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SUFFICIENT CONDITIONS FOR ERROR BOUNDS*

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Abstract. For a lower semicontinuous (l.s.c.) inequality system on a Banach space, it is shown that error bounds hold, provided every element in an abstract subdifferential of the constraint function at each point outside the solution set is norm bounded away from zero. A sufficient condition for a global error bound to exist is also given for an l.s.c. inequality system on a real normed linear space. It turns out that a global error bound closely relates to metric regularity, which is useful for presenting sufficient conditions for an l.s.c. system to be regular at sets. Under the generalized Slater condition, a continuous convex system on R^n is proved to be metrically regular at bounded sets.

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1. Introduction.

Let X be a real normed linear space and C a nonempty closed subset of X. Let $f_i, |g_j| : X \to (-\infty, +\infty]$ be lower semicontinuous (l.s.c.) for each $i = 1, \dots, r$ and $j = 1, \dots, s$. Denote the solution set of an l.s.c. (inequality) system by

$$S := \{x \in C : f_1(x) \le 0, \dots, f_r(x) \le 0; g_1(x) = 0, \dots, g_s(x) = 0\},$$

which is assumed to be nonempty. *The distance function* $d_S : X \to R$ is defined by

$$d_S(x) = \inf\{\|x - s\| : s \in S\}.$$

The set S is said to have a *global error bound* if there exists a constant $\mu > 0$ such that

$$d_S(x) \le \mu(\,\|F(x)_+\| + \|G(x)\|\,) \quad \forall x \in C,$$

where $F(x)_+ = (f_1(x)_+, \dots, f_r(x)_+) \in R^r$ with $f_i(x)_+ := \max\{f_i(x), 0\}$ for $i = 1, \dots, r$, $G(x) = (g_1(x), \dots, g_s(x)) \in R^s$ and $\|\cdot\|$ is the usual Euclidean norm. The set S is said to have a *local error bound* if there exist constants $\mu > 0$ and $\delta > 0$ such that

$$d_S(x) \le \mu(\,\|F(x)_+\| + \|G(x)\|\,) \quad \forall x \in C \text{ with } \|(\,F(x)_+, G(x)\,)\| < \delta.$$

Apparently if the set S has a global (local) error bound, then functions involved provide a global (local) error estimate for the distance from any point x to the solution set S. Because this kind of estimation has many important applications in optimization, many sufficient conditions for error bounds to exist have been given since Hoffman [10] proved that a global error bound always holds for any linear inequality systems on R^n. The reader is referred to [1, 2, 4, 5, 6, 9, 11, 13, 14, 15, 16, 17, 18, 21, 22, 23] and the references therein for the results on error bounds.

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It is worth pointing out that there are two important classes of conditions in these results. One contains the Slater condition (explicitly or implicitly), which closely relates to the points inside the solution set S of a system, while the other is expressed by various subdifferentials of functions at the points outside S. Since the latter includes subdifferentials, one can use the knowledge of nonsmooth analysis to study this issue more effectively.

To the authors' knowledge, it is Ioffe [11] who first used Ekeland's variational principle and the sum rule to prove the existence of a global error bound (as well as metric regularity at a point) for a Lipschitz continuous equality system under the condition that any element in the Clarke subdifferential of the constraint function at each point outside the solution set be norm bounded away from zero. Using Ioffe's method, Ye [22] and Jourani [13] have sharpened the result of Ioffe by replacing the Clarke subdifferential with the limiting subdifferential in R^n and a partial subdifferential in a general Banach space, respectively. In a Hilbert space, Clarke et al. [4, Theorem 3.3.1] have weakened Ioffe's condition using the proximal subdifferential instead of the Clarke subdifferential (see also Ye [23, Claim]). Since the proximal subdifferential does not satisfy the sum rule, the result of Clarke et al. was proved not by Ioffe's method but by the decrease principle. We note that Wu [21] has used a different tool, that is, the fuzzy sum rule (instead of the sum rule), to prove that the Clarke subdifferential in Ioffe's condition can be replaced by the proximal subdifferential for an l.s.c. system on R^n. This method is in fact suitable for various subdifferentials on a Banach space no matter whether they satisfy the fuzzy sum rule or the sum rule since the latter always implies the former. Therefore in this paper, we introduce an abstract subdifferential which satisfies the fuzzy sum rule and then take advantage of this method to show that the Clarke subdifferential in Ioffe's condition can really be replaced by such an abstract subdifferential which includes many subdifferentials in the nonsmooth analysis literature. These results unify and extend those stated in this paragraph. In fact, for an l.s.c. system they have provided sufficient conditions not only for a global error bound but also for a local error bound as well as for metric regularity.

This paper is organized as follows. In section 2, we introduce the concept of ∂_ω-subdifferential and show that several common subdifferentials in nonsmooth analysis are ∂_ω-subdifferentials. In section 3, we use the ∂_ω-subdifferential to present sufficient conditions for error bounds to exist for l.s.c. inequality systems on Banach spaces. Section 4 is devoted to a sufficient condition for a global error bound to hold for a general inequality system on a real normed linear space. With this result we extend those of Deng [5, 6] to an l.s.c. convex system on a real normed linear space. In section 5, relations between error bound and metrical regularity are revealed, and some sufficient conditions are given for a continuous convex system to be metrically regular at a nonempty set. In particular, we prove that a generalized Slater condition is sufficient for a continuous convex system to be metrically regular at any bounded sets in R^n.

Throughout this paper, B and $\overline{B}$, respectively, denote the open unit ball and its closure of X, while B^* and $\overline{B^*}$ are, respectively, the open unit ball and its closure of the dual space X^*. For a nonempty closed subset C of X, ψ_C and $N_C(x)$ denote the indicator function of C and the (Clarke) normal cone to C at $x \in C$, respectively. For an extended real-valued function f defined on X, its epigraph is written as

$$epi\, f := \{(x, r) \in dom\, f \times R : f(x) \le r\},$$

where $dom\, f := \{x \in X : f(x) < +\infty\}$.

2. ∂_ω-subdifferentials. Here we introduce the concept of ∂_ω-subdifferentials for l.s.c. functions, which unifies that of several common subdifferentials in the nonsmooth analysis literature.

DEFINITION 2.1. *Let X be a Banach space and $f : X \to (-\infty, +\infty]$ be l.s.c. at $x \in dom\, f$. A subset of X^*, denoted by $\partial_\omega f(x)$, is called a* ∂_ω-subdifferential of f *at x if it has the following properties:*

(ω_1) *$\partial_\omega g(x) = \partial_\omega f(x)$ if $g = f$ in a neighborhood of x.*
(ω_2) *$0 \in \partial_\omega f(x)$ when f attains a local minimum at x.*
(ω_3) *If f is convex and Lipschitz of rank L near x, then $\partial_\omega f(x) \subseteq L\overline{B^*}$.*
(ω_4) *The fuzzy sum rule holds; that is, if $g : X \to (-\infty, +\infty]$ is Lipschitz near x, then for any $\xi \in \partial_\omega(f+g)(x)$ and any $\delta > 0$, there exist $x_1, x_2 \in x + \delta B$ such that*

$$f(x_1) \in f(x) + \delta B_1,\ g(x_2) \in g(x) + \delta B_1,\ \text{and}\ \xi \in \partial_\omega f(x_1) + \partial_\omega g(x_2) + \delta B^*,$$

where $B_1 = (-1, 1)$.

The following commonly used subdifferentials turn out to be ∂_ω-subdifferentials.

Example 2.1. Let X be a Banach space and $f : X \to (-\infty, +\infty]$ be l.s.c. at $x \in dom\, f$. The *Clarke–Rockafellar generalized derivative* of f at x in the direction $v \in X$ is defined as follows:

$$f^\circ(x; v) := \lim_{\epsilon \to 0^+} \limsup_{\substack{y \xrightarrow{f} x \\ t \to 0^+}} \inf_{w \in v + \epsilon B} \frac{f(y + tw) - f(y)}{t},$$

where $y \xrightarrow{f} x$ signifies that y and $f(y)$ converge to x and $f(x)$, respectively. The *generalized gradient* of f at x is the subset of X^* given by

$$\partial f(x) = \begin{cases} \{\xi \in X^* : f^\circ(x; v) \geq \langle \xi, v \rangle \quad \forall v \in X\} & \text{if } f^\circ(x; 0) \neq -\infty; \\ \emptyset & \text{if } f^\circ(x; 0) = -\infty. \end{cases}$$

By the above definition (see Clarke [3, Proposition 2.1.2 (a) and Corollary 1 of Theorem 2.9.8]), $\partial f(x)$ satisfies properties (ω_1)–(ω_4).

Example 2.2. Let X be an Asplund space, i.e., a Banach space such that every continuous convex function is Fréchet differentiable at each point of some G_δ dense subset of this space (which includes all reflexive Banach spaces). Let $f : X \to (-\infty, +\infty]$ be l.s.c. at $x \in dom\, f$. The *Fréchet subdifferential* of f at x, denoted by $\partial_F f(x)$, is the set

$$\left\{ \xi \in X^* : \liminf_{\|h\| \to 0} \frac{f(x + h) - f(x) - \langle \xi, h \rangle}{\|h\|} \geq 0 \right\}.$$

Based on the definition, Ioffe [12, Proposition 1], and Fabian [8, Theorem 3], $\partial_F f(x)$ is a ∂_ω-subdifferential of f at x.

Example 2.3. Let H be a Hilbert space and $f : H \to (-\infty, +\infty]$ be l.s.c. at $x \in dom\, f$. A vector $\xi \in H^*$ is called a *proximal subgradient* of f at x provided that there exist positive numbers M and δ such that

$$f(y) \geq f(x) + \langle \xi, y - x \rangle - M\|y - x\|^2 \quad \forall y \in x + \delta B.$$

The set of all such ξ is denoted by $\partial^\pi f(x)$ and is referred to as the *proximal subdifferential* of f at x. It follows that $\partial^\pi f(x)$ satisfies properties (ω_1)–(ω_4) from the above inequality and Clarke et al. [4, Theorems 1.7.3 and 1.8.3].

Remark 2.1. (i) For a convex function, all subdifferentials in Examples 2.1–2.3 coincide with the subdifferentials in the sense of convex analysis.

(ii) The Fréchet subdifferential contains only the Fréchet derivative whenever a function is Fréchet differentiable, and the proximal subdifferential includes only the Fréchet derivative when a function is Fréchet differentiable and its Fréchet derivative is locally Lipschitz continuous.

(iii) In an Asplund space, one has

$$\partial_F f(x) \subseteq \partial f(x),$$

while in a Hilbert space, the following inclusions hold:

$$\partial^\pi f(x) \subseteq \partial_F f(x) \subseteq \partial f(x).$$

3. Sufficient conditions for an l.s.c. system. Consider a simple inequality system

$$f(x) \le 0,$$

where f is a locally Lipschitz function defined on R. If the solution set $S := \{x \in R : f(x) \le 0\}$ is nonempty, then the inequality $d_S(x) \le \mu f(x)_+$ holds automatically for any $x \in S$ and any $\mu > 0$. To look for a sufficient condition for this inequality to hold for some $\mu > 0$ and any point $x \in R\backslash S$, we can take one point $x_0 \in S$ such that $f(x_0) = 0$ and $f(y) > 0$ for any $y \in (x_0, x] = \{tx_0 + (1-t)x : t \in [0, 1)\}$. By the Lebourg mean-value theorem [3, Theorem 2.3.7], there exist $z \in (x_0, x]$ and $\xi \in \partial f(z)$ such that

$$f(x) - f(x_0) = \xi \cdot (x - x_0),$$

from which $f(x)_+ = \|\xi\| \cdot \|x - x_0\| \ge \|\xi\| d_S(x)$. Therefore if $\|\xi\| \ge \mu^{-1}$ holds for some $\mu > 0$ and any $\xi \in \partial f(x)$ for each $x \in R\backslash S$, then $d_S(x) \le \mu f(x)_+$ holds for any $x \in R$.

For an l.s.c. function f defined on a Banach space X, will the existence of a positive constant μ such that

$$\|\xi\| \ge \mu^{-1} \quad \forall \xi \in \partial_\omega f(x) \qquad \forall x \in X\backslash S$$

also imply the existence of a global error bound? The following theorem gives an affirmative answer.

THEOREM 3.1. *Let $f : X \to (-\infty, +\infty]$ be an l.s.c. function on a Banach space X. Suppose that $x_0 \in S := \{x \in X : f(x) \le 0\}$ and there exist $\mu > 0$ and $0 < \epsilon \le \infty$ such that*

$$\|\xi\|_* \ge \mu^{-1} \quad \forall \xi \in \partial_\omega f(x)$$

for any x with $0 < f(x) < \epsilon$ (or $\|x - x_0\| < \epsilon$ and $0 < f(x) < +\infty$). Then we have

$$d_S(x) \le \mu f(x)_+ \quad \forall x \in X \text{ with } f(x) < \epsilon/2 \text{ (or } \|x - x_0\| < \epsilon/2).$$

Proof. We need only to prove the conclusion for the case where $0 < \epsilon < +\infty$, since for the case $\epsilon = +\infty$ we can obtain the corresponding result by taking the limit from the former one.

Suppose that there were u such that $f(u) < \epsilon/2$ (or $u \in x_0 + (\epsilon/2)B$) and

$$d_S(u) > \mu f(u)_+.$$

Then $u \notin S$ and hence $0 < f(u) < +\infty$. Besides, we can choose $\alpha > 0$ and $t > 1$ such that

$$f(u) \leq \frac{\epsilon}{2+\alpha} < \frac{\epsilon}{2} \quad \left(\text{or } \|u - x_0\| \leq \frac{\epsilon}{2+\alpha} < \frac{\epsilon}{2}\right) \text{ and } d_S(u) > t\mu f(u) := \gamma. \tag{3.1}$$

Thus $f(u)_+ = f(u) = \gamma(t\mu)^{-1}$ and hence

$$f(u)_+ \leq \inf_{v \in X} f(v)_+ + \gamma(t\mu)^{-1}.$$

Note that the function $f(\cdot)_+$ is l.s.c. and bounded below. Applying Ekeland's variational principle [7] to $f(\cdot)_+$ with $\sigma = \gamma(t\mu)^{-1}$ and $\lambda = \gamma$, we find $x \in X$ satisfying

$$\|x - u\| \leq \gamma, \tag{3.2}$$

$$f(v)_+ + (t\mu)^{-1}h(v) \geq f(x)_+ \quad \forall v \in X, \tag{3.3}$$

where $h(v) := \|v - x\|$.

From (3.1), (3.2), and (3.3), we have

$$x \in X, x \notin S \quad \text{and} \quad 0 < f(x) < +\infty. \tag{3.4}$$

On the other hand, (3.3) implies that the function

$$f(v)_+ + (t\mu)^{-1}h(v)$$

attains its minimum on X at x. Hence by property (ω_2) in Definition 2.1,

$$0 \in \partial_\omega[f(x)_+ + (t\mu)^{-1}h(x)]. \tag{3.5}$$

Since f is l.s.c. and $0 < f(x)$, there exists $\delta_1 > 0$ such that

$$0 < f(y) \quad \forall y \in x + \delta_1 B.$$

Thus by property (ω_1) in Definition 2.1 and (3.5),

$$0 \in \partial_\omega(f + (t\mu)^{-1}h)(x). \tag{3.6}$$

Let $\delta := \min\{f(x), (1 - t^{-1})\mu^{-1}, \delta_1, \alpha\epsilon(2+\alpha)^{-1}\}$. Then by property (ω_4) in Definition 2.1 and (3.6), there exist x_1 and x_2 both in $x + \delta B$ such that

$$f(x) - \delta < f(x_1) < f(x) + \delta$$

and

$$0 \in \partial_\omega f(x_1) + \partial_\omega((t\mu)^{-1}h)(x_2) + \delta B^*.$$

The inequalities mean that $x_1 \in x + \delta B$ and $0 < f(x_1) < +\infty$. The inclusion, by property (ω_3) in Definition 2.1, implies that there exists

$$\xi \in \partial_\omega f(x_1)$$

such that

$$\|\xi\|_* < (t\mu)^{-1} + \delta \leq (t\mu)^{-1} + (1 - t^{-1})\mu^{-1} = \mu^{-1},$$

which contradicts the assumption since

$$\begin{aligned} 0 < f(x_1) &< f(x) + \delta \le f(u)_+ + (t\mu)^{-1}\|u - x\| + \delta \\ &\le f(u) + (t\mu)^{-1}\gamma + \delta = 2f(u) + \delta \\ &\le \frac{2\epsilon}{2+\alpha} + \frac{\alpha\epsilon}{2+\alpha} = \epsilon \end{aligned}$$

$$\Bigg(\text{or } \|x_1 - x_0\| \le \|x_1 - x\| + \|x - u\| + \|u - x_0\| < \delta + \gamma + \frac{\epsilon}{2+\alpha}$$
$$\le \frac{\alpha\epsilon}{2+\alpha} + d_S(u) + \frac{\epsilon}{2+\alpha} \le \frac{(1+\alpha)\epsilon}{2+\alpha} + \|u - x_0\|$$
$$\le \frac{(1+\alpha)\epsilon}{2+\alpha} + \frac{\epsilon}{2+\alpha} = \epsilon\Bigg). \qquad \square$$

Remark 3.1. In terms of the proximal subdifferential in a Hilbert space, Clarke et al. [4, Theorem 3.3.1] indicates that the inequality $d_S(x) \le \mu f(x)_+$ holds if x is sufficiently near x_0 and $0 < f(x)$ is sufficiently small. (For more discussion about Clarke et al. [4, Theorem 3.3.1], see Ye [23, Claim].) Theorem 3.1 guarantees the inequality to be true if x is sufficiently near x_0 (or $0 < f(x)$ is sufficiently small).

If X is an Asplund space and f is Fréchet differentiable, the Fréchet subdifferential can be taken as $\partial_\omega f$. Theorem 3.1 applied in this case gives the following corollary. Note that a Fréchet differentiable function may not be Lipschitz continuous. The result cannot obtained by Ioffe [11, Theorem 1 or Corollary 1.1].

Corollary 3.2. *Let $f : X \to (-\infty, +\infty]$ be l.s.c. on an Asplund space X. Assume that $x_0 \in S := \{x \in X : f(x) \le 0\}$ and that there exist $\mu > 0$ and $0 < \epsilon \le \infty$ such that f is Fréchet differentiable at any x with $0 < f(x) < \epsilon$ (or $\|x - x_0\| < \epsilon$ and $0 < f(x) < +\infty$), and*

$$\|\nabla f(x)\|_* \ge \mu^{-1}.$$

Then we have

$$d_S(x) \le \mu f(x)_+ \quad \forall x \in X \text{ with } f(x) < \epsilon/2 \text{ (or } \|x - x_0\| < \epsilon/2).$$

The result in Theorem 3.1 for a single inequality system can easily be extended to a system including equalities, inequalities, and an abstract constraint $x \in C$ as follows.

Theorem 3.3. *Let C be a closed subset of X and each $f_i, |g_j| : X \to (-\infty, +\infty]$ be l.s.c. for $i = 1, \ldots, r$ and $j = 1, \ldots, s$. Assume that*

$$x_0 \in S := \{x \in C : f_1(x) \le 0, \ldots, f_r(x) \le 0;\ g_1(x) = 0, \ldots, g_s(x) = 0\},$$

and denote

$$f(x) = \max\{f_1(x), \ldots, f_r(x);\ |g_1(x)|, \ldots, |g_s(x)|\}.$$

Suppose that there exist $\mu > 0$ and $0 < \epsilon \le \infty$ such that

$$\|\xi\|_* \ge \mu^{-1}$$

whenever $\xi \in \partial_\omega(f + \psi_C)(x)$ for any $x \in C$ with $0 < f(x) < \epsilon$ (or $\|x - x_0\| < \epsilon$ and $0 < f(x) < +\infty$). Then we have

$$d_S(x) \le \mu f(x)_+ \le \mu(\|F(x)_+\| + \|G(x)\|)$$

for any $x \in C$ *with* $f(x) < \epsilon/2$ (*or* $\|x - x_0\| < \epsilon/2$).

Proof. By Theorem 3.1, it suffices to check that f is l.s.c.

For any $x \in X$, denote $F_i(x) = f_i(x)$ for $i = 1, \ldots, r$, and $F_i(x) = |g_{i-r}(x)|$ for $i = r+1, \ldots, r+s$. Then for each $1 \leq i \leq r+s$, $F_i(x)$ is l.s.c.,

$$\begin{aligned} \liminf_{y \to x} f(y) &= \liminf_{y \to x} \, \max\{F_i(y) : 1 \leq i \leq r+s\} \\ &\geq \liminf_{y \to x} F_i(y) \, \geq F_i(x), \end{aligned}$$

and hence

$$\liminf_{y \to x} f(y) \geq f(x) \quad \forall x \in X,$$

which implies that f is l.s.c. $\square$

Remark 3.2. (i) We have proved Theorem 3.3 based on Theorem 3.1, while Theorem 3.1 can be obtained from Theorem 3.3 by taking $C = X$, $r = 1$, and $s = 0$ in it. Therefore they are equivalent to each other. Besides, for the cases $\epsilon = +\infty$ and $\epsilon < +\infty$, Theorems 3.1 and 3.3 both give the corresponding sufficient conditions for global error bounds and local error bounds, respectively.

(ii) Theorem 3.3 has extended Ioffe [11, Theorem 1 and Corollary 1.1] from a Lipschitz equality system to an l.s.c. inequality system. It is also an extension of Wu [21, Theorem 4.19] in which $X = R^n, r = 1, s = 0, \epsilon = +\infty$, and $\partial_\omega = \partial^\pi$.

Theorem 3.3 is stated in terms of any ∂_ω-subdifferentials; however, to simplify checking the conditions, we often try to use smaller ∂_ω-subdifferentials (such as the proximal subdifferential in a Hilbert space and the Fréchet subdifferential in an Asplund space) or some ∂_ω-subdifferentials with better properties (for example, the Clarke subdifferential). Besides, in Theorem 3.3, only $|g_i|$ is required to be l.s.c. no matter whether g is. These points are illustrated in the following example.

Example 3.1. Consider the function $g : R \to R$ given by

$$g(x) = \begin{cases} \;\; 1 - |x| & \text{if } x \text{ is a rational number;} \\ -1 + |x| & \text{if } x \text{ is a irrational number.} \end{cases}$$

Take $C = R$. Then $S = \{x \in R : g(x) = 0\} = \{-1, 1\}, \psi_C(x) = 0$, and

$$|g(x)| = |1 - |x|| = \begin{cases} 1 - |x| & \text{if } |x| \leq 1; \\ |x| - 1 & \text{if } |x| > 1 \end{cases}$$

is l.s.c. (in fact it is Lipschitz of rank 1). It is easy to find

$$\begin{aligned} &\partial^\pi |g(x)| = \{-1\} \quad \text{for} \quad x < -1 \quad \text{or} \quad 0 < x < 1, \\ &\partial^\pi |g(x)| = \{1\} \quad \text{for} \quad -1 < x < 0 \quad \text{or} \quad 1 < x, \quad \text{and} \\ &\partial^\pi |g(0)| = \emptyset. \end{aligned}$$

For any $x \in C$ with $g(x) \neq 0$, since

$$\partial^\pi(|g| + \psi_C)(x) = \partial^\pi |g(x)| \, \subseteq \{-1, 1\},$$

we have $\|\xi\| = 1$ for any $\xi \in \partial^\pi(|g| + \psi_C)(x)$. Thus, by Theorem 3.3,

$$d_S(x) \leq |g(x)| = |1 - |x|| \quad \forall x \in R.$$

Remark 3.3. Note $d_S(0) = 1 = |g(0)|$ in this example. Thus $\mu = 1$ is the smallest constant such that the above inequality holds for any x in R. Besides, to use Theorem 3.3 to find a global error bound, we cannot use the Clarke subdifferential since if we choose it as a ∂_ω-subdifferential, then $\partial_\omega g(0) = \partial g(0) = [-1, 1]$ and it is impossible to find a μ to satisfy the condition in Theorem 3.3.

Let Y be a real normed linear space and $F : X \times Y \to (-\infty, +\infty]$ be l.s.c. For any fixed $y \in Y$, the partial subdifferential $\partial_x F(x, y)$ at $(x, y) \in X \times Y$ in x defined in Jourani [13] is in fact a ∂_ω-subdifferential of $F(x, y)$ at x (denoted by $\partial_\omega^x F(x, y)$) when $F(x, y)$ is considered as a function of the first variable. Since we use the fuzzy sum rule in the definition of ∂_ω-subdifferential instead of the sum rule as in that of the partial subdifferential, ∂_ω-subdifferentials include more subdifferentials in nonsmooth analysis than partial subdifferentials. For example, for the case $F(x, y) = f(x)\ \forall y \in Y$ the proximal subdifferential $\partial^\pi F(x, y) = \partial^\pi f(x)$ is a ∂_ω-subdifferential but not a partial subdifferential.

Now applying Theorem 3.3 to a function F defined on $X \times Y$, we have the following result of which Jourani [13, Theorem 2.4] is a special case when we take $C = X \times Y$ and $\epsilon = +\infty$.

THEOREM 3.4. *Let $F : X \times Y \to (-\infty, +\infty]$ satisfy that for each $y \in Y$ the function $F(\cdot, y)$ is l.s.c. Let C be a nonempty closed subset of $X \times Y$. Assume that for $y \in Y$ the set*

$$S(y) := \{x \in X : (x, y) \in C \text{ and } F(x, y) \leq 0\}$$

is nonempty and that there exist $\mu > 0$ and $0 < \epsilon \leq \infty$ such that

$$\|\xi\|_* \geq \mu^{-1} \quad \forall \xi \in \partial_\omega^x (F + \psi_C)(x, y)$$

for any $x \in X$ with $(x, y) \in C$ and $0 < F(x, y) < \epsilon$. Then we have

$$d_{S(y)}(x) \leq \mu F(x, y)_+ \quad \forall x \in X \text{ with } (x, y) \in C \text{ and } F(x, y) < \epsilon/2.$$

Proof. For $y \in Y$ in the assumption, denote

$$f(\cdot) := F(\cdot, y) \text{ and } C(y) := \{x \in X : (x, y) \in C\}.$$

Upon applying Theorem 3.3 to the solution set

$$S(y) = \{x \in C(y) : f(x) \leq 0\}$$

we obtain the inequality desired. ∎

4. Sufficient conditions for a general system. In this section we suppose that X is a real normed linear space. Motivated by a note of a referee of Deng [6, Corollary 2], we present the following condition to guarantee the existence of a global error bound for a general inequality system.

THEOREM 4.1. *Let f be an extended real-valued function on a subset C of X and $S = \{x \in C : f(x) \leq 0\}$ be nonempty. Suppose that there exist a unit vector u in X and a constant $\mu > 0$ such that for any $\lambda > 0$,*

$$x + \lambda u \in C \text{ and } \sup_{\lambda > 0} \frac{f(x + \lambda u) - f(x)}{\lambda} \leq -\mu^{-1} \tag{4.1}$$

for any $x \in C \backslash S$ with $f(x) < +\infty$. Then

$$d_S(x) \leq \mu f(x)_+ \quad \forall x \in C.$$

Proof. It suffices to show that the inequality holds for $x \in C \backslash S$ with $f(x) < +\infty$. Now for such an x, $0 < f(x) < +\infty$, $x + \lambda u \in C$, and $f(x + \lambda u) \leq f(x) - \mu^{-1}\lambda$ for any $\lambda > 0$, so by taking $\lambda = \mu f(x)$, we have $f(x + \lambda u) \leq 0$, i.e., $x + \lambda u \in S$. Thus $d_S(x) \leq \|\lambda u\| = \mu f(x)$. ◻

Remark 4.1. It is easy to see that C in Theorem 4.1 must be unbounded since for $x \in C$ with $f(x) < +\infty$ and any $\lambda > 0$, $x + \lambda u$ must be in C.

Recall that for a nonempty closed convex subset C of X, *the recession cone* of C, denoted by C^∞, is the set

$$C^\infty = \left\{ x \in X : \exists \{\mu_i\} \subseteq (0, +\infty) \ \&\ \{x_i\} \subseteq C \text{ s.t. } \lim_{i \to \infty} \mu_i = 0 \text{ and } \lim_{i \to \infty} \mu_i x_i = x \right\}.$$

According to Rockafellar [19, Theorem 2A(c)], C^∞ can equivalently be expressed as

$$C^\infty = \{x \in X : C + \{x\} \subseteq C\}.$$

For an l.s.c. and proper convex function $f : X \to (-\infty, +\infty]$, since its epigraph is a closed convex subset of $X \times R$, one can use the recession cone of *epi* f to define *the recession function of* f, denoted by f^∞, i.e.,

$$epi(f^\infty) = (epi\, f)^\infty.$$

We refer to [20] for examples of recession functions.

Similar to Deng [5, 6], we use the recession function to give the following sufficient condition for a global error bound.

Corollary 4.2. *Let C be a closed convex subset of X and each $f_i : X \to (-\infty, +\infty]$ be l.s.c. proper convex for $i \in I = \{1, \ldots, r\}$. Assume that $S = \{x \in C : f_i(x) \leq 0, i \in I\}$ is nonempty and denote $f(x) := \max\{f_i(x) : i \in I\}$. Suppose that there exist a unit vector $u \in C^\infty$ and a constant $\mu > 0$ such that $f_i^\infty(u) \leq -\mu^{-1}$ for each $i \in I$. Then for any $1 \leq p \leq +\infty$,*

$$d_S(x) \leq \mu f(x)_+ \leq \mu \|F(x)_+\|_p \quad \forall x \in C,$$

where $\|\cdot\|_p$ denotes the p-norm on R^r.

Proof. Since $S = \{x \in C : f(x) \leq 0\}$, we need only to check that the conditions in Theorem 4.1 are satisfied for C and f.

First, by Rockafellar [19, Theorem 2A(a)], the inclusion $u \in C^\infty$ implies that $x + \lambda u$ must be in C for each $x \in C$ and any $\lambda \geq 0$. Besides, according to Rockafellar [19, Corollary 3C(a)], for each $i \in I$,

$$f_i^\infty(u) = \sup_{\lambda > 0} \frac{f_i(x + \lambda u) - f_i(x)}{\lambda} \quad \forall x \in dom\, f_i.$$

So if $f_i^\infty(u) \leq -\mu^{-1}$, then for any $\lambda > 0$,

$$f_i(x + \lambda u) \leq f_i(x) - \lambda \mu^{-1} \quad \forall x \in dom\, f_i.$$

Hence for any $x \in dom\, f$ and any $\lambda > 0$,

$$f(x + \lambda u) \leq f(x) - \lambda \mu^{-1}.$$

In particular, for any $x \in C \backslash S$ with $f(x) < +\infty$,

$$\sup_{\lambda > 0} \frac{f(x + \lambda u) - f(x)}{\lambda} \leq -\mu^{-1}.$$

Therefore, for any $1 \le p \le +\infty$, by Theorem 4.1,

$$d_S(x) \le \mu f(x)_+ \le \mu \|F(x)_+\|_p \quad \forall x \in C. \qquad \square$$

Remark 4.2. Note that each f_i in Corollary 4.2 is an l.s.c. and convex function on a real normed linear space. So it is an improvement on Deng [5, Theorem 2.3], in which X is a reflexive Banach space and each f_i is a continuous and convex function for $i = 1, \ldots, r$. Besides, Deng [6, Corollary 2] can be obtained as a special case of Corollary 4.2 where $p = 1$ and f_1 is a continuous and convex function on a Banach space X. Furthermore, Corollary 4.2 not only extends Jourani [13, Theorem 3.3] but also proves that condition (i) in it is redundant.

5. Global error bounds and metric regularity. In Deng [6] close relations between global error bounds and metric regularity are revealed for a continuous and convex inequality system. Most of them turn out to be true for an l.s.c. convex inequality system, and some of them can further be refined. To show this we recall the concept of metric regularity and introduce that of uniformly metric regularity.

DEFINITION 5.1. *Let f be an extended real-valued function on X, C be a subset of X, and $S = \{x \in C : f(x) \le 0\}$ be nonempty. The system*

$$f(x) \le 0, \ x \in C, \tag{5.1}$$

is said to be metrically regular *at a nonempty set $S_1 \subseteq S$ if there exist constants $\delta > 0$ and $\mu(\delta) > 0$ such that*

$$d_S(x) \le \mu(\delta) f(x)_+ \quad \forall x \in C \text{ with } d_{S_1}(x) \le \delta.$$

When $S_1 = \{z\} \subseteq S$, we simply say that the system (5.1) *is* metrically regular *at z. In particular, the system* (5.1) *is said to be* uniformly metrically regular *at S if it is metrically regular at each point of S with the same $\delta > 0$ and $\mu(\delta) > 0$.*

Obviously for any $\emptyset \ne S_1 \subseteq S_2$ we have $d_{S_1}(x) \ge d_{S_2}(x)$ for any $x \in X$, so if the system (5.1) is metrically regular at S_2, then it must also be metrically regular at S_1.

As the referees of this paper pointed out, the notion of metric regularity is related to moving sets, and the equivalence between error bound and (the very definition of) metric regularity usually fails to hold. The following result states the relations between global error bounds and metric regularity for an l.s.c. inequality system.

THEOREM 5.2. *Let f be an l.s.c. extended real-valued function on X and $S = \{x \in X : f(x) \le 0\}$ be nonempty. Consider the following statements:*

(a) *There is a constant $\mu > 0$ such that $d_S(x) \le \mu f(x)_+$ for any $x \in X$.*
(b) *The system* (5.1) *is metrically regular at any nonempty set $S_1 \subseteq S$.*
(c) *The system* (5.1) *is metrically regular at S.*
(d) *The system* (5.1) *is uniformly metrically regular at S.*
(e) *The system* (5.1) *is metrically regular at each point of S.*

Then the following implications hold:

(i) (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$(e).
(ii) *If f is convex,* (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d).
(iii) *If f is convex and S is compact,* (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e).

Proof. Since the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) in (i) are obvious, it suffices to show (d) $\Rightarrow$ (b) for (i), (d) $\Rightarrow$ (a) for (ii), and (e) $\Rightarrow$ (a) for (iii). But since the last implication was proved in Deng [6, Corollary 4] (assuming that $X = R^n$, f is continuous and convex, and S is bounded) and the proof is still valid with the hypothesis in this theorem, it remains to prove (d) $\Rightarrow$ (b) for (i) and (d) $\Rightarrow$ (a) for (ii).

(d) $\Rightarrow$ (b) for (i): We suppose that statement (d) is true. Then there are constants $\delta > 0$ and $\mu(\delta) > 0$ such that for each $z \in S$,

$$d_S(x) \leq \mu(\delta) f(x)_+ \text{ whenever } \|x - z\| \leq \delta.$$

Hence for any nonempty subset S_1 of S, we have $d_S(x) \leq \mu(\delta) f(x)_+$ for any x with $d_{S_1}(x) \leq \delta/2$ since for such an x we can find a point $x_1 \in S_1$ such that $\|x - x_1\| \leq \delta$. This proves that statement (b) holds.

(d) $\Rightarrow$(a) for (ii): Suppose that f is an l.s.c. proper convex function, that (d) holds, and that $\delta > 0$ is the constant in the definition of uniformly metric regularity. Then S is closed, $d_S(x) > 0$ for any $x \in X\backslash S$. Thus for any fixed $x \in X\backslash S$ and any $\epsilon > 0$, there exists $\overline{x} \in S$ such that $\|x - \overline{x}\| \leq d_S(x) + \epsilon$. If $\|x - \overline{x}\| \leq \delta$, we already have the inequality $d_S(x) \leq \mu(\delta) f(x)_+$. If $\|x - \overline{x}\| > \delta$, taking $\lambda := \delta / \|x - \overline{x}\|$ and $y = \lambda x + (1 - \lambda)\overline{x}$, we have

$$\|y - \overline{x}\| = \lambda \|x - \overline{x}\| = \delta,$$

which implies $d_S(y) \leq \mu(\delta) f(y)_+$. Besides, by the convexity of f, $f(y) \leq \lambda f(x)$. Hence

$$\begin{aligned} d_S(x) &\leq \|x - \overline{x}\| = \|y - \overline{x}\|/\lambda = [\|x - \overline{x}\| - \|y - x\|]/\lambda \\ &\leq [d_S(x) + \epsilon - \|y - x\|]/\lambda \leq [d_S(y) + \epsilon]/\lambda \\ &\leq [\mu(\delta) f(y)_+ + \epsilon]/\lambda \leq [\mu(\delta)\lambda f(x) + \epsilon]/\lambda \\ &= \mu(\delta) f(x)_+ + \epsilon/\lambda = \mu(\delta) f(x)_+ + \epsilon[d_S(x) + \epsilon]/\delta. \end{aligned}$$

This explains that statement (a) is true since $\epsilon > 0$ and x are arbitrary. $\square$

Remark 5.1. Deng [6] proved the implications (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) for a continuous convex system on a Banach space, and the implication (e) $\Rightarrow$ (a) when $X = R^n$ and S is bounded. Theorem 5.2 has extended these results to an l.s.c. system and contains the new equivalent statement (d). Furthermore Theorem 5.2 is allowed to be applied to an l.s.c. extended real-valued function f defined on a closed convex subset C of X to obtain an equivalent result whose statement is the same as that of Theorem 5.2 with the set $\{x \in X : f(x) \leq 0\}$ and the inequality "$d_S(x) \leq \mu f(x)_+$ for any $x \in X$" replaced by $\{x \in C : f(x) \leq 0\}$ and "$d_S(x) \leq \mu f(x)_+$ for any $x \in C$," respectively.

In the rest of this paper, we use Theorems 3.1 and 5.2 to give some sufficient conditions for l.s.c. systems to be metrically regular at sets.

Proposition 5.3. *Let $f : X \to R$ be l.s.c. Assume that $S = \{x \in X : f(x) \leq 0\}$ is nonempty and that there exist $\mu > 0$ and $0 < \varepsilon \leq \infty$ such that for each $z \in S$,*

$$\|\xi\|_* \geq \mu^{-1}$$

whenever $\xi \in \partial_\omega f(x)$ for any $x \in X$ with $0 < f(x)$ and $\|x - z\| < \varepsilon$. Then the system (5.1) *is metrically regular at S. If f is in addition convex, then there is a constant $\mu > 0$ such that*

$$d_S(x) \leq \mu f(x)_+ \quad \forall x \in X.$$

Proof. According to Theorem 5.2, it suffices to show that the system (5.1) is metrically regular at S.

By Theorem 3.1, the inequality

$$d_S(x) \le \mu f(x)_+$$

holds for each $z \in S$ and any $x \in X$ with $\|x - z\| < \varepsilon/2$, i.e., the system (5.1) is uniformly metrically regular at S. Hence by implication (i) of Theorem 5.2, the system (5.1) is metrically regular at S. □

The following proposition indicates that if the solution set is compact and contains no stationary points for ∂_ω-subdifferentials with some limiting property, then the system is metrically regular at the solution set.

PROPOSITION 5.4. *Let $f : X \to R$ be continuous. Assume that*

$$S = \{x \in X : f(x) \le 0\}$$

is nonempty and compact and that for each $z \in S$, $0 \notin \partial_\omega f(z)$ and $\partial_\omega f$ satisfies that $\xi \in \partial_\omega f(z)$ if $x_n \to z$, $\xi_n \in \partial_\omega f(x_n)$, and $\xi_n \to \xi$. Then the system (5.1) *is metrically regular at S, and hence there is a constant $\mu > 0$ such that*

$$d_S(x) \le \mu f(x)_+ \quad \forall x \in X.$$

Proof. Based on relation (iii) in Theorem 5.2, we need only to prove statement (e) in Theorem 5.2. Let $z \in S$ be fixed. Then by Theorem 3.1 it is enough to show that there exist $\mu > 0$ and $\epsilon > 0$ such that

$$\|\xi\|_* \ge \mu^{-1} \quad \forall \xi \in \partial_\omega f(x)$$

for any x with $\|x - z\| < \epsilon$ and $0 < f(x) < +\infty$. In fact, if this were not true, then there would exist sequences $\{x_n\}$ and $\{\xi_n\}$ such that $x_n \to z$, $\xi_n \in \partial_\omega f(x_n)$, and $\xi_n \to 0$. But this would lead to $0 \in \partial_\omega f(z)$, which contradicts the assumption. □

We now consider a convex system which also includes an abstract constraint. In the following proposition we prove that the generalized Slater condition is sufficient for metric regularity.

PROPOSITION 5.5. *Let $f_i : X \to R$ be locally Lipschitz and convex for $i \in I = \{1, \dots, r\}$, and let C be a closed and convex subset of X. Let $N \cup L$ be a partition of the index set I such that f_i is linear for each $i \in L$. Denote*

$$f(x) = \max\{f_i(x), |f_j(x)| : i \in N, j \in L\}.$$

Suppose that there exist

$$x^*, x_0 \in S := \{x \in C : f_i(x) \le 0, i \in N; f_j(x) = 0, j \in L\}$$

such that $f_i(x^*) < 0$ for each $i \in N$ and $\{-\nabla f_i(x_0) : i \in L\}$ is C-linearly independent, i.e.,

$$-\sum_{i \in L} \lambda_i \nabla f_i(x_0) \in N_C(x_0) \textit{ implies } \quad \lambda_i = 0 \quad \forall i \in L.$$

Then there exist positive numbers δ and μ such that

(i) $\|\xi\|_* \ge \mu^{-1} \quad \forall \xi \in \partial f(x) + N_C(x)$ *for any $x \in C$ with $\|x - x_0\| < \delta$ and $0 < f(x)$;*

(ii) $d_S(x) \le \mu f(x)_+$ *for any $x \in C$ with $\|x - x_0\| < \delta/2$, i.e., the system* (5.1) *is metrically regular at x_0.*

Moreover, if $X = R^n$ and $\{-\nabla f_i(x) : i \in L\}$ is C-linearly independent for each $x \in S$, then for any bounded subset $\Omega \subseteq R^n$ there exist $\delta > 0$ and $\mu > 0$ such that

$$d_S(x) \leq \mu f(x)_+ \text{ for any } x \in C \cap (\Omega + \delta \overline{B}),$$

i.e., the system (5.1) *is metrically regular at* Ω.

Proof. Since f is Lipschitz near x and ψ_C is finite at x and both functions are convex, by Clarke [3, Corollary 1 of Theorem 2.9.8 and Proposition 2.4.12],

$$\partial(f + \psi_C)(x) = \partial f(x) + \partial \psi_C(x) = \partial f(x) + N_C(x).$$

Hence by applying Theorem 3.1 to the function $f + \psi_C$, statement (ii) follows from statement (i). So for statements (i) and (ii), it suffices to prove statement (i).

Suppose that statement (i) were not true. Then there would exist sequences $\{x_k\} \subseteq C$ and $\xi_k \in \partial f(x_k) + N_C(x_k)$ such that $x_k \to x_0$, $\xi_k \to 0$, and $0 < f(x_k)$ for each k. By Clarke [3, Proposition 2.3.12 and Theorem 2.3.9], for each x_k there exists a set of numbers $\lambda_i^{(k)}$ such that

$$\begin{gathered}
\lambda_i^{(k)} \geq 0 \quad \forall i \in N, \ \sum_{i \in N} \lambda_i^{(k)} + \sum_{i \in L} |\lambda_i^{(k)}| = 1, \\
\xi_k \in \sum_{i \in N} \lambda_i^{(k)} \partial f_i(x_k) \ + \sum_{i \in L} \lambda_i^{(k)} \nabla f_i(x_k) + N_C(x_k), \text{ and} \\
\lambda_i^{(k)}(f_i(x_k) - f(x_k)) = 0 \quad \forall i \in N, \ \lambda_i^{(k)}(|f_i(x_k)| - f(x_k)) = 0 \quad \forall i \in L.
\end{gathered}$$

Since each sequence $\{\lambda_i^{(k)}\}$ is bounded by 1 for each i, we can assume that $\lambda_i^{(k)} \to \lambda_i$ for each $i \in N \cup L$ as $k \to +\infty$. We denote the index of binding constraints at x_0 by $I(x_0) = \{i \in N : f_i(x_0) = 0\}$. Taking the limit as $k \to \infty$ gives

$$\begin{gathered}
\lambda_i \geq 0 \quad \forall\ i \in I(x_0), \quad \lambda_i = 0 \quad \forall i \in N \backslash I(x_0), \\
\sum_{i \in N} \lambda_i + \sum_{i \in L} |\lambda_i| = 1, \text{ and} \\
0 \in \sum_{i \in I(x_0)} \lambda_i \partial f_i(x_0) \ + \sum_{i \in L} \lambda_i \nabla f_i(x_0) + N_C(x_0),
\end{gathered}$$

where the inclusion follows from the fact that $\partial f_i(x_k)$ is the subdifferential of f_i at x_k and $N_C(x_k)$ is the normal cone to C at x_k in the sense of convex analysis. Since by assumption $\{-\nabla f_i(x_0) : i \in L\}$ is C-linearly independent, this inclusion implies that there is at least one $i_0 \in I(x_0)$ such that $\lambda_{i_0} > 0$, from which we would obtain a contradiction.

In fact, if we use the above λ_i to define the function

$$g(y) = \sum_{i \in I(x_0)} \lambda_i f_i(y) + \sum_{i \in L} \lambda_i f_i(y) + \psi_C(y),$$

then g is convex, and by the sum rule of subdifferentials (in the sense of convex analysis) we have

$$0 \in \sum_{i \in I(x_0)} \lambda_i \partial f_i(x_0) + \sum_{i \in L} \lambda_i \nabla f_i(x_0) + N_C(x_0) = \partial g(x_0),$$

which means that g attains its global minimum at x_0. Therefore this together with the continuity of g yields

$$0 = g(x_0) \le g(x^*) \le \lambda_{i_0} f_{i_0}(x^*) < 0.$$

This is a contradiction.

Now suppose that $X = R^n$. Let $\delta > 0$ be the positive number stated in (i). Then for any fixed bounded set Ω we can take $\epsilon > \delta$ such that $\Omega + \delta\overline{B} \subseteq \overline{B}(x_0, \epsilon/2)$. By Theorem 3.3, it suffices to show that there exists $\mu > 0$ such that $\|\xi\| \ge \mu^{-1} \quad \forall\xi \in \partial f(x) + N_C(x)$ for any $x \in C$ with $\delta \le \|x - x_0\| \le \epsilon$ and $0 < f(x)$.

Suppose that there exist sequences $\{x_k\} \subseteq C$ and $\xi_k \in \partial f(x_k)+N_C(x_k)$ such that $\delta \le \|x_k - x_0\| \le \epsilon$, $0 < f(x_k)$, and $\xi_k \to 0$ as $k \to +\infty$. Since $\{x_k\}$ lies in a compact set, we can assume that x_k converges to some point $\overline{x} \in C$ with $\delta \le \|\overline{x} - x_0\| \le \epsilon$. Taking the limit for $\xi_k \in \partial f(x_k)+N_C(x_k)$, we have $0 \in \partial f(\overline{x})+N_C(\overline{x}) \subseteq \partial(f+\psi_C)(\overline{x})$ by the sum rule of subdifferentials in the sense of convex analysis. This means that f attains its global minimum over C at $\overline{x}$ since $f + \psi_C$ is convex. Note that f is continuous and $f(x_k)$ is positive. Thus

$$0 = f(x^*) \ge f(\overline{x}) = \lim_{k\to+\infty} f(x_k) \ge 0.$$

Thus $\overline{x} \in S$. But by statement (i) there exist positive numbers δ and μ such that

$$\|\xi\| \ge \mu^{-1} \quad \forall\xi \in \partial f(x) + N_C(x)$$

for any $x \in C$ with $\|x - \overline{x}\| < \delta$ and $0 < f(x)$. This contradicts the properties of subsequences $\{x_k\}$ and $\{\xi_k\}$. $\square$

Remark 5.2. In Proposition 5.5, the Slater condition $f_i(x^*) < 0$ for $i \in N$ is important to guarantee that (i) and (ii) hold. Without this condition, (i) and (ii) may fail. One simple example is the function $f(x) = x^2$ with $S = \{x \in R : f(x) \le 0\} = \{0\}$. On the other hand, statement (i) is a local property, i.e., without additional conditions, property (i) cannot generally be extended to all points outside the neighborhood. For example, for any $n \ge 2$, the function

$$f(x) = \begin{cases} x - 1 & \text{if } x \ge 0; \\ -\sqrt{2-(x+1)^2} & \text{if } -1 - \sqrt{\frac{2}{n^2+1}} < x < 0; \\ -\frac{x}{n} - \frac{1+\sqrt{2(n^2+1)}}{n} & \text{if } x \le -1 - \sqrt{\frac{2}{n^2+1}} \end{cases}$$

is differentiable and convex with $f(0) = -1$ and $f(1) = 0$. The inequality in statement (i) of Proposition 5.5 holds for $x_0 = 1$, $\delta = 1$, and $\mu = 1$. But for any $x < -1 - \sqrt{2(n^2+1)}$, $f(x) > 0$ and $|f'(x)| = 1/n < 1/\mu$.

REFERENCES

[1] A. Auslender and J. P. Crouzeix, *Global regularity theorems*, Math. Oper. Res., 13 (1988), pp. 243–253.

[2] J. V. Burke and P. Tseng, *A unified analysis of Hoffman's bound via Fenchel duality*, SIAM J. Optim., 6 (1996), pp. 265–282.

[3] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.

[4] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer-Verlag, New York, 1998.

[5] S. Deng, *Computable error bounds for convex inequality systems in reflexive Banach spaces*, SIAM J. Optim., 7 (1997), pp. 274–279.

[6] S. Deng, *Global error bounds for convex inequality systems in Banach spaces*, SIAM J. Control Optim., 36 (1998), pp. 1240–1249.
[7] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., 47 (1974), pp. 324–353.
[8] M. Fabian, *Subdifferentiability and trustworthiness in the light of a new variational principle of Borwein and Preiss*, Acta Univ. Carolin. Math. Phys., 30 (1989), pp. 51–56.
[9] M. C. Ferris and J. S. Pang, *Nondegenerate solutions and related concepts in affine variational inequalities*, SIAM J. Control Optim., 34 (1996), pp. 244–263.
[10] A. J. Hoffman, *On approximate solutions of systems of linear inequalities*, J. Research Nat. Bur. Standards, 49 (1952), pp. 263–265.
[11] A. D. Ioffe, *Regular points of Lipschitz functions*, Trans. Amer. Math. Soc., 251 (1979), pp. 61–69.
[12] A. D. Ioffe, *Proximal analysis and approximate subdifferentials,* J. London Math. Soc., 41 (1990), pp. 175–192.
[13] A. Jourani, *Hoffman's error bound, local controllability, and sensitivity analysis*, SIAM J. Control Optim., 38 (2000), pp. 947–970.
[14] A. S. Lewis and J. S. Pang, *Error bounds for convex inequality systems,* in Generalized Convexity, Generalized Monotonicity: Recent Results, J.-P. Crouzeix, J.-E. Martinez-Legaz, and M. Volle, eds., Kluwer, Dordrecht, 1998, pp. 75–110.
[15] W. Li, *Abadie's constraint qualification, metric regularity, and error bounds for differentiable convex inequalities*, SIAM J. Optim., 7 (1997), pp. 966–978.
[16] X.-D. Luo and Z.-Q. Luo, *Extension of Hoffman's error bound to polynomial systems*, SIAM J. Optim., 4 (1994), pp. 383–392.
[17] O. L. Mangasarian, *A condition number for differentiable convex inequalities*, Math. Oper. Res., 10 (1985), pp. 175–179.
[18] S. M. Robinson, *An application of error bounds for convex programming in a linear space*, SIAM J. Control, 13 (1975), pp. 271–273.
[19] R. T. Rockafellar, *Level sets and continuity of conjugate convex functions*, Trans. Amer. Math. Soc., 123 (1966), pp. 46–63.
[20] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, NJ, 1970.
[21] Z. Wu, *Subdifferentials and Their Applications,* Masters thesis, The University of Victoria, Victoria, BC, Canada, 1997.
[22] J. J. Ye, *New uniform parametric error bounds*, J. Optim. Theory Appl., 98 (1998), pp. 197–219.
[23] J. J. Ye, *Constraint qualifications and necessary optimality conditions for optimization problems with variational inequality constraints*, SIAM J. Optim., 10 (2000), pp. 943–962.