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# STABILITY ANALYSIS FOR PARAMETRIC MATHEMATICAL PROGRAMS WITH GEOMETRIC CONSTRAINTS AND ITS APPLICATIONS*

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**Abstract.** This paper studies stability for parametric mathematical programs with geometric constraints. We show that, under the no nonzero abnormal multiplier constraint qualification and the second-order growth condition or second-order sufficient condition, the locally optimal solution mapping and stationary point mapping are nonempty-valued and continuous with respect to the perturbation parameter and, under some suitable conditions, the stationary pair mapping is calm. Furthermore, we apply the above results to parametric mathematical programs with equilibrium constraints. In particular, we show that the M-stationary pair mapping is calm with respect to the perturbation parameter if the M-multiplier second-order sufficient condition is satisfied, and the S-stationary pair mapping is calm if the S-multiplier second-order sufficient condition is satisfied and the bidegenerate index set is empty.

**Key words.** mathematical program with geometric constraints, mathematical program with equilibrium constraints, variational analysis, stationarity, stability, calmness

**AMS subject classifications.** 49K40, 90C30, 90C31, 90C33

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## 1. Introduction.

Consider the following parametric mathematical program with geometric constraints:

$$(\mathrm{MPGC}_p) \qquad \min_{x} \quad f(x,p) \qquad \text{s.t.} \quad F(x,p) \in \Lambda, \tag{1.1}$$

where $f : \Re^{n_1+n_2} \to \Re$ and $F : \Re^{n_1+n_2} \to \Re^l$ are both twice continuously differentiable functions and $\Lambda \subseteq \Re^l$ is a nonempty closed set. Problem (1.1) is very general. It includes as special cases the standard nonlinear programs and the problem considered by Robinson [26] where $\Lambda$ is assumed to be a closed convex cone. When the data of the problems are subject to small perturbations, the stability of solutions and multipliers is an important issue. To the best of our knowledge, most research on this issue has been devoted to the case where $\Lambda$ is assumed to be closed and convex; see, e.g., Kojima [18], Robinson [26], and Bonnans and Shapiro [4]. However, restricting $\Lambda$ in problem (1.1) to be convex significantly reduces the applicability of the model since many practical optimization problems can be formulated as problem (1.1) with a nonconvex set $\Lambda$. For example, the disjunctive programming problem [7] where $\Lambda$ is a union of finitely many convex sets is such a problem, whereas problems such as mathematical programs with equilibrium constraints [22, 25], mathematical programs with vertical complementarity constraints [29], and mathematical programs with vanishing constraints [1, 15] can be reformulated as disjunctive programming problems.

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For the case where $\Lambda$ is nonconvex, Levy and Mordukhovich [19] and Mordukhovich [23] have studied Aubin's pseudo-Lipschitz continuity of stationary point mapping and stationary pair mapping by making use of the advanced tools of variational analysis and coderivatives of set-valued mappings [23, 28]. In this paper, we focus on the continuity of locally optimal solution mapping and stationary point mapping, and the calmness of stationary pair mapping. Our incentive for studying the general problem (1.1) comes from the fact that the parametric mathematical program with equilibrium constraints (MPEC),

$$
\begin{array}{lll}
(\mathrm{MPEC}_p) & \min\limits_{x} & f(x,p) \\
& \text{s.t.} & g(x,p) \le 0,\ h(x,p) = 0, \\
& & 0 \le G(x,p) \ \perp\ H(x,p) \ge 0,
\end{array}
\tag{1.2}
$$

where $f$ is the same as above, $g : \Re^{n_1+n_2} \to \Re^{m_1}$, $h : \Re^{n_1+n_2} \to \Re^{m_2}$, $G, H : \Re^{n_1+n_2} \to \Re^{m}$ are all twice continuously differentiable functions, and $a \perp b$ means that vector $a$ is perpendicular to vector $b$, can be reformulated as a special case of problem (1.1) by letting

$$
F(x,p) := \begin{pmatrix} g(x,p) \\ h(x,p) \\ \Psi(x,p) \end{pmatrix}, \quad \Lambda := \Re_-^{m_1} \times \{0\}^{m_2} \times C^m,
\tag{1.3}
$$

where $\Re_-$ denotes the nonpositive orthant $\{x \in \Re \mid x \le 0\}$ and

$$
\Psi(x,p) := \begin{pmatrix} -G_1(x,p) \\ -H_1(x,p) \\ \vdots \\ -G_m(x,p) \\ -H_m(x,p) \end{pmatrix}, \quad C := \{\, (a,b) \in \Re^2 \mid 0 \le -a \perp -b \ge 0\}.
\tag{1.4}
$$

We call the nonconvex cone $\Lambda$ defined as in (1.3) the *MPEC cone*. Here, the minus signs in $\Psi$ are used only for convenience of the subsequent analysis.

It is well known that MPECs play a very important role in many fields such as engineering design, economic equilibria, transportation science, multilevel games, and mathematical programming itself. However, this kind of problem is generally difficult to deal with because their constraints fail to satisfy the standard Mangasarian–Fromovitz constraint qualification (MFCQ) at any feasible point [34]. A lot of research has been done during the last two decades to study the optimality conditions for MPECs, including the Clarke (C-), Mordukhovich (M-), strong (S-), and Bouligand (B-) stationarity conditions; see, e.g., [10, 11, 17, 29, 31, 32, 33, 34]. At the same time, algorithms for solving MPECs have been proposed by using a number of approaches such as the sequential quadratic programming approach, penalty function approach, relaxation approach, and active set identification approach; see, e.g., [8, 21, 22, 25] and the references therein.

Compared with the developments on optimality conditions and algorithms, little research has been done with the stability for ($\mathrm{MPEC}_p$). Lignola and Morgan [20] studied the existence and continuity of the approximate globally optimal solutions to parametric Stackelberg problems in a topological frame. Scheel and Scholtes [29] considered the stability of C-stationarity and B-stationarity and showed that, under the upper level strict complementarity and some regularity conditions, both the C-stationary and B-stationary points are locally unique for (MPEC$_p$). Izmailov [14] studied some kind of semicontinuity and Lipschitz continuity of locally optimal solution mapping under some mild conditions. More recently, Jongen, Shikhman, and Steffensen [16] and Shikhman [30] studied the stability for C-stationarity and S-stationarity and showed that, under suitable conditions, the C-/S-stationary points are strongly stable in the sense of Kojima [18]. A natural question is, are there any stability results for the M-stationarity?

When $\Lambda$ is a closed convex cone, Robinson [26] showed that if the second-order sufficient condition and Robinson's constraint qualification hold at a locally optimal solution of (MPGC$_p$), the set of locally stationary points is nonempty and continuous for sufficiently smooth perturbations of constraints and objective function and, if $\Lambda$ reduces to a polyhedral cone, the locally optimal solutions and stationary pairs obey some type of Lipschitz condition. In this paper, we first extend various stability results of Robinson [26] to the case where $\Lambda$ is a closed set and then apply the obtained results to (MPEC$_p$). Note that when $F$ and $\Lambda$ are defined as in (1.3), it can be shown that the standard stationarity for (MPGC$_p$) reduces to the M-stationarity for (MPEC$_p$).

The organization of the paper can be summarized as follows. In section 2, we give some useful terminologies and results. In section 3, we show that under the no nonzero abnormal multiplier constraint qualification and either second-order growth condition or second-order sufficient condition, the locally optimal solution mapping and stationary point mapping are nonempty-valued and continuous in the sense of set-valued analysis with respect to the perturbation parameter. Section 4 is devoted to the calmness of stationary pair mapping, which means that the mapping possesses some kind of Lipschitz continuity. When $\Lambda$ reduces to a polyhedral cone, the calmness result given in section 4 improves Theorem 4.2 of [26] in that no constraint qualification is required here. It also extends Lemma 2 of [12] from the linear perturbation case to the smooth perturbation case. In section 5, we apply the results in sections 3 and 4 to (MPEC$_p$). In particular, we show that the M-stationary pair mapping is calm with respect to the perturbation parameter under the M-multiplier second-order sufficient condition, which complements the recent work of Jongen, Shikhman, and Steffensen [16], and the S-stationary pair mapping is calm if the S-multiplier second-order sufficient condition is satisfied and the bidegenerate index set is empty. The results can be applied to mathematical programs with vertical complementarity constraints [29] and mathematical programs with vanishing constraints [1, 15] in a similar manner.

Throughout the paper, all vectors are viewed as column vectors. Moreover, we denote by $\|\cdot\|$ the Euclidean norm and denote by $\mathcal{B}_\delta(x) := \{y \in \Re^n \mid \|y - x\| < \delta\}$ and $\bar{\mathcal{B}}_\delta(x) := \{y \in \Re^n \mid \|y - x\| \leq \delta\}$ the open and closed balls centered at $x$ with radius $\delta > 0$, respectively. Given a set $\Omega \subseteq \Re^n$ and a point $x \in \Re^n$, the distance from $x$ to $\Omega$ is denoted by

$$\mathrm{dist}(x, \Omega) := \inf\{\|y - x\| \mid y \in \Omega\}.$$

For a mapping $\Phi : \Re^n \to \Re^m$ and a vector $x \in \Re^n$, $\nabla\Phi(x)$ denotes the transposed Jacobian of $\Phi$ at $x$ and $\mathrm{gph}\,\Phi$ denotes the graph of $\Phi$, i.e., $\mathrm{gph}\,\Phi := \{(z, v) \in \Re^{n+m} \mid v \in \Phi(z)\}$. Given a matrix $A$, $[A]_i$ denotes the transposed vector of its $i$th row vector. In addition, for simplicity, $x' \to_\Omega x$ means $x' \to x$ with $x' \in \Omega$.

**2. Preliminaries.** In this section, we review some basic concepts and results, which will be used later on.

**2.1. Variational analysis.** We next give some background materials on variational analysis. See [5, 6, 23, 24, 28] for more details.

Let $\Phi : \Re^n \rightrightarrows \Re^m$ be a set-valued mapping and let $\Omega \subseteq \Re^n$ be a nonempty set. We denote by $\limsup_{x \to_\Omega \bar{x}} \Phi(x)$ the *Painlevé–Kuratowski upper limit* with respect to $\Omega$, i.e.,

$$\limsup_{x \to_\Omega \bar{x}} \Phi(x) := \{v \in \Re^m \mid \exists x_k \to_\Omega \bar{x}, v_k \to v \text{ with } v_k \in \Phi(x_k) \text{ for each } k\}.$$

DEFINITION 2.1. *The* tangent cone *of $\Omega$ at $x^* \in \Omega$ is a closed cone defined by*

$$\mathcal{T}_\Omega(x^*) := \{\, d \mid d = \lim_{k \to \infty} t_k(x^k - x^*) \text{ with } t_k \geq 0 \text{ and } x^k \to_\Omega x^*\}.$$

*The* regular normal cone *(also known as the Frechét normal cone) of $\Omega$ at $x^* \in \Omega$ is a closed cone defined by*

$$\widehat{\mathcal{N}}_\Omega(x^*) := \{d \mid d^T(x - x^*) \leq o(\|x - x^*\|) \text{ for each } x \in \Omega\},$$

*where $\frac{o(\alpha)}{\alpha} \to 0$ as $\alpha \downarrow 0$. The* limiting normal cone *(also known as the Mordukhovich normal cone or basic normal cone) of $\Omega$ at $x^* \in \Omega$ is a closed cone defined by*

$$\mathcal{N}_\Omega(x^*) := \limsup_{x \to_\Omega x^*} \widehat{\mathcal{N}}_\Omega(x).$$

By straightforward calculation, we can obtain the formulas for the regular normal cone and limiting normal cone of the set $C$ defined in (1.4) as follows (see, e.g., [17, 31]).

PROPOSITION 2.1. *For any $(a, b) \in C$, we have*

$$\widehat{\mathcal{N}}_C(a, b) = \left\{ (d_1, d_2) \;\middle|\; \begin{array}{c} d_1 \in \Re, d_2 = 0 \text{ if } a = 0 > b \\ d_1 = 0, d_2 \in \Re \text{ if } a < 0 = b \\ d_1 \geq 0, d_2 \geq 0 \text{ if } a = b = 0 \end{array} \right\},$$
$$\mathcal{N}_C(a, b) = \left\{ (d_1, d_2) \;\middle|\; \begin{array}{r} d_1 \in \Re, d_2 = 0 \text{ if } a = 0 > b \\ d_1 = 0, d_2 \in \Re \text{ if } a < 0 = b \\ \text{either } d_1 > 0, d_2 > 0 \text{ or } d_1 d_2 = 0 \text{ if } a = b = 0 \end{array} \right\}.$$

**2.2. Set-valued mappings and semicontinuity.** Let $\Omega$ be an open subset of $\Re^n$.

DEFINITION 2.2. *A set-valued mapping $\Phi : \Omega \rightrightarrows \Re^m$ is said to be* outer semicontinuous *or* closed *at $\bar{x} \in \Omega$ if*

$$\limsup_{x \to_\Omega \bar{x}} \Phi(x) = \Phi(\bar{x}).$$

*$\Phi$ is said to be* calm *at $(\bar{x}, \bar{y}) \in \operatorname{gph} \Phi$ if there exist $\delta > 0$ and $\kappa > 0$ such that*

$$\Phi(x) \cap \mathcal{B}_\delta(\bar{y}) \subseteq \Phi(\bar{x}) + \kappa \|x - \bar{x}\| \bar{\mathcal{B}}_1(0) \quad \forall x \in \mathcal{B}_\delta(\bar{x}).$$

*$\Phi$ is said to be* locally bounded *at $\bar{x} \in \Omega$ or* uniformly compact *near $\bar{x} \in \Omega$ if there is $\delta > 0$ such that the closure of $\bigcup_{x \in \mathcal{B}_\delta(\bar{x})} \Phi(x)$ is compact. $\Phi$ is said to be* upper semicontinuous *at $\bar{x} \in \Omega$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\Phi(\mathcal{B}_\delta(\bar{x})) \subset \Phi(\bar{x}) + \epsilon \mathcal{B}_1(0).$$

$\Phi$ *is said to be* lower semicontinuous *at* $\bar{x} \in \Omega$ *if, for any open set* $V$ *intersecting* $\Phi(\bar{x})$*, there exists a neighborhood* $U$ *of* $\bar{x}$ *such that* $\Phi(x)$ *intersects* $V$ *for each* $x \in U$. $\Phi$ *is said to be* continuous *at* $\bar{x}$ *if* $\Phi$ *is both upper semicontinuous and lower semicontinuous at* $\bar{x}$.

*Remark* 2.1. Although the term "calmness" was coined in [28], the concept of calmness for a set-valued mapping was first introduced by Ye and Ye in [33] under the term "pseudo upper Lipschitz continuity," which comes from the fact that it is a weaker condition than either Aubin's pseudo-Lipschitz continuity or Robinson's upper Lipschitz continuity. Hence, the calmness can be considered as a kind of weak Lipschitz continuity.

We will need the following result; see, e.g., [28, Theorem 5.19].

PROPOSITION 2.2. *Let* $\Phi : \Omega \rightrightarrows \Re^m$ *be uniformly compact near* $\bar{x}$. *Then* $\Phi$ *is outer semicontinuous at* $\bar{x}$ *if and only if* $\Phi$ *is upper semicontinuous at* $\bar{x}$ *and* $\Phi(\bar{x})$ *is closed.*

**2.3. Optimality conditions for (MPEC$_p$).** We next review some popular stationarity concepts for (MPEC$_p$). In order to facilitate the notation, for a given feasible point $x^*$ of (MPEC$_p$), we let

$$\begin{aligned} I_g^* &:= \{\, i \mid g_i(x^*,p^*) = 0\},\\ I_{g^-}^* &:= \{\, i \mid g_i(x^*,p^*) < 0\},\\ \mathcal{I}^* &:= \{\, i \mid G_i(x^*,p^*) = 0 < H_i(x^*,p^*)\},\\ \mathcal{J}^* &:= \{\, i \mid G_i(x^*,p^*) = 0 = H_i(x^*,p^*)\},\\ \mathcal{K}^* &:= \{\, i \mid G_i(x^*,p^*) > 0 = H_i(x^*,p^*)\}. \end{aligned}$$

Obviously, $\{\mathcal{I}^*, \mathcal{J}^*, \mathcal{K}^*\}$ is a partition of $\{1, 2, \dots, m\}$. Moreover, given $r \geq 0$ and $p \in \Re^{n_2}$, we define the *generalized MPEC-Lagrangian* of (MPEC$_p$) as

$$L^r_{\mathrm{MPEC}}(x,p;\lambda,\mu,u,v) := rf(x,p) + g(x,p)^T\lambda + h(x,p)^T\mu - G(x,p)^T u - H(x,p)^T v.$$

DEFINITION 2.3 (see [29, 32, 33]). *Let* $x^*$ *be a given feasible point of* (MPEC$_{p^*}$).

(1) *We say that* $x^*$ *is* generalized Mordukhovich stationary *(generalized M-stationary) to* (MPEC$_{p^*}$) *if there exist vectors* $(r, \lambda, \mu, u, v) \neq 0$ *with* $r \geq 0$ *such that*

$$\left\{\begin{array}{l} \nabla_x L^r_{\mathrm{MPEC}}(x^*,p^*;\lambda,\mu,u,v) = 0,\\ \lambda \geq 0, g(x^*,p^*)^T\lambda = 0,\\ u_i = 0, \qquad i \in \mathcal{K}^*,\\ v_i = 0, \qquad i \in \mathcal{I}^*,\\ \textit{either } u_i v_i = 0 \textit{ or } u_i > 0, v_i > 0, \qquad i \in \mathcal{J}^*. \end{array}\right. \tag{2.1}$$

(2) *We say that* $x^*$ *is* generalized strongly stationary *(generalized S-stationary) to* (MPEC$_{p^*}$) *if there exist vectors* $(r, \lambda, \mu, u, v) \neq 0$ *with* $r \geq 0$ *such that*

$$\left\{\begin{array}{l} \nabla_x L^r_{\mathrm{MPEC}}(x^*,p^*;\lambda,\mu,u,v) = 0,\\ \lambda \geq 0, g(x^*,p^*)^T\lambda = 0,\\ u_i = 0, \qquad i \in \mathcal{K}^*,\\ v_i = 0, \qquad i \in \mathcal{I}^*,\\ u_i \geq 0, v_i \geq 0, \qquad i \in \mathcal{J}^*. \end{array}\right. \tag{2.2}$$

(3) *Given any* $r \geq 0$, *we denote by* $\mathcal{M}^r_M(x^*,p^*)$ *and* $\mathcal{M}^r_S(x^*,p^*)$ *the sets of generalized multipliers* $(\lambda, \mu, u, v)$ *such that* $(r, \lambda, \mu, u, v) \neq 0$ *satisfies* (2.1) *and* (2.2),

*respectively. If there exists* $(\lambda^*, \mu^*, u^*, v^*) \in \mathcal{M}_M^r(x^*, p^*)$ *(or* $\mathcal{M}_S^r(x^*, p^*)$*) with* $r = 1$*, we say that* $x^*$ *is* M-stationary *(or* S-stationary*) to* (MPEC$_{p^*}$).

For a given $x \in \Re^{n_1}$ and index sets $\mathcal{I}_1 \subseteq \{1, \ldots, m_2\}$ and $\mathcal{I}_2, \mathcal{I}_3 \subseteq \{1, \ldots, m\}$, we let

$$\mathcal{G}\,(x, p^*; \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3) := \{\nabla_x h_j(x, p^*), \nabla_x G_\imath(x, p^*), \nabla_x H_\jmath(x, p^*) \mid j \in \mathcal{I}_1, \imath \in \mathcal{I}_2, \jmath \in \mathcal{I}_3\}.$$

Definition 2.4 (see [11, 29, 31, 32]). *Let* $x^*$ *be a given feasible point of* (MPEC$_{p^*}$).

(1) *We say that the* MPEC no nonzero abnormal multiplier constraint qualification *(MPEC-NNAMCQ) holds at* $x^*$ *if there is no nonzero multiplier* $(\lambda, \mu, u, v)$ *such that*

$$(2.3)\quad \begin{cases} \nabla_x g(x^*, p^*)\lambda + \nabla_x h(x^*, p^*)\mu - \nabla_x G(x^*, p^*)u - \nabla_x H(x^*, p^*)v = 0, \\ \lambda \geq 0, g(x^*, p^*)^T \lambda = 0, \\ u_i = 0, \qquad i \in \mathcal{K}^*, \\ v_i = 0, \qquad i \in \mathcal{I}^*, \\ \textit{either } u_i v_i = 0 \textit{ or } u_i > 0, v_i > 0, \qquad i \in \mathcal{J}^*. \end{cases}$$

(2) *We say that the* MPEC linear constraint qualification *holds if all functions* $\{g, h, G, H\}$ *are linear with respect to* $x$.

(3) *Let* $\mathcal{I}_1 \subseteq \{1, \ldots, m_2\}$*,* $\mathcal{I}_2 \subseteq \mathcal{I}^*$*, and* $\mathcal{I}_3 \subseteq \mathcal{K}^*$ *be such that* $\mathcal{G}\,(x^*, p^*; \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ *is a basis for* $\operatorname{span} \mathcal{G}\,(x^*, p^*; \{1, \ldots, m_2\}, \mathcal{I}^*, \mathcal{K}^*)$*. We say that the* MPEC relaxed constant positive linear dependence *(MPEC-RCPLD) condition holds at* $x^*$ *if there exists* $\delta > 0$ *such that*

– $\mathcal{G}\,(x, p^*; \{1, \ldots, m_2\}, \mathcal{I}^*, \mathcal{K}^*)$ *has the same rank for each* $x \in \mathcal{B}_\delta(x^*)$*;*

– *for each* $\mathcal{I}_4 \subseteq I_g^*$ *and* $\mathcal{I}_5, \mathcal{I}_6 \subseteq \mathcal{J}^*$*, if there exist vectors* $\{\lambda, \mu, u, v\}$ *with* $\lambda_i \geq 0$ *for each* $i \in \mathcal{I}_4$*, either* $u_l v_l = 0$ *or* $u_l > 0, v_l > 0$ *for each* $l \in \mathcal{J}^*$*, which are not all zero, such that*

$$\sum_{i \in \mathcal{I}_4} \lambda_i \nabla_x g_i(x^*, p^*) + \sum_{j \in \mathcal{I}_1} \mu_j \nabla_x h_j(x^*, p^*) - \sum_{\imath \in \mathcal{I}_2 \cup \mathcal{I}_5} u_\imath \nabla G_\imath(x^*, p^*) - \sum_{\jmath \in \mathcal{I}_3 \cup \mathcal{I}_6} v_\jmath \nabla_x H_\jmath(x^*, p^*) = 0;$$

*then, for any* $x \in \mathcal{B}_\delta(x^*)$*, the vectors*

$$\{\nabla_x g_i(x, p^*)\}_{i \in \mathcal{I}_4},\ \{\nabla_x h_j(x, p^*)\}_{j \in \mathcal{I}_1}, \quad \{\nabla_x G_\imath(x, p^*)\}_{\imath \in \mathcal{I}_2 \cup \mathcal{I}_5},\ \{\nabla_x H_\jmath(x, p^*)\}_{\jmath \in \mathcal{I}_3 \cup \mathcal{I}_6}$$

*are linearly dependent.*

The MPEC-RCPLD was introduced in [11] to show the isolatedness of an M-stationary point, and it was shown to be a constraint qualification for the M-stationarity in [10]. The MPEC-RCPLD is a much weaker condition than the MPEC-NNAMCQ and MPEC linear constraint qualification. In the case where there is no complementarity constraint, the MPEC-RCPLD reduces to the relaxed constant positive linear dependence (RCPLD) condition introduced recently in [2] for standard nonlinear programs.

**3. Stability analysis for (MPGC$_p$).** In this section, we consider the stability for (MPGC$_p$). We denote by $\mathcal{X}(p)$ the feasible region of (MPGC$_p$). Moreover, the *Lagrangian function* of (MPGC$_p$) is defined as

$$L(x, p; y) := f(x, p) + F(x, p)^T y$$

and the *critical cone* of (MPGC$_p$) at $x^* \in \mathcal{X}(p^*)$ is defined as

$$\mathcal{C}(x^*,p^*) := \{d \mid \nabla_x f(x^*,p^*)^T d \le 0, \nabla_x F(x^*,p^*)^T d \in \mathcal{T}_\Lambda(F(x^*,p^*))\}.$$

Then the stationarity system of (MPGC$_p$) can be written as

$$\left\{\begin{array}{l} \nabla_x L(x,p;y) = 0, \\ y \in \mathcal{N}_\Lambda(F(x,p)). \end{array}\right. \tag{3.1}$$

We define the *multiplier mapping* $\mathcal{M} : \Re^{n_1+n_2} \rightrightarrows \Re^l$, the *stationary point mapping* $\mathcal{S} : \Re^{n_2} \rightrightarrows \Re^{n_1}$, and the *locally optimal solution mapping* $\mathcal{O} : \Re^{n_2} \rightrightarrows \Re^{n_1}$ as follows:

$$\begin{aligned} \mathcal{M}(x,p) &:= \{y \mid (x,y,p) \text{ satisfies } (3.1)\}, \\ \mathcal{S}(p) &:= \{x \mid (x,y,p) \text{ satisfies } (3.1) \text{ for some } y\}, \\ \mathcal{O}(p) &:= \{x \mid x \text{ is a locally optimal solution of } (\mathrm{MPGC}_p)\}. \end{aligned}$$

For standard nonlinear programs with equality and inequality constraints, it is well known that the MFCQ holds at a feasible point if and only if the set of Lagrange multipliers is nonempty and bounded; see, e.g., [4, 9]. For (MPGC$_p$), we consider the following constraint qualification, which is equivalent to the MFCQ in the case of standard nonlinear programs and equivalent to Robinson's constraint qualification when $\Lambda$ reduces to a closed convex set; see, e.g., [28, Exercise 6.39] and [4, Proposition 2.97].

DEFINITION 3.1 (see [28]). *We say that the* no nonzero abnormal multiplier constraint qualification *(NNAMCQ) holds at $x^* \in \mathcal{X}(p^*)$ if there is no nonzero multiplier $y \in \Re^l$ such that*

$$\left\{\begin{array}{l} \nabla_x F(x^*,p^*)y = 0, \\ y \in \mathcal{N}_\Lambda(F(x^*,p^*)). \end{array}\right.$$

It is well known that, under the NNAMCQ, the set $\mathcal{M}(x^*,p^*)$ is nonempty and compact for any fixed $x^* \in \mathcal{X}(p^*)$. We next show that the multiplier mapping is locally bounded (i.e., uniformly compact) and upper semicontinuous with respect to $(x,p)$. This extends Theorem 2.3 of Robinson [26] in that $\Lambda$ is only assumed to be a closed set here.

THEOREM 3.1. *If the NNAMCQ holds at $x^* \in \mathcal{X}(p^*)$, then there exists $\delta > 0$ such that the set-valued mappings $\mathcal{M}$ and $\mathcal{S} \cap \bar{\mathcal{B}}_\delta(x^*)$ are locally bounded and upper semicontinuous on $\mathcal{B}_\delta(x^*) \times \mathcal{B}_\delta(p^*)$ and $\mathcal{B}_\delta(p^*)$, respectively.*

*Proof.* We first show that $\mathcal{M}$ is locally bounded at $(x^*,p^*)$. Suppose to the contrary that there exist sequences $\{x^k\} \to x^*$, $\{p^k\} \to p^*$, and $\|y^k\| \to \infty$ with $y^k \in \mathcal{M}(x^k,p^k)$. This implies

$$\left\{\begin{array}{l} 0 = \nabla_x f(x^k,p^k) + \nabla_x F(x^k,p^k)y^k, \\ y^k \in \mathcal{N}_\Lambda(F(x^k,p^k)). \end{array}\right. \tag{3.2}$$

Assume without loss of generality that $y^k/\|y^k\| \to y^*$ with $\|y^*\| = 1$. It follows from (3.2) and the outer semicontinuity of $\mathcal{N}_\Lambda$ (see, e.g., [28, Proposition 6.6]) that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \Big( \frac{\nabla_x f(x^k,p^k)}{\|y^k\|} + \frac{\nabla_x F(x^k,p^k)y^k}{\|y^k\|} \Big) = \nabla_x F(x^*,p^*)y^*, \\ y^* &= \lim_{k\to\infty} \frac{y^k}{\|y^k\|} \in \limsup_{k\to\infty} \mathcal{N}_\Lambda(f(x^k,p^k)) = \mathcal{N}_\Lambda(F(x^*,p^*)). \end{aligned}$$

This together with $\|y^*\| = 1$ contradicts the fact that the NNAMCQ holds at $x^* \in \mathcal{X}(p^*)$, and hence the set-valued mapping $\mathcal{M}$ is locally bounded at $(x^*, p^*)$; i.e., there exist a number $\delta > 0$ and a compact set $K$ such that

$$\mathcal{M}(x,p) \subseteq K \quad \forall (x,p) \in \mathcal{B}_\delta(x^*) \times \mathcal{B}_\delta(p^*). \tag{3.3}$$

It follows from (3.3) that $\mathcal{M}$ is locally bounded at each point $(x,p) \in \mathcal{B}_\delta(x^*) \times \mathcal{B}_\delta(p^*)$. Since $\nabla_x f$ and $\nabla_x F$ are continuous and $\mathcal{N}_\Lambda$ is outer semicontinuous, it is easy to verify that the set-valued mapping $\mathcal{M}$ is outer semicontinuous at every point $(x,p) \in \mathcal{B}_\delta(x^*) \times \mathcal{B}_\delta(p^*)$. Therefore, by virtue of Proposition 2.2, $\mathcal{M}$ is upper semicontinuous on $\mathcal{B}_\delta(x^*) \times \mathcal{B}_\delta(p^*)$ as well.

Now consider the set-valued mapping $\mathcal{S} \cap \bar{\mathcal{B}}_\delta(x^*)$. Since

$$\mathcal{S}(p) \cap \bar{\mathcal{B}}_\delta(x^*) = \left\{ x \in \bar{\mathcal{B}}_\delta(x^*) \;\middle|\; \begin{array}{l} \exists y \in \mathcal{N}_\Lambda(F(x,p)) \\ \text{s.t. } \nabla_x f(x,p) + \nabla_x F(x,p) y = 0 \end{array} \right\}, \tag{3.4}$$

it is clear that $\mathcal{S} \cap \bar{\mathcal{B}}_\delta(x^*)$ is locally bounded at every $\bar{p} \in \mathcal{B}_\delta(p^*)$. So it remains to show that $\mathcal{S} \cap \bar{\mathcal{B}}_\delta(x^*)$ is outer semicontinuous at every $\bar{p} \in \mathcal{B}_\delta(p^*)$. Suppose that $x^k \in \mathcal{S}(p^k) \cap \bar{\mathcal{B}}_\delta(x^*)$ and $(x^k, p^k) \to (\bar{x}, \bar{p})$. Then there exists $y^k \in \mathcal{M}(x^k, p^k)$. It follows from (3.3) that $y^k \in K$ for each $k$ sufficiently large. Without loss of generality, we assume $y^k \to \bar{y}$. From the outer semicontinuity of $\mathcal{N}_\Lambda$, we have $\bar{y} \in \mathcal{N}_\Lambda(F(\bar{x}, \bar{p}))$ and $\nabla_x f(\bar{x}, \bar{p}) + \nabla_x F(\bar{x}, \bar{p}) \bar{y} = 0$. Thus we have $\bar{x} \in \mathcal{S}(\bar{p}) \cap \bar{\mathcal{B}}_\delta(x^*)$, which implies that $\mathcal{S} \cap \bar{\mathcal{B}}_\delta(x^*)$ is outer semicontinuous at $\bar{p}$. Therefore, by Proposition 2.2, $\mathcal{S} \cap \bar{\mathcal{B}}_\delta(x^*)$ is upper semicontinuous on $\mathcal{B}_\delta(p^*)$ as well. This completes the proof. $\square$

Since for standard nonlinear programs the NNAMCQ (or, equivalently, the MFCQ) is a necessary and sufficient condition for compactness of the set of multipliers, a natural question to ask is whether the compactness of the multiplier set $\mathcal{M}(x^*, p^*)$ at $x^* \in \mathcal{X}(p^*)$ implies the NNAMCQ at $x^*$. The following example shows that the answer is negative.

*Example* 3.1. Consider the problem

$$\begin{array}{ll} \min & 2x_1 \\ \text{s.t.} & F(x) := \begin{pmatrix} x_2 - x_1^2 \\ x_1 \\ -x_2 \end{pmatrix} \in \Lambda := \{0\} \times C, \end{array}$$

where $C$ is the same as in (1.4). Clearly, the only optimal solution of the above problem is $x^* = (0,0)$. The stationarity system at $x^* = (0,0)$ is

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mu \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \mu \\ u \\ v \end{bmatrix} \in \Re \times \mathcal{N}_C(0,0).$$

By Proposition 2.1, we have

$$\mathcal{N}_C(0,0) = \{(u,v) \mid \text{ either } u > 0, v > 0 \text{ or } uv = 0\}.$$

It is easy to verify that the set of multipliers is equal to the singleton $\{(0,-2,0)\}$, which is nonempty and bounded. However, there exists a nonzero vector $y = (\mu, u, v) = (1,0,1) \neq 0$ such that

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mu \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \mu \\ u \\ v \end{bmatrix} \in \Re \times \mathcal{N}_C(0,0),$$

and hence the NNAMCQ does not hold at $x^* = (0,0)$.

It is well known that the NNAMCQ at $x^* \in \mathcal{X}(p^*)$ implies the existence of local error bound at $x^*$. In fact, as shown in the next lemma, this property is robust with respect to small perturbation; i.e., under the NNAMCQ at $x^* \in \mathcal{X}(p^*)$, the local error bound property holds for all points in a neighborhood of $(x^*, p^*)$.

LEMMA 3.1. *Suppose that the NNAMCQ holds at $x^* \in \mathcal{X}(p^*)$. Then there exist $\delta > 0$ and $\kappa > 0$ such that*

$$\operatorname{dist}(x, \mathcal{X}(p)) \leq \kappa \operatorname{dist}(F(x,p), \Lambda) \qquad \forall x \in \mathcal{B}_\delta(x^*),\ \forall p \in \mathcal{B}_\delta(p^*).$$

*Proof.* Let $\mathcal{S}_p(x)$ be a set-valued mapping defined by $\mathcal{S}_p(x) := F(x,p) - \Lambda$ and let its inverse be defined by $\mathcal{S}_p^{-1}(u) := \{x \mid u \in \mathcal{S}_p(x)\}$. It is not hard to see that $\mathcal{S}_p^{-1}(0) = \{x \mid F(x,p) \in \Lambda\} = \mathcal{X}(p)$ and

$$x \in \mathcal{S}_p^{-1}(u) \iff x \in \mathcal{S}_{p^*}^{-1}(u + F(x,p^*) - F(x,p)).$$

Since the NNAMCQ holds at $x^* \in \mathcal{X}(p^*)$, it follows from [28, Example 9.44] (or the Mordukhovich criterion) that there exists $\delta_1 > 0$ such that

$$\operatorname{dist}(x, \mathcal{S}_{p^*}^{-1}(u)) \leq \kappa \operatorname{dist}(u, \mathcal{S}_{p^*}(x)) \qquad \forall x \in \mathcal{B}_{\delta_1}(x^*),\ \forall u \in \mathcal{B}_{\delta_1}(0). \tag{3.5}$$

Pick $\delta \in (0, \delta_1]$ such that if $u \in \mathcal{B}_\delta(0), x \in \mathcal{B}_\delta(x^*)$, and $p \in \mathcal{B}_\delta(p^*)$, then $\|u + F(x,p^*) - F(x,p)\| \leq \delta_1$, i.e., $u + F(x,p^*) - F(x,p) \in \mathcal{B}_{\delta_1}(0)$. It then follows from (3.5) that, for each $u \in \mathcal{B}_\delta(0), x \in \mathcal{B}_\delta(x^*)$, and $p \in \mathcal{B}_\delta(p^*)$,

$$\begin{aligned}
\operatorname{dist}(x, \mathcal{S}_p^{-1}(u)) &= \operatorname{dist}(x, \mathcal{S}_{p^*}^{-1}(u + F(x,p^*) - F(x,p))) \\
&\leq \kappa \operatorname{dist}(u + F(x,p^*) - F(x,p), \mathcal{S}_{p^*}(x)) \\
&= \kappa \operatorname{dist}(u, \mathcal{S}_p(x)).
\end{aligned}$$

Therefore, the desired result is obtained by letting $u = 0$. The proof is complete. ∎

We next show that if $x^* \in \mathcal{X}(p^*)$ satisfies the NNAMCQ and second-order growth condition, then the locally optimal solution exists under small perturbation. This improves Theorem 3.1 of Robinson [26], where the same conclusion was shown under the stronger assumption that $\Lambda$ is a closed convex cone and the second-order sufficient condition holds, because the second-order sufficient condition implies the second-order growth condition when $\Lambda$ reduces to a closed convex set (see, e.g., [4, Theorem 3.63]).

THEOREM 3.2. *Suppose that the NNAMCQ and second-order growth condition hold at $x^* \in \mathcal{X}(p^*)$; i.e., there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ such that*

$$f(x,p^*) \geq f(x^*,p^*) + \frac{1}{2}\epsilon_0 \|x - x^*\|^2 \qquad \forall x \in \mathcal{X}(p^*) \cap \mathcal{B}_{\delta_0}(x^*).$$

*Then, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $p \in \mathcal{B}_\delta(p^*)$, then* (MPGC$_p$) *has a locally optimal solution in $\mathcal{B}_\epsilon(x^*)$.*

*Proof.* Since the NNAMCQ holds at $x^*$, by Lemma 3.1, there exist $\delta_1 > 0$ and $\kappa > 0$ such that

$$\operatorname{dist}(x, \mathcal{X}(p)) \leq \kappa \operatorname{dist}(F(x,p), \Lambda) \qquad \forall x \in \mathcal{B}_{\delta_1}(x^*), \forall p \in \mathcal{B}_{\delta_1}(p^*). \tag{3.6}$$

By the second-order growth condition, there exist $\epsilon_0 > 0$ and $\delta_2 \in (0, \delta_1)$ such that

$$f(x,p^*) \geq f(x^*,p^*) + \frac{1}{2}\epsilon_0 \|x - x^*\|^2 \qquad \forall x \in \mathcal{X}(p^*) \cap \mathcal{B}_{\delta_2}(x^*). \tag{3.7}$$

Let $\epsilon > 0$ and $\delta_3 := \frac{1}{2}\min(\delta_2, \epsilon)$. Due to the uniform continuity of $f$ and $F$ over $\mathcal{B}_{2\delta_3}(x^*) \times \mathcal{B}_{\delta_3}(p^*)$, there exist $\alpha \in (0, \delta_3/2)$ and $\delta \in (0, \min(\delta_3, \alpha/\kappa))$ such that, for each $p \in \mathcal{B}_\delta(p^*)$ and $x^1, x^2 \in \mathcal{B}_{2\delta_3}(x^*)$ with $\|x^1 - x^2\| \le \alpha$,

$$|f(x^1, p^*) - f(x^2, p)| < \gamma := \frac{\epsilon_0 \delta_3^2}{16} \tag{3.8}$$

and

$$\kappa \|F(x^1, p^*) - F(x^1, p)\| \le \min\left(\alpha, \frac{1}{2}\delta_3\right). \tag{3.9}$$

Let $p \in \mathcal{B}_\delta(p^*)$. Then, by (3.6) and (3.9), we have

$$\operatorname{dist}(x^*, \mathcal{X}(p)) \le \kappa \operatorname{dist}(F(x^*, p), \Lambda) \le \kappa \|F(x^*, p) - F(x^*, p^*)\| \le \min\left(\alpha, \frac{1}{2}\delta_3\right). \tag{3.10}$$

It is not hard to see from (3.10) that $\mathcal{X}(p) \cap \mathcal{B}_{\delta_3}(x^*)$ is nonempty. Since $f$ is continuous in $x$ and $\mathcal{X}(p) \cap \bar{\mathcal{B}}_{\delta_3}(x^*)$ is nonempty compact, there exists $x_p^* \in \mathcal{X}_p \cap \bar{\mathcal{B}}_{\delta_3}(x^*)$ such that

$$f(x_p^*, p) \le f(x, p) \quad \forall x \in \mathcal{X}(p) \cap \bar{\mathcal{B}}_{\delta_3}(x^*). \tag{3.11}$$

It suffices to show that $\|x_p^* - x^*\| < \delta_3$ for each $p \in \mathcal{B}_\delta(p^*)$, which implies that $x_p^*$ is a locally optimal solution of (MPGC$_p$). In fact, by (3.10), there exists $x' \in \mathcal{X}(p)$ such that $\|x^* - x'\| \le \alpha$. We then have from (3.8) that

$$|f(x^*, p^*) - f(x', p)| \le \gamma$$

and hence

$$f(x', p) \le f(x^*, p^*) + |f(x^*, p^*) - f(x', p)| \le f(x^*, p^*) + \gamma. \tag{3.12}$$

This together with (3.11) implies that

$$f(x_p^*, p) \le f(x^*, p^*) + \gamma. \tag{3.13}$$

On the other hand, for each $x \in \mathcal{X}(p)$ with $\|x - x^*\| = \delta_3$, we have from (3.6) and (3.9) that

$$\operatorname{dist}(x, \mathcal{X}(p^*)) \le \kappa \operatorname{dist}(F(x, p^*), \Lambda) \le \kappa \|F(x, p^*) - F(x, p)\| \le \alpha.$$

Therefore, there exists $x^0 \in \mathcal{X}(p^*)$ satisfying $\|x^0 - x\| \le \alpha$ and hence

$$\frac{\delta_3}{2} \le \|x - x^*\| - \|x - x^0\| \le \|x^0 - x^*\| \le \|x - x^*\| + \|x - x^0\| \le 2\delta_3.$$

It then follows from (3.7) and (3.8) that $|f(x^0, p^*) - f(x, p)| < \gamma$ and

$$f(x^0, p^*) \ge f(x^*, p^*) + \frac{1}{2}\epsilon_0 \|x^0 - x^*\|^2 \ge f(x^*, p^*) + \frac{1}{8}\epsilon_0 \delta_3^2 = f(x^*, p^*) + 2\gamma.$$

This together with (3.8) indicates that, for each $x \in \mathcal{X}(p)$ with $p \in \mathcal{B}_\delta(p^*)$ and $\|x - x^*\| = \delta_3$,

$$f(x, p) \ge f(x^0, p^*) - \|f(x^0, p^*) - f(x, p)| > f(x^*, p^*) + \gamma. \tag{3.14}$$

It follows from (3.13) and (3.14) that $\|x_p^* - x^*\| < \delta_3$. The proof is completed by noting that $\delta_3 < \epsilon$. $\square$

We now study conditions under which a stationary point of (MPGC$_p$) is isolated. We will require $\Lambda$ to satisfy one of the following two assumptions.

*Assumption* 3.1.

(A1) If $a^k \to a^*$ and $b^k \to b^*$ with $a^k \in \mathcal{N}_\Lambda(b^k)$, there exists an infinite subset $\mathbb{K} \subseteq \mathbb{N}$ such that $(a^k)^T(b^* - b^k) \le 0$ and $(a^*)^T(b^k - b^*) \le 0$ for each $k \in \mathbb{K}$.

(A2) If $a^k \to a^*$ and $b^k \to b^*$ with $a^k \in \mathcal{N}_\Lambda(b^k)$, there exists an infinite subset $\mathbb{K} \subseteq \mathbb{N}$ such that $(a^k)^T(b^* - b^k) \le 0$ and $(a^*)^T b^k = 0$ for each $k \in \mathbb{K}$.

Note that (A2) implies (A1) and, by the outer semicontinuity of $\mathcal{N}_\Lambda$, $a^k \to a^*$ and $b^k \to b^*$ with $a^k \in \mathcal{N}_\Lambda(b^k)$ imply $a^* \in \mathcal{N}_\Lambda(b^*)$. If $\Lambda$ is a closed convex set, then $\Lambda$ must satisfy (A1). In fact, by the definition of a normal cone, we have $(a^k)^T(b^* - b^k) \le 0$ and $(a^*)^T(b^k - b^*) \le 0$. Moreover, a sufficient condition for both (A1) and (A2) to hold is that $a^k \circ b^k = 0$, where $\circ$ stands for the Hadamard product. Indeed, we can show that $a_i^k b_i^* = 0$ and $a_i^* b_i^k = 0$ for $i = 1, \ldots, l$. If $a_i^* \neq 0$, then $a_i^k \neq 0$ for each $k$ sufficient large, which implies that $b_i^k = 0$ for such $k$ since $a_i^k b_i^k = 0$. Similarly, we have $a_i^k b_i^* = 0$ for $i = 1, \ldots, l$. Hence, it follows that if $\Lambda$ is a union of finitely many half spaces $\Re_-^{l_1^i} \times \{0\}^{l_2^i}$, then it satisfies (A1) and (A2) by $a^k \circ b^k = 0$. Thus, since the MPEC cone defined in (1.3) is a union of finitely many half spaces, it satisfies (A1) and (A2) automatically.

In general, stability analysis for optimization problems requires some second-order sufficient conditions. In what follows, we give several kinds of second-order sufficient conditions for (MPGC$_p$).

Definition 3.2. *We say that the* strong second-order sufficient condition *(SSOSC) for* (MPGC$_p$) *holds at* $x^* \in \mathcal{X}(p^*)$ *if* $\mathcal{M}(x^*, p^*) \neq \emptyset$ *and, for every* $y^* \in \mathcal{M}(x^*, p^*)$,

$$d^T \nabla_x^2 L(x^*, p^*; y^*) d > 0 \qquad \forall d \in \mathcal{C}(x^*, p^*) \backslash \{0\}.$$

*We say that the* second-order sufficient condition *(SOSC) for* (MPGC$_p$) *holds at* $(x^*, y^*)$ *with* $y^* \in \mathcal{M}(x^*, p^*)$ *if*

$$d^T \nabla_x^2 L(x^*, p^*; y^*) d > 0 \qquad \forall d \in \mathcal{C}(x^*, p^*) \backslash \{0\}.$$

Definition 3.3. *We say that the* multiplier-stability *for* (MPGC$_p$) *holds at* $x^* \in \mathcal{X}(p^*)$ *if, for any stationary point sequence* $\{x^k\} \subseteq \mathcal{S}(p^*) \backslash \{x^*\}$ *converging to* $x^*$*, there exists a multiplier sequence* $\{y^k \in \mathcal{M}(x^k, p^*)\}$ *containing a bounded subsequence.*

Note that the sequence $\{y^k\}$ in Definition 3.3 may not be unique. It is clear from Theorem 3.1 that if the NNAMCQ holds at $x^* \in \mathcal{X}(p^*)$, then $x^*$ is multiplier-stable. For $\Lambda$ with some special structures, there may exist some weak conditions to ensure the multiplier-stability. For example, the MPEC-RCPLD at $x^*$ is sufficient to ensure the multiplier-stability when $\Lambda$ is the MPEC cone; see the proof of Theorem 5.1 in [11]. Consequently, if $F$ is linear with respect to $x$ and $\Lambda$ is the MPEC cone, the MPEC linear constraint qualification holds and hence the multiplier-stability holds at every feasible point. In fact, when $F$ is linear with respect to $x$ and $\Lambda$ is a union of polyhedral sets satisfying some kind of separability, the multiplier-stability holds. Let $\Lambda := \cup_{\nu=1}^N \Lambda_\nu$.

– We say that $\Lambda$ is *locally separable* at $x$ if, for any sequence $\{x^k\}$ converging to $x$, there exist $\nu_0 \in \{1, \ldots, N\}$ and a subsequence $\{x^k\}_{k \in \mathbb{K}}$ such that $F(x^k, p^*) \in \Lambda_{\nu_0} \backslash \cup_{\nu \neq \nu_0} \Lambda_\nu$, $k \in \mathbb{K}$.

For simplicity, assume that $F(x^k, p^*) := Ax^k + a$ and $\Lambda_\nu := \{x \mid B^\nu x \le b^\nu\}$, $\nu = 1, \ldots, N$. Let $x \in \mathcal{X}(p^*)$. Pick an arbitrary sequence $\{x^k\}$ converging to $x$ with the associated multiplier sequence $\{y^k\}$ satisfying

$$\nabla_x f(x^k, p^*) + A^T y^k = 0, \quad y^k \in \mathcal{N}_\Lambda(Ax^k + a). \tag{3.15}$$

By the separability assumption, we may assume without loss of generality that $Ax^k + a \in \Lambda_{\nu_0} \setminus \cup_{\nu \neq \nu_0} \Lambda_\nu$ for every $k$. Then we have from (3.15) that

$$\nabla_x f(x^k, p^*) + A^T y^k = 0, \quad y^k \in \mathcal{N}_{\Lambda_{\nu_0}}(Ax^k + a). \tag{3.16}$$

It follows from (3.16) and [28, Theorem 6.46] that

$$\begin{aligned} &\nabla_x f(x^k, p^*) + A^T y^k = 0, \quad y^k = (B^{\nu_0})^T \lambda^k, \\ &0 \leq \lambda^k \ \perp \ B^{\nu_0}(Ax^k + a) - b^{\nu_0} \leq 0. \end{aligned}$$

Let $\mathcal{J}^k := \{i \mid [B^{\nu_0}]_i^T (Ax^k + a) - b_i^{\nu_0} = 0\}$. Then we have

$$\nabla_x f(x^k, p^*) + \sum_{i \in \mathcal{J}^k} \lambda_i^k [M]_i = 0, \quad \lambda_i^k \geq 0 \ (i \in \mathcal{J}^k), \quad M := B^{\nu_0} A.$$

By Carathéodory's theorem for cone hulls (see, e.g., [3, Proposition 1.3.1]), we may assume without loss of generality that $\{[M]_i, i \in \mathcal{J}^k\}$ are linearly independent. Assume further that $\mathcal{J}^k \equiv \mathcal{J}$ (otherwise, we can choose a subsequence). Then we have

$$\nabla_x f(x^k, p^*) + \sum_{i \in \mathcal{J}} \lambda_i^k [M]_i = 0, \quad \lambda_i^k \geq 0 \ (i \in \mathcal{J}).$$

It is not hard to see that $\{\lambda^k\}$ is bounded by setting $\lambda_i^k = 0$ if $i \notin \mathcal{J}$. Obviously, $\{y^k\}$ is bounded by noting that $y^k = (B^{\nu_0})^T \lambda^k$. Therefore, the multiplier-stability holds at $x$ if $\Lambda$ is locally separable at $x \in \Lambda$.

Robinson [26, Theorem 2.4] showed the isolatedness of a stationary point under the assumptions that $\Lambda$ is a convex cone and the SSOSC holds, while Bonnans and Shapiro [4, Proposition 4.52] showed the isolatedness when $\Lambda$ is a polyhedral cone and the second-order growth condition holds. In what follows, we show the isolatedness under much weaker conditions, and hence our result improves both Robinson's and Bonnans and Shapiro's results.

Since (A1) must hold if $\Lambda$ is convex and Robinson's constraint qualification implies the multiplier-stability, the following result improves Theorem 2.4 of Robinson [26] in that $\Lambda$ is only needed to be a closed set satisfying (A1) instead of a closed convex cone in [26].

THEOREM 3.3. *Let $x^* \in \mathcal{S}(p^*)$ and let the set $\Lambda$ satisfy* (A1) *in Assumption* 3.1. *Suppose that the multiplier-stability and SSOSC hold at $x^*$. Then there exists a neighborhood $V$ of $x^*$ containing no other stationary point of* (MPGC$_p$) *for $p = p^*$.*

*Proof.* Suppose to the contrary that there exists a sequence $\{x^k\} \subseteq \mathcal{S}(p^*) \setminus \{x^*\}$ converging to $x^*$. Since the multiplier-stability holds at $x^*$, we may assume without loss of generality that there exists a multiplier sequence $\{y^k\} \to y^*$ such that

$$\left\{ \begin{array}{l} 0 = \nabla_x f(x^k, p^*) + \nabla_x F(x^k, p^*) y^k, \\ y^k \in \mathcal{N}_\Lambda(F(x^k, p^*)). \end{array} \right. \tag{3.17}$$

We further assume $(x^k - x^*)/\|x^k - x^*\| \to d^0$ with $\|d^0\| = 1$. It is clear from the outer semicontinuity of $\mathcal{N}_\Lambda$ that $y^* \in \mathcal{N}_\Lambda(F(x^*, p^*))$ and hence $y^* \in \mathcal{M}(x^*, p^*)$. Note that, for each $k$,

$$\Lambda \ni F(x^k, p^*) = F(x^*, p^*) + \nabla_x F(x^*, p^*)^T (x^k - x^*) + o(\|x^k - x^*\|)$$

and hence

$$\nabla_x F(x^*, p^*)^T d^0 = \lim_{k \to \infty} \frac{F(x^k, p^*) - F(x^*, p^*)}{\|x^k - x^*\|} \in \mathcal{T}_\Lambda(F(x^*, p^*)). \tag{3.18}$$

Since $\Lambda$ satisfies (A1), there exists an infinite subset $\mathbb{K}$ such that, for each $k \in \mathbb{K}$,

$$(y^*)^T(F(x^k,p^*) - F(x^*,p^*)) \leq 0, \quad (y^k)^T(F(x^*,p^*) - F(x^k,p^*)) \leq 0. \tag{3.19}$$

Noting that $\{y^k\}$ is bounded, we have that, for each $k \in \mathbb{K}$,

$$(y^k)^T F(x^k,p^*) - (y^k)^T F(x^*,p^*) = (y^k)^T \nabla_x F(x^*,p^*)^T (x^k - x^*) + o(\|x^k - x^*\|).$$

It follows from (3.19) that, for each $k \in \mathbb{K}$,

$$(y^k)^T \nabla_x F(x^*,p^*)^T (x^k - x^*) + o(\|x^k - x^*\|) \geq 0.$$

Dividing it by $\|x^k - x^*\|$ and taking a limit, we obtain

$$(y^*)^T \nabla_x F(x^*,p^*)^T d^0 \geq 0.$$

It follows from (3.17) that

$$0 = \nabla_x f(x^k,p^*)^T(x^k - x^*) + (y^k)^T \nabla_x F(x^k,p^*)^T (x^k - x^*). \tag{3.20}$$

Dividing it by $\|x^k - x^*\|$ and taking a limit, we have $\nabla_x f(x^*,p^*)^T d^0 \leq 0$. Since (3.18) holds, this indicates that $d^0 \in \mathcal{C}(x^*,p^*)$ and hence, by the SSOSC at $x^*$,

$$(d^0)^T \nabla_x^2 L(x^*,p^*;y^*) d^0 > 0. \tag{3.21}$$

It follows from (3.1), (3.17), (3.19), and the twice continuous differentiability of $F$ that, for each $k \in \mathbb{K}$,

$$\begin{aligned} 0 &\geq -(y^k - y^*)^T(F(x^k,p^*) - F(x^*,p^*)) \\ &= (x^k - x^*)^T \nabla_x L(x^k,p^*;y^k) - (y^k - y^*)^T(F(x^k,p^*) - F(x^*,p^*)) \\ &= (x^k - x^*)^T \nabla_x L(x^k,p^*;y^*) - (y^k - y^*)^T(F(x^k,p^*) - F(x^*,p^*) \\ &\quad - \nabla_x F(x^k,p^*)^T(x^k - x^*)) \\ &= (x^k - x^*)^T \nabla_x^2 L(x^*,p^*;y^*)(x^k - x^*) \\ &\quad - \frac{1}{2}(y^k - y^*)^T(D_x^2 F(x^*,p^*)(x^k - x^*, x^k - x^*)) \\ &\quad + o(\|x^k - x^*\|^2), \end{aligned}$$

where $D_x^2 F(x^*,p^*)$ can be regarded as the second-order derivative of $F$ at $x^* \in \mathcal{X}(p^*)$. Dividing the above inequality by $\|x^k - x^*\|^2$ and taking a limit, we can get a contradiction to (3.21). This completes the proof. ∎

The following result improves Proposition 4.52 of Bonnans and Shapiro [4] in that $\Lambda$ does not need to be a polyhedral cone.

THEOREM 3.4. *Let $x^* \in \mathcal{S}(p^*)$ and let the set $\Lambda$ satisfy* (A2) *in Assumption* 3.1. *Suppose that the multiplier-stability and second-order growth condition hold at $x^*$. Then there exists a neighborhood $V$ of $x^*$ containing no other stationary point of* (MPGC$_p$) *for $p = p^*$.*

*Proof.* Note that the SSOSC assumption in Theorem 3.3 is used to show (3.21). For this theorem, it is sufficient to show (3.21) under the second-order growth condition at $x^*$. In fact, by the second-order growth condition, there exist $\delta > 0$ and $c > 0$ such that

$$f(x,p^*) \geq f(x^*,p^*) + c\|x - x^*\|^2 \qquad \forall x \in \mathcal{X}(p^*) \cap \mathcal{B}_\delta(x^*). \tag{3.22}$$

Since $y^k \in \mathcal{N}_\Lambda(F(x^k, p^*))$ and $\Lambda$ satisfies (A2), there exists an infinite subset $\mathbb{K}$ such that

$$(y^*)^T F(x^k, p^*) = 0 \qquad \forall k \in \mathbb{K}.$$

This implies $(y^*)^T F(x^*, p^*) = 0$ and hence $L(x^k, p^*; y^*) = f(x^k, p^*)$, $L(x^*, p^*; y^*) = f(x^*, p^*)$. Thus, by (3.1) and (3.22), we have that, for each $k \in \mathbb{K}$,

$$\begin{aligned} c\|x^k - x^*\|^2 &\le L(x^k, p^*; y^*) - L(x^*, p^*; y^*) \\ &= \frac{1}{2}(x^k - x^*)^T \nabla_x^2 L(x^*, p^*; y^*)(x^k - x^*) + o(\|x^k - x^*\|^2), \end{aligned}$$

which implies (3.21). This completes the proof. $\square$

Since the NNAMCQ implies the multiplier-stability, we have the following result immediately.

COROLLARY 3.1. *Let $x^* \in \mathcal{S}(p^*)$ and the NNAMCQ hold at $x^*$. Suppose that either* (A1) *and the SSOSC hold at $x^*$ or* (A2) *and the second-order growth condition hold at $x^*$. Then there exists a neighborhood $V$ of $x^*$ containing no other stationary point of* (MPGC$_p$) *for $p = p^*$.*

It is interesting that, even for standard nonlinear programs, our result improves the classical result. Note that the RCPLD is a very weak constraint qualification and implies the multiplier-stability when $\Lambda = \Re^{l_1}_- \times \{0\}^{l_2}$. It is weaker than the relaxed constant rank condition. It holds, for example, if all constraint functions are linear with respect to $x$.

COROLLARY 3.2. *Consider the standard nonlinear program, i.e., the set $\Lambda := \Re^{l_1}_- \times \{0\}^{l_2}$. Let $x^* \in \mathcal{S}(p^*)$. Suppose that the RCPLD and second-order growth condition hold at $x^*$. Then there exists a neighborhood $V$ of $x^*$ containing no other stationary point of* (MPGC$_p$) *for $p = p^*$.*

The isolatedness of a stationary point has been shown under either the SSOSC or the second-order growth condition. When $\Lambda$ reduces to a closed convex set, the SOSC implies the second-order growth condition; see, e.g., [4, Theorem 3.63]. For a nonconvex set $\Lambda$, does the SSOSC imply the second-order growth condition or the other way around? The answer is negative. We now give some examples to show that the SSOSC neither implies nor is implied by the second-order growth condition. The first example shows that the SSOSC holds but the second-order growth condition does not hold.

*Example* 3.2. Consider the problem

$$\begin{aligned} \min \quad & (x_1 - 1)^2 + x_2^2 \\ \text{s.t.} \quad & (-x_1, -x_2) \in C, \end{aligned}$$

where $C$ is the same as in (1.4). It is easy to see that $x^* = (x_1^*, x_2^*) = (0, 0)$ is a stationary point with the unique multiplier $(u, v) = (-2, 0) \in \mathcal{N}_C(0, 0)$ (see Proposition 2.1) but it is not a locally optimal solution, which implies that the second-order growth condition does not hold at $x^*$. However, it is easy to verify that

$$\nabla_x^2 L(x^*; u, v) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

which is a positive definite matrix. Hence the SSOSC holds at $x^*$.

The following example shows that the second-order growth condition holds but the SSOSC does not hold.

*Example* 3.3. Consider the problem

$$\begin{array}{ll} \min & x_1^2 + x_2^2 \\ \text{s.t.} & (x_1^2, x_2^2) \in C. \end{array}$$

Obviously, $x^* = (0,0)$ is the global optimal solution and the second-order growth condition holds at $x^*$. On the other hand, the critical cone $\mathcal{C}(x^*) = \Re^2$ and, for any $(u,v)$,

$$\nabla_x^2 L(x^*; u, v) = \left[ \begin{array}{cc} 2(1+u) & 0 \\ 0 & 2(1+v) \end{array} \right].$$

Picking a multiplier $(u,v) = (-2,0)$ and $d^0 = (1,0)^T$, we have

$$(d^0)^T \nabla_x^2 L(x^*; u, v) d^0 = -2 < 0.$$

Therefore, the SSOSC does not hold at $x^*$.

The rest of this section is devoted to studying the continuity of locally optimal solutions and stationary points. The following result improves Theorem 3.2 of Robinson [26] in that $\Lambda$ does not need to be a closed convex cone.

THEOREM 3.5. *Let $x^* \in \mathcal{S}(p^*)$ and let the set $\Lambda$ satisfy* (A1) *in Assumption* 3.1. *Suppose that the NNAMCQ, SSOSC, and second-order growth condition hold at $x^*$. Then there exist $\delta > 0$ and $\epsilon > 0$ such that both $\mathcal{O} \cap \mathcal{B}_\delta(x^*)$ and $\mathcal{S} \cap \mathcal{B}_\delta(x^*)$ are continuous at $p^*$ and, for each $p \in \mathcal{B}_\epsilon(p^*)$, $\emptyset \neq \mathcal{O}(p) \cap \mathcal{B}_\delta(x^*) \subseteq \mathcal{S}(p) \cap \mathcal{B}_\delta(x^*)$.*

*Proof.* Since the NNAMCQ and SSOSC hold at $x^*$, by Corollary 3.1, $x^*$ is an isolated stationary point; i.e., there exists $\delta_1 > 0$ such that

$$\mathcal{S}(p^*) \cap \mathcal{B}_{\delta_1}(x^*) = \{x^*\}.$$

Since the second-order growth condition holds at $x^*$ and the NNAMCQ persists under small perturbations (see, e.g., [28, proof of Theorem 6.14]), there exist $\epsilon_1 > 0$ and $\delta_2 \in (0, \delta_1)$ such that, for any $\epsilon' \in (0, \epsilon_1]$ and $\delta' \in (0, \delta_2]$,

$$\mathcal{O}(p^*) \cap \mathcal{B}_{\delta'}(x^*) = \mathcal{S}(p^*) \cap \mathcal{B}_{\delta'}(x^*) = \{x^*\}, \tag{3.23}$$

$$\mathcal{O}(p) \cap \mathcal{B}_{\delta'}(x^*) \subseteq \mathcal{S}(p) \cap \mathcal{B}_{\delta'}(x^*) \quad \forall p \in \mathcal{B}_{\epsilon'}(p^*). \tag{3.24}$$

Moreover, it follows from Theorem 3.1 that there exists $\delta \in (0, \delta_2)$ such that $\mathcal{S} \cap \mathcal{B}_\delta(x^*)$ is upper semicontinuous at $p^*$. For such $\delta$, since the NNAMCQ and second-order growth condition hold at $x^*$, it follows from Theorem 3.2 that there exists $\epsilon \in (0, \epsilon_1)$ such that

$$\mathcal{O}(p) \cap \mathcal{B}_\delta(x^*) \neq \emptyset \quad \forall p \in \mathcal{B}_\epsilon(p^*). \tag{3.25}$$

This, together with (3.23)–(3.24), implies that $\mathcal{O} \cap \mathcal{B}_\delta(x^*)$ is upper semicontinuous at $p^*$.

We next show that $\mathcal{O} \cap \mathcal{B}_\delta(x^*)$ and $\mathcal{S} \cap \mathcal{B}_\delta(x^*)$ are both lower semicontinuous at $p^*$. To this end, let $S \subseteq \mathcal{B}_\delta(x^*)$ be an arbitrary open set with $\mathcal{O}(p^*) \cap \mathcal{B}_\delta(x^*) \cap S \neq \emptyset$. It is obvious from (3.23) that $\mathcal{O}(p^*) \cap \mathcal{B}_\delta(x^*) = \{x^*\} \subseteq S$. Thus, we have from Theorem 3.2 that there exists a neighborhood $V$ of $p^*$ such that, for each $p \in V$, $\mathcal{O}(p) \cap \mathcal{B}_\delta(x^*) \cap S \neq \emptyset$. Hence, $\mathcal{O} \cap \mathcal{B}_\delta(x^*)$ is lower semicontinuous at $p^*$. This, together with (3.23)–(3.24), implies that $\mathcal{S} \cap \mathcal{B}_\delta(x^*)$ is lower semicontinuous at $p^*$.

Therefore, both $\mathcal{S} \cap \mathcal{B}_\delta(x^*)$ and $\mathcal{O} \cap \mathcal{B}_\delta(x^*)$ are continuous at $p^*$. The proof is completed by noting the condition (3.25). $\square$

By virtue of Theorems 3.2 and 3.4, we can get the following result in a way similar to Theorem 3.5.

THEOREM 3.6. *Let $x^* \in \mathcal{S}(p^*)$ and let the set $\Lambda$ satisfy* (A2) *in Assumption* 3.1*. Suppose that the NNAMCQ and second-order growth condition hold at $x^*$. Then there exist $\delta > 0$ and $\epsilon > 0$ such that both $\mathcal{O} \cap \mathcal{B}_\delta(x^*)$ and $\mathcal{S} \cap \mathcal{B}_\delta(x^*)$ are continuous at $p^*$ and, for each $p \in \mathcal{B}_\epsilon(p^*)$, $\emptyset \neq \mathcal{O}(p) \cap \mathcal{B}_\delta(x^*) \subseteq \mathcal{S}(p) \cap \mathcal{B}_\delta(x^*)$.*

**4. Calmness of stationary point mapping.** Theorems 3.5 and 3.6 show that the stationary point mapping near $x^*$ is continuous at $p^*$ under some suitable conditions, but it gives no measure of how it depends on $p$. In this section, we make some additional assumptions on the set $\Lambda$ and show that the stationary pair mapping $p \to \{(x, y) \mid (p, x, y) \text{ satisfies (3.1)}\}$ is calm with respect to $p$ at $(p^*, x^*, y^*)$ under mild conditions.

Our study on calmness for (MPGC$_p$) is based on its linearized problem, and the linearized problem itself is also important just like the importance of the linear and quadratic programming problems to the study of nonlinear programming problems. Thus, we first consider the following quadratic program with linear geometric constraints:

$$(4.1) \qquad (\mathrm{QP})_{(a,q)} \qquad \begin{array}{rl} \min\limits_{x} & \frac{1}{2} x^T Q x + x^T q \\ \text{s.t.} & Ax + a \in \Lambda, \end{array}$$

where $Q$ is a symmetric matrix, and $A$ and $\{a, q\}$ are a matrix and vectors with appropriate dimensions. Here, vectors $\{a, q\}$ are considered to be perturbation parameters. The stationarity system of $(\mathrm{QP})_{(a,q)}$ at a feasible point $x$ with multiplier $y$ can be written as

$$(4.2) \qquad Qx + A^T y + q = 0, \quad y \in \mathcal{N}_\Lambda(Ax + a).$$

Note that the SSOSC for $(\mathrm{QP})_{(a,q)}$ at a feasible point $x$ is independent of the multipliers, that is,

$$d^T Q d > 0 \qquad \forall d \in \mathcal{C}(x) \setminus \{0\},$$

where $\mathcal{C}(x) := \{d \mid (Qx + q)^T d \leq 0, A^T d \in \mathcal{T}_\Lambda(Ax + a)\}$ is the critical cone. In order to study the calmness of stationary pair mapping for $(\mathrm{QP})_{(a,q)}$, we further make the following assumptions on the set $\Lambda$.

*Assumption* 4.1.

(A3) The set-valued mapping $\mathcal{N}_\Lambda : \Re^l \rightrightarrows \Re^l$ is polyhedral [27]; i.e., the graph of $\mathcal{N}_\Lambda$ is a union of finitely many polyhedral sets.

(A4) Given $y^* \in \mathcal{N}_\Lambda(F(x^*, p^*))$, there exists $\delta > 0$ such that $\mathcal{N}_\Lambda(F(x, p)) \cap \mathcal{B}_\delta(y^*) \subseteq \mathcal{N}_\Lambda(F(x^*, p^*))$ for each $(x, p) \in \mathcal{B}_\delta(x^*, p^*)$.

(A3) holds when $\Lambda$ is a union of polyhedral sets (e.g., the MPEC cone). For simplicity, we assume that $\Lambda$ is a union of two polyhedral sets, i.e., $\Lambda := A \cup B$ with $A$ and $B$ to be polyhedral. Let $a \in \Lambda$ and assume without loss of generality that $a \in A \cap B$. We have

$$\begin{aligned} \mathcal{N}_{A\cup B}(a) &= \limsup_{a' \to_{A\cup B}(a)} \hat{\mathcal{N}}_{A\cup B}(a') \\ &= \hat{\mathcal{N}}_{A\cup B}(a) \cup \limsup_{a' \to_{A\setminus B} a} \hat{\mathcal{N}}_A(a') \cup \limsup_{a' \to_{B\setminus A} a} \hat{\mathcal{N}}_B(a') \cup \limsup_{a' \to_{A\cap B} a} \hat{\mathcal{N}}_{A\cup B}(a'). \end{aligned}$$

By the polyhedron assumption of $A$ and $B$, we see that $\mathcal{N}_{A\cup B}(a)$ is a union of finitely many polyhedral sets and, by the polyhedron assumption again, its graph is a union of finitely many polyhedral sets. Thus, (A3) holds. (A4) holds when $\Lambda$ is a polyhedral set. Moreover, we can show that (A4) also holds if $\Lambda$ is an MPEC cone discussed in the next section.

The following result extends Lemma 1 of Hager and Gowda [12] from the case where $\Lambda = \Re^{l_1}_- \times \{0\}^{l_2}$ by noting that the multiplier-stability automatically holds in the case.

THEOREM 4.1. *Suppose that $x^*$ is a stationary point of* $(\mathrm{QP})_{(a^*,q^*)}$*; i.e., there exists $y^*$ such that $(x^*, q^*, a^*, y^*)$ satisfies* (4.2)*. Assume that $x^*$ is multiplier-stable and the assumption* (A3) *holds. Assume also that either* (A1) *and the SSOSC hold at $x^*$ or* (A2) *and the second-order growth condition hold at $x^*$. Then there exist $\beta > 0$ and $\delta > 0$ such that if $(x, y)$ is a solution of* (4.2) *with $(q, a) \in \mathcal{B}_\delta(q^*, a^*)$ and $x \in \mathcal{B}_\delta(x^*)$, then*

$$\|x - x^*\| \le \beta\|(q, a) - (q^*, a^*)\|.$$

*Proof.* Recall that any finite composition of polyhedral set-valued mappings is polyhedral [27]. By (A3), the set $\{(x, a, y) \mid y \in \mathcal{N}_\Lambda(Ax + a)\}$ is a union of finitely many polyhedral sets. Define a set-valued mapping by

$$\mathcal{F}(q, a) := \{(x, y) \mid (q, a, x, y) \text{ satisfies } (4.2)\}.$$

It is not hard to see that its graph

$$\operatorname{gph}\mathcal{F} := \{(q, a, x, y) \mid (q, a, x, y) \text{ satisfies } (4.2)\}$$

is a union of finitely many polyhedral sets. Thus, $\mathcal{F}$ is a polyhedral set-valued mapping. Let $P_1$ be the projection operator defined by $P_1(x, y) := x$. It follows that $P_1 \circ \mathcal{F}$ is a polyhedral set-valued mapping.

By [27, Proposition 1], a polyhedral set-valued mapping is upper Lipschitz continuous; i.e., there exist $\beta > 0$ independent of $(q^*, a^*)$ and $\delta_1 > 0$ such that

$$P_1 \circ \mathcal{F}(q, a) \subseteq P_1 \circ \mathcal{F}(q^*, a^*) + \beta\|(q, a) - (q^*, a^*)\|\bar{\mathcal{B}}_1(0) \quad \forall (q, a) \in \mathcal{B}_{\delta_1}(q^*, a^*). \tag{4.3}$$

Moreover, by Theorems 3.3 and 3.4, the stationary point $x^*$ is isolated; i.e., there exists $\delta_2 \in (0, \delta_1)$ such that

$$P_1 \circ \mathcal{F}(q^*, a^*) \cap \mathcal{B}_{\delta_2}(x^*) = \{x^*\}. \tag{4.4}$$

Letting $\delta := \delta_2/2$, we have from (4.3) and (4.4) that, for each $x \in P_1 \circ \mathcal{F}(q, a) \cap \mathcal{B}_\delta(x^*)$ and $(q, a) \in \mathcal{B}_\delta(q^*, a^*)$,

$$\begin{aligned} \|x - x^*\| &= \operatorname{dist}(x, P_1 \circ \mathcal{F}(q^*, a^*) \cap \mathcal{B}_{2\delta}(x^*)) \\ &= \operatorname{dist}(x, P_1 \circ \mathcal{F}(q^*, a^*)) \le \beta\|(q, a) - (q^*, a^*)\|. \end{aligned}$$

This completes the proof. ◻

We next extend the above results to $(\mathrm{MPGC}_p)$. To this end, we first give an error estimate for a union of polyhedral sets by using the well-known result in [13].

LEMMA 4.1. *Let $S := \cup_{i=1}^N S_i$, where $S_i := \{x \mid A_i x \le a^i, B_i x = b^i\}$ is nonempty for each $i$. Then there exists $c > 0$ such that, for each $x$,*

$$\operatorname{dist}(x, S) = \min_{1\le i\le N} \operatorname{dist}(x, S_i) \le c \min_{1\le i\le N}(\|\max\{0, A_i x - a^i\}\| + \|B_i x - b^i\|).$$

*Proof.* Since each $S_i$ is polyhedral, it follows from [13] that, for each $i$, there exists $c_i > 0$ such that

$$\operatorname{dist}(x, S_i) \le c_i \left(\|\max\{0, A_i x - a^i\}\| + \|B_i x - b^i\|\right).$$

Since $S$ is closed, we have $\operatorname{dist}(x, S) = \min_{1\le i\le N} \operatorname{dist}(x, S_i)$. Letting $c = \max_{1\le i\le N}\ c_i$, we can get the desired result. ∎

The following result shows the calmness of stationary pair mapping for (MPGC$_p$). Recall that the multiplier-stability automatically holds when $\Lambda$ is a polyhedral set and $F$ is linear with respect to $x$. It is an improvement of Robinson's result given in [26, section 4] in that no constraint qualification is required here when $\Lambda$ reduces to a polyhedral cone. It also extends Lemma 2 of Hager and Gowda [12] from the case where $\Lambda = \Re^{l_1}_- \times \{0\}^{l_2}$ and perturbation is additive.

THEOREM 4.2. *Let $x^* \in \mathcal{X}(p^*)$ and $y^* \in \mathcal{M}(x^*, p^*)$. Suppose that* (A1) *and* (A3)–(A4) *are satisfied, the SOSC holds at $(x^*, y^*)$, and $x^*$ is multiplier-stable for* $(\mathrm{QP})_{(a^*,q^*)}$ *defined below. Then there exist $\delta > 0$ and $\kappa > 0$ such that if $y \in \mathcal{M}(x,p)$ with $(x,y,p) \in \mathcal{B}_\delta(x^*, y^*, p^*)$, then*

$$\|x - x^*\| + \operatorname{dist}(y, \mathcal{M}(x^*, p^*)) \le \kappa\|p - p^*\|.$$

*Proof.* Consider the following quadratic problem with geometric constraints:

$$(\mathrm{QP})_{(a^*,q^*)} \qquad \begin{array}{ll} \displaystyle\min_x & \displaystyle\frac{1}{2}x^TQx + x^Tq^* \\ \text{s.t.} & Ax + a^* \in \Lambda, \end{array}$$

where

$$\begin{cases} Q := \nabla_x^2 L(x^*, p^*; y^*), \\ q^* := \nabla_x f(x^*, p^*) - \nabla_x^2 L(x^*, p^*; y^*)x^*, \\ A := \nabla_x F(x^*, p^*)^T, \\ a^* := F(x^*, p^*) - \nabla_x F(x^*, p^*)^T x^*. \end{cases}$$

It is easy to check that $(x^*, y^*)$ is a stationary pair of (MPGC$_{p^*}$) and the SOSC holds at $(x^*, y^*)$ if and only if $(x^*, y^*)$ is a stationary pair of $(\mathrm{QP})_{(a^*,q^*)}$ and the SOSC holds at $(x^*, y^*)$. Moreover, $y \in \mathcal{M}(x,p)$ if and only if

$$Qx + A^Ty + q = 0, \qquad y \in \mathcal{N}_\Lambda(Ax + a),$$

where $q := \nabla_x L(x,p;y) - (Qx + A^Ty)$ and $a := F(x,p) - Ax$.

It is obvious that $(q,a) \to (q^*, a^*)$ as $(x,y) \to (x^*, y^*)$ and $p \to p^*$. Thus, we have from Theorem 4.1 that there exist $\delta_1 > 0$ and $\beta > 0$ such that, for each $(x,y) \in \mathcal{B}_{\delta_1}(x^*, y^*)$ and $p \in \mathcal{B}_{\delta_1}(p^*)$,

$$\|x - x^*\| \le \beta\|(q,a) - (q^*, a^*)\|. \tag{4.5}$$

Note that

$$\begin{aligned} \|q - q^*\| &= \|\nabla_x L(x,p;y) - \nabla_x L(x^*, p^*; y) - Q(x - x^*)\| \\ &\le \|\nabla_x L(x,p;y) - \nabla_x L(x^*, p^*; y) - \nabla(\nabla_x)_{(x,p)} L(x^*, p^*; y)(x - x^*, p - p^*)\| \\ &\quad + \|\nabla_x^2 L(x^*, p^*; y) - Q\|\|x - x^*\| + \|\nabla_{xp}^2 L(x^*, p^*; y)\|\|p - p^*\|. \end{aligned}$$

By Taylor's theorem, for $(x,p,y)$ near $(x^*,p^*,y^*)$, it holds that

$$\begin{aligned}&\nabla_x L(x,p;y)-\nabla_x L(x^*,p^*;y)-\nabla(\nabla_x)_{(x,p)}L(x^*,p^*;y)(x-x^*,p-p^*)\\&\quad=o(\|(x-x^*,p-p^*)\|).\end{aligned}$$

Hence, for any given $\epsilon'>0$, there exist $\delta'>0$ and $\kappa'$ such that, for each $(x,y)\in\mathcal{B}_{\delta'}(x^*,y^*)$ and $p\in\mathcal{B}_{\delta'}(p^*)$,

$$\|q-q^*\|\le\epsilon'\|x-x^*\|+\kappa'\|p-p^*\|.$$

Let $\epsilon:=\frac{1}{2\beta}$. It follows from the second-order differentiability of $\{f,F\}$ with respect to $(x,p)$ that there exist $\delta_2\in(0,\delta_1)$ and $\kappa_1>0$ such that, for each $(x,y)\in\mathcal{B}_{\delta_2}(x^*,y^*)$ and $p\in\mathcal{B}_{\delta_2}(p^*)$,

$$\|(q,a)-(q^*,a^*)\|\le\epsilon\|x-x^*\|+\kappa_1\|p-p^*\|. \tag{4.6}$$

We then have from (4.5) and (4.6) that, for each $(x,y)\in\mathcal{B}_{\delta_2}(x^*,y^*)$ and $p\in\mathcal{B}_{\delta_2}(p^*)$,

$$\|x-x^*\|\le 2\beta\kappa_1\|p-p^*\|. \tag{4.7}$$

We next consider the multiplier part. Note that

$$\mathcal{M}(x^*,p^*):=\{y\mid\nabla_x L(x^*,p^*;y)=0,\ y\in\mathcal{N}_\Lambda(F(x^*,p^*))\},$$

where $\nabla_x L(x^*,p^*;y)$ is a linear mapping in $y$ and the set $\mathcal{N}_\Lambda(F(x^*,p^*))$ is a union of finitely many polyhedral sets by the assumption (A3). Thus, $\mathcal{M}(x^*,p^*)$ is a union of finitely many polyhedral sets. By the assumption (A4), there exists $\delta\in(0,\delta_2)$ such that if $y\in\mathcal{N}_\Lambda(F(x,p))$ with $(x,y,p)\in\mathcal{B}_\delta(x^*,y^*,p^*)$, then $y\in\mathcal{N}_\Lambda(F(x^*,p^*))$. By Lemma 4.1 and the Lipschitz continuity of $\nabla_x L$ on $\mathcal{B}_\delta(x^*,y^*)\times\mathcal{B}_\delta(p^*)$, there exist $c>0$ and $c'>0$ such that, for each $(x,y,p)\in\mathcal{B}_\delta(x^*,y^*,p^*)$ with $y\in\mathcal{M}(x,p)$,

$$\begin{aligned}\operatorname{dist}(y,\mathcal{M}(x^*,p^*))&\le c\|\nabla_x L(x^*,p^*;y)\|\\&\le c\|\nabla_x L(x,p;y)\|+c\|\nabla_x L(x^*p^*;y)-\nabla_x L(x,p;y)\|\\&\le cc'\|x-x^*\|+cc'\|p-p^*\|.\end{aligned} \tag{4.8}$$

Therefore, from (4.7) and (4.8), we have

$$\|x-x^*\|+\operatorname{dist}(y,\mathcal{M}(x^*,p^*))\le(cc'+2\beta\kappa_1+2cc'\beta\kappa_1)\|p-p^*\|.$$

The proof is completed by setting $\kappa:=cc'+2\beta\kappa_1+2cc'\beta\kappa_1$. ◻

**5. Applications to (MPEC$_p$).** In this section, we apply the results obtained in sections 3 and 4 to (MPEC$_p$) by reformulating the problem as a mathematical program with geometric constraints (MPGC$_p$) where the function $F$ and the set $\Lambda$ are defined as in (1.3). Before our discussion, we emphasize that (MPEC$_p$) can also be rewritten as a special case of (MPGC$_p$) with a convex set $\Lambda$ by letting

$$F(x,p):=\begin{pmatrix}g(x,p)\\h(x,p)\\-G(x,p)\\-H(x,p)\\G(x,p)\circ H(x,p)\end{pmatrix},\quad\Lambda:=\Re^{m_1}_-\times\{0\}^{m_2}\times\Re^m_-\times\Re^m_-\times\Re^m_-,$$

but this formulation fails to satisfy the most commonly used constraint qualifications, and hence the results given in section 3 cannot be applied. In what follows, we let $F$ and $\Lambda$ be defined as in (1.3). Then the generalized MPEC-Lagrangian function can be rewritten as

$$L^r_{\text{MPEC}}(x,p;y) = rf(x,p) + F(x,p)^T y, \quad y := (\lambda, \mu, u_1, v_1, \ldots, u_m, v_m).$$

Based on the formulas for normal cones of set $C$ in Proposition 2.1, we can easily get the following properties.

PROPOSITION 5.1. *Let $x^*$ be a given feasible point of* (MPEC$_{p^*}$).

(1) *The generalized M-stationarity condition at $x^*$ is equivalent to*

$$r\nabla_x f(x^*,p^*) + \nabla_x F(x^*,p^*)y = 0, \qquad y \in \mathcal{N}_\Lambda(F(x^*,p^*)).$$

(2) *The generalized S-stationarity condition at $x^*$ is equivalent to*

$$r\nabla_x f(x^*,p^*) + \nabla_x F(x^*,p^*)y = 0, \qquad y \in \widehat{\mathcal{N}}_\Lambda(F(x^*,p^*)).$$

(3) *The MPEC-NNAMCQ at $x^*$ is equivalent to*

$$\nabla_x F(x^*,p^*)y = 0, \quad y \in \mathcal{N}_\Lambda(F(x^*,p^*)) \quad \Longrightarrow \quad y = 0.$$

We now give several kinds of second-order sufficient conditions for (MPEC$_p$) in terms of M-multipliers and S-multipliers.

DEFINITION 5.1 (see [11]). *Let $x^*$ be a given feasible point of* (MPEC$_{p^*}$). *We say that the* M-multiplier strong second-order sufficient condition *(M-SSOSC) holds at $x^*$ if $\mathcal{M}^1_M(x^*,p^*) \neq \emptyset$ and, for every $y^* \in \mathcal{M}^1_M(x^*,p^*)$,*

$$d^T \nabla^2_x L^1_{\text{MPEC}}(x^*,p^*;y^*)d > 0 \qquad \forall d \in \mathcal{C}(x^*,p^*)\backslash\{0\}.$$

*We say that the* S-multiplier refined second-order sufficient condition *(S-RSOSC) holds at $x^*$ if, for every $d \in \mathcal{C}(x^*,p^*)\backslash\{0\}$, there exist $r \geq 0$ and $y^* \in \mathcal{M}^r_S(x^*,p^*)$ such that*

$$d^T \nabla^2_x L^r_{\text{MPEC}}(x^*,p^*;y^*)d > 0.$$

DEFINITION 5.2. *We say that the* M-multiplier second-order sufficient condition *(M-SOSC) holds at $(x^*,y^*)$ with $y^* \in \mathcal{M}^1_M(x^*,p^*)$ if*

$$d^T \nabla^2_x L^1_{\text{MPEC}}(x^*,p^*;y^*)d > 0 \qquad \forall d \in \mathcal{C}(x^*,p^*)\backslash\{0\}.$$

*We say that the* S-multiplier second-order sufficient condition *(S-SOSC) holds at $(x^*,y^*)$ with $y^* \in \mathcal{M}^1_S(x^*,p^*)$ if*

$$d^T \nabla^2_x L^1_{MPEC}(x^*,p^*;y^*)d > 0 \qquad \forall d \in \mathcal{C}(x^*,p^*)\backslash\{0\}.$$

Note that, from Proposition 5.1, the M-stationarity for (MPEC$_p$) is equivalent to the stationarity for (MPGC$_p$), the M-SSOSC for (MPEC$_p$) is equivalent to the SSOSC for (MPGC$_p$), and the M-SOSC for (MPEC$_p$) is equivalent to the SOSC for (MPGC$_p$). We next consider the M-multiplier mapping $\mathcal{M}^1_M : \Re^{n_1+n_2} \to \Re^{m_1+m_2+2m}$, the M-stationary point mapping $\mathcal{S}_M : \Re^{n_2} \to \Re^{n_1}$, and the locally optimal solution mapping $\mathcal{O} : \Re^{n_2} \to \Re^{n_1}$.

Since, by Proposition 5.1, the MPEC-NNAMCQ for (MPEC$_p$) is equivalent to the NNAMCQ for (MPGC$_p$), we can obtain the next theorem from Theorem 3.1 immediately. Note that Izmailov [14, Theorem 6] considered the upper semicontinuity of $\mathcal{O}$ near $x^*$ under the MFCQ for a branch of MPEC. The following result shows the upper semicontinuity of $\mathcal{S}$ near $x^*$. Note that the MPEC-NNAMCQ neither implies nor is implied by the MFCQ for a branch of MPEC.

THEOREM 5.1. *Let $x^*$ be a given feasible point of* (MPEC$_{p^*}$). *If the MPEC-NNAMCQ holds at $x^*$, then there exists $\delta > 0$ such that the M-multiplier mapping $\mathcal{M}_M^1$ and the M-stationary point mapping $\mathcal{S}_M \cap \bar{\mathcal{B}}_\delta(x^*)$ are locally bounded and upper semicontinuous on $\mathcal{B}_\delta(x^*) \times \mathcal{B}_\delta(p^*)$ and $\mathcal{B}_\delta(p^*)$, respectively.*

We next show that if the S-RSOSC and MPEC-NNAMCQ hold at a locally optimal solution, then the local optimal solution persists under small perturbation. To this end, we first introduce a lemma, which is an improvement of Theorem 7 of Scheel and Scholtes [29] in that $r$ can be taken as zero here. This improvement is significant since, for MPECs, a locally optimal solution is always a generalized S-stationary point with $r = 0$, and hence the S-RSOSC is not a strong condition; see, e.g., [11] for more details.

LEMMA 5.1 (see [11]). *Let $x^*$ be a given feasible point of* (MPEC$_{p^*}$). *If the S-RSOSC holds at $x^*$, then $x^*$ satisfies the second-order growth condition; i.e., there exist $\delta > 0$ and $c > 0$ such that*

$$f(x, p^*) \geq f(x^*, p^*) + c\|x - x^*\|^2 \qquad \forall x \in \mathcal{X}(p^*) \cap \mathcal{B}_\delta(x^*).$$

From Theorem 3.2 and Lemma 5.1, we can get the persistence of locally optimal solutions for MPECs as follows.

THEOREM 5.2. *Let $x^*$ be a given feasible point of* (MPEC$_{p^*}$). *Suppose that the MPEC-NNAMCQ and S-RSOSC hold at $x^*$. Then, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $p \in \mathcal{B}_\delta(p^*)$,* (MPEC$_p$) *has a locally optimal solution in $\mathcal{B}_\varepsilon(x^*)$.*

As stated in section 3, both (A1) and (A2) are satisfied for the MPEC cone. Moreover, the MPEC-RCPLD implies the multiplier-stability; see, e.g., [11, Theorem 5.1]. Note that the MPEC-RCPLD is a very weak MPEC constraint qualification since either the MPEC linear constraint qualification or the MPEC-NNAMCQ implies the MPEC-RCPLD. In consequence, we have the isolatedness of M-stationary point from Theorems 3.3 and 3.4 immediately.

THEOREM 5.3. *Let $x^* \in \mathcal{X}(p^*)$ be an M-stationary point of* (MPEC$_{p^*}$). *Suppose that the MPEC-RCPLD holds at $x^*$. If the M-SSOSC or second-order growth condition holds at $x^*$, then there exists a neighborhood $V$ of $x^*$ containing no other M-stationary point of* (MPEC$_{p^*}$).

We further have the continuity of locally optimal solution mapping and M-stationary point mapping from Theorem 3.6 immediately.

THEOREM 5.4. *Suppose that the MPEC-NNAMCQ and second-order growth condition hold at $x^* \in \mathcal{X}(p^*)$. Then there exist $\delta > 0$ and $\epsilon > 0$ such that both $\mathcal{O} \cap \mathcal{B}_\delta(x^*)$ and $\mathcal{S}_M \cap \mathcal{B}_\delta(x^*)$ are continuous at $p^*$ and, for each $p \in \mathcal{B}_\epsilon(p^*)$, $\emptyset \neq \mathcal{O}(p) \cap \mathcal{B}_\delta(x^*) \subseteq \mathcal{S}_M(p) \cap \mathcal{B}_\delta(x^*)$.*

The following corollary follows from Theorem 5.4 and Lemma 5.1.

COROLLARY 5.1. *Suppose that the MPEC-NNAMCQ and S-RSOSC hold at $x^* \in \mathcal{X}(p^*)$. Then there exist $\delta > 0$ and $\epsilon > 0$ such that both $\mathcal{O} \cap \mathcal{B}_\delta(x^*)$ and $\mathcal{S}_M \cap \mathcal{B}_\delta(x^*)$ are continuous at $p^*$ and, for each $p \in \mathcal{B}_\epsilon(p^*)$, $\emptyset \neq \mathcal{O}(p) \cap \mathcal{B}_\delta(x^*) \subseteq \mathcal{S}_M(p) \cap \mathcal{B}_\delta(x^*)$.*

We next apply the calmness result for (MPGC$_p$) in Theorem 4.2 to (MPEC$_p$). Since the MPEC cone is a finite union of half spaces, (A1)–(A3) hold. The following result shows that (A4) also holds.

LEMMA 5.2. *There exists $\delta > 0$ such that $\mathcal{N}_\Lambda(F(x,p)) \subset \mathcal{N}_\Lambda(F(x^*,p^*))$ for each $(x,p) \in \mathcal{B}_\delta(x^*,p^*)$.*

*Proof.* By the continuity assumption, there exists $\delta > 0$ such that, for each $(x,p) \in \mathcal{B}_\delta(x^*,p^*)$ with $x \in \mathcal{X}(p)$,

$$\begin{cases} I_{g-}^* \subseteq I_{g-} := \{i \mid g_i(x,p) < 0\}, \\ \mathcal{I}^* \subseteq \mathcal{I} := \{i \mid G_i(x,p) = 0 < H_i(x,p)\}, \\ \mathcal{K}^* \subseteq \mathcal{K} := \{i \mid G_i(x,p) > 0 = H_i(x,p)\}, \\ \mathcal{J}^* \supseteq \mathcal{J} := \{i \mid G_i(x,p) = 0 = H_i(x,p)\}. \end{cases} \tag{5.1}$$

Let $y \in \mathcal{N}_\Lambda(F(x,p))$ with $(x,p) \in \mathcal{B}_\delta(x^*,p^*)$. From Proposition 2.1 and [28, Proposition 6.41], we have

$$\begin{cases} \lambda \geq 0, \quad \lambda_{\mathcal{I}_{g-}} = 0, \\ u_\mathcal{K} = 0, \quad v_\mathcal{I} = 0, \\ u_i \geq 0, v_i \geq 0 \text{ or } u_i = 0 \text{ or } v_i = 0, \quad i \in \mathcal{J}. \end{cases} \tag{5.2}$$

Since $\mathcal{I} \cup \mathcal{J} \cup \mathcal{K} = \mathcal{I}^* \cup \mathcal{J}^* \cup \mathcal{K}^*$, then $i \in \mathcal{J}^* \backslash \mathcal{J}$ implies $i \in \mathcal{I} \cup \mathcal{K}$. Note that $y \in \mathcal{N}_\Lambda(F(x^*,p^*))$ means

$$\begin{cases} \lambda \geq 0, \quad \lambda_{\mathcal{I}_{g-}^*} = 0, \\ u_{\mathcal{K}^*} = 0, \quad v_{\mathcal{I}^*} = 0, \\ u_i \geq 0, v_i \geq 0 \text{ or } u_i = 0 \text{ or } v_i = 0, \quad i \in \mathcal{J}^*. \end{cases} \tag{5.3}$$

It follows from (5.1)–(5.3) that if $y \in \mathcal{N}_\Lambda(F(x,p))$ with $(x,p) \in \mathcal{B}_\delta(x^*,p^*)$, then $y \in \mathcal{N}_\Lambda(F(x^*,p^*))$. The proof is complete. ∎

As shown above, the MPEC cone $\Lambda$ satisfies all the assumptions (A1)–(A4). Moreover, from the comments after Definition 3.3, we know that the multiplier-stability for quadratic programs with geometric constraints holds at every feasible point. Therefore, we have the following calmness result for (MPEC$_p$) from Theorem 4.2 immediately. Note that no constraint qualification is required here.

THEOREM 5.5. *Suppose that $x^* \in \mathcal{X}(p^*)$ and there exists $y^* \in \mathcal{M}_M^1(x^*,p^*)$ such that the M-SOSC holds at $(x^*,y^*)$ for $p = p^*$. Then there exist $\delta > 0$ and $\kappa > 0$ such that if $y \in \mathcal{M}_M^1(x,p)$ with $p \in \mathcal{B}_\delta(p^*)$ and $(x,y) \in \mathcal{B}_\delta(x^*,y^*)$, then*

$$\|x - x^*\| + \operatorname{dist}(y, \mathcal{M}_M^1(x^*,p^*)) \leq \kappa \|p - p^*\|. \tag{5.4}$$

Jongen, Shikhman, and Steffensen [16] gave a counterexample to show that the C-stationary points are strongly stable in the sense of Kojima [18], but the M-stationary points are not strongly stable. We next show that although the counterexample given in [16] does not satisfy the M-SOSC at the given point, the error estimate (5.4) still holds.

*Example* 5.1 (see [16]). Consider the problem

$$\begin{aligned} \min_{x,y} \quad & -x - (y+t)^2 \\ \text{s.t.} \quad & 0 \leq x \perp y \geq 0. \end{aligned}$$

It is not difficult to see that

– for $t = 0$, $(x_0, y_0) = (0,0)$ is an M-stationary point with a unique M-multiplier vector $(u_0, v_0) = (-1, 0)$;

– for $t < 0$, $(x_t, y_t) = (0, -t)$ is an M-stationary point with a unique M-multiplier $(u_t, v_t) = (-1, 0)$;

– for $t > 0$, there is no M-stationary point.

It is obvious that

$$\|(x_t, y_t) - (x_0, y_0)\| + \|(u_t, v_t) - (u_0, v_0)\| = \|t\|,$$

which implies that the error estimate (5.4) holds. In addition, it is easy to verify that the M-SOSC does not hold at $(x_0, y_0, u_0, v_0)$.

We now study the calmness of S-stationary pair mapping $p \to \{(x, y) \mid \nabla_x L(x, p; y) = 0,\ y \in \hat{\mathcal{N}}_\Lambda(F(x, p))\}$. Note that, in contrast to Theorem 5.5, the bidegenerate index set $\{i \in \mathcal{J}^* \mid u_i^* = v_i^* = 0\} = \emptyset$ is necessary in the following calmness result.

THEOREM 5.6. *Suppose that $x^* \in \mathcal{X}(p^*)$ and there exists $y^* \in \mathcal{M}_S^1(x^*, p^*)$ such that $\{i \in \mathcal{J}^* \mid u_i^* = v_i^* = 0\} = \emptyset$ and the S-SOSC holds at $(x^*, y^*)$. Then there exist $\delta > 0$ and $\kappa > 0$ such that if $y \in \mathcal{M}_S^1(x, p)$ with $p \in \mathcal{B}_\delta(p^*)$ and $(x, y) \in \mathcal{B}_\delta(x^*, y^*)$, then*

$$\|x - x^*\| + \operatorname{dist}(y, \mathcal{M}_S^1(x^*, p^*)) \le \kappa\|p - p^*\|.$$

*Proof.* Consider the problem $(\mathrm{QP})_{(a^*, q^*)}$ defined in the proof of Theorem 4.2. It is easy to check that $(x^*, y^*)$ is an S-stationary pair of $(\mathrm{MPEC}_{p^*})$ and the S-SOSC holds at $x^*$ if and only if $(x^*, y^*)$ is an S-stationary pair of $(\mathrm{QP})_{(a^*, q^*)}$ and the S-SOSC holds at $x^*$. Note that, from Proposition 5.1, $\mathcal{M}_S^1(x^*, p^*)$ is the solution set of the system

$$\nabla_x L(x^*, p^*; y) = 0, \quad y \in \hat{\mathcal{N}}_\Lambda(F(x^*, p^*)).$$

It is not hard to see from Proposition 2.1 that the set $\hat{\mathcal{N}}_\Lambda(F(x^*, p^*))$ is a union of finitely many polyhedral sets and hence $\mathcal{M}(x^*, p^*)$ is a union of finitely many polyhedral sets. From the proof of Theorem 4.2, it suffices to show that if $y \in \hat{\mathcal{N}}_\Lambda(F(x, p))$, then $y \in \hat{\mathcal{N}}_\Lambda(F(x^*, p^*))$ when $(x, y, p)$ is sufficiently close to $(x^*, y^*, p^*)$. As in Lemma 5.2, by the continuity assumption, there exists $\delta_1 > 0$ such that, for each $(x, p) \in \mathcal{B}_{\delta_1}(x^*, p^*)$ with $x \in \mathcal{X}(p)$,

$$I_{g-}^* \subseteq I_{g-}, \quad \mathcal{I}^* \subseteq \mathcal{I}, \quad \mathcal{K}^* \subseteq \mathcal{K}, \quad \mathcal{J}^* \supseteq \mathcal{J}, \tag{5.5}$$

where $I_{g-}, \mathcal{I}, \mathcal{J}, \mathcal{K}$ are the same as in Lemma 5.2. Define the index sets

$$\begin{aligned} \mathcal{J}_{++}^* &:= \{i \in \mathcal{J}^* \mid u_i^* > 0, v_i^* > 0\}, \\ \mathcal{J}_{0+}^* &:= \{i \in \mathcal{J}^* \mid u_i^* = 0, v_i^* > 0\}, \\ \mathcal{J}_{+0}^* &:= \{i \in \mathcal{J}^* \mid u_i^* > 0, v_i^* = 0\}. \end{aligned}$$

Since $\{i \in \mathcal{J}^* \mid u_i^* = v_i^* = 0\} = \emptyset$, it is obvious that $\mathcal{J}_{++}^*$, $\mathcal{J}_{0+}^*$, and $\mathcal{J}_{+0}^*$ constitute a partition of $\mathcal{J}^*$. From Proposition 2.1 and [28, Proposition 6.41], $y \in \hat{\mathcal{N}}_\Lambda(F(x, p))$ if and only if

$$\begin{cases} \lambda \ge 0, \quad \lambda_{\mathcal{I}_{g-}} = 0, \\ u_\mathcal{K} = 0, \quad v_\mathcal{I} = 0, \\ u_i \ge 0, \ v_i \ge 0 \quad \forall i \in \mathcal{J}. \end{cases} \tag{5.6}$$

We can show that there exists $\delta \in (0, \delta_1)$ independent of $i$ such that, for each $y \in \mathcal{B}_\delta(y^*) \cap \hat{\mathcal{N}}_\Lambda(F(x, p))$,

$$u_i \ge 0, \ v_i \ge 0 \quad \forall i \in \mathcal{J}^* = \mathcal{J}_{++}^* \cup \mathcal{J}_{0+}^* \cup \mathcal{J}_{+0}^*. \tag{5.7}$$

In fact, if $i \in \mathcal{J}^*_{++}$, we have $u_i > 0$ and $v_i > 0$ by the continuity. If $i \in \mathcal{J}^*_{0+}$, we have $v_i > 0$ by the continuity, which implies $i \in \mathcal{K} \cup \mathcal{J}$ by (5.6). If $i \in \mathcal{K}$, we have $u_i = 0$ from (5.6) and, if $i \in \mathcal{J}$, we have $u_i \geq 0$. In a similar way, we can show $u_i \geq 0$ and $v_i \geq 0$ when $i \in \mathcal{J}^*_{+0}$.

Note that $y \in \hat{\mathcal{N}}_\Lambda(F(x^*, p^*))$ means

$$\left\{ \begin{array}{l} \lambda \geq 0, \quad \lambda_{\mathcal{I}^*_{g-}} = 0, \\ u_{\mathcal{K}^*} = 0, \quad v_{\mathcal{I}^*} = 0, \\ u_i \geq 0, \; v_i \geq 0 \;\; \forall i \in \mathcal{J}^*. \end{array} \right. \tag{5.8}$$

It follows from (5.5)–(5.8) that if $y \in \hat{\mathcal{N}}_\Lambda(F(x,p))$ with $(x,p,y) \in \mathcal{B}_\delta(x^*, p^*, y^*)$, then $y \in \mathcal{N}_\Lambda(F(x^*, p^*))$. This completes the proof. ∎

**6. Concluding remarks.** We have shown a number of results related to the stability for (MPGC$_p$) and applied these results to the special case (MPEC$_p$). Actually, there is no difficulty in applying the results for (MPGC$_p$) to the parametric mathematical program with vertical complementarity constraints, which was first considered in [29],

$$\begin{array}{ll} \min & f(x,p) \\ \text{s.t.} & g(x,p) \leq 0, \; h(x,p) = 0, \\ & \min\{F_1(x,p), \ldots, F_\ell(x,p)\} = 0, \end{array} \tag{6.1}$$

and the parametric mathematical program with vanishing constraints, which was first considered in [1],

$$\begin{array}{ll} \min & f(x,p) \\ \text{s.t.} & g(x,p) \leq 0, \; h(x,p) = 0, \\ & H(x,p) \geq 0, \; G(x,p) \circ H(x,p) \leq 0, \end{array} \tag{6.2}$$

where $\{f, g, h, G, H\}$ are the same as before and $F_i : \Re^{n_1+n_2} \to \Re^m$ $(i = 1, \ldots, \ell)$ are all twice continuously differentiable functions. In fact, by introducing

$$C := \{(a_1, \ldots, a_l) \mid \; \min\{a_1, \ldots, a_l\} = 0\}$$

or

$$C := \{(a, b) \mid b \geq 0, \; ab \leq 0\}$$

instead of the cone $C$ in (1.3), we can rewrite (6.1) or (6.2) as a problem with geometric constraints like (1.1). In particular, it is not difficult to verify that the assumptions (A1)–(A4) hold for these two cases, and the multiplier-stability also holds when the involved constraints are linear with respect to $x$. Therefore, in a similar way to section 5, we can establish stability analysis for stationarity systems of (6.1) and (6.2) accordingly.

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