 = \xi_1 + \xi_2 + \xi_3 + \xi_4.$$

That is, there exist $(a, -\lambda) \in R^{n+m+1}, (b, -\gamma) \in R^{n+m+d}, c \in R^{n+m}, (d, -\eta) \in R^{n+m+m}$ not all zero such that

$$(a, -\lambda) \in N((\bar{x}, \bar{y}, f(\bar{x}, \bar{y})), epi f), \tag{3.1}$$

$$(b, -\gamma) \in N((\bar{x}, \bar{y}, 0), epi \psi), \tag{3.2}$$

$$c \in N((\bar{x}, \bar{y}), C), \tag{3.3}$$

$$(d, -\eta) \in N((\bar{x}, \bar{y}, 0), gph \varphi) \text{ where } \varphi(x, y) := F(x, y) + N(y, \Omega), \tag{3.4}$$

and

$$0 = a + b + c + d. \tag{3.5}$$

By the definition of epigraph, inclusion (3.1) implies that $\lambda \geq 0$. Since f is Lipschitz near $(\bar{x}, \bar{y})$, either $a = 0, \lambda = 0$, or $\lambda > 0$ and $(\frac{a}{\lambda}, -1) \in N((\bar{x}, \bar{y}, f(\bar{x}, \bar{y})), epi f)$, which by definition implies that $\frac{a}{\lambda} \in \partial f(\bar{x}, \bar{y})$. Hence (3.1) implies that

$$\lambda \geq 0, a \in \lambda \partial f(\bar{x}, \bar{y}). \tag{3.6}$$

Similarly, inclusion (3.2) implies that $\gamma \geq 0$. Let $M := \{i : \psi_i(\bar{x}, \bar{y}) = 0\}$ be the index set of the binding constraints. Inclusion (3.2) implies that $(b, -\gamma_M) \in N((\bar{x}, \bar{y}, 0), gph \psi_M)$, which is equivalent to $b \in D^* \psi_M(\bar{x}, \bar{y})(\gamma_M)$ by definition of coderivatives. By virtue of Proposition 2.5, we have $D^* \psi_M(\bar{x}, \bar{y})(\gamma_M) = \partial \langle \psi_M, \gamma_M \rangle (\bar{x}, \bar{y})$. Therefore we have

$$\gamma \geq 0, \langle \psi(\bar{x}, \bar{y}), \gamma \rangle = 0, b \in \partial \langle \psi, \gamma \rangle (\bar{x}, \bar{y}). \tag{3.7}$$

By definition of coderivatives, (3.4) implies that $d \in D^* \varphi(\bar{x}, \bar{y}, 0)(\eta)$. By Corollary 2.9, we have

$$\begin{aligned} & d \in D^* \varphi(\bar{x}, \bar{y}, 0)(\eta) \\ & \subseteq \partial \langle F, \eta \rangle (\bar{x}, \bar{y}) + \{0\} \times D^* N_\Omega(\bar{y}, -F(\bar{x}, \bar{y}))(\eta). \end{aligned} \tag{3.8}$$

The conclusion of the theorem follows from inclusions (3.6), (3.7), (3.3), (3.8), and (3.5). $\square$

Remark. In the case of ordinary mathematical programming problems, $\Omega = R^m$, Theorem 3.2 is a limiting subgradient version of the generalized Lagrange multiplier rules in Clarke [3, Theorem 6.1.1] and was obtained by Mordukhovich in [12, Theorem 1(b)].

The following constraint qualification called *no nonzero abnormal multiplier constraint qualification* (NNAMCQ) follows from the Fritz John type necessary condition.

COROLLARY 3.3. *Let* $(\bar{x}, \bar{y})$ *be a local solution of* (OPVIC). *Assume that condition* (NNAMCQ) *is satisfied, i.e., there is no nonzero vector* $(\gamma, \eta) \in R^d_+ \times R^m$ *such that*

$$\begin{aligned} &0 \in \partial\langle \psi, \gamma\rangle(\bar{x}, \bar{y}) + \partial\langle F, \eta\rangle(\bar{x}, \bar{y}) \\ &\qquad +\{0\} \times D^* N_\Omega(\bar{y}, -F(\bar{x}, \bar{y}))(\eta) + N((\bar{x}, \bar{y}), C), \\ &\langle \psi(\bar{x}, \bar{y}), \gamma\rangle = 0 \end{aligned} \tag{3.9}$$

is satisfied at $(\bar{x}, \bar{y})$. *Then* $\lambda > 0$ *in the conclusion of Theorem* 3.2.

Proof. By Theorem 3.2, there exists $\lambda \geq 0$, $\eta \in R^m$, $\gamma \in R^d_+$ not all zero such that

$$\begin{aligned} &0 \in \lambda\partial f(\bar{x}, \bar{y}) + \partial\langle \psi, \gamma\rangle(\bar{x}, \bar{y}) + \partial\langle F, \eta\rangle(\bar{x}, \bar{y}) \\ &\qquad +\{0\} \times D^* N_\Omega(\bar{y}, -F(\bar{x}, \bar{y}))(\eta) + N((\bar{x}, \bar{y}), C), \\ &\langle \psi(\bar{x}, \bar{y}), \gamma\rangle = 0. \end{aligned} \tag{3.10}$$

The case $\lambda = 0$ is impossible under condition (NNAMCQ). Indeed, if $\lambda = 0$ in the above condition, then the inclusion (3.10) coincides with inclusion (3.9). But this is impossible since (γ, η) is nonzero. $\square$

It is well known that the calmness condition (see, e.g., Clarke [3]) is the weakest constraint qualification for nonlinear programming problems with Lipschitz problem data. We now extend the concept to the setting of (OPVIC).

DEFINITION 3.4. *Let* $(\bar{x}, \bar{y})$ *be a local solution to* (OPVIC). (GP) *is said to be calm at* $(\bar{x}, \bar{y})$ *provided that there exist* $\epsilon > 0$ *and* $\mu > 0$ *such that* $\forall\ (p, q) \in \epsilon B$ $\forall\ (x, y) \in B_\epsilon(\bar{x}, \bar{y})$ *satisfying*

$$\begin{aligned} &\psi(x, y) + p \leq 0, (x, y) \in C, \\ &q \in F(x, y) + N(y, \Omega) \end{aligned}$$

it follows that

$$f(\bar{x}, \bar{y}) \leq f(x, y) + \mu\|(p, q)\|.$$

LEMMA 3.5. *Let* $(\bar{x}, \bar{y})$ *be a local solution to* (GP), *where* (GP) *is calm at* $(\bar{x}, \bar{y})$. *Then* $(\bar{x}, \bar{y}, 0)$ *is a local solution to the following problem:*

$$\begin{aligned} \min \quad & f(x, y) + d\mu \max\{\psi_i(x, y), 0, i = 1, \ldots, d\} + \mu\|q\| \\ s.t. \quad & (x, y, q) \in gph\Phi \cap C \times R^m, \end{aligned}$$

where Φ *is a set-valued map defined by* $\Phi(x, y) := F(x, y) + N(y, \Omega)$.

Proof. By definition of the calmness,

$$\begin{aligned} &f(\bar{x}, \bar{y}) \leq f(x, y) + \mu(\|p\| + \|q\|) \quad \forall (x, y, p, q) \\ &\text{ s.t. } \psi(x, y) + p \leq 0, (x, y, q) \in gph\Phi \cap C \times R^m, (x, y) \in B_\epsilon(\bar{x}, \bar{y}), (p, q) \in \epsilon B. \end{aligned}$$

Since

$$\psi_i(x, y) - \psi_i^+(x, y) \leq 0, i = 1, \ldots, d$$

taking $p_i = -\psi_i^+(x, y)$, we have for (x, y) in a neighborhood of $(\bar{x}, \bar{y})$ and q near 0,

$$\begin{aligned} f(\bar{x}, \bar{y}) &\leq f(x, y) + \mu\left(\sum_{i=1}^d \psi_i^+(x, y) + \|q\|\right) \\ &\leq f(x, y) + d\mu \max\{\psi_i(x, y), 0, i = 1, \ldots, d\} + \mu\|q\|. \end{aligned}$$

Notice that $\max\{\psi_i(\bar{x},\bar{y}), 0, i=1,\ldots,d\} = 0$. The proof is complete. ∎

THEOREM 3.6. *Let* $(\bar{x},\bar{y})$ *be a local solution of* (OPVIC). *Suppose that* (GP) *is calm at* $(\bar{x},\bar{y})$. *Then* λ *can be taken as* 1 *in the conclusion of Theorem* 3.2.

Proof. By Lemma 3.5, $(\bar{x},\bar{y},0)$ is a local solution to the new (OPVIC):

$$\begin{array}{ll} \min & \tilde{f}(x,y,q) \\ \text{s.t.} & 0 \in \tilde{F}(x,y,q) + N(y,\Omega), \end{array}$$

where $\tilde{f}(x,y,q) := f(x,y) + d\mu \max\{\psi_i(x,y), 0, i=1,\ldots,d\} + \mu\|q\|$ and $\tilde{F}(x,y,q) := -q + F(x,y)$.

We now prove that condition (NNAMCQ) is satisfied. Indeed, it is easy to see that the inclusion (3.9) for the new (OPVIC) is

$$0 \in \partial\langle F,\eta\rangle(\bar{x},\bar{y}) \times \{-\eta\} + \{0\} \times D^*N_\Omega(\bar{y}, -F(\bar{x},\bar{y}))(\eta) \times \{0\} + N((\bar{x},\bar{y}), C) \times \{0\},$$

which is only satisfied by the zero vector $\eta = 0$.

Applying Corollary 3.3, there exists $\eta \in R^m$ such that

$$\begin{aligned} &0 \in \partial\tilde{f}(\bar{x},\bar{y},0) + \partial\langle F,\eta\rangle(\bar{x},\bar{y}) \times \{-\eta\} \\ &+\{0\} \times D^*N_\Omega(\bar{y}, -F(\bar{x},\bar{y}))(\eta) \times \{0\} + N((\bar{x},\bar{y}), C) \times \{0\}. \end{aligned}$$

Note that

$$\tilde{f}(x,y,q) = f(x,y) + d\mu h(\psi(x,y)) + \mu\|q\|,$$

where $h : R^d \to R$ is defined by $h(u) := \max\{u_1,\ldots,u_d,0\}$. By the sum rule and the chain rule in Proposition 2.3,

$$\partial\tilde{f}(\bar{x},\bar{y},0) \subseteq \partial f(\bar{x},\bar{y}) \times \{0\} + d\mu \cup \{\partial\langle\eta,\psi\rangle(\bar{x},\bar{y}) : \eta \in \partial h(\psi(\bar{x},\bar{y}))\} + \mu(\{0\}\times\{0\}\times B).$$

The proof of the theorem is completed after calculating the subgradient of the convex function h at $\psi(\bar{x},\bar{y})$, i.e.,

$$\partial h(\psi(\bar{x},\bar{y})) = \left\{\gamma \in R^d : \gamma_i \geq 0, \gamma_i\psi_i(\bar{x},\bar{y}) = 0, i=1,\ldots,d \text{ and } \sum_{i=1}^d \gamma_i = 1\right\}. \qquad ∎$$

Remark. In the case of ordinary mathematical programming problems, $\Omega = R^m$, Theorem 3.6 can be considered as a limiting subgradient version of the generalized Lagrange multiplier rules in Clarke [3, Proposition 6.4.4].

Note that Theorems 3.2 and 3.6 involve the coderivative $D^*N_\Omega(\bar{y}, -F(\bar{x},\bar{y}))(\eta)$. By the definition of coderivatives,

$$\xi \in D^*N_\Omega(\bar{y}, -F(\bar{x},\bar{y}))(\eta) \Longleftrightarrow (\xi, -\eta) \in N((\bar{y}, -F(\bar{x},\bar{y})), gphN_\Omega).$$

Hence calculation of the coderivative $D^*N_\Omega(\bar{y}, -F(\bar{x},\bar{y}))(\eta)$ depends on calculation of the limiting normal cone $N((\bar{y}, -F(\bar{x},\bar{y})), gphN_\Omega)$. In the case $\Omega = R^m_+$, the limiting normal cone $N((\bar{y}, -F(\bar{x},\bar{y})), gphN_\Omega)$ can be calculated explicitly by using the following proposition.

PROPOSITION 3.7. *For any* $(\bar{y},\bar{z}) \in gphN_{R^m_+}$, *define*

$$\begin{aligned} L := L(\bar{y},\bar{z}) &:= \{i \in \{1,2,\cdots,m\} : \bar{y}_i > 0, \bar{z}_i = 0\}, \\ I_+ := I_+(\bar{y},\bar{z}) &:= \{i \in \{1,2,\cdots,m\} : \bar{y}_i = 0, \bar{z}_i < 0\}, \\ I_0 := I_0(\bar{y},\bar{z}) &:= \{i \in \{1,2,\cdots,m\} : \bar{y}_i = 0, \bar{z}_i = 0\}. \end{aligned}$$

Then

$$N((\bar{y},\bar{z}), gphN_{R^m_+}) = \{(\alpha,-\beta)\in R^{2m} : \alpha_L = 0, \beta_{I_+} = 0$$
$$\forall i \in I_0,\ \ either\ \alpha_i\beta_i = 0\ or\ \alpha_i < 0\ and\ \beta_i < 0\}.$$

Proof. The proof of the above proposition follows from [25, Proposition 2.7] and the definition of the limiting normal cones. ◻

In many applications, Ω can be chosen as $\Omega = R^a \times R^b_+$ for some nonnegative integers a, b with $a + b = m$. Let $y = (z, u), F(x, y) = (G(x, y), H(x, y))$. Since $N_{R^a}(z) = \{0\}$ is a constant map, we have

$$D^*N_\Omega(\bar{y}, -F(\bar{x},\bar{y}))(\alpha,\beta) = \{0\} \times D^*N_{R^b_+}(\bar{u}, -H(\bar{x},\bar{z},\bar{u}))(\beta).$$

Again the limiting normal cone $N((\bar{u}, -H(\bar{x},\bar{z},\bar{u})), gphN_{R^b_+})$ can be calculated by using Proposition 3.7.

In the case where Ω is a polyhedral convex set, a calculation of the limiting normal cone to the graph of the normal cone to the set Ω was first given in the proof of [6, Theorem 2] and stated in [20, Proposition 4.4].

4. Constraint qualifications. In this section we study sufficient conditions for the calmness, introduce some constraint qualifications, and discuss the relationships between them.

DEFINITION 4.1. *We say that the constraint system* (CS) *for* (OPVIC)

$$\text{(CS)} \qquad \begin{aligned} &\psi(x,y) \le 0, (x,y) \in C,\\ &0 \in F(x,y) + N(y,\Omega) \end{aligned}$$

has a local error bound at a point $(\bar{x},\bar{y})$ *if there exist positive constants* μ, δ, *and* ϵ *such that*

$$d((x,y),\Sigma(0,0)) \le \mu\|(p,q)\| \quad \forall\ (p,q) \in \epsilon B, \qquad (x,y) \in \Sigma(p,q) \cap B_\delta(\bar{x},\bar{y}), \tag{4.1}$$

where

$$\Sigma(p,q) := \{(x,y) \in C : \psi(x,y) + p \le 0, q \in F(x,y) + N(y,\Omega)\} \tag{4.2}$$

is the set of solutions to the perturbed generalized equation.

Note that (CS) has a local error bound at a point $(\bar{x},\bar{y})$ if and only if $\Sigma(p,q)$ is pseudoupper-Lipschitz continuous around $(0,0,\bar{x},\bar{y})$ in the terminology of [26, Definition 2.8]. $\Sigma(p,q)$ being either pseudo-Lipschitz continuous around $(0,0,\bar{x},\bar{y})$ (see Definition 2.6) or upper-Lipschitz continuous (see Definition 2.7) at $(\bar{x},\bar{y})$ implies that (CS) has a local error bound at $(\bar{x},\bar{y})$.

We now prove that the existence of a local error bound for the constraint system of (OPVIC) at a solution $(\bar{x},\bar{y})$ implies that (OPVIC) is calm at $(\bar{x},\bar{y})$.

PROPOSITION 4.2. *Suppose that* (CS) *has a local error bound at* $(\bar{x},\bar{y})$, *a local solution to* (OPVIC). *Then* (GP) *is calm at* $(\bar{x},\bar{y})$.

Proof. Since (CS) has a local error bound at $(\bar{x},\bar{y})$, there exist positive numbers μ,δ,ϵ such that (4.1) is satisfied. Let $(p,q) \in \epsilon B$, $(x,y) \in \Sigma(p,q) \cap B_\delta(\bar{x},\bar{y})$ and $(x^*,y^*) \in \Sigma(0,0)$ be the projection of the vector (x,y). Then

$$
\begin{aligned}
f(\bar{x},\bar{y}) &\le f(x^*,y^*) \quad \text{since } (\bar{x},\bar{y}) \text{ solves (OPVIC)}\\
&= f(x,y) + (f(x^*,y^*) - f(x,y))\\
&\le f(x,y) + L_f\|(x^*,y^*) - (x,y)\| \quad \text{where } L_f \text{ is the Lipschitz constant of } f\\
&= f(x,y) + L_f d((x,y),\Sigma(0,0))\\
&\le f(x,y) + L_f\mu\|(p,q)\| \text{ by virtue of (4.1).}
\end{aligned}
$$

The proof is complete. □

We now study sufficient conditions for existence of a local error bound that are easier to verify. Recall that a set-valued map is called a polyhedral multifunction if its graph is a union of finitely many polyhedral convex sets. This class of set-valued maps is closed under (finite) addition, scalar multiplication, and (finite) composition. By Robinson [23, Proposition 1], a polyhedral multifunction is upper-Lipschitz. Hence the following result provides a sufficient condition for existence of a local error bound.

THEOREM 4.3. *Suppose that the mappings ψ, F are affine, C is polyhedral, and Ω is a polyhedral convex set. Then the solution map for the perturbed generalized equation* (4.2) *is upper-Lipschitz at any feasible solution of* (OPVIC) *and hence* (CS) *has a local error bound at any feasible solution of* (OPVIC).

Proof. Since the graph of N_Ω is a finite union of polyhedral convex sets, N_Ω is polyhedral. Hence $(\psi,F)+R^d_+\times N_\Omega$ (as the sum of polyhedral maps (ψ,F), $R^d_+\times N_\Omega$) is polyhedral, and so therefore is its inverse map

$$S(p,q) := \{(x,y) : \psi(x,y)+p \le 0, q \in F(x,y)+N(y,\Omega)\}.$$

That is, the graph

$$gphS := \{(x,y,p,q) : \psi(x,y)+p \le 0, q \in F(x,y)+N(y,\Omega)\}$$

is a union of polyhedral convex sets. Since

$$
\begin{aligned}
gph\Sigma &= \{(x,y,p,q) \in C\times R^d\times R^m : \psi(x,y)+p\le 0, q\in F(x,y)+N(y,\Omega)\}\\
&= (C\times R^d\times R^m)\cap gphS,
\end{aligned}
$$

which is also a union of polyhedral convex sets, Σ is also a polyhedral multifunction. By [23, Proposition 1], Σ is upper-Lipschitz. Hence (CS) has a local error bound at any feasible point. □

Remark. The result in the case $\Omega = R^m$ is actually the well-known error bound result for linear systems due to Hoffman [7]. In this case, the above result recovers the well-known result in nonlinear programming that no other constraint qualification is needed when the constraint system is linear.

We now prove that condition (NNAMCQ) defined in Corollary 3.3 is a sufficient condition for existence of a local error bound.

THEOREM 4.4. *Assume that condition* (NNAMCQ) *is satisfied at $(\bar{x},\bar{y})$. Then the solution map for the perturbed generalized equation* (4.2) *is pseudo-Lipschitz continuous around $(0,0,\bar{x},\bar{y})$ and hence* (CS) *has a local error bound at $(\bar{x},\bar{y})$.*

Proof. By virtue of [16, Proposition 3.5], it suffices to prove that

$$D^*\Sigma(0,0,\bar{x},\bar{y})(0,0) = \{(0,0)\}.$$

Suppose that $(\gamma,-\eta)\in D^*\Sigma(0,0,\bar{x},\bar{y})(0,0)$, which means by the definition of coderivatives that

$$(\gamma,-\eta,0,0)\in N((0,0,\bar{x},\bar{y}), gph\Sigma).$$

By the definition of limiting normal cones, there are sequences $(p^k, q^k, x^k, y^k) \to (0, 0, \bar{x}, \bar{y})$ and $(\gamma^k, -\eta^k, \alpha^k, \beta^k) \to (\gamma, -\eta, 0, 0)$ with

$$(\gamma^k, -\eta^k, \alpha^k, \beta^k) \in N^\pi((p^k, q^k, x^k, y^k), gph\Sigma).$$

For each k by the definition of proximal normal cones, there are $M > 0$ such that $\forall\ (p, q, x, y) \in gph\Sigma$,

$$\langle(\gamma^k, -\eta^k, \alpha^k, \beta^k), (p, q, x, y) - (p^k, q^k, x^k, y^k)\rangle \leq M\|(p, q, x, y) - (p^k, q^k, x^k, y^k)\|^2.$$

That is, (p^k, q^k, x^k, y^k) is a solution to the optimization problem

$$\begin{aligned} &\min\langle-(\gamma^k, -\eta^k, \alpha^k, \beta^k), (p, q, x, y)\rangle + M\|(p, q, x, y) - (p^k, q^k, x^k, y^k)\|^2 \\ &\text{s.t. } \psi(x, y) + p \leq 0, (x, y) \in C, \\ &\qquad q \in F(x, y) + N(y, \Omega). \end{aligned}$$

Inclusion (3.9) for the above problem is

$$\begin{aligned} &0 \in \{(\gamma, 0)\} \times \partial\langle\psi, \gamma\rangle(x^k, y^k) + \{(0, -\eta)\} \times \partial\langle F, \eta\rangle(x^k, y^k) \\ &\qquad + \{(0, 0, 0)\} \times D^*N_\Omega(y^k, q^k - F(x^k, y^k))(\eta) + \{(0, 0)\} \times N((x, y), C), \\ &\langle\psi(x^k, y^k) + p^k, \gamma\rangle = 0, \end{aligned}$$

which is only satisfied by $\gamma = 0, \eta = 0$ and hence (NNAMCQ) is satisfied at (p^k, q^k, x^k, y^k). Applying Corollary 3.3, there exist $\tilde{\gamma}^k \in R^d, \tilde{\eta}^k \in R^m$ such that

$$\begin{aligned} &0 \in -(\gamma^k, -\eta^k, \alpha^k, \beta^k) + \{(\tilde{\gamma}^k, 0)\} \times \partial\langle\psi, \tilde{\gamma}^k\rangle(x^k, y^k) + \{(0, -\tilde{\eta}^k)\} \times \partial\langle F, \tilde{\eta}^k\rangle(x^k, y^k) \\ &\qquad + \{(0, 0, 0)\} \times D^*N_\Omega(y^k, q^k - F(x^k, y^k))(\tilde{\eta}^k) + \{(0, 0)\} \times N((x^k, y^k), C), \\ &\langle\psi(x^k, y^k) + p^k, \tilde{\gamma}^k\rangle = 0. \end{aligned}$$

That is,

$$\begin{aligned} &(\alpha^k, \beta^k) \in \partial\langle\psi, \gamma^k\rangle(x^k, y^k) + \partial\langle F, \eta^k\rangle(x^k, y^k) \\ &\qquad + \{0\} \times D^*N_\Omega(y^k, q^k - F(x^k, y^k))(\eta^k) + N((x^k, y^k), C), \\ &\langle\psi(x^k, y^k) + p^k, \gamma^k\rangle = 0. \end{aligned}$$

Taking limits as $k \to \infty$ by virtue of Lipschitz continuity of ψ and F, we have

$$\begin{aligned} &0 \in \partial\langle\psi, \gamma\rangle(\bar{x}, \bar{y}) + \partial\langle F, \eta\rangle(\bar{x}, \bar{y}) + \{0\} \times D^*N_\Omega(\bar{y}, -F(\bar{x}, \bar{y}))(\eta) + N((\bar{x}, \bar{y}), C), \\ &\langle\psi(\bar{x}, \bar{y}), \gamma\rangle = 0. \end{aligned}$$

Consequently, by condition (NNAMCQ), $(\gamma, \eta) = (0, 0)$ and hence Σ is pseudo-Lipschitz continuous around $(0, 0, \bar{x}, \bar{y})$. □

In the case of the nonlinear programming problem (i.e, when $\Omega = R^m$), condition (NNAMCQ), with the limiting subgradient replaced by the Clarke generalized gradient, is equivalent to the generalized Mangasarian–Fromovitz constraint qualification (see, e.g., [24, Proposition 3.1] and [8]). We now extend the equivalence to the case where $\Omega = R^a \times R^b_+$. The result was proved by Outrata [18, Proposition 3.3] for the case where $\Omega = R^m_+$, ψ is independent of y and there are no abstract constraints. Note that our result improves the one in [18] in that no extra assumption such as (A) in [18] is needed for the inequality constraints. However, the proof technique is the same as that in [18]. Hence we only sketch the proof.

PROPOSITION 4.5. *Assume that* $\Omega = R^a \times R^b_+$ *with* a, b *nonnegative integers and* $a + b = m$, $C = D \times R^b$, *where* D *is a closed subset of* R^{n+a}. *Let* $y = (z, u)$ *and* $F(x, y) = (G(x, y), H(x, y))$ *and suppose all mappings* ψ, G, H *are* C^1. *We say that the generalized Mangasarian–Fromovitz constraint qualification* (GMFCQ) *is satisfied at* $(\bar{x}, \bar{y})$ *if*

(i) *for every partition of* I_0 *into sets* P, Q, R *with* $R \neq \emptyset$, *there exist vectors* $k \in intT_C((\bar{x}, \bar{z}), D), h \in R^b$ *such that* $h_{I_+} = 0, h_Q = 0, h_R \geq 0$,

$$\begin{aligned}
&\nabla_{x,z}\psi_M(\bar{x}, \bar{z}, \bar{u})k + \nabla_u \psi_M(\bar{x}, \bar{z}, \bar{u})h \leq 0,\\
&\nabla_{x,z}G(\bar{x}, \bar{z}, \bar{u})k + \nabla_u G(\bar{x}, \bar{z}, \bar{u})h = 0,\\
&\nabla_{x,z}H_{L\cup P}(\bar{x}, \bar{z}, \bar{u})k + \nabla_u H_{L\cup P}(\bar{x}, \bar{z}, \bar{u})h = 0,\\
&\nabla_{x,z}H_R(\bar{x}, \bar{z}, \bar{u})k + \nabla_u H_R(\bar{x}, \bar{z}, \bar{u})h \geq 0,
\end{aligned}$$

and either $h_i > 0$ *or*

$$\nabla_{x,z}H_i(\bar{x}, \bar{z}, \bar{u})k + \nabla_u H_i(\bar{x}, \bar{z}, \bar{u})h > 0 \text{ for some } i \in R;$$

(ii) *for every partition of* I_0 *into the sets* P, Q, *the matrix*

$$\begin{bmatrix} \nabla_{x,z}G(\bar{x}, \bar{z}, \bar{u}) & \nabla_u G_{L\cup P}(\bar{x}, \bar{z}, \bar{u}) \\ \nabla_{x,z}H_{L\cup P}(\bar{x}, \bar{z}, \bar{u}) & \nabla_u H_{L\cup P, L\cup P}(\bar{x}, \bar{z}, \bar{u}) \end{bmatrix}$$

has full row rank and there exist vectors $k \in intT_C((\bar{x}, \bar{z}), D), h \in R^b$ *such that*

$$\begin{aligned}
&h_{I_+} = 0, h_Q = 0,\\
&\nabla_{x,z}\psi_M(\bar{x}, \bar{z}, \bar{u})k + \nabla_u \psi_M(\bar{x}, \bar{z}, \bar{u})h < 0,\\
&\nabla_{x,z}G(\bar{x}, \bar{z}, \bar{u})k + \nabla_u G(\bar{x}, \bar{z}, \bar{u})h = 0,\\
&\nabla_{x,z}H_{L\cup P}(\bar{x}, \bar{z}, \bar{u})k + \nabla_u H_{L\cup P}(\bar{x}, \bar{z}, \bar{u})h = 0,
\end{aligned}$$

where $T_C((\bar{x}, \bar{z}), D)$ *denotes the Clarke tangent cone of* D *at* $(\bar{x}, \bar{z})$, $M := \{i : \psi_i(\bar{x}, \bar{z}, \bar{u}) = 0\}$ *is the index set of binding inequality constraints, and*

$$\begin{aligned}
L := L(\bar{x}, \bar{z}, \bar{u}) &:= \{i \in \{1, 2, \cdots, b\} : \bar{u}_i > 0, H_i(\bar{x}, \bar{z}, \bar{u}) = 0\},\\
I_+ := I_+(\bar{x}, \bar{z}, \bar{u}) &:= \{i \in \{1, 2, \cdots, b\} : \bar{u}_i = 0, H_i(\bar{x}, \bar{z}, \bar{u}) > 0\},\\
I_0 := I_0(\bar{x}, \bar{z}, \bar{u}) &:= \{i \in \{1, 2, \cdots, b\} : \bar{u}_i = 0, H_i(\bar{x}, \bar{z}, \bar{u}) = 0\}.
\end{aligned}$$

Then (GMFCQ) *implies* (NNAMCQ) *and under the assumption that* $\mathrm{int}T_C((\bar{x}, \bar{z}), D) \neq \emptyset$ (GMFCQ) *is equivalent to* (NNAMCQ) *with limiting normal cone of* D *replaced by the Clarke normal cone of* D.

Proof. Let $\eta = (\alpha, \beta)$. Then the condition (NNAMCQ) is equivalent to saying that there is no nonzero vector $(\gamma, \alpha, \beta) \in R^d_+ \times R^a \times R^b$ such that

$$\begin{aligned}
0 \in\ & \nabla\psi_M(\bar{x}, \bar{z}, \bar{u})^\top \gamma_M + \nabla G(\bar{x}, \bar{z}, \bar{u})^\top \alpha + \nabla H(\bar{x}, \bar{z}, \bar{u})^\top \beta\\
& + \{0\} \times \{0\} \times D^* N_{R^b_+}(\bar{u}, -H(\bar{x}, \bar{z}, \bar{u}))(\beta) + N((\bar{x}, \bar{z}), D) \times \{0\},
\end{aligned}$$

where $A^\top$ denotes the transpose of a matrix A. That is, there is no $(\gamma, \alpha, \beta) \neq 0$ such that $\gamma \geq 0$ and

$$\begin{aligned}
&-\nabla_{x,z}\psi_M(\bar{x}, \bar{z}, \bar{u})^\top \gamma_M - \nabla_{x,z}G(\bar{x}, \bar{z}, \bar{u})^\top \alpha - \nabla_{x,z}H(\bar{x}, \bar{z}, \bar{u})^\top \beta \in N((\bar{x}, \bar{z}), D),\\
&(-\nabla_u \psi_M(\bar{x}, \bar{z}, \bar{u})^\top \gamma_M - \nabla_u G(\bar{x}, \bar{z}, \bar{u})^\top \alpha - \nabla_u H(\bar{x}, \bar{z}, \bar{u})^\top \beta, -\beta)\\
&\qquad \in N((\bar{u}, -H(\bar{x}, \bar{z}, \bar{u})), gphN_{R^b_+}).
\end{aligned}$$

Let $(w, -\beta) \in N((\bar{u}, -H(\bar{x}, \bar{z}, \bar{u})), gphN_{R^b_+})$. Then, by Proposition 3.7, $w_L = 0, \beta_{I_+} = 0$ and for any $i \in I_0$, either $w_i\beta_i = 0$ or $w_i < 0, \beta_i < 0$. So I_0 splits into the sets

$$P := \{i \in I_0 : w_i = 0\}, \quad Q := \{i \in I_0 : \beta_i = 0\}, \quad R := \{i \in I_0 : w_i < 0, \beta_i < 0\}.$$

Using this partition, condition (NNAMCQ) is equivalent to the following two conditions:

(i) For every partition of I_0 into the sets P, Q, R with $R \neq \emptyset$ there are no vectors $\gamma_M, w, \alpha, \beta_{L\cup P\cup R}$ satisfying the system

$$\begin{aligned}
&-\nabla_{x,z}\psi_M(\bar{x}, \bar{z}, \bar{u})^\top \gamma_M - \nabla_{x,z}G(\bar{x}, \bar{z}, \bar{u})^\top \alpha \\
&\qquad\qquad -\nabla_{x,z}H_{L\cup P\cup R}(\bar{x}, \bar{z}, \bar{u})^\top \beta_{L\cup P\cup R} \in N((\bar{x}, \bar{z}), D) \\
&-\nabla_u\psi_{M,L\cup P}(\bar{x}, \bar{z}, \bar{u})^\top \gamma_M - \nabla_u G_{A,L\cup P}(\bar{x}, \bar{z}, \bar{u})^\top \alpha \\
&\qquad -\nabla_u H_{L\cup P\cup R,L\cup P}(\bar{x}, \bar{z}, \bar{u})^\top \beta_{L\cup P\cup R} = 0, \\
&w_{I_+\cup Q\cup R} = -\nabla_u\psi_{M,I_+\cup Q\cup R}(\bar{x}, \bar{z}, \bar{u})^\top \gamma_M - \nabla_u G_{A,I_+\cup Q\cup R}(\bar{x}, \bar{z}, \bar{u})^\top \alpha \\
&\qquad\qquad -\nabla_u H_{L\cup P\cup R,I_+\cup Q\cup R}(\bar{x}, \bar{z}, \bar{u})^\top \beta_{L\cup P\cup R}, \\
&\gamma_M \geq 0, w_R < 0, \beta_R < 0;
\end{aligned}$$

(ii) For every partition of I_0 into the sets P, Q there are no vectors $\gamma_M, w, \alpha, \beta_{L\cup P}$ satisfying the system

$$\begin{aligned}
&-\nabla_{x,z}\psi_M(\bar{x}, \bar{z}, \bar{u})^\top \gamma_M - \nabla_{x,z}G(\bar{x}, \bar{z}, \bar{u})^\top \alpha \\
&\qquad\qquad -\nabla_{x,z}H_{L\cup P}(\bar{x}, \bar{z}, \bar{u})^\top \beta_{L\cup P} \in N((\bar{x}, \bar{z}), D) \\
&-\nabla_u\psi_{M,L\cup P}(\bar{x}, \bar{z}, \bar{u})^\top \gamma_M - \nabla_u G_{A,L\cup P}(\bar{x}, \bar{z}, \bar{u})^\top \alpha \\
&\qquad\qquad -\nabla_u H_{L\cup P,L\cup P}(\bar{x}, \bar{z}, \bar{u})^\top \beta_{L\cup P} = 0, \\
&w_{I_+\cup Q} = -\nabla_u\psi_{M,I_+\cup Q}(\bar{x}, \bar{z}, \bar{u})^\top \gamma_M - \nabla_u G_{A,I_+\cup Q}(\bar{x}, \bar{z}, \bar{u})^\top \alpha \\
&\qquad\qquad -\nabla_u H_{L\cup P,I_+\cup Q}(\bar{x}, \bar{z}, \bar{u})^\top \beta_{L\cup P}, \\
&\gamma_M \geq 0,
\end{aligned}$$

where A denotes the index set $A := \{1, 2, \cdots, a\}$.

In the case where D is an open set, as in Outrata [18], the results follow from applying Motzkin's and Tucker's theorems of alternatives and the general case follows from applying the convex separation theorem. ∎

Remark. Note that in the case where $\Omega = R^m$, (OPVIC) is an ordinary nonlinear programming problem with equality, inequality constraints, and abstract constraints and (GMFCQ) is reduced to the condition that the matrix $\nabla F(\bar{x}, \bar{y})$ has full row rank and there exist vectors $k \in \mathrm{int}T_C((\bar{x}, \bar{y}), C)$ such that

$$\begin{aligned}
&\nabla\psi_M(\bar{x}, \bar{y})k < 0, \\
&\nabla F(\bar{x}, \bar{y})k = 0,
\end{aligned}$$

which is the generalized Mangasarian–Fromovitz constraint qualification for the nonlinear programming problems (see, e.g., Jourani [8]). Note that we can also deal with the case where the mappings ψ, F are not smooth but Lipschitz continuous only by replacing the gradient ∇ by the Clarke gradient ∂_C without any difficulty. The smoothness in the assumption is just for the easy exposition.

The following theorem extends a sufficient condition in [4, Theorem 3.3.1] for existence of a local error bound of an equality system to (CS). Note that as in the proof of [4, Theorem 3.3.8], we can prove that (NNAMCQ) is stronger than the following sufficient condition for existence of an local error bound.

THEOREM 4.6. *Let $(\bar{x}, \bar{y}) \in \Sigma(0,0)$, where Σ is the solution map (4.2). Assume that the bounded constraint qualification condition (Bounded CQ) is satisfied at $(\bar{x}, \bar{y})$, i.e., there exist constants $\mu > 0$, $0 < \epsilon \leq \infty$, such that*

$$\begin{aligned}\mu^{-1} \leq \inf\{\|\xi\| :& \xi \in \partial\langle\psi, e_1\rangle(x,y) + \partial\langle F, e_2\rangle(x,y) \\ & +\{0\} \times D^* N_\Omega(y, q - F(x,y))(e_2) + N((x,y), C), \\ & \langle\psi(x,y) + p, e_1\rangle = 0, \|(e_1, e_2)\| = 1, e_1 \geq 0, \\ & (p,q) \neq 0, (x,y) \in \Sigma(p,q) \cap B_\epsilon(\bar{x}, \bar{y})\}.\end{aligned}$$

Then if $\epsilon < \infty$ $\forall\ 0 < \delta < \epsilon$,

$$d((x,y), \Sigma(0,0)) \leq \mu\|(p,q)\| \ \forall\ (x,y) \in \Sigma(p,q) \cap B_\delta(\bar{x}, \bar{y}), (p,q) \in (\epsilon - \delta)\mu^{-1}B$$

and if $\epsilon = \infty$,

$$d((x,y), \Sigma(0,0)) \leq \mu\|(p,q)\| \ \forall\ (x,y) \in \Sigma(p,q).$$

Proof. Observe that

$$\begin{aligned}\Sigma(0,0) &= \{(x,y) : 0 \in \Phi(x,y)\} \\ &= \{(x,y) : d(0, \Phi(x,y)) = 0\},\end{aligned}$$

where $\Phi(x,y) := (-\psi(x,y), F(x,y)) + R^d_- \times N(y, \Omega) + \Delta_C(x,y)$ and Δ_C is the indicator mapping of set C defined by

$$\Delta_C(x,y) := \begin{cases} \{0\} & \text{if } (x,y) \in C, \\ \emptyset & \text{if } (x,y) \notin C. \end{cases}$$

It is obvious that the following claim will be useful.

Claim. Suppose the function $f(x) : R^n \to R \cup \{+\infty\}$ is nonnegative and lower semicontinuous. Let x_0 be a solution of $S = \{x : f(x) = 0\}$. Suppose that for some $\mu > 0, 0 < \epsilon \leq \infty$,

$$\|\xi\| \geq \mu^{-1} \ \forall\ \xi \in \partial^\pi f(x), 0 < f(x) < \infty, x \in B_\epsilon(x_0).$$

If $\epsilon < \infty$, then $\forall\ 0 < \delta < \epsilon$,

$$d(x, S) \leq \mu f(x) \qquad \forall x \in B_\delta(x_0), f(x) < (\epsilon - \delta)\mu^{-1}$$

and if $\epsilon = \infty$, then

$$d(x, S) \leq \mu f(x) \qquad \forall x \in R^n.$$

Proof of the claim. Taking $V = B_\epsilon(x_0)$ in [4, Theorem 3.3.1],

$$\min\{d(x, B_\epsilon(x_0)^C), d(x, S)\} \leq \mu f(x) \quad \forall x \in B_\epsilon(x_0),$$

where Ω^C denotes the complement of a set Ω.

Let $0 < \delta < \epsilon$ and $x \in B_\delta(x_0)$. Then obviously, $d(x, B_\epsilon(x_0)^C) > \epsilon - \delta$. Hence for all $x \in B_\delta(x_0)$ satisfying $f(x) < (\epsilon - \delta)\mu^{-1}$,

$$d(x, S) = \min\{d(x, B_\epsilon(x_0)^C), d(x, S)\} \leq \mu f(x) < \epsilon - \delta.$$

In the case $\epsilon = \infty$, $d(x, B_\infty(x_0)^C) = \infty$, hence

$$d(x, S) \leq \mu f(x) \quad \forall x.$$

The proof of the claim is complete.

Observe that

$$d(0, \Phi(x, y)) := \inf\{\|(p, q)\| : (p, q) \in \Phi(x, y)\} = \inf\{\|(p, q)\| + \Psi_{gph\Phi}(x, y, p, q)\},$$

where Ψ_E denotes the indicator function of set E. By the statement and the proof of [9, Theorem 5A.2], the function $(x, y) \to d(0, \Phi(x, y))$ is lower semicontinuous and

$$\begin{aligned}\partial^\pi d(0, \Phi(x, y)) \subseteq \{(\gamma, \eta) : (\gamma, \eta, 0, 0) \in \partial^\pi g(x, y, p, q)\\ \text{for some } (p, q) \text{ such that } d(0, \Phi(x, y)) = \|(p, q)\| + \Psi_{gph\Phi}(x, y, p, q)\},\end{aligned}$$

where $g(x, y, p, q) := \|(p, q)\| + \Psi_{gph\Phi}(x, y, p, q)$. At the point $(x, y, p, q) \in gph\Phi$ such that $0 < d(0, \Phi(x, y)) = \|(p, q)\|$, $\|(p, q)\|$ is smooth and the subgradient is the unit sphere S_{d+m}. By the sum rule Proposition 2.3, we have

$$\partial^\pi g(x, y, p, q) = \{0\} \times \{0\} \times S_{d+m} + N^\pi((x, y, p, q), gph\Phi).$$

Hence

$$\begin{aligned}\partial^\pi d(0, \Phi(x, y)) \subseteq \{(\gamma, \eta) : (\gamma, \eta, 0, 0) \in \{0\} \times \{0\} \times S_{d+m} + N^\pi((x, y, p, q), gph\Phi)\\ \text{for some } (p, q) \text{ such that } d(0, \Phi(x, y)) = \|(p, q)\| + \Psi_{gph\Phi}(x, y, p, q)\}.\end{aligned}$$

For any $(\gamma, \eta, 0, 0) \in \{0\} \times \{0\} \times S_{d+m} + N^\pi((x, y, p, q), gph\Phi)$, there exists $(e_1, e_2) \in S_{d+m}$ such that $(\gamma, \eta, e_1, -e_2) \in N^\pi((x, y, p, q), gph\Phi)$. By definition of the proximal normal cone, there exists $M > 0$ such that $\forall\ (x', y', p', q') \in gph\Phi$,

$$\langle(\gamma, \eta, e_1, -e_2), (x', y', p', q') - (x, y, p, q)\rangle \leq M\|(x', y', p', q') - (x, y, p, q)\|^2.$$

That is, (x, y, p, q) is an optimal solution to

$$\begin{aligned}\min \quad & \langle -(\gamma, \eta, e_1, -e_2), (x', y', p', q')\rangle + M\|(x', y', p', q') - (x, y, p, q)\|^2\\ \text{s.t.} \quad & \psi(x', y') + p' \leq 0, (x', y') \in C,\\ & q' \in F(x', y') + N(y', \Omega).\end{aligned}$$

One can easily verify that (NNAMCQ) for the above problem is satisfied. Applying Corollary 3.3, we conclude that

$$\begin{aligned}(\gamma, \eta) \in\ & \partial\langle\psi, e_1\rangle(x, y) + \partial\langle F, e_2\rangle(x, y)\\ & +\{0\} \times D^*N_\Omega(y, q - F(x, y))(e_2) + N(x, y, C),\\ & e_1 \geq 0, \langle\psi(x, y) + p, e_1\rangle = 0.\end{aligned}$$

Hence,

$$\begin{aligned}\partial^\pi d(0, \Phi(x, y)) \subseteq & \{(\gamma, \eta) : (\gamma, \eta) \in \partial\langle\psi, e_1\rangle(x, y) + \partial\langle F, e_2\rangle(x, y)\\ & +\{0\} \times D^*N_\Omega(y, q - F(x, y))(e_2) + N((x, y), C)\\ & \text{for some } (e_1, e_2) \in S_{d+m} \text{ such that } e_1 \geq 0, \langle p + \psi(x, y), e_1\rangle = 0,\\ & \text{and some } (p, q) \text{ such that } d(0, \Phi(x, y)) = \|(p, q)\| + \Psi_{gph\Phi}(x, y, p, q)\}.\end{aligned}$$

The proof of the theorem is completed after applying the claim. $\square$

Now consider the case where the abstract constraint is independent of y, i.e., $C = D \times R^m$, there is no inequality constraint and $\forall\, x$ near $\bar{x}$ the solution map

$$y(x) := \{y \in R^m : 0 \in F(x,y) + N(y,\Omega)\}$$

is single-valued and Lipschitz on a neighborhood of $\bar{y}$. Then it is obvious that a local solution $(\bar{x},\bar{y})$ of (OPVIC) is also a local solution to the problem of minimizing $f(x,y(x))$ over D, and hence no other constraint qualifications are needed. A sufficient condition for the existence of such a Lipschitz continuous single-valued map is the strong regularity of the generalized equation

$$0 \in F(\bar{x},y) + N(y,\Omega) \tag{4.3}$$

at $\bar{y}$ in the sense of Robinson [22]. Indeed, in the following theorem we will show that strong regularity is stronger than the constraint qualification (NNAMCQ). The reader is referred to [22] for conditions of strong regularity. Since (4.3) is strongly regular in particular if F is locally strongly monotone in y uniformly in x, the following condition is weaker than the one in [26, Theorem 3.2 (b)]. Note that the result for the case $\Omega = R^m_+$ was proved by Outrata [18] using a different proof.

THEOREM 4.7. *Let $(\bar{x},\bar{y})$ be a solution to the generalized equation. Assume that $F(x,y)$ is C^1 around $(\bar{x},\bar{y})$ and the generalized equation* (4.3) *is strongly regular at $\bar{y}$. Then the constraint qualification* (NNAMCQ) *is satisfied at $(\bar{x},\bar{y})$.*

Proof. Let $y := x$ and $f(q,y) := -q + F(\bar{x},y)$ in [22, Theorem 2.1 and Corollary 2.2]. Since the generalized equation (4.3) is strongly regular at $\bar{y}$, there exist neighborhoods N of 0 and W of $\bar{y}$, and a single-valued function $y(q) : N \to W$, such that for any $q \in N$, $y(q)$ is the unique solution in W of the inclusion

$$q \in F(\bar{x},y) + N(y,\Omega).$$

Further, $y(q)$ is Lipschitz continuous near 0. That is $\Sigma_{\bar{x}}(q) := \{y \in R^m : q \in F(\bar{x},y) + N(y,\Omega)\}$ is pseudo-Lipschitz continuous around $(0,\bar{y})$. Note that from [15, Theorem 5.8], $\Sigma_{\bar{x}}(q)$ is pseudo-Lipschitz continuous around $(0,\bar{y})$ if and only if there is no nonzero vector $\eta \in R^m$ such that

$$0 \in \nabla_y F(\bar{x},\bar{y})^\top \eta + D^* N_\Omega(\bar{y}, -F(\bar{x},\bar{y}))(\eta).$$

Therefore there is no nonzero vector $\eta \in R^m$ such that

$$\begin{aligned} &0 \in \nabla_x F(\bar{x},\bar{y})^\top \eta + N(\bar{x},D),\\ &0 \in \nabla_y F(\bar{x},\bar{y})^\top \eta + D^* N_\Omega(\bar{y}, -F(\bar{x},\bar{y}))(\eta). \end{aligned}$$

That is, (NNAMCQ) is satisfied. ◻

COROLLARY 4.8. *The following conditions are constraint qualifications:*

(1) [calmness constraint qualification (calmness CQ)]*: The problem* (GP) *is calm at $(\bar{x},\bar{y})$.*
(2) [error bound CQ]*:* (CS) *has a local error bound at $(\bar{x},\bar{y})$.*
(3) [linear constraint qualification (linear CQ)]*: The mappings ψ, F are affine, C is polyhedral, and Ω is a polyhedral convex set.*
(4) [strongly regular constraint qualification (SRCQ)]: *There is no inequality constraint $\psi(x,y) \le 0$. F is C^1 around the optimal solution $(\bar{x},\bar{y})$. $C = D \times R^m$, where D is a closed subset of R^n. The generalized equation*

$$0 \in F(\bar{x},y) + N(y,\Omega)$$

is strongly regular at $\bar{y}$.

(5) [no nonzero abnormal multiplier constraint qualification (NNAMCQ)]*: There is no nonzero vector* $(\gamma,\eta) \in R^d_+ \times R^m$ *such that*

$$0 \in \partial\langle\psi,\gamma\rangle(\bar{x},\bar{y}) + \partial\langle F,\eta\rangle(\bar{x},\bar{y}) + \{0\} \times D^* N_\Omega(\bar{y},-F(\bar{x},\bar{y}))(\eta) + N((\bar{x},\bar{y}),C)$$
$$\langle\psi(\bar{x},\bar{y}),\gamma\rangle = 0.$$

(6) [generalized Mangasarian–Fromovitz constraint qualification (GMFCQ)]*: Stated as in Proposition* 4.5.

(7) [bounded constraint qualification (bounded CQ)]*: There exist constants* $\mu > 0$, $0 < \epsilon \leq \infty$, *such that*

$$\begin{aligned}\mu^{-1} \leq \inf\{\|\xi\| :& \xi \in \partial\langle\psi,e_1\rangle(x,y) + \partial\langle F,e_2\rangle(x,y)\\ &+\{0\} \times D^* N_\Omega(y,q-F(x,y))(e_2) + N((x,y),C),\\ &\langle\psi(x,y)+p,e_1\rangle = 0, \|(e_1,e_2)\| = 1, e_1 \geq 0,\\ &(p,q) \neq 0, (x,y) \in \Sigma(p,q) \cap B_\epsilon(\bar{x},\bar{y})\}.\end{aligned}$$

In summary, we have proved the following relationships between the constraint qualifications:

$$\begin{array}{ccc}
 & \text{(SRCQ)} & \\
 & \Downarrow & \\
 & \text{(NNAMCQ)} & \Longleftarrow \text{(GMFCQ) when } \Omega = R^a \times R^b_+ \\
 & \Downarrow & \\
\text{Linear CQ} & \text{Bounded CQ} & \\
\Downarrow & \Downarrow & \\
\text{Error Bound CQ} & & \\
\Downarrow & & \\
\text{Calmness CQ.} & &
\end{array}$$

5. Applications to bilevel programming problems. The purpose of this section is to illustrate applications of the results obtained in the previous sections to the bilevel programming problems defined as follows:

$$\text{(BP)} \qquad \text{minimize } f(x,z) \qquad \text{s.t. } \psi(x,z) \leq 0, (x,z) \in D \text{ and } z \in S(x),$$

where $S(x)$ is the set of solutions of the problem (P_x):

$$(P_x) \qquad \text{minimize } g(x,z) \qquad \text{s.t. } \varphi(x,z) \leq 0$$

and $f : R^{n+a} \to R$, $\psi : R^{n+a} \to R^d$, $\varphi : R^{n+a} \to R^b$. For simplicity, we assume all functions f, g, ψ, φ are smooth enough.

Let $z \in S(x)$. If a certain constraint qualification holds for the lower level problem (P_x) at z, then there exists $u \in R^b$ such that

$$\begin{aligned}\nabla_z g(x,z) + u\nabla_z\varphi(x,z) = 0, \varphi(x,z) \leq 0,\\ u \geq 0, \langle u,\varphi(x,z)\rangle = 0,\end{aligned}$$

where $u\nabla_z\varphi(x,z) := \sum u_k \nabla_z\varphi_k(x,z)$. It is easy to see that the above Kuhn–Tucker conditions for (P_x) can be written as the generalized equation

$$0 \in ((\nabla_z g + u\nabla_z\varphi)^t(x,z), -\varphi(x,z)) + N((z,u), R^a \times R^b_+),$$

where a^t denotes the transpose of a vector a. Hence the original bilevel programming problem becomes an (OPVIC).

Applying Theorems 3.2 and 3.6 we now derive necessary optimality conditions for (BP).

THEOREM 5.1. *Assume that f and ψ are C^1, g, φ are twice continuously differentiable around $(\bar{x}, \bar{z})$. Further assume that g is pseudoconvex in z, φ is quasi-convex in z. Let $(\bar{x}, \bar{z})$ solve the problem* (BP). *For each feasible solution (x, z) of* (BP) *suppose that a certain constraint qualification holds for (P_x) at z and $\bar{u}$ is a corresponding multiplier associated with $(\bar{x}, \bar{z})$, i.e.,*

$$0 = \nabla_z g(\bar{x}, \bar{z}) + \bar{u}\nabla_z \varphi(\bar{x}, \bar{z}), \quad \bar{u} \geq 0, \quad \langle \varphi(\bar{x}, \bar{z}), \bar{u} \rangle = 0.$$

Then there exist $\lambda \geq 0$, $\gamma \in R^d_+$, $\alpha \in R^a, \beta \in R^b$ not all zero such that

$$0 \in \lambda \nabla f(\bar{x}, \bar{z}) + \gamma \nabla \psi(\bar{x}, \bar{z}) + \alpha \nabla(\nabla_z g + \bar{u}\nabla_z \varphi)^t(\bar{x}, \bar{z}) - \beta \nabla \varphi(\bar{x}, \bar{z}) + N((\bar{x}, \bar{z}), D),$$
$$\langle \psi(\bar{x}, \bar{z}), \gamma \rangle = 0, \quad (-\nabla_z \varphi(\bar{x}, \bar{z})\alpha, -\beta) \in N(\bar{u}, \varphi(\bar{x}, \bar{z})), gphN_{R^b_+}).$$

λ can be taken as 1 if one of the following constraint qualifications hold:

(a) *$\nabla_z g, \psi, \varphi$ are affine mappings and D is polyhedral.*

(b) *There is no nonzero vector $(\gamma, \alpha, \beta) \in R^d_+ \times R^a \times R^b$ such that*

$$0 \in \gamma \nabla \psi(\bar{x}, \bar{z}) + \alpha \nabla(\nabla_z g + \bar{u}\nabla_z \varphi)^t(\bar{x}, \bar{z}) - \beta \nabla \varphi(\bar{x}, \bar{z}) + N((\bar{x}, \bar{z}), D),$$
$$\langle \psi(\bar{x}, \bar{z}), \gamma \rangle = 0, \quad (-\nabla_z \varphi(\bar{x}, \bar{z})\alpha, -\beta) \in N((\bar{u}, \varphi(\bar{x}, \bar{z})), gphN_{R^b_+}).$$

(c) *There exist $\mu > 0$ and $\epsilon > 0$ such that*

$$\begin{aligned}\mu^{-1} \leq &\inf\{\|(\xi_1, \xi_2)\| : \\ &\xi_1 \in e_1 \nabla \psi(x, z) + e_2 \nabla(\nabla_z g + u\nabla_z \varphi)^t(x, z) - e_3 \nabla \varphi(x, z) + N((x, z), D), \\ &(\xi_2 - \nabla_z \varphi(x, z)e_2, -e_3) \in N((u, q + \varphi(x, z)), gphN_{R^b_+}), \\ &\langle \psi(x, z) + p, e_1 \rangle = 0, \|(e_1, e_2, e_3)\| = 1, e_1 \geq 0, \\ &(p, q) \neq 0, (x, z, u) \in \Sigma(p, q) \cap B_\epsilon(\bar{x}, \bar{z}, \bar{u})\},\end{aligned}$$

where

$$\begin{aligned}\Sigma(p, q) := \{&(x, z, u) \in C \times R^b : \psi(x, z) + p \leq 0, \\ &q \in ((\nabla_z g + u\nabla_z \varphi)^t(x, z), -\varphi(x, z)) + N((z, u), R^a \times R^b_+)\}.\end{aligned}$$

(d) *$D = E \times R^a$, where E is a closed subset of R^n and there is no inequality constraint $\psi(x, z) \leq 0$. Furthermore the strong second order sufficient condition and the linear independence of binding constraints hold for the lower level problem $P_{\bar{x}}$ at $\bar{z}$, i.e., for any nonzero v such that*

$$\nabla_z \varphi_i(\bar{x}, \bar{z})^t v = 0, \quad i \in L,$$

$\langle v, (\nabla^2_z g(\bar{x}, \bar{z}) + \bar{u}\nabla^2_z \varphi(\bar{x}, \bar{z}))v \rangle > 0$, and gradients of the binding constraints $\{\nabla_z \varphi_i(\bar{x}, \bar{z}), i \in L \cup I_0\}$ are linearly independent, where

$$\bar{u}\nabla^2_z \varphi(\bar{x}, \bar{z}) := \sum \bar{u}_i \nabla^2_z \varphi_i(\bar{x}, \bar{z})$$

and

$$\begin{aligned}L := L(\bar{x}, \bar{z}, \bar{u}) &:= \{i : \bar{u}_i > 0, \varphi_i(\bar{x}, \bar{z}) = 0\}, \\ I_0 := I_0(\bar{x}, \bar{z}, \bar{u}) &:= \{i : \bar{u}_i = 0, \varphi_i(\bar{x}, \bar{z}) = 0\}, \\ I_+ := I_+(\bar{x}, \bar{z}, \bar{u}) &:= \{i : \bar{u}_i = 0, \varphi_i(\bar{x}, \bar{z}) < 0\}.\end{aligned}$$

Proof. Since the objective function of the lower level problem g is pseudoconvex in z and the constraint φ is quasi-convex in z, by Theorem 4.2.11 of Bazaraa and Shetty [2] the Kuhn–Tucker condition is a necessary and sufficient condition for optimality. Therefore from the discussion preceding Theorem 5.1 we know that $(\bar{x}, \bar{z})$ is a solution of the following problem:

$$\begin{array}{ll} \min & f(x,z) \\ \text{s.t.} & 0 \in ((\nabla_z g + u\nabla_z \varphi)^t(x,z), -\varphi(x,z)) + N((z,u), R^a \times R^b_+), \\ & \psi(x,z) \le 0, (x,z) \in C. \end{array} \tag{5.1}$$

Condition (a) is the linear constraint qualification (Linear CQ). Condition (b) is the no nonzero abnormal multiplier constraint qualification (NNAMCQ). Condition (c) is the bounded constraint qualification (Bounded CQ). Condition (d) is a sufficient condition for the strong regularity of the generalized equation (5.1) by virtue of [22, Theorem 4.1]. □

Remark. In the case where $D = \{(x,z) : h(x,z) \le 0\}$ and $h(x,z) : R^{n+a} \to R^q$, if h is an affine mapping, it is known that

$$N((\bar{x},\bar{z}), D) = \{\zeta \nabla h(\bar{x},\bar{z}) : \zeta \in R^q_+, \langle h(\bar{x},\bar{z}), \zeta\rangle = 0\}.$$

In this case, the necessary optimality condition becomes the existence of $\lambda \ge 0$, $\gamma \in R^d_+$, $\alpha \in R^a, \beta \in R^b$ not all zero and $\zeta \in R^q_+$ such that

$$\begin{aligned} &0 = \lambda\nabla f(\bar{x},\bar{z}) + \gamma\nabla\psi(\bar{x},\bar{z}) + \alpha\nabla(\nabla_z g + \bar{u}\nabla_z\varphi)^t(\bar{x},\bar{z}) - \beta\nabla\varphi(\bar{x},\bar{z}) + \zeta\nabla h(\bar{x},\bar{z}), \\ &\langle h(\bar{x},\bar{z}), \zeta\rangle = 0, \langle \psi(\bar{x},\bar{z}), \gamma\rangle = 0, \\ &(-\nabla_z\varphi(\bar{x},\bar{z})\alpha, -\beta) \in N(\bar{u}, \varphi(\bar{x},\bar{z})), gphN_{R^b_+}). \end{aligned}$$

Hence incorporating an abstract constraint in (OPVIC) can be used as a useful device to handle linear and nonlinear constraints separately.

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