FIRST-ORDER OPTIMALITY CONDITIONS FOR MATHEMATICAL PROGRAMS WITH SECOND-ORDER CONE COMPLEMENTARITY CONSTRAINTS*

JANE J. YE[†] AND JINCHUAN ZHOU[‡]

Abstract. In this paper we consider a mathematical program with second-order cone complementarity constraints (SOCMPCC). The SOCMPCC generalizes the mathematical program with complementarity constraints (MPCC) in replacing the set of nonnegative reals by second-order cones. There are difficulties in applying the classical Karush–Kuhn–Tucker (KKT) condition to the SOCM-PCC directly since the usual constraint qualification such as Robinson's constraint qualification never holds if it is considered as an optimization problem with a convex cone constraint. Using various reformulations and recent results on the exact formula for the proximal/regular and limiting normal cone, we derive necessary optimality conditions in the forms of the strong, Mordukhovich, and Clarke (resp., S, M, and C) stationary conditions under certain constraint qualifications. We also show that, unlike the MPCC, the classical KKT condition of the SOCMPCC is in general not equivalent to the S-stationary condition unless the dimension of each second-order cone is not more than 2. Finally, we show that reformulating an MPCC as an SOCMPCC produces new and weaker necessary optimality conditions.

Key words. mathematical program with second-order cone complementarity constraints, necessary optimality conditions, constraint qualifications, S-stationary conditions, M-stationary conditions, C-stationary conditions

1. Introduction. In this paper we consider the following mathematical program with second-order cone complementarity constraints (SOCMPCC or MPSOCC):

$$\text{(SOCMPCC)} \qquad \begin{aligned} \min \ & f(z) \\ \text{s.t.} \ & h(z) = 0, \quad g(z) \leq 0, \\ & \mathcal{K}_i \ni G_i(z) \perp H_i(z) \in \mathcal{K}_i, \quad i = 1, \dots, J, \end{aligned}$$

where $a \perp b$ means that the vector a is perpendicular to vector b. Throughout the paper we assume that $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, $h : \mathbb{R}^n \to \mathbb{R}^q$, $G_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$, $H_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$ are all continuously differentiable and $\mathcal{K}_i$ is an m_i-dimensional second-order cone defined as

$$\mathcal{K}_i := \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m_i - 1} \mid x_1 \geq \|x_2\|\},$$

where $\| \cdot \|$ denotes the Euclidean norm and, when $m_i = 1$, $\mathcal{K}_i$ stands for the set of nonnegative reals $\mathbb{R}_+$. In particular, SOCMPCC with all $m_i = 1$ for $i = 1, \dots, J$ coincides with the mathematical program with complementarity constraints (MPCC),

†Department of Mathematics and Statistics, University of Victoria, Victoria, BC, V8W 2Y2 Canada (janeye@uvic.ca).

‡Department of Mathematics, School of Science, Shandong University of Technology, Zibo 255049, People's Republic of China (jinchuanzhou@163.com).

which has received a lot of attention in the last twenty years or so [9, 13]. The generalization from MPCC to SOCMPCC has many important applications. We briefly review two of them. In practice it is more realistic to assume that an optimization problem involves uncertainty. An approach to optimization under uncertainty is robust optimization. For example, consider a robust bilevel programming problem where for a fixed upper level decision variable x, the lower level problem is replaced by its robust counterpart:

$$P_x: \quad \min_y \{f(x, y, \zeta) : g(x, y, \zeta) \leq 0 \quad \forall \zeta \in \mathcal{U}\},$$

where $\mathcal{U}$ is some "uncertainty set" in the space of the data. It is well known (see [2]) that if the uncertainty set $\mathcal{U}$ is given by a system of conic quadratic inequalities, then the deterministic counterpart of the problem P_x is a second-order cone program. If this second-order cone program can be equivalently replaced by its Karush–Kuhn–Tucker (KKT) condition, then it yields an SOCMPCC. Another application of SOCMPCC is in modeling an inverse quadratic programming problem over the second-order cone, in which the parameters in a given second-order cone quadratic programming problem need to be adjusted as little as possible so that a known feasible solution becomes optimal (see [28] for details).

It is known that if an MPCC is treated as a nonlinear program with equality and inequality constraints, then Mangasarian–Fromovitz constraint qualification (MFCQ) fails to hold at each feasible point of the feasible region; see [26, Proposition 1.1]. This causes great difficulties in applying classical theories and algorithms in nonlinear programs directly to MPCCs. To remedy this problem, several variants of stationary conditions such as the strong (S), Mordukhovich (M), Clarke (C) stationary conditions have been proposed, and constraint qualifications under which a local minimizer is an S-, M-, C-stationary point have been studied; see, e.g., [17, 23] for a detailed discussion. For a mathematical program with semidefinite cone complementarity constraints (SDCMPCC), the matrix analogue of the MPCC, it was shown in [6] that Robinson's CQ—which is the usual constraint qualification for an optimization problem with a convex cone constraint—fails to hold at each feasible point, and the corresponding S-, M-, C-stationary conditions were proposed and the constraint qualifications under which a local minimizer is an S-, M-, C-stationary point have been studied.

The same difficulties exist for SOCMPCC. Notice that the cone complementarity constraint

$$(1) \quad \mathcal{K} \ni G(z) \perp H(z) \in \mathcal{K},$$

where $G, H : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathcal{K}$ is the m-dimensional second-order cone, amounts to the following convex cone constraints:

$$\langle G(z), H(z) \rangle \leq 0, \quad G(z) \in \mathcal{K}, \quad H(z) \in \mathcal{K}.$$

In this paper we show that if SOCMPCC is regarded as an optimization problem with a convex cone constraint, then Robinson's CQ fails to hold at each feasible point of SOCMPCC.

So far there are only a few papers devoted to the study of SOCMPCC [8, 14, 18, 19, 20, 27, 28, 29], and [18, 19, 20, 27, 28, 29] mainly study numerical algorithms, which are not the main purpose of this paper. To the best of our knowledge, the problem SOCMPCC was studied for the first time by Outrata and Sun in [14]. The approach taken was to consider the cone complementarity constraint (1) as

$$(G(z) - H(z), G(z)) \in \mathrm{gph}\Pi_\mathcal{K},$$

where $\mathrm{gph}\Pi_{\mathcal{K}}$ is the graph of the metric projection operator onto the second-order cone $\mathcal{K}$. By computing the limiting normal cone to $\mathrm{gph}\Pi_{\mathcal{K}}$ or, equivalently, the limiting coderivative of the metric projection $\Pi_{\mathcal{K}}(\cdot)$, an M-stationary condition was shown to be necessary for optimality under the condition that there be no nonzero abnormal multipliers (see [14, Theorem 6]). The same reformulation was further taken by Zhang, Zhang, and Wu in [27] to define M- and S-stationary conditions in terms of the regular and the limiting coderivative of the metric projection onto the second-order cone, respectively (see [27, Definitions 3.6 and 3.7]). Moreover, a B-stationary condition is defined in [27, Definition 3.3], and it was shown that under the SOCMPCC-LICQ, the B-stationarity is equivalent to the S-stationarity [27, Lemma 3.2]. Moreover, in [27, Definition 3.5] the C-stationary condition was proposed to be the nonsmooth KKT condition involving the Clarke generalized gradient for problem SOCMPCC where the cone complementarity constraint (1) is reformulated as a nonsmooth equation constraint:

$$G(z) - \Pi_{\mathcal{K}}(G(z) - H(z)) = 0.$$

However these optimality conditions are not in forms that are analogues to the S-, M-, and C-stationary conditions for MPCCs, and they are not explicit due to the existence of coderivatives or a Clarke subdifferential of the metric projection onto the second-order cone in these formulas.

Notice that the second-order cone complementarity constraint (1) can be reformulated as a nonconvex cone constraint:

$$(G(z), H(z)) \in \Omega,$$

where

$$\Omega := \{(x, y) \mid x \in \mathcal{K}, \ y \in \mathcal{K}, \ x^T y = 0\}$$

is called the second-order cone complementarity set (or complementarity cone since it is a cone). Note that Ω is nonconvex due to the existence of complementarity conditions. If the exact expression for the regular and the limiting normal cones of second-order cone complementarity sets can be derived, then the corresponding stationary conditions would be the suitable generalization of the S- and M-stationary conditions. The first attempt in this direction was initiated by Liang, Zhu, and Lin in [8], where they tried to derive exact expressions for the regular and the limiting normal cones of the second-order cone complementary set by using the relationships between the metric projection operator and the second-order cone complementary set. Unfortunately, there are some gaps in their expressions of the regular and the limiting normal cones, mainly on the boundary points, which result in gaps in their proposed expressions for the S-, M-, and C-stationary conditions. In a recent paper [25], we fill in this gap and establish the correct exact expressions for the regular and limiting normal cone of the second-order cone complementary set. Furthermore, we show that the regular and the proximal normal cones to the second-order cone complementary set coincide with each other. Using these exact expressions for the regular and the limiting normal cone of the second-order cone complementary set, in this paper we propose S-, M-, and C-stationary conditions for SOCMPCC in a form analogous to that of the S-, M- and C-stationary conditions for MPCCs.

It is well known that for MPCC, the classical KKT condition is equivalent to the S-stationary condition (see, e.g., [7]). For SDCMPCC it was shown in [6] that in general the classical KKT condition is stronger than the S-stationary condition, but these two conditions may not be equivalent. It is natural to ask whether or not the classical KKT

condition is equivalent to the S-stationary condition for SOCMPCC. In this paper we show that for SOCMPCC, in general the classical KKT condition is a stronger condition than the S-stationary condition, while these two concepts coincide when the dimension of each second-order cone $\mathcal{K}_i$ is not more than 2. Moreover, an example is given to illustrate that an S-stationary point may not be a classical KKT point when one of the second-order cones $\mathcal{K}_i$ has dimension greater than 2. Since in general the classical KKT condition and the S-stationary condition are different, we introduce a new stationary point concept called the K-stationary point, which is equivalent to the classical KKT point. Furthermore we have derived an exact expression for the set of all multipliers satisfying the K-stationary condition and shown that it is just a subset of the regular normal cone of the second-order cone complementarity set.

We summarize our main contributions as follows:

- We show that Robinson's CQ fails to hold at every feasible point of SOCM-PCC if the SOCMPCC is treated as an optimization problem with a convex cone constraint.
- We obtain the precise description for the S-, M-, and C-stationary conditions in the forms that are analogues of the associated stationary conditions for MPCCs and show that they are necessary for optimality under the corresponding Clarke calmness conditions. We also show that the S-stationary condition is a necessary optimality condition for a local minimum if the SOCMPCC-LICQ holds. Moreover, we show that for the case where all mappings are affine and the dimension of each second-order cone is less than or equal to 2, a local minimal solution of SOCMPCC must be an M-stationary point without any further constraint qualification.
- We derive the relationships between various stationary conditions and show that in general the K-stationary condition is stronger than the S-stationary condition but not equivalent, and these two concepts coincide when the dimension of all $\mathcal{K}_i$ is less than or equal to 2.
- We obtain the relationship between various Clarke calmness conditions for the general optimization problem with symmetric cone complementarity constraints. Such results are new even for the case of MPCCs.
- We establish the relationship of various stationary points between MPCC and its SOCMPCC reformulation.

We organize our paper as follows. Section 2 contains the preliminaries. In section 3, we show that Robinson's CQ never holds if SOCMPCC is considered as an optimization problem with a convex cone constraint. The K-stationary condition is introduced and studied in this section. In sections 4, 5, and 6, we give the explicit expressions for the S-, M- and C-stationary conditions and propose some constraint qualifications for them to be necessary for optimality. Section 7 gives the connections among various stationary conditions and various Clarke calmness conditions. In section 8, we reformulate MPCC as SOCMPCC and obtain some new and weaker necessary optimality conditions for MPCCs.

The following notation will be used throughout the paper. We denote by I and O the identity and zero matrix of appropriate dimensions, respectively. For a matrix A, we denote its transpose by A^T. The inner product of two vectors x, y is denoted by $x^T y$ or $\langle x, y \rangle$. For $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1}$, we write its reflection about the x_1 axis as $\hat{x} := (x_1, -x_2)$. Denote by $\mathbb{R}x$ the set $\{tx \mid t \in \mathbb{R}\}$. $\mathbb{R}_+ x$ and $\mathbb{R}_{++} x$, where $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_{++} := (0, \infty)$, are similarly defined. For a set C, denote by $\text{int} C$, $\text{cl} C$, $\text{bd} C$, $\text{co} C$, and C^c its interior, closure, boundary, convex hull, and complement, respectively. The polar cone of a vector v is $v^\circ := \{x \mid x^T v \leq 0\}$. Given a point $x \in \mathbb{R}^n$

and $\varepsilon > 0$, $B_\varepsilon(x)$ denotes an open ball centered at x with radius ε, while B denotes the open unit ball center at the origin of an appropriate dimension. For a differentiable mapping $H : \mathbb{R}^n \to \mathbb{R}^m$ and a vector $x \in \mathbb{R}^n$, we denote by $\mathcal{J}H(x)$ the Jacobian matrix of H at x and $\nabla H(x) := \mathcal{J}H(x)^T$. The graph of a set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, is denoted by $\mathrm{gph}\Phi$, i.e., $\mathrm{gph}\Phi := \{(z,v) \in \mathbb{R}^n \times \mathbb{R}^m \mid v \in \Phi(z)\}$.

2. Preliminaries. In this section we review some basic concepts in variational analysis and then specialize it to the second-order cone and the second-order cone complementarity set.

2.1. Background in variational analysis. First we summarize some background material on variational analysis which will be used throughout the paper. Detailed discussions on these subjects can be found in [4, 5, 11, 12, 16].

Let C be a nonempty subset of $\mathbb{R}^n$. Given $x^* \in \mathrm{cl}C$, the proximal normal cone of C at x^* is defined as

$$N_C^\pi(x^*) := \{v \in \mathbb{R}^n \mid \exists\, M > 0, \text{ such that } \langle v, x - x^* \rangle \le M\|x - x^*\|^2 \ \ \forall\, x \in C\},$$

and the regular/Fréchet normal cone is

$$\widehat{N}_C(x^*) := \{v \in \mathbb{R}^n \mid \langle v, x - x^* \rangle \le o(\|x - x^*\|)\ \forall x \in C\},$$

where $o(\cdot)$ means that $o(\alpha)/\alpha \to 0$ as $\alpha \to 0$. The limiting/Mordukhovich normal cone is defined as the outer limit of either the proximal normal cone or the regular normal cone, i.e.,

$$\begin{aligned} N_C(x^*) &:= \{\lim_{i \to \infty} \zeta_i \mid \zeta_i \in N_C^\pi(x_i),\ \ x_i \to x^*,\ \ x_i \in C\} \\ &= \{\lim_{i \to \infty} \zeta_i \mid \zeta_i \in \widehat{N}_C(x_i),\ \ x_i \to x^*,\ \ x_i \in C\}. \end{aligned}$$

PROPOSITION 2.1 (change of coordinates [16, Exercise 6.7]). *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be smooth and set $D \subset \mathbb{R}^m$. Suppose that $\nabla F(x^*)$ has full column rank m at a point $x^* \in \mathcal{F} := \{x \in \mathbb{R}^n \mid F(x) \in D\}$. Then*

$$\begin{aligned} \widehat{N}_\mathcal{F}(x^*) &= \{\nabla F(x^*)y \mid y \in \widehat{N}_D(F(x^*))\}, \\ N_\mathcal{F}(x^*) &= \{\nabla F(x^*)y \mid y \in N_D(F(x^*))\}. \end{aligned}$$

Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map and $(x^*, y^*) \in \mathrm{gph}\Phi$. The regular coderivative and the limiting (Mordukhovich) coderivative of Φ at (x^*, y^*) are the set-valued mappings defined by

$$\begin{aligned} \widehat{D}^*\Phi(x^*, y^*)(v) &:= \{u \in \mathbb{R}^n \mid (u, -v) \in \widehat{N}_{\mathrm{gph}\Phi}(x^*, y^*)\}, \\ D^*\Phi(x^*, y^*)(v) &:= \{u \in \mathbb{R}^n \mid (u, -v) \in N_{\mathrm{gph}\Phi}(x^*, y^*)\}, \end{aligned}$$

respectively. We omit y^* in the coderivative notation if the set-valued map Φ is single-valued at x^*.

For a single-valued Lipschitz continuous map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, the B(ouligand)-subdifferential $\partial_B\Phi$ is defined as

$$\partial_B\Phi(x) = \{\lim_{k \to \infty} \mathcal{J}\Phi(x_k) \mid x_k \to x, \Phi \text{ is differentiable at } x_k\}.$$

It is known that $\mathrm{co}\partial_B\Phi(x) = \partial^c\Phi(x)$, the Clarke generalized Jacobian of Φ at x (see [4]). Moreover if Φ is a continuously differentiable single-valued map, then

$$\widehat{D}^*\Phi(x^*) = D^*\Phi(x^*) = \nabla\Phi(x^*).$$

2.2. Background in variational analysis associated with the second-order cone. Let $\mathcal{K}$ be the m-dimensional second-order cone. The topological interior and the boundary of $\mathcal{K}$ are

$$\text{int}\mathcal{K} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1} | \, x_1 > \|x_2\|\}$$
$$\text{and } \text{bd}\mathcal{K} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1} | \, x_1 = \|x_2\|\},$$

respectively.

PROPOSITION 2.2. *For any $x, y \in \text{bd}\mathcal{K}\backslash\{0\}$, the following equivalence holds:*

$$x^T y = 0 \Longleftrightarrow y = k\hat{x} \text{ with } k = y_1/x_1 > 0 \Longleftrightarrow y = k\hat{x} \text{ with } k \in \mathbb{R}_{++}.$$

Proof. Suppose that $x, y \in \text{bd}\mathcal{K}\backslash\{0\}$ and $x^T y = 0$. Then

(2) $$x_1 = \|x_2\| > 0, \ y_1 = \|y_2\| > 0, \ x^T y = x_1 y_1 + x_2^T y_2 = 0,$$

which implies that $-x_2^T y_2 = x_1 y_1 = \|x_2\| \|y_2\|$. Hence there exists a positive constant k such that $y_2 = -kx_2$. It follows from (2) that $k = y_1/x_1$ and hence $y = k\hat{x}$. The rest of the proof follows from [8, Lemma 2.3]. $\qquad\square$

The exact formula of the regular normal cone and limiting normal cone of Ω have been established in [25].

PROPOSITION 2.3 (see [25, Theorem 3.1]). *Let (x, y) be in the m-dimensional second-order cone complementarity set Ω. Then*

$$\widehat{N}_\Omega(x,y) = \begin{cases} \{(u,v)|u \in \mathbb{R}^m, \ v = 0\} & \text{if } x = 0, \ y \in \text{int}\mathcal{K}, \\ \{(u,v)|u = 0, \ v \in \mathbb{R}^m\} & \text{if } x \in \text{int}\mathcal{K}, \ y = 0, \\ \{(u,v)|u \perp x, \ v \perp y, \ x_1\hat{u} + y_1 v \in \mathbb{R}x\} & \text{if } x,y \in \text{bd}\mathcal{K}\backslash\{0\}, \ x^T y = 0, \\ \{(u,v)|u \in \hat{y}^\circ, \ v \in \mathbb{R}_-\hat{y}\} & \text{if } x = 0, \ y \in \text{bd}\mathcal{K}\backslash\{0\}, \\ \{(u,v)|u \in \mathbb{R}_-\hat{x}, \ v \in \hat{x}^\circ\} & \text{if } x \in \text{bd}\mathcal{K}\backslash\{0\}, \ y = 0, \\ \{(u,v)|u \in -\mathcal{K}, \ v \in -\mathcal{K}\} & \text{if } x = 0, \ y = 0. \end{cases}$$

PROPOSITION 2.4 (see [25, Theorem 3.3]). *Let (x, y) be in the m-dimensional second-order cone complementarity set Ω. Then*

$$N_\Omega(x,y) = \widehat{N}_\Omega(x,y) = \begin{cases} \{(u,v)|u \in \mathbb{R}^m, \ v = 0\} & \text{if } x = 0, \ y \in \text{int}\mathcal{K}, \\ \{(u,v)|u = 0, \ v \in \mathbb{R}^m\} & \text{if } x \in \text{int}\mathcal{K}, \ y = 0, \\ \{(u,v)|u \perp x, \ v \perp y, \ x_1\hat{u} + y_1 v \in \mathbb{R}x\} & \text{if } x,y \in \text{bd}\mathcal{K}\backslash\{0\}. \end{cases}$$

For $x = 0, y \in \text{bd}\mathcal{K}\backslash\{0\}$,

$$N_\Omega(x,y) = \{(u,v)|u \in \mathbb{R}^m, \ v = 0 \ \text{ or } \ u \perp \hat{y}, \ v \in \mathbb{R}\hat{y} \ \text{ or } \ \langle u, \hat{y}\rangle \leq 0, \ v \in \mathbb{R}_-\hat{y}\};$$

for $x \in \text{bd}\mathcal{K}\backslash\{0\}, y = 0$,

$$N_\Omega(x,y) = \{(u,v)|u = 0, v \in \mathbb{R}^m \ \text{ or } \ u \in \mathbb{R}\hat{x}, \ v \perp \hat{x} \ \text{ or } \ u \in \mathbb{R}_-\hat{x}, \langle v, \hat{x}\rangle \leq 0\};$$

for $x = y = 0$,

$$\begin{aligned} N_\Omega(x,y) = \{(u,v)| \ & u \in -\mathcal{K}, v \in -\mathcal{K} \text{ or } u \in \mathbb{R}^m, v = 0 \text{ or } u = 0, v \in \mathbb{R}^m \\ & \text{or } u \in \mathbb{R}_-\xi, \ v \in \xi^\circ \ \text{ or } \ u \in \xi^\circ, v \in \mathbb{R}_-\xi \\ & \text{or } u \perp \xi, \ v \perp \hat{\xi}, \ \alpha\hat{u} + (1-\alpha)v \in \mathbb{R}\xi \text{ for some } \alpha \in [0,1] \text{ and } \xi \in C\}, \end{aligned}$$

where

$$C := \{(1, w)| \ w \in \mathbb{R}^{m-1}, \ \|w\| = 1\}.$$

3. Failure of Robinson's CQ and the classical KKT condition. Note that $G_i(z), H_i(z) \in \mathcal{K}_i$ implies that $G_i(z)^T H_i(z) \geq 0$ for $i = 1, \ldots, J$. Hence SOCMPCC can be rewritten as an optimization problem with a convex cone constraint:

(K-SOCMPCC)
$$\min \ f(z)$$
$$\text{s.t.} \quad g(z) \leq 0, \ h(z) = 0,$$
$$\langle G(z), H(z) \rangle \leq 0,$$
$$(G(z), H(z)) \in \widetilde{\mathcal{K}} \times \widetilde{\mathcal{K}},$$

where $G(z) := (G_1(z), \ldots, G_J(z))$, $H(z) := (H_1(z), \ldots, H_J(z))$, and $\widetilde{\mathcal{K}} := \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_J$. We denote $\tau := \sum_{j=1}^{J} m_j$.

For a general optimization problem with a cone constraint such as K-SOCMPCC, the following Robinson's CQ is considered to be a usual constraint qualification:

$\nabla h_i(z^*)(i = 1, \ldots, q)$ are linearly independent,

$$\exists \, d \in \mathbb{R}^n \text{ such that } \begin{cases} \nabla h_i(z^*)^T d = 0, \quad i = 1, \ldots, q, \\ g(z^*) + \nabla g(z^*)^T d \in \text{int} \mathbb{R}_+^p, \\ \left(\nabla H(z^*) G(z^*) + \nabla G(z^*) H(z^*) \right)^T d < 0, \\ G(z^*) + \nabla G(z^*)^T d \in \text{int} \widetilde{\mathcal{K}}, \\ H(z^*) + \nabla H(z^*)^T d \in \text{int} \widetilde{\mathcal{K}}. \end{cases}$$

It is well known that the MFCQ never holds for MPCCs. We now show that Robinson's CQ never holds for the K-SOCMPCC.

PROPOSITION 3.1. *For K-SOCMPCC, Robinson's CQ fails to hold at every feasible solution of SOCMPCC.*

Proof. Any feasible solution z^* of SOCMPCC must be a solution to the following mathematical program with a convex cone constraint:

$$\min \ \langle G(z), H(z) \rangle$$
$$\text{s.t.} \ G(z) \in \widetilde{\mathcal{K}}, \quad H(z) \in \widetilde{\mathcal{K}}.$$

By the Fritz John necessary optimality condition, there exist $\lambda_0 \geq 0, \lambda^G \in \mathbb{R}^\tau, \lambda^H \in \mathbb{R}^\tau$ not all equal to zero such that

$$0 = \lambda_0 \nabla \langle G, H \rangle(z^*) + \nabla G(z^*) \lambda^G + \nabla H(z^*) \lambda^H, \ \lambda^G \in N_{\widetilde{\mathcal{K}}}(G(z^*)), \ \lambda^H \in N_{\widetilde{\mathcal{K}}}(H(z^*)).$$

It is clear that $(0, 0, 0, \lambda_0, \lambda^G, \lambda^H)$ is a singular Lagrange multiplier of K-SOCMPCC. By [3, Propositions 3.16(ii) and 3.19(iii)], a singular Lagrange multiplier exists if and only if Robinson's CQ does not hold. Therefore we conclude that Robinson's CQ does not hold at z^* for K-SOCMPCC. $\qquad \square$

For a feasible point z of SOCMPCC, define the following index sets:

$$I_g(z) := \{i| \ g_i(z) = 0\},$$
$$I_G(z) := \{i| \ G_i(z) = 0\}, \quad I_G^+(z) := \{i| \ G_i(z) \in \text{int}\mathcal{K}_i\},$$
$$I_H(z) := \{i| \ H_i(z) = 0\}, \quad I_H^+(z) := \{i| \ H_i(z) \in \text{int}\mathcal{K}_i\},$$
$$B_H(z) := \{i| \ H_i(z) \in \text{bd}\mathcal{K}_i \backslash \{0\}\},$$
$$B_G(z) := \{i| \ G_i(z) \in \text{bd}\mathcal{K}_i \backslash \{0\}\}.$$

For simplicity we may omit the dependence of z in the above index sets and denote $G_i(z), H_i(z)$ by G_i, H_i and $\widehat{G}_i(z), \widehat{H}_i(z)$ by $\widehat{G}_i, \widehat{H}_i$, respectively, if there is no confusion.

Now we introduce a new concept of stationary point for SOCMPCC, called the K-stationary point, and we show that the K-stationary condition (3) is equivalent to the classical KKT conditions (4).

DEFINITION 3.2 (K-stationary point). *Let z^* be a feasible solution of SOCMPCC. We say that z^* is a K-stationary point of SOCMPCC if there exists a multiplier $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ such that the following K-stationary condition holds:*

$$
(3) \quad
\begin{cases}
\nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum\limits_{i=1}^{J} \nabla G_i(z^*)\lambda_i^G + \sum\limits_{i=1}^{J} \nabla H_i(z^*)\lambda_i^H = 0, \\
\lambda^g \geq 0, \ g(z^*)^T \lambda^g = 0, \\
\lambda_i^G \in \mathbb{R}^{m_i}, \ \lambda_i^H = 0 \qquad if \ \ i \in I_G \cap I_H^+, \\
\lambda_i^G = 0, \ \lambda_i^H \in \mathbb{R}^{m_i} \qquad if \ \ i \in I_G^+ \cap I_H, \\
\lambda_i^G \in \mathbb{R}\widehat{G}_i(z^*), \ \lambda_i^H \in \mathbb{R}\widehat{H}_i(z^*) \qquad if \ \ i \in B_G \cap B_H, \\
\lambda_i^G \in -\mathcal{K}_i + \mathbb{R}_+ H_i(z^*), \ \lambda_i^H \in \mathbb{R}_- \widehat{H}_i(z^*) \qquad if \ \ i \in I_G \cap B_H, \\
\lambda_i^G \in \mathbb{R}_- \widehat{G}_i(z^*), \ \lambda_i^H \in -\mathcal{K}_i + \mathbb{R}_+ G_i(z^*) \qquad if \ \ i \in B_G \cap I_H, \\
\lambda_i^G \in -\mathcal{K}_i, \ \lambda_i^H \in -\mathcal{K}_i \qquad if \ \ i \in I_G \cap I_H.
\end{cases}
$$

DEFINITION 3.3. *We say that K-SOCMPCC is Clarke calm at a feasible solution z^* if there exist positive ε and μ such that, for all $(r,s,t,p) \in \varepsilon B$ and for all $z \in B_\varepsilon(z^*) \cap \mathcal{F}_K(r,s,t,p)$, one has*

$$
f(z) - f(z^*) + \mu\|(r,s,t,p)\| \geq 0,
$$

where

$$
\mathcal{F}_K(r,s,t,p)
$$
$$
:= \left\{ z \mid h(z) + r = 0, g(z) + s \leq 0, \langle G(z), H(z) \rangle + t \leq 0, (G(z), H(z)) + p \in \widetilde{\mathcal{K}} \times \widetilde{\mathcal{K}} \right\}.
$$

THEOREM 3.4. *Let z^* be a local optimal solution of SOCMPCC. Suppose that the problem K-SOCMPCC is Clarke calm at z^*. Then z^* is a K-stationary point. Moreover the K-stationary condition is equivalent to the classical KKT condition (4).*

Proof. Since the problem K-SOCMPCC is Clarke calm at z^*, by the classical necessary optimality condition (see, e.g., [6, Theorem 2.2]), there exists $(\lambda^g, \lambda^h, a, b, \gamma) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^\tau \times \mathbb{R}^\tau \times \mathbb{R}$ such that the classical KKT condition holds:

$$
(4) \quad
\begin{cases}
\nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum\limits_{i=1}^{J} \nabla G_i(z^*)a_i + \sum\limits_{i=1}^{J} \nabla H_i(z^*)b_i \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \gamma \nabla(G^T H)(z^*) = 0, \\
\lambda^g \geq 0, \ g(z^*)^T \lambda^g = 0, \\
G_i(z^*) \in \mathcal{K}_i, \ -a_i \in \mathcal{K}_i, \ G_i(z^*)^T a_i = 0, \qquad i = 1, \ldots, J, \\
H_i(z^*) \in \mathcal{K}_i, \ -b_i \in \mathcal{K}_i, \ H_i(z^*)^T b_i = 0, \qquad i = 1, \ldots, J, \\
\gamma \geq 0.
\end{cases}
$$

We now show the equivalence of the classical KKT condition (4) and the K-stationary condition (3).

Let $\lambda^G := a + \gamma H(z^*)$ and $\lambda^H := b + \gamma G(z^*)$. We first show that $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ satisfies (3). Consider the following cases.

- $i \in I_G \cap I_H^+$. Then $G_i(z^*) = 0, H_i(z^*) \in \operatorname{int}\mathcal{K}_i$. By (4), $b_i = 0$ and hence $\lambda_i^H = b_i + \gamma G_i(z^*) = 0$.

- $i \in I_G^+ \cap I_H$. Similarly to Case 1, we can show that $\lambda_i^G = 0$.
- $i \in B_G \cap B_H$. Then $H_i(z^*), G_i(z^*) \in bd\mathcal{K}_i \backslash \{0\}$ and $H_i(z^*) \perp G_i(z^*)$. Since $-a_i \perp G_i(z^*)$ and $-a_i \in \mathcal{K}_i$ by (4), then $-a_i \in \mathbb{R}_+ \widehat{G}_i(z^*)$. Similarly, $-b_i \in \mathbb{R}_+ \widehat{H}_i(z^*)$ by $-b_i \perp H_i(z^*)$ and $-b_i \in \mathcal{K}_i$. It follows from Proposition 2.2 that $H_i(z^*) \in \mathbb{R}_{++} \widehat{G}_i(z^*)$ and $G_i(z^*) \in \mathbb{R}_{++} \widehat{H}_i(z^*)$. So $\lambda_i^G = a_i + \gamma H_i(z^*) \in \mathbb{R}\widehat{G}_i(z^*)$ and $\lambda_i^H = b_i + \gamma G_i(z^*) \in \mathbb{R}\widehat{H}_i(z^*)$.
- $i \in I_G \cap B_H$. Then $G_i(z^*) = 0, H_i(z^*) \in bd\mathcal{K}_i \backslash \{0\}$. Since $-a_i, -b_i \in \mathcal{K}_i$ and $-b_i \perp H_i(z^*)$ by (4), then $-b_i \in \mathbb{R}_+ \widehat{H}_i(z^*)$. Hence $\lambda_i^G = a_i + \gamma H_i(z^*) \in -\mathcal{K}_i + \mathbb{R}_+ H_i(z^*)$ and $\lambda_i^H = b_i + \gamma G_i(z^*) = b_i \in \mathbb{R}_- \widehat{H}_i(z^*)$.
- $i \in B_G \cap I_H$. Similarly to the above case, we have $\lambda_i^G = a_i + \gamma H_i(z^*) = a_i \in \mathbb{R}_- \widehat{G}_i(z^*)$ and $\lambda_i^H = b_i + \gamma G_i(z^*) \in -\mathcal{K}_i + \mathbb{R}_+ G_i(z^*)$.
- $i \in I_G \cap I_H$. Then $G_i(z^*) = H_i(z^*) = 0$. By (4), we have $\lambda_i^G = a_i \in -\mathcal{K}_i$ and $\lambda_i^H = b_i \in -\mathcal{K}_i$.

Hence $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ satisfies (3).

Conversely, take $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ satisfying (3). Let $a := \lambda^G - \gamma H(z^*)$ and $b := \lambda^H - \gamma G(z^*)$, where $\gamma > 0$. We now show that $(\lambda^g, \lambda^h, a, b, \gamma)$ satisfies (4) if γ is sufficiently large. Consider the following cases.

- $i \in I_G \cap I_H^+$. Then $a_i = \lambda_i^G - \gamma H_i(z^*) = \gamma\big[\frac{\lambda_i^G}{\gamma} - H_i(z^*)\big]$ and $b_i = \lambda_i^H - \gamma G_i(z^*) = \lambda_i^H = 0$. Since $H_i(z^*) \in int\mathcal{K}_i$, $\lambda_i^G/\gamma - H_i(z^*) \in -\mathcal{K}_i$ when γ is sufficiently large. Hence $a_i \in -\mathcal{K}_i$ and $b_i = 0 \in -\mathcal{K}_i$.
- $i \in I_G^+ \cap I_H$. Similarly to Case 1, we can show that $a_i = 0 \in -\mathcal{K}_i$ and $b_i \in -\mathcal{K}_i$.
- $i \in B_G \cap B_H$. Then $G_i(z^*), H_i(z^*) \in bd\mathcal{K}_i \backslash \{0\}$. By Proposition 2.2, $G_i(z^*) = k\widehat{H}_i(z^*)$ for some $k > 0$, which in turn implies that $H_i(z^*) = \widehat{G}_i(z^*)/k$. By (3), $\lambda_i^G = t_1 \widehat{G}_i(z^*)$ and $\lambda_i^H = t_2 \widehat{H}_i(z^*)$ for some $t_1, t_2 \in \mathbb{R}$. Hence $-a_i = \gamma H_i(z^*) - \lambda_i^G = (\gamma/k - t_1)\widehat{G}_i(z^*) \in \mathcal{K}_i$ and $-b_i = \gamma G_i(z^*) - \lambda_i^H = (\gamma k - t_2)\widehat{H}_i(z^*) \in \mathcal{K}_i$, provided that γ is sufficiently large. In addition, $\langle a_i, G_i(z^*)\rangle = (t_1 - \gamma/k)\langle \widehat{G}_i(z^*), G_i(z^*)\rangle = 0$ and $\langle b_i, H_i(z^*)\rangle = (t_2 - \gamma k)\langle \widehat{H}_i(z^*), H_i(z^*)\rangle = 0$.
- $i \in I_G \cap B_H$. It follows from (3) that $\lambda_i^G = t_1 H_i(z^*) - \xi_i$ and $\lambda_i^H = -t_2 \widehat{H}_i(z^*)$ for some $t_1, t_2 \geq 0$ and $\xi_i \in \mathcal{K}_i$. Hence $-a_i = \gamma H_i(z^*) - \lambda_i^G = (\gamma - t_1)H_i(z^*) + \xi_i \in \mathcal{K}_i$ as $\gamma \geq t_1$. Similarly, $b_i = \lambda_i^H - \gamma G_i(z^*) = \lambda_i^H = -t_2 \widehat{H}_i(z^*) \in -\mathcal{K}_i$. In addition, $\langle a_i, G_i(z^*)\rangle = 0$ since $G_i(z^*) = 0$ and $\langle b_i, H_i(z^*)\rangle = \langle -t_2 \widehat{H}_i(z^*), H_i(z^*)\rangle = 0$.
- $i \in B_G \cap I_H$. The argument is similar to the above case.
- $i \in I_G \cap I_H$. Then $G_i(z^*) = H_i(z^*) = 0$ and $a_i = \lambda_i^G \in -\mathcal{K}_i$ and $b_i = \lambda_i^H \in -\mathcal{K}_i$.

Hence $(\lambda^g, \lambda^h, a, b, \gamma)$ satisfies (4).   □

4. S-stationary conditions. For MPCC, it is known (see Ye [21, Theorem 3.2]) that the S-stationary condition is equivalent to the stationary condition derived by using the proximal normal cone of the complementarity set. In this vector case, the regular normal cone is the same as the proximal normal cone. For SDCM-PCC, it was shown that the regular normal cone is the same as the proximal normal cone, and the S-stationary condition is defined by using the proximal normal cone [6]. Similarly, in [25] it was verified that the regular normal cone coincides with the proximal normal cone for the second-order cone complementarity set, and hence we can define the S-stationary condition below. First we introduce the concept of

weak (W) stationary points. Note that when the dimension $m_i \leq 2$, the condition $(G_i(z^*))_1 \hat{\lambda}_i^G + (H_i(z^*))_1 \lambda_i^H \in \mathbb{R} G_i(z^*)$ is redundant and can be omitted.

DEFINITION 4.1 (W-stationary point). *Let z^* be a feasible solution of SOCM-PCC. We say that z^* is a weak stationary point of SOCMPCC if there exist a multiplier $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ such that*

$$
(5) \quad
\begin{cases}
\nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum\limits_{i=1}^{J} \nabla G_i(z^*)\lambda_i^G + \sum\limits_{i=1}^{J} \nabla H_i(z^*)\lambda_i^H = 0, \\
\lambda^g \geq 0, \ g(z^*)^T \lambda^g = 0, \\
\lambda_i^G \in \mathbb{R}^{m_i}, \ \lambda_i^H = 0 \quad if \ \ i \in I_G \cap I_H^+, \\
\lambda_i^G = 0, \ \lambda_i^H \in \mathbb{R}^{m_i} \quad if \ \ i \in I_G^+ \cap I_H, \\
\lambda_i^G \perp G_i(z^*), \ \lambda_i^H \perp H_i(z^*), \ (G_i(z^*))_1 \hat{\lambda}_i^G + (H_i(z^*))_1 \lambda_i^H \in \mathbb{R} G_i(z^*) \\
\hspace{7cm} if \ \ i \in B_G \cap B_H.
\end{cases}
$$

DEFINITION 4.2 (S-stationary point). *Let z^* be a feasible solution of SOCMPCC. We say that z^* is a strong stationary point of SOCMPCC if there exist a multiplier $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ such that*

$$
\begin{cases}
0 = \nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum\limits_{i=1}^{J} \nabla G_i(z^*)\lambda_i^G + \sum\limits_{i=1}^{J} \nabla H_i(z^*)\lambda_i^H, \\
\lambda^g \geq 0, \ g(z^*)^T \lambda^g = 0, \\
(\lambda_i^G, \lambda_i^H) \in \widehat{N}_{\Omega_i}(G_i(z^*), H_i(z^*)), \quad i = 1, \ldots, J,
\end{cases}
$$

or equivalently such that (5) and the following conditions hold:

$$
\begin{cases}
\lambda_i^H \in \mathbb{R}_- \widehat{H}_i(z^*), \ \langle \lambda_i^G, \widehat{H}_i(z^*) \rangle \leq 0 & if \ \ i \in I_G(z^*) \cap B_H(z^*), \\
\lambda_i^G \in \mathbb{R}_- \widehat{G}_i(z^*), \ \langle \lambda_i^H, \widehat{G}_i(z^*) \rangle \leq 0 & if \ \ i \in B_G(z^*) \cap I_H(z^*), \\
\lambda_i^G \in -\mathcal{K}_i, \ \lambda_i^H \in -\mathcal{K}_i & if \ \ i \in I_G(z^*) \cap I_H(z^*).
\end{cases}
$$

DEFINITION 4.3. *Let z^* be a feasible solution of SOCMPCC. We say that SOCMPCC-LICQ holds at z^*, provided that the gradient vectors*

$$
\nabla g_i(z^*)(i \in I_g(z^*)), \ \nabla h_i(z^*) \ (i = 1, \ldots, q),
$$
$$
\nabla G_i(z^*)(i \in I_G(z^*) \cup B_G(z^*)), \ \nabla H_i(z^*) \ (i \in I_H(z^*) \cup B_H(z^*))
$$

are linearly independent.

In the following theorem we show that under SOCMPCC-LICQ, a local optimal solution of SOCMPCC must be an S-stationary point.

THEOREM 4.4. *Let z^* be a local optimal solution of SOCMPCC. If SOCMPCC-LICQ holds at z^*, then z^* is an S-stationary point.*

Proof. Since z^* is a local optimal solution, it is also a local optimal solution of the problem with the same objective function and with the inactive constraints $g_i(z) < 0$, $i \notin I_g(z^*)$, $H_i(z) \in \text{int} \mathcal{K}_i$, $i \in I_1, G_i(z) \in \text{int} \mathcal{K}_i$, $i \in I_2$, deleted from the feasible region, i.e., z^* is a local optimal solution to the problem:

$$
\begin{aligned}
\min \ & f(z) \\
\text{s.t.} \ & h(z) = 0, \ g_i(z) \leq 0, \ \ i \in I_g(z^*), \\
& G_i(z) = 0, \ i \in I_1, \ H_i(z) = 0, \ i \in I_2, \\
& \mathcal{K}_i \ni G_i(z) \perp H_i(z) \in \mathcal{K}_i, \ \ i \in (I_1 \cup I_2)^c,
\end{aligned}
$$

where $I_1 := I_G(z^*) \cap I_H^+(z^*)$, $I_2 := I_G^+(z^*) \cap I_H(z^*)$. Then

$$0 \in \nabla f(z^*) + \widehat{N}_{\mathcal{F}}(z^*),$$

where $\mathcal{F} := \{z | F(z) \in D\}$ is the feasible region of the above problem with

$$F(z) := \big(h(z), g_{I_g}(z), G_{I_1}(z), H_{I_2}(z), G_{(I_1 \cup I_2)^c}(z), H_{(I_1 \cup I_2)^c}(z)\big),$$
$$D := \{0\} \times \mathbb{R}_-^{I_g} \times \{0\}^{I_1} \times \{0\}^{I_2} \times \Omega_{(I_1 \cup I_2)^c},$$

and

$$\Omega_{(I_1 \cup I_2)^c} := \{(u_i, v_i) | \mathcal{K}_i \ni u_i \perp v_i \in \mathcal{K}_i, \ i \in (I_1 \cup I_2)^c\}.$$

By SOCMPCC-LICQ, $\nabla F(z^*)$ has a full column rank. The desired result follows from Propositions 2.1 and 2.3 by letting $\lambda_i^g = 0$ for $i \notin I_g(z^*)$, $\lambda_i^H = 0$ for $i \in I_1$, and $\lambda_i^G = 0$ for $i \in I_2$, i.e., letting the multiplies corresponding to the deleted constraints be zero. $\qquad\square$

5. M-stationary conditions. In this section we study the M-stationary condition for SOCMPCC. For this purpose we rewrite the SOCMPCC as an optimization problem with a nonconvex cone constraint:

(M-SOCMPCC)
$$\min \ f(z)$$
$$\text{s.t.} \ \ h(z) = 0,$$
$$g(z) \le 0,$$
$$(G_i(z), H_i(z)) \in \Omega_i, \quad i = 1, \ldots, J,$$

where $\Omega_i := \{(x, y) | x \in \mathcal{K}_i, \ y \in \mathcal{K}_i, \ x \perp y\}$.

As in the MPCC case, we will show that the M-stationary condition introduced below is the KKT condition of M-SOCMPCC by using the limiting normal cone. Note that when the dimension $m_i \le 2$, the condition $\alpha_i \hat{\lambda}_i^G + (1 - \alpha_i)\lambda_i^H \in \mathbb{R}\xi_i$ for some $\alpha_i \in [0, 1]$ is redundant and can be omitted.

DEFINITION 5.1 (M-stationary point). *Let z^* be a feasible solution of SOCM-PCC. We say that z^* is an M-stationary point of SOCMPCC if there exist multipliers $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ such that*

$$\begin{cases} 0 = \nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum\limits_{i=1}^J \nabla G_i(z^*)\lambda_i^G + \sum\limits_{i=1}^J \nabla H_i(z^*)\lambda_i^H, \\ \lambda^g \ge 0, \ g(z^*)^T \lambda^g = 0, \\ (\lambda_i^G, \lambda_i^H) \in N_{\Omega_i}(G_i(z^*), H_i(z^*)), \quad i = 1, \ldots, J, \end{cases}$$

or equivalently such that (5) and the following conditions hold:

$$\begin{cases} \lambda_i^G \in \mathbb{R}^{m_i}, \ \lambda_i^H = 0 \ \text{ or } \ \lambda_i^G \perp \widehat{H}_i(z^*), \ \lambda_i^H \in \mathbb{R}\widehat{H}_i(z^*) \ \text{ or} \\ \lambda_i^H \in \mathbb{R}_- \widehat{H}_i(z^*), \ \langle \lambda_i^G, \widehat{H}_i(z^*) \rangle \le 0 & \text{if } i \in I_G \cap B_H, \\ \lambda_i^G = 0, \lambda_i^H \in \mathbb{R}^{m_i} \ \text{ or } \ \lambda_i^G \in \mathbb{R}\widehat{G}_i(z^*), \ \lambda_i^H \perp \widehat{G}_i(z^*) \ \text{ or} \\ \lambda_i^G \in \mathbb{R}_- \widehat{G}_i(z^*), \ \langle \lambda_i^H, \widehat{G}_i(z^*) \rangle \le 0 & \text{if } i \in B_G \cap I_H, \\ \lambda_i^G \in -\mathcal{K}_i, \lambda_i^H \in -\mathcal{K}_i \ \text{ or } \ \lambda_i^H = 0, \lambda_i^G \in \mathbb{R}^{m_i} \ \text{ or} \\ \lambda_i^G = 0, \lambda_i^H \in \mathbb{R}^{m_i} \ \text{ or} \\ \lambda_i^G \in \mathbb{R}_- \xi_i, \ \lambda_i^H \in \xi_i^\circ \ \text{ or } \ \lambda_i^H \in \mathbb{R}_- \xi_i, \ \lambda_i^G \in \xi_i^\circ \ \text{ or} \\ \lambda_i^G \perp \xi_i, \ \lambda_i^H \perp \hat{\xi}_i, \ \alpha_i \hat{\lambda}_i^G + (1 - \alpha_i)\lambda_i^H \in \mathbb{R}\xi_i \\ \text{for some } \alpha_i \in [0, 1] \ \text{and some } \xi_i \in C_i & \text{if } i \in I_G \cap I_H, \end{cases}$$

where $C_i := \{(1, w) \in \mathbb{R} \times \mathbb{R}^{m_i - 1} | \ \|w\| = 1\}$.

DEFINITION 5.2. *We say that M-SOCMPCC is Clarke calm at a feasible solution z^* if there exist positive ε and μ such that, for all $(r, s, p) \in \varepsilon B$ and for all $z \in B_\varepsilon(z^*) \cap \mathcal{F}_M(r, s, p)$, one has*

$$f(z) - f(z^*) + \mu\|(r, s, p)\| \geq 0,$$

where

$$\mathcal{F}_M(r, s, p) := \big\{z \mid h(z) + r = 0, g(z) + s \leq 0, (G_i(z), H_i(z)) + p_i \in \Omega_i, i = 1, \ldots, J\big\}.$$

THEOREM 5.3. *Let z^* be a local optimal solution of SOCMPCC. Suppose that the problem M-SOCMPCC is Clarke calm at z^*. Then z^* is an M-stationary point of SOCMPCC.*

Proof. By [6, Theorem 2.2], there exists a multiplier $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ such that

$$0 = \nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^{J} \nabla G_i(z^*)\lambda_i^G + \sum_{i=1}^{J} \nabla H_i(z^*)\lambda_i^H,$$

$$\lambda^g \geq 0, \ g(z^*)^T\lambda^g = 0, \ (\lambda_i^G, \lambda_i^H) \in N_{\Omega_i}(G_i(z^*), H_i(z^*)), \ i = 1, \ldots, J,$$

and so the desired result follows from using the expression of the limiting normal cone in Proposition 2.4. □

DEFINITION 5.4. *Let z^* be a feasible solution of SOCMPCC. We say that the constraint system of M-SOCMPCC has a local error bound at z^* if there exist $\mu, \varepsilon > 0$ such that*

$$dist(z, \mathcal{F}_M(0, 0, 0)) \leq \mu\|(r, s, p)\| \quad \forall (r, s, p) \in \varepsilon B \text{ and } z \in \mathcal{F}_M(r, s, p) \cap B_\varepsilon(z^*).$$

Note that the constraint system of M-SOCMPCC has a local error bound at z^* if and only if the set-valued mapping $\mathcal{F}_M(r, s, p)$ is calm [16] (or pseudo-upper-Lipschitz continuous using the terminology of [24]) around $(0, 0, 0, z^*)$. Hence $\mathcal{F}_M(r, s, p)$ being either pseudo-Lipschitz continuous [1] around $(0, 0, 0, z^*)$ or upper-Lipschitz continuous [15] at z^* implies that the constraint system of M-SOCMPCC has a local error bound at z^*.

The proposition below is an easy consequence of Clarke's exact penalty principle [4, Proposition 2.4.3] and the calmness of the constraint system. See [22, Proposition 4.2] for a proof.

PROPOSITION 5.5. *If the objective function is Lipschitz near z^* and $\mathcal{F}_M(r, s, p)$ is calm at $(0, 0, 0, z^*)$, then the problem M-SOCMPCC is calm at z^*.*

DEFINITION 5.6 (SOCMPCC-NNAMCQ). *Let z^* be a local optimal solution of SOCMPCC. We say that SOCMPCC-No Nonzero Abnormal Multiplier Constraint Qualification (SOCMPCC-NNAMCQ) holds at z^* if there is no nonzero vector $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ such that the following conditions hold:*

$$\begin{cases} 0 = \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^{J} \nabla G_i(z^*)\lambda_i^G + \sum_{i=1}^{J} \nabla H_i(z^*)\lambda_i^H, \\ \lambda^g \geq 0, \ g(z^*)^T\lambda^g = 0, \\ (\lambda_i^G, \lambda_i^H) \in N_{\Omega_i}(G_i(z^*), H_i(z^*)), \quad i = 1, \ldots, J. \end{cases}$$

THEOREM 5.7. *Let z^* be a local optimal solution of SOCMPCC. Then z^* is an M-stationary point under one of the following constraint qualifications:*

  (i) *The SOCMPCC-NNAMCQ holds at z^*.*

  (ii) *All mappings h, g, G, H are affine and $m_i \leq 2$ for $i = 1, \ldots, J$.*

Proof. By Theorem 5.3 and Proposition 5.5, it suffices to show the calmness of $\mathcal{F}_M$.

(i) Similarly to [22, Theorem 4.4], we can show that under SOCMPCC-NNAMCQ, the constraint system of M-SOCMPCC is pseudo-Lipschitz continuous around $(0, 0, 0, z^*)$ and hence has a local error bound at z^*.

(ii) Since, when $m_i \leq 2$, the second-order cone $\mathcal{K}_i$ is polyhedral, hence the second-order cone complementarity set Ω_i is a union of finitely many polyhedral convex sets. Since all mappings h, g, G, H are affine, the graph of the set-valued mapping $\mathcal{F}_M$ is a union of polyhedral convex sets, and hence $\mathcal{F}_M$ is a polyhedral set-valued mapping. By [15, Proposition 1], $\mathcal{F}_M$ is upper-Lipschitz, and hence the local error bound condition holds at z^*. ∎

6. C-stationary conditions. In this section, we consider the C-stationary condition by reformulating SOCMPCC as a nonsmooth problem:

(C-SOCMPCC)
$$\begin{aligned} \min \;\; & f(z) \\ \text{s.t.} \;\; & h(z) = 0, \\ & g(z) \leq 0, \\ & G_i(z) - \Pi_{\mathcal{K}_i}\big(G_i(z) - H_i(z)\big) = 0, \;\; i = 1, \ldots, J. \end{aligned}$$

As in the MPCC case, the C-stationary condition introduced below is the nonsmooth KKT condition of C-SOCMPCC by using the Clarke generalized gradient.

DEFINITION 6.1 (C-stationary point). *Let z^* be a feasible solution of SOCMPCC. We say that z^* is a C-stationary point of SOCMPCC if there exists a multiplier $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ such that (5) and the following conditions hold:*

$$\begin{cases} \lambda_i^H \in \mathbb{R}\widehat{H}_i(z^*) & \textit{if } \;\; i \in I_G \cap B_H, \\ \lambda_i^G \in \mathbb{R}\widehat{G}_i(z^*) & \textit{if } \;\; i \in B_G \cap I_H, \\ \langle \lambda_i^G, \lambda_i^H \rangle \geq 0 & \forall \; i = 1, \ldots, J. \end{cases}$$

We present the first-order optimality condition of SOCMPCC in terms of C-stationary conditions in the following result.

DEFINITION 6.2. *We say that C-SOCMPCC is Clarke calm at a feasible solution z^* if there exist positive ε and μ such that, for all $(r, s, \alpha) \in \varepsilon B$ and for all $z \in B_\varepsilon(z^*) \cap \mathcal{F}_C(r, s, \alpha)$, one has*

$$f(z) - f(z^*) + \mu \|(r, s, \alpha)\| \geq 0,$$

where

$$\mathcal{F}_C(r, s, \alpha)$$
$$:= \big\{z \,|\, h(z) + r = 0, g(z) + s \leq 0, G_i(z) - \Pi_{\mathcal{K}_i}\big(G_i(z) - H_i(z)\big) + \alpha_i = 0, i = 1, \ldots, J\big\}.$$

THEOREM 6.3. *Let z^* be a local optimal solution of SOCMPCC. Suppose that the problem C-SOCMPCC is Clarke calm at z^*. Then z^* is a C-stationary point of SOCMPCC.*

Proof. Since the problem is calm, by the Clarke nonsmooth KKT condition (see [4, Proposition 6.4.4]), there exist $\lambda^h \in \mathbb{R}^q$, $\lambda^g \in \mathbb{R}^p$, and $\beta_i \in \mathbb{R}^{m_i}$ $(i = 1, \ldots, J)$ such that

(6) $$0 \in \partial_z^c L(z^*, \lambda^h, \lambda^g, \beta), \quad \lambda^g \geq 0, \quad \text{and} \quad \langle \lambda^g, g(z^*) \rangle = 0,$$

where ∂_z^c denotes the Clarke generalized gradient with respect to z and

$$L(z, \lambda^h, \lambda^g, \beta) := f(z) + \langle \lambda^h, h(z) \rangle + \langle \lambda^g, g(z) \rangle + \sum_{i=1}^{J} \langle \beta_i, G_i(z) - \Pi_{\mathcal{K}_i}(G_i(z) - H_i(z)) \rangle.$$

Consider the Clarke generalized gradient of the nonsmooth function

$$S_i(z) := \langle \beta_i, \Pi_{\mathcal{K}_i}(G_i(z) - H_i(z)) \rangle.$$

Applying the Jacobian chain rule [4, Theorem 2.6.6] yields

$$\partial^c S_i(z^*) \subseteq \beta_i^T \partial^c \Pi_{\mathcal{K}_i}(G_i(z^*) - H_i(z^*))(\mathcal{J}G_i(z^*) - \mathcal{J}H_i(z^*)).$$

Therefore, since any element of the Clarke generalized gradient of the metric projection operator to a closed convex set is self-adjoint (see, e.g., [10, Proposition 1(a)]), we know from (6) that there exists $A_i \in \partial^c \Pi_{\mathcal{K}_i}(G_i(z^*) - H_i(z^*))$ such that

$$\nabla f(z^*) + \nabla h(z^*)\lambda^h + \nabla g(z^*)\lambda^g$$

(7) $$+ \sum_{i=1}^{J} \nabla G_i(z^*)\beta_i - \sum_{i=1}^{J} (\nabla G_i(z^*) - \nabla H_i(z^*))A_i\beta_i = 0.$$

Define $\lambda_i^G := \beta_i - A_i\beta_i$ and $\lambda_i^H := A_i\beta_i$. Then it follows from (6) and (7) that

$$0 = \nabla f(z^*) + \nabla h(z^*)\lambda^h + \nabla g(z^*)\lambda^g + \sum_{i=1}^{J} \nabla G_i(z^*)\lambda_i^G + \sum_{i=1}^{J} \nabla H_i(z^*)\lambda_i^H,$$

$$\lambda^g \geq 0, \ \langle \lambda^g, g(z^*) \rangle = 0.$$

We now continue to show that (5) holds. Notice that for $i \in (I_G \cap I_H^+) \cup (I_G^+ \cap I_H) \cup (B_G \cap B_H)$, $\Pi_{\mathcal{K}_i}(\cdot)$ is continuously differentiable at $G_i(z^*) - H_i(z^*)$. Hence $A_i = \mathcal{J}\Pi_{\mathcal{K}_i}(G_i(z^*) - H_i(z^*))$. Since

$$\lambda_i^H = A_i\beta_i = A_i\Big[(I_i - A_i)\beta_i + A_i\beta_i\Big] = A_i(\lambda_i^G + \lambda_i^H),$$

where I_i denotes the m_i-dimensional identity matrix, it follows that

$$-\lambda_i^H = \widehat{D}^* \Pi_{\mathcal{K}_i}(G_i(z^*) - H_i(z^*))(-\lambda_i^G - \lambda_i^H).$$

Hence $(\lambda_i^G, \lambda_i^H) \in \widehat{N}_{\Omega_i}(G_i(z^*), H_i(z^*))$ by [25, Proposition 2.1] for $i \in (I_G \cap I_H^+) \cup (I_G^+ \cap I_H) \cup (B_G \cap B_H)$. Using the regular normal cone formula given in Proposition 2.3 yields

$$\begin{cases} \lambda_i^G \in \mathbb{R}^{m_i}, \lambda_i^H = 0 \ \text{ if } \ i \in I_G \cap I_H^+, \\ \lambda_i^G = 0, \ \lambda_i^H \in \mathbb{R}^{m_i} \ \text{ if } \ i \in I_G^+ \cap I_H, \\ \lambda_i^G \perp G_i(z^*), \ \lambda_i^H \perp H_i(z^*), \ (G_i(z^*))_1 \hat{\lambda}_i^G + (H_i(z^*))_1 \lambda_i^H \in \mathbb{R}G_i(z^*) \\ \hspace{6cm} \text{if } \ i \in B_G \cap B_H, \end{cases}$$

which implies that (5) holds. Now consider the case where $i \in I_G \cap B_H$. In this case $G_i(z^*) - H_i(z^*) = -H_i(z^*) \in -bd\mathcal{K}_i \setminus \{0\}$, and hence by [14, Lemma 1(iii)] we have

$$A_i \in co\partial_B \Pi_{\mathcal{K}_i}\big(G_i(z^*) - H_i(z^*)\big) = co\left\{O, \frac{1}{2\big(H_i(z^*)\big)_1^2}\widehat{H}_i(z^*)\widehat{H}_i(z^*)^T\right\},$$

which from $\lambda_i^H = A_i\beta_i$ implies $\lambda_i^H \in \mathbb{R}\widehat{H}_i(x^*)$. In the case where $i \in B_G \cap I_H$, $G_i(z^*) - H_i(z^*) = G_i(z^*) \in bd\mathcal{K}_i \setminus \{0\}$, and hence by [14, Lemma 1(ii)] we have

$$A_i \in co\partial_B \Pi_{\mathcal{K}_i}\big(G_i(z^*) - H_i(z^*)\big) = co\left\{I, I - \frac{1}{2\big(G_i(z^*)\big)_1^2}\widehat{G}_i(z^*)\widehat{G}_i(z^*)^T\right\}.$$

It follows from $\lambda_i^G = (I_i - A_i)\beta_i$ that $\lambda_i^G \in \mathbb{R}\widehat{G}_i(z^*)$. Moreover, from [10, Proposition 1(c)], we know that

$$\langle A_i\beta_i, \beta_i - A_i\beta_i \rangle \geq 0,$$

which implies that $\langle \lambda_i^G, \lambda_i^H \rangle \geq 0$ for all $i = 1, \ldots, J$. The proof of the theorem is complete. $\qquad\square$

7. Connections between various stationary points. In this section, we discuss the relationships among various stationary points and the Clarke calmness conditions for various reformulations given in the previous sections. First, we give the following result.

PROPOSITION 7.1. *Let (x, y) be in the m-dimensional second-order cone complementarity set Ω. Then*

(8)
$$\left(N_{\mathcal{K}}(x) + \mathbb{R}_+ y, N_{\mathcal{K}}(y) + \mathbb{R}_+ x\right)$$
$$= \begin{cases} (\mathbb{R}^m, 0) & if \quad x = 0, \ y \in int\mathcal{K}, \\ (0, \mathbb{R}^m) & if \quad x \in int\mathcal{K}, \ y = 0, \\ (\mathbb{R}\hat{x}, \mathbb{R}\hat{y}) & if \quad x, y \in bd\mathcal{K}\setminus\{0\}, \ x^T y = 0, \\ (-\mathcal{K} + \mathbb{R}_+ y, \mathbb{R}_-\hat{y}) & if \quad x = 0, \ y \in bd\mathcal{K}\setminus\{0\}, \\ (\mathbb{R}_-\hat{x}, -\mathcal{K} + \mathbb{R}_+ x) & if \quad x \in bd\mathcal{K}\setminus\{0\}, \ y = 0, \\ (-\mathcal{K}, -\mathcal{K}) & if \quad x = 0, \ y = 0, \end{cases}$$

and

(9)
$$\left(N_{\mathcal{K}}(x) + \mathbb{R}_+ y, N_{\mathcal{K}}(y) + \mathbb{R}_+ x\right) \subset \widehat{N}_\Omega(x, y).$$

Proof. We will prove (8) and (9) simultaneously.
Consider the following cases.

- Let $x = 0$ and $y \in int\mathcal{K}$. For any $z \in \mathbb{R}^m$, since $y \in int\mathcal{K}$, there exists $t > 0$ such that $y - tz \in \mathcal{K}$. Hence $z \in \frac{-\mathcal{K}+y}{t} \subset -\mathcal{K} + \mathbb{R}_+ y$. Because $z \in \mathbb{R}^m$ is arbitrary, we have $-\mathcal{K} + \mathbb{R}_+ y = \mathbb{R}^m$. Thus

$$\left(N_{\mathcal{K}}(x) + \mathbb{R}_+ y, N_{\mathcal{K}}(y) + \mathbb{R}_+ x\right) = (-\mathcal{K} + \mathbb{R}_+ y, 0) = (\mathbb{R}^m, 0) = \widehat{N}_\Omega(x, y).$$

- Let $x \in int\mathcal{K}$ and $y = 0$. Then, similar to the above case, we can show that

$$\left(N_{\mathcal{K}}(x) + \mathbb{R}_+ y, N_{\mathcal{K}}(y) + \mathbb{R}_+ x\right) = (0, -\mathcal{K} + \mathbb{R}_+ x) = (0, \mathbb{R}^m) = \widehat{N}_\Omega(x, y).$$

- Let $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and $x^T y = 0$. It follows from Proposition 2.2 that $y \in \mathbb{R}_+ \hat{x}$ and $x \in \mathbb{R}_+ \hat{y}$. Note that $N_\mathcal{K}(x) = \mathbb{R}_- \hat{x}$ and $N_\mathcal{K}(y) = \mathbb{R}_- \hat{y}$. This implies

  $$(10) \quad \big(N_\mathcal{K}(x) + \mathbb{R}_+ y, N_\mathcal{K}(y) + \mathbb{R}_+ x\big) = (\mathbb{R}_- \hat{x} + \mathbb{R}_+ \hat{x}, \mathbb{R}_- \hat{y} + \mathbb{R}_+ \hat{y}) = (\mathbb{R}\hat{x}, \mathbb{R}\hat{y}),$$

  since $\mathbb{R} = \mathbb{R}_- + \mathbb{R}_+$. For $(u, v) \in (\mathbb{R}\hat{x}, \mathbb{R}\hat{y})$, we have $u \perp x$, $v \perp y$, and $x_1 \hat{u} + y_1 v \in \mathbb{R}x$ due to $\hat{y} \in \mathbb{R}_+ x$. Comparing the formula given in (10) and Proposition 2.3 yields

  $$\big(N_\mathcal{K}(x) + \mathbb{R}_+ y, N_\mathcal{K}(y) + \mathbb{R}_+ x\big) \subset \widehat{N}_\Omega(x, y).$$

- Let $x = 0$ and $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Then by Proposition 2.3 we have

  $$(11) \qquad \big(N_\mathcal{K}(x) + \mathbb{R}_+ y, N_\mathcal{K}(y) + \mathbb{R}_+ x\big) = (-\mathcal{K} + \mathbb{R}_+ y, \mathbb{R}_- \hat{y}) \subset \widehat{N}_\Omega(x, y),$$

  since $\langle -w + \beta y, \hat{y}\rangle = \langle -w, \hat{y}\rangle \le 0$ for all $w \in \mathcal{K}$ and $\beta \in \mathbb{R}_+$.

- Let $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and $y = 0$. Similarly as in the above case we can show that

  $$(N_\mathcal{K}(x) + \mathbb{R}_+ y, N_\mathcal{K}(y) + \mathbb{R}_+ x) = (\mathbb{R}_- \hat{x}, -\mathcal{K} + \mathbb{R}_+ x) \subset \widehat{N}_\Omega(x, y).$$

- Let $(x, y) = (0, 0)$. Then by Proposition 2.3 we have

  $$(N_\mathcal{K}(x) + \mathbb{R}_+ y, N_\mathcal{K}(y) + \mathbb{R}_+ x) = (-\mathcal{K}, -\mathcal{K}) = \widehat{N}_\Omega(x, y). \qquad \square$$

Comparing Proposition 7.1 and (3) leads to the following expression of the K-stationary condition immediately.

COROLLARY 7.2. *A feasible solution z^* is a K-stationary point of SOCMPCC if and only if there exists a multiplier $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ such that*

$$\begin{cases} \nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^J \nabla G_i(z^*)\lambda_i^G + \sum_{i=1}^J \nabla H_i(z^*)\lambda_i^H = 0, \\ \lambda^g \ge 0, \ g(z^*)^T \lambda^g = 0, \\ (\lambda_i^G, \lambda_i^H) \in \Big(N_{\mathcal{K}_i}(G_i(z^*)) + \mathbb{R}_+ H_i(z^*), N_{\mathcal{K}_i}(H_i(z^*)) + \mathbb{R}_+ G_i(z^*)\Big), \quad i = 1, \ldots, J. \end{cases}$$

In the following proposition we show that (9) becomes an equality when the dimension of $\mathcal{K}$ is less than or equal to 2.

PROPOSITION 7.3. *If $\mathcal{K}$ is the m-dimensional second-order cone with $m \le 2$, then for $(x, y) \in \Omega$,*

$$\left(N_\mathcal{K}(x) + \mathbb{R}_+ y, N_\mathcal{K}(y) + \mathbb{R}_+ x\right) = \widehat{N}_\Omega(x, y).$$

Proof. If $m = 1$, then the possible cases are $x = 0, y \in \mathrm{int}\mathcal{K}$ or $x \in \mathrm{int}\mathcal{K}, y = 0$ or $x = y = 0$. In these three cases, according to (8) and the formula of the regular normal cone given in Proposition 2.3 we have

$$\left(N_\mathcal{K}(x) + \mathbb{R}_+ y, N_\mathcal{K}(y) + \mathbb{R}_+ x\right) = \widehat{N}_\Omega(x, y).$$

If $m = 2$, according to the proof of Proposition 7.1 it only needs to show

$$\left(N_\mathcal{K}(x) + \mathbb{R}_+ y, N_\mathcal{K}(y) + \mathbb{R}_+ x\right) \supset \widehat{N}_\Omega(x, y)$$

for $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ or $x = 0, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ or $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}, y = 0$.

- Let $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Take $(u, v) \in \widehat{N}_\Omega(x, y)$. Then it follows from $u \perp x$ and $v \perp y$ that $u \in \mathbb{R}\hat{x}, v \in \mathbb{R}\hat{y}$. Since $\big(N_\mathcal{K}(x) + \mathbb{R}_+ y, N_\mathcal{K}(y) + \mathbb{R}_+ x\big) = (\mathbb{R}\hat{x}, \mathbb{R}\hat{y})$ by (10), then

$$\left(N_\mathcal{K}(x) + \mathbb{R}_+ y, N_\mathcal{K}(y) + \mathbb{R}_+ x \right) \supset \widehat{N}_\Omega(x, y).$$

- Let $x = 0$ and $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. According to Proposition 2.3 and (11), it suffices to show that $\hat{y}^\circ \subset -\mathcal{K} + \mathbb{R}_+ y$. Let $u \in \hat{y}^\circ$, i.e., $u_1 y_1 - u_2 y_2 \le 0$. Since $y_1 = |y_2|$ due to the assumption that $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, consider the following two cases. If $y_1 = y_2$, then $u_1 \le u_2$. Let $t > 0$ be sufficiently large so that $ty_1 - u_2 \ge 0$. Then $ty_1 - u_1 \ge ty_1 - u_2 = |ty_1 - u_2| = |ty_2 - u_2|$. This means $ty - u \in \mathcal{K}$, i.e., $u \in -\mathcal{K} + \mathbb{R}_+ y$. If $y_1 = -y_2$, then $u_1 + u_2 \le 0$. Let $t > 0$ be sufficiently large so that $ty_1 + u_2 \ge 0$. Then $ty_1 - u_1 \ge ty_1 + u_2 = |ty_1 + u_2| = |-ty_2 + u_2| = |ty_2 - u_2|$. Hence $ty - u \in \mathcal{K}$, i.e., $u \in -\mathcal{K} + \mathbb{R}_+ y$. In both cases, we have shown that $\hat{y}^\circ \subset -\mathcal{K} + \mathbb{R}_+ y$.

- Let $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and $y = 0$. The proof is similar to the above case. □

The following result follows from Propositions 7.1 and 7.3 and Corollary 7.2.

COROLLARY 7.4. *A K-stationary point is an S-stationary point. Moreover if the dimension of every $\mathcal{K}_i$ is less than or equal to 2, then an S-stationary point is a K-stationary point.*

It is well known that the KKT conditions and the S-stationary conditions are equivalent for MPCC. However, for SOCMPCC, according to Proposition 7.3 and Example 1 below, this equivalence holds only for the case where all $m_i \le 2$ but may fail to hold as $m_i \ge 3$ for some $i \in 1, \ldots, J$. Since the S-stationary point is defined in terms of the regular normal cone and the M-stationary point is defined in terms of the limiting normal cone, it is obvious that an S-stationary point must be an M-stationary point. However, unlike MPCC, it is not so easy to see that an M-stationary point must be an C-stationary point. We now verify this implication.

THEOREM 7.5. *An M-stationary point must be a C-stationary point.*

Proof. It suffices to show that for every $(u, v) \in N_\Omega(x, y)$, one has $\langle u, v \rangle \ge 0$. The cases where $x = 0$, $y \in \mathrm{int}\mathcal{K}$ or $x \in \mathrm{int}\mathcal{K}$, $y = 0$ or $x = 0$, $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ or $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, $y = 0$ are clear. It suffices to prove for the cases where $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and where $x = y = 0$. Let $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Then by Proposition 2.4, $x_1 \hat{u} + y_1 v = \beta x$ for some $\beta \in \mathbb{R}$ and $u \perp x$. Since $y_1 = \|y_2\| \neq 0$, it follows that $v = \frac{\beta x - x_1 \hat{u}}{y_1}$. Hence

$$\langle u, v \rangle = \frac{1}{y_1}\langle u, \beta x - x_1 \hat{u} \rangle = -\frac{x_1}{y_1}\langle u, \hat{u} \rangle = -\frac{x_1}{y_1}(u_1^2 - \|u_2\|^2) \ge 0,$$

where the last inequality follows from the fact that $|u_1| \le \|u_2\|$ since $u_1 = -\bar{x}_2^T u_2$ due to $x \perp u$. Now consider the case where $x = y = 0$. In this case, we only need to consider the case where there exists $\alpha \in [0, 1]$, $\beta \in \mathbb{R}$, and $\xi \in C$ such that $\alpha \hat{u} + (1 - \alpha)v = \beta \xi$, $u \perp \xi$, $v \perp \hat{\xi}$. If $\alpha = 0$, then $v = \beta \xi$ and hence $u \perp v$. If $\alpha = 1$, then $\hat{u} = \beta \xi$, i.e., $u = \beta \hat{\xi}$, which in turn implies $u \perp v$. If $\alpha \in (0, 1)$, then $\alpha \hat{u} + (1 - \alpha)v = \beta \xi$ implies that $v = \frac{\beta \xi - \alpha \hat{u}}{1 - \alpha}$. Hence

$$\langle u, v \rangle = \frac{1}{1 - \alpha}\langle u, \beta \xi - \alpha \hat{u} \rangle = -\frac{\alpha}{1 - \alpha}\langle u, \hat{u} \rangle = -\frac{\alpha}{1 - \alpha}(u_1^2 - \|u_2\|^2) \ge 0,$$

where the last inequality follows from the fact that $u \perp \xi$ and $\xi = (1, w)$ with $\|w\| = 1$. □

We can now summarize the relation between various stationary points as follows:

$$\text{K-stationary point} \implies \text{S-stationary point} \implies \text{M-stationary point}$$
$$\implies \text{C-stationary point} \implies \text{W-stationary point.}$$

The following examples show that the reverse relationships between various stationary points may not hold in general. In all examples, we use the notation $\mathcal{K}^m$ to denote the m-dimensional second-order cone.

Example 1 (S-stationary but not K-stationary). Consider the SOCMPCC

$$\min \; f(z) := z_1^2 + z_2^2 - z_3$$
$$\text{s.t.} \; g(z) := z_3^2 \leq 0,$$
$$G(z) := \begin{pmatrix} -z_1 + 1 \\ z_2 + 1 \\ z_3 \end{pmatrix} \in \mathcal{K}^3,$$
$$H(z) := \begin{pmatrix} z_1 \\ z_2 \\ -z_3 \end{pmatrix} \in \mathcal{K}^3,$$
$$G(z) \perp H(z).$$

It is obvious that the optimal solution is $z^* = (0,0,0)$. The index sets are all empty except for $I_g(z^*)$, $B_G(z^*)$, $I_H(z^*)$. Hence the S-stationary condition is

(12)
$$\begin{cases} \nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla G(z^*)\lambda^G + \nabla H(z^*)\lambda^H = 0, \\ \lambda^g \geq 0, \; g(z^*)^T \lambda^g = 0, \\ \lambda^G \in \mathbb{R}_- \widehat{G}(z^*), \; \langle \lambda^H, \widehat{G}(z^*) \rangle \leq 0, \end{cases}$$

and the K-stationary condition is

(13)
$$\begin{cases} \nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla G(z^*)\lambda^G + \nabla H(z^*)\lambda^H = 0, \\ \lambda^g \geq 0, \; g(z^*)^T \lambda^g = 0, \\ \lambda^G \in \mathbb{R}_- \widehat{G}(z^*), \; \lambda^H \in -\mathcal{K}^3 + \mathbb{R}_+ G(z^*). \end{cases}$$

Take $\lambda^G = (-1, 1, 0)$, $\lambda^H = (-1, -1, -1)$. Then the first condition in the S-stationary condition (12) holds:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \lambda^g \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_1^G \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2^G \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3^G \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$+ \lambda_1^H \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2^H \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3^H \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Moreover $\lambda^G = -1(1, -1, 0) \in \mathbb{R}_- \widehat{G}(z^*)$ and $\langle \lambda^H, \widehat{G}(z^*) \rangle = 0$, and so the third condition in the S-stationary condition (12) holds. However, z^* is not a K-stationary point. In fact, let $\lambda^G \in \mathbb{R}_- \widehat{G}(z^*)$, i.e., $\lambda^G = (\lambda_1^G, \lambda_2^G, \lambda_3^G) = t(1, -1, 0)$ for some $t \leq 0$; then $\lambda^H = (\lambda_1^H, \lambda_2^H, \lambda_3^H) = (t, t, -1)$ by the first condition in the K-stationary condition (13). But $-\lambda^H + \eta G(z^*) = (-t, -t, 1) + \eta(1, 1, 0) \notin \mathcal{K}^3$ for all $\eta \geq 0$, which means $\lambda^H \notin -\mathcal{K}^3 + \mathbb{R}_+ G(z^*)$. Hence z^* is not a K-stationary point. This example demonstrates that the K-stationary point and S-stationary point may be different when the dimension of one of the second-order cones is more than 2.

Example 2 (M-stationary but not S-stationary). Consider the SOCMPCC

$$\min \ f(z) := -z_1 + z_2^2$$

$$\text{s.t.} \ \ G(z) := \left(\begin{array}{c} z_1 + 1 \\ z_2 + 1 \end{array} \right) \in \mathcal{K}^2,$$

$$H(z) := \left(\begin{array}{c} z_1 \\ z_2 \end{array} \right) \in \mathcal{K}^2,$$

$$G(z) \perp H(z).$$

The optimal solution is $z^* = (0,0)$. The index sets are all empty except for $B_G(z^*), I_H(z^*)$. Hence the M-stationary condition is

(14)
$$\begin{cases} \nabla f(z^*) + \nabla G(z^*) \lambda^G + \nabla H(z^*) \lambda^H = 0, \\ \lambda^G = 0, \lambda^H \in \mathbb{R}^2 \text{ or } \lambda^G \in \mathbb{R}\widehat{G}(z^*), \lambda^H \perp \widehat{G}(z^*) \\ \text{or } \lambda^G \in \mathbb{R}_-\widehat{G}(z^*), \ \langle \lambda^H, \widehat{G}(z^*) \rangle \le 0, \end{cases}$$

and the S-stationary condition is

$$\begin{cases} \nabla f(z^*) + \nabla G(z^*) \lambda^G + \nabla H(z^*) \lambda^H = 0, \\ \lambda^G \in \mathbb{R}_-\widehat{G}(z^*), \ \langle \lambda^H, \widehat{G}(z^*) \rangle \le 0. \end{cases}$$

Since G, H are affine and $m = 2$, z^* must be an M-stationary point by Theorem 5.7. In fact, let $\lambda^G = (1/2, -1/2)$ and $\lambda^H = (1/2, 1/2)$. Then the first condition in the M-stationary condition (14) holds:

(15)
$$\left(\begin{array}{c} 0 \\ 0 \end{array} \right) = \left(\begin{array}{c} -1 \\ 0 \end{array} \right) + \lambda_1^G \left(\begin{array}{c} 1 \\ 0 \end{array} \right) + \lambda_2^G \left(\begin{array}{c} 0 \\ 1 \end{array} \right) + \lambda_1^H \left(\begin{array}{c} 1 \\ 0 \end{array} \right) + \lambda_2^H \left(\begin{array}{c} 0 \\ 1 \end{array} \right)$$

and the second condition in the M-stationary condition (14) holds:

$$\lambda^G = (1/2, -1/2) \in \mathbb{R}\widehat{G}(z^*) \text{ and } \lambda^H = (1/2, 1/2) \perp \widehat{G}(z^*).$$

Hence the M-stationary condition holds. However, z^* is not an S-stationary point. If (λ^G, λ^H) satisfies (15) with $\lambda^G \in \mathbb{R}_-(1, -1)$ and $\langle \lambda^H, (1, -1) \rangle \le 0$, then $\lambda_1^G + \lambda_1^H = 1$, $\lambda_2^G + \lambda_2^H = 0$, $\lambda_1^G = -\lambda_2^G \le 0$, and $\lambda_1^H \le \lambda_2^H$. So $\lambda_1^G = 1 - \lambda_1^H \ge 1 - \lambda_2^H = 1 + \lambda_2^G = 1 - \lambda_1^G$. This implies $\lambda_1^G \ge 1/2$, which contradicts $\lambda_1^G \le 0$.

Example 3 (C-stationary but not M-stationary). Consider the SOCMPCC

$$\min \ f(z) := -z_1 + z_2 - z_3$$

$$\text{s.t.} \ \ G(z) := \left(\begin{array}{c} z_1 + 1 \\ z_1 + z_2 - z_3 \\ z_1 \end{array} \right) \in \mathcal{K}^3,$$

$$H(z) := \left(\begin{array}{c} z_1 \\ -z_1 - z_3^2 \\ z_2 - 1 \end{array} \right) \in \mathcal{K}^3,$$

$$G(z) \perp H(z).$$

We can show that the only feasible solution is $(0, 1, 0)$, and hence it is also the unique optimal solution. Indeed, since $H(z) = (z_1, -z_1 - z_3^2, z_2 - 1) \in \mathcal{K}^3$, by definition of the second-order cone we have

$$z_1 \ge \sqrt{(z_1 + z_3^2)^2 + (z_2 - 1)^2},$$

which implies that $z_1 \geq 0$ and

(16)
$$z_1^2 \geq (z_1 + z_3^2)^2 + (z_2 - 1)^2 \geq (z_1 + z_3^2)^2.$$

Hence we obtain $z_3 = 0$. Plugging $z_3 = 0$ into (16), we obtain $z_2 = 1$. Similarly from $G(z) = (z_1 + 1, z_1 + z_2 - z_3, z_1) \in \mathcal{K}^3$ and $z_3 = 0$, we obtain $z_1 = 0$. The index sets are all empty except for $B_G(z^*), I_H(z^*)$. Hence the C-stationary condition is

(17)
$$\begin{cases} \nabla f(z^*) + \nabla G(z^*)\lambda^G + \nabla H(z^*)\lambda^H = 0, \\ \lambda^G \in \mathbb{R}\widehat{G}(z^*), \ \langle \lambda^G, \lambda^H \rangle \geq 0, \end{cases}$$

and the M-stationary condition is

$$\begin{cases} \nabla f(z^*) + \nabla G(z^*)\lambda^G + \nabla H(z^*)\lambda^H = 0, \\ \lambda^G = 0, \lambda^H \in \mathbb{R}^3 \text{ or } \lambda^G \in \mathbb{R}\widehat{G}(z^*), \lambda^H \perp \widehat{G}(z^*) \\ \text{or } \lambda^G \in \mathbb{R}_-\widehat{G}(z^*), \ \langle \lambda^H, \widehat{G}(z^*) \rangle \leq 0. \end{cases}$$

Take $\lambda^G = (1, -1, 0)$ and $\lambda^H = (2, 1, 0)$. Then the first condition in the C-stationary condition (17) holds:

(18)
$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \lambda_1^G \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2^G \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda_3^G \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$+ \lambda_1^H \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2^H \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3^H \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and $\lambda^G = \widehat{G}(z^*) \in \mathbb{R}\widehat{G}(z^*), \langle \lambda^G, \lambda^H \rangle = \lambda_1^H - \lambda_2^H = 1 > 0$. So z^* is a C-stationary point. However, z^* is not an M-stationary point. Indeed, from (18), it is clear that λ^G must be nonzero. For $\lambda^G \in \mathbb{R}\widehat{G}(z^*) = t(1, -1, 0)$ for some $t \in \mathbb{R}$, it then follows from (18) that $t = 1$ and $\lambda_1^H - \lambda_2^H = 1$. Thus $\lambda^G \in \mathbb{R}_+\widehat{G}(z^*)$ but $\langle \lambda^H, \widehat{G}(z^*) \rangle \neq 0$. So z^* is not an M-stationary point.

Example 4 (W-stationary but not C-stationary). Consider the SOCMPCC

$$\min \ f(z) := -z_1 + z_2^2 - z_3$$
$$\text{s.t. } G(z) := \begin{pmatrix} z_1 + z_2 \\ z_1 + z_2 + z_3 \\ z_3 \end{pmatrix} \in \mathcal{K}^3,$$
$$H(z) := \begin{pmatrix} z_1 \\ -z_1 - z_1^2 \\ z_2 - 1 \end{pmatrix} \in \mathcal{K}^3,$$
$$G(z) \perp H(z).$$

We can show that the feasible region is $\{z = (z_1, z_2, z_3) | z_1 = 0, z_2 = 1, -1 \leq z_3 \leq 0\}$. Indeed, from $H(z) \in \mathcal{K}^3$, we have $z_1 \geq 0$ and $z_1^2 \geq (z_1 + z_1^2)^2 + (z_2 - 1)^2 \geq (z_1 + z_1^2)^2$. Hence $z_1 = 0$ and $z_2 = 1$. Then $H(z) = (0, 0, 0)$ and $G(z) = (1, 1 + z_3, z_3) \in \mathcal{K}^3$, which implies $z_3 \in [-1, 0]$. Hence the optimal solution is $z^* = (0, 1, 0)$. The index sets are all empty except for $B_G(z^*), I_H(z^*)$. Hence the W-stationary condition is

(19)
$$\nabla f(z^*) + \nabla G(z^*)\lambda^G + \nabla H(z^*)\lambda^H = 0,$$

and the C-stationary condition is

$$\begin{cases} \nabla f(z^*) + \nabla G(z^*)\lambda^G + \nabla H(z^*)\lambda^H = 0, \\ \lambda^G \in \mathbb{R}\widehat{G}(z^*), \ \langle \lambda^G, \lambda^H \rangle \geq 0. \end{cases}$$

Let $\lambda^G = (-1, 1, 0)$ and $\lambda^H = (2, 1, -2)$ or $\lambda^G = (1, 0, 1)$ and $\lambda^H = (1, 1, -3)$. Then the W-stationary condition (19) holds:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + \lambda_1^G \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2^G \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_3^G \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(20)
$$+ \lambda_1^H \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2^H \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3^H \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

However, z^* is not a C-stationary point. Indeed, for $\lambda^G \in \mathbb{R}\widehat{G}(z^*)$, i.e., $\lambda^G = t(1, -1, 0)$ for some $t \in \mathbb{R}$, it then follows from (20) that $t = -1$, $\lambda_1^H - \lambda_2^H = 1 > 0$, and $\lambda_3^H = -2$. Thus $\langle \lambda^G, \lambda^H \rangle = t(\lambda_1^H - \lambda_2^H) = -1 < 0$. Hence z^* is not a C-stationary point.

To study the relationship between the Clarke calmness conditions for the various reformulations, we consider the following general optimization problem with cone complementarity constraints:

(P) $$\min \ f(z)$$
$$\text{s.t.} \ \mathcal{K} \ni G(z) \perp H(z) \in \mathcal{K},$$

where $\mathcal{K}$ is a convex symmetric cone of a finite-dimensional space X and G, H are continuous. For simplicity we omit the standard inequality and equality constraints. Let $t \in \mathbb{R}$ and $\alpha, \beta \in X$. Consider the following perturbed feasible regions of (P):

$$\mathcal{F}_K(t, \alpha, \beta) := \{z \in X | \langle G(z), H(z) \rangle + t \leq 0, (G(z), H(z)) + (\alpha, \beta) \in \mathcal{K} \times \mathcal{K}\},$$
$$\mathcal{F}_M(\alpha, \beta) := \{z \in X | (G(z), H(z)) + (\alpha, \beta) \in \Omega\},$$
$$\mathcal{F}_C(\alpha) := \{z \in X | G(z) - \Pi_{\mathcal{K}}(G(z) - H(z)) + \alpha = 0\}.$$

PROPOSITION 7.6. *Let z^* be a feasible solution of problem* (P).
(a) *Suppose that there exist positive ε_1 and μ_1 such that, for all $(t, \alpha, \beta) \in \varepsilon_1 B$ and for all $z \in B_{\varepsilon_1}(z^*) \cap \mathcal{F}_K(t, \alpha, \beta)$, one has*

(21) $$f(z) - f(z^*) + \mu_1 \|(t, \alpha, \beta)\| \geq 0.$$

Then there exist positive ε_2, μ_2 such that for all $(\alpha, \beta) \in \varepsilon_2 B$ and for all $z \in B_{\varepsilon_2}(z^*) \cap \mathcal{F}_M(\alpha, \beta)$, one has

(22) $$f(z) - f(z^*) + \mu_2 \|(\alpha, \beta)\| \geq 0.$$

(b) *Suppose that there exist positive ε_1 and μ_1 such that, for all (α, β) in $\varepsilon_1 B$ and for all $z \in B_{\varepsilon_1}(z^*) \cap \mathcal{F}_M(\alpha, \beta)$, one has*

(23) $$f(z) - f(z^*) + \mu_1 \|(\alpha, \beta)\| \geq 0.$$

Then for all $\alpha \in \varepsilon_2 B$ and for all $z \in B_{\varepsilon_2}(z^*) \cap \mathcal{F}_C(\alpha)$, one has

(24) $$f(z) - f(z^*) + \mu_2 \|\alpha\| \geq 0,$$

where $\varepsilon_2 = \varepsilon_1/\sqrt{2}$ and $\mu_2 = \sqrt{2}\mu_1$.

Proof. (a) Suppose that $z \in \mathcal{F}_M(\alpha, \beta)$, i.e., $(G(z), H(z)) + (\alpha, \beta) \in \Omega$. Then it is easy to verify that

$$\langle G(z), H(z) \rangle + t_M \leq 0, \quad (G(z), H(z)) + (\alpha, \beta) \in \mathcal{K} \times \mathcal{K}$$

with

$$t_M := \langle G(z), \beta \rangle + \langle H(z), \alpha \rangle + \langle \alpha, \beta \rangle.$$

That is, $z \in \mathcal{F}_K(t_M, \alpha, \beta)$. Now suppose that there exist positive ε_1 and μ_1 such that, for all (t, α, β) in $\varepsilon_1 B$ and $z \in B_{\varepsilon_1}(z^*) \cap \mathcal{F}_K(t, \alpha, \beta)$, (21) holds. Then by the continuity of G, H, one can find positive $\varepsilon_2 < \varepsilon_1$ and μ_2 such that for all $z \in B_{\varepsilon_2}(z^*)$ and (α, β) in $\varepsilon_2 B$,

$$(t_M, \alpha, \beta) = \big(\langle G(z), \beta \rangle + \langle H(z), \alpha \rangle + \langle \alpha, \beta \rangle, \alpha, \beta \big) \in \varepsilon_1 B$$

and

$$\mu_2 \left\| (\alpha, \beta) \right\| \geq \mu_1 \left\| (t_M, \alpha, \beta) \right\|.$$

Combining these and (21) with $t = t_M$ ensures that (22) holds.

(b) Suppose that $z \in \mathcal{F}_C(\alpha)$. Then $G(z) - \Pi_\mathcal{K}(G(z) - H(z)) + \alpha = 0$, which can be rewritten as $G(z) + \alpha = \Pi_\mathcal{K}(G(z) + \alpha - H(z) - \alpha)$, i.e.,

$$(G(z), H(z)) + (\alpha, \alpha) \in \Omega.$$

That is, $z \in \mathcal{F}_M(\alpha, \alpha)$. Now suppose that there exist positive ε_1 and μ_1 such that, for all (α, β) in $\varepsilon_1 B$ and for all $z \in B_{\varepsilon_1}(z^*) \cap \mathcal{F}_M(\alpha, \beta)$, (23) holds. Let $\varepsilon_2 = \varepsilon_1/\sqrt{2}$ and $\mu_2 = \sqrt{2}\mu_1$. Then for all $\alpha \in \varepsilon_2 B$ and $z \in B_{\varepsilon_2}(z^*) \cap \mathcal{F}_C(\alpha)$, we have $(\alpha, \alpha) \in \varepsilon_1 B$ and hence

$$f(z) - f(z^*) + \mu_2 \|\alpha\| = f(z) - f(z^*) + \mu_1 \|(\alpha, \alpha)\| \geq 0,$$

i.e., (24) holds.   □

DEFINITION 7.7 (K-Clarke calmness). *We say that a feasible solution z^* of* (P) *is K-Clarke calm if there exist positive ε and μ such that, for all $(t, \alpha, \beta) \in \varepsilon B$ and for all $z \in B_\varepsilon(z^*) \cap \mathcal{F}_K(t, \alpha, \beta)$, one has*

$$f(z) - f(z^*) + \mu \|(t, \alpha, \beta)\| \geq 0.$$

DEFINITION 7.8 (M-Clarke calmness). *We say that a feasible solution z^* of* (P) *is M-Clarke calm if there exist positive ε and μ such that, for all $(\alpha, \beta) \in \varepsilon B$ and for all $z \in B_\varepsilon(z^*) \cap \mathcal{F}_M(\alpha, \beta)$, one has*

$$f(z) - f(z^*) + \mu \|(\alpha, \beta)\| \geq 0.$$

DEFINITION 7.9 (C-Clarke calmness). *We say that a feasible solution z^* of* (P) *is C-Clarke calm if there exist positive ε and μ such that, for all $\alpha \in \varepsilon B$ and for all $z \in B_\varepsilon(z^*) \cap \mathcal{F}_C(\alpha)$, one has*

$$f(z) - f(z^*) + \mu \|\alpha\| \geq 0.$$

According to Proposition 7.6, the following implications hold.

THEOREM 7.10. *K-Clarke calmness* $\Longrightarrow$ *M-Clarke calmness* $\Longrightarrow$ *C-Clarke calmness.*

8. New optimality conditions for MPCC via SOCMPCC. Consider the vector MPCC

(MPCC)
$$\min \ f(z)$$
$$\text{s.t.} \ \ h(z) = 0, \ \ g(z) \leq 0,$$
$$0 \leq G_i(z) \perp H_i(z) \geq 0, \ \ i = 1, \ldots, J,$$

where $G_i(z), H_i(z) : \mathbb{R}^n \to \mathbb{R}$. We reformulate MPCC as the associated SOCMPCC:

(a-SOCMPCC)
$$\min \ f(z)$$
$$\text{s.t.} \ \ h(z) = 0, \ \ g(z) \leq 0,$$
$$\mathcal{K}_i \ni \widetilde{G}_i(z) \perp \widetilde{H}_i(z) \in \mathcal{K}_i, \ \ i = 1, \ldots, J,$$

where $\widetilde{G}_i(z) := (G_i(z), 0, \ldots, 0) \in \mathbb{R}^{m_i}$ and $\widetilde{H}_i(z) := (G_i(z), 0, \ldots, 0) \in \mathbb{R}^{m_i}$ for $i = 1, \ldots, J$.

Note that the index sets $B_{\widetilde{G}}(z), B_{\widetilde{H}}(z)$ are always empty. For the convenience of the discussion we recall the definition of W-, S-, M-, and C-stationary conditions for the problem a-SOCMPCC. A feasible solution of a-SOCMPCC is a W-stationary point if there exists $(\lambda^g, \lambda^h, \tilde{\lambda}^G, \tilde{\lambda}^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^\tau \times \mathbb{R}^\tau$ such that

(25)
$$\begin{cases} \nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^J \nabla \widetilde{G}_i(z^*)\tilde{\lambda}_i^G + \sum_{i=1}^J \nabla \widetilde{H}_i(z^*)\tilde{\lambda}_i^H = 0, \\ \lambda^g \geq 0, \ g(z^*)^T \lambda^g = 0, \\ \tilde{\lambda}_i^H = 0 \ \ if \ \ i \in I_{\widetilde{G}}(z^*) \cap I_{\widetilde{H}}^+(z^*), \\ \tilde{\lambda}_i^G = 0 \ \ if \ \ i \in I_{\widetilde{G}}^+(z^*) \cap I_{\widetilde{H}}(z^*). \end{cases}$$

It is an S-, M-, and C-stationary point if (25) holds and for all $i \in I_{\widetilde{G}}(z^*) \cap I_{\widetilde{H}}(z^*)$

$$\tilde{\lambda}_i^G \in -\mathcal{K}_i, \ \tilde{\lambda}_i^H \in -\mathcal{K}_i;$$

$$\begin{cases} \tilde{\lambda}_i^G \in -\mathcal{K}_i, \tilde{\lambda}_i^H \in -\mathcal{K}_i \ or \ \tilde{\lambda}_i^H = 0, \tilde{\lambda}_i^G \in \mathbb{R}^{m_i} \ or \\ \tilde{\lambda}_i^G = 0, \tilde{\lambda}_i^H \in \mathbb{R}^{m_i} \ or \\ \tilde{\lambda}_i^G \in \mathbb{R}_- \xi_i, \ \tilde{\lambda}_i^H \in \xi_i^\circ \ or \ \tilde{\lambda}_i^H \in \mathbb{R}_- \xi_i, \ \tilde{\lambda}_i^G \in \xi_i^\circ \ or \\ \tilde{\lambda}_i^G \perp \xi_i, \ \tilde{\lambda}_i^H \perp \hat{\xi}_i, \ \alpha_i \hat{\tilde{\lambda}}_i^G + (1 - \alpha_i)\tilde{\lambda}_i^H \in \mathbb{R}\xi_i \\ for \ some \ \alpha_i \in [0, 1] \ and \ some \ \xi_i \in C_i; \end{cases}$$

$$\langle \tilde{\lambda}_i^G, \tilde{\lambda}_i^H \rangle \geq 0$$

hold, respectively.

Let us discuss the relationship of the various stationary points between MPCC and its associated SOCMPCC reformulations.

THEOREM 8.1. *The following statements hold:*
(a) *If z^* is an S-stationary point of vector MPCC with $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^J \times \mathbb{R}^J$, then z^* is an S-stationary point of the associated SOCMPCC with $(\lambda^g, \lambda^h, \tilde{\lambda}^G, \tilde{\lambda}^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^\tau \times \mathbb{R}^\tau$, where $\tilde{\lambda}_i^G = (\lambda_i^G, 0, \ldots, 0) \in \mathbb{R}^{m_i}$ and $\tilde{\lambda}_i^H = (\lambda_i^H, 0, \ldots, 0) \in \mathbb{R}^{m_i}$ for $i = 1, \ldots, J$. Conversely, if z^* is an S-stationary point of the associated SOCMPCC with $(\lambda^g, \lambda^h, \tilde{\lambda}^G, \tilde{\lambda}^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^\tau \times \mathbb{R}^\tau$, then z^* is an S-stationary point of vector MPCC with $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^J \times \mathbb{R}^J$, where $\lambda_i^G = (\tilde{\lambda}_i^G)_1$ and $\lambda_i^H = (\tilde{\lambda}_i^H)_1$ for $i = 1, \ldots, J$.*

(b) *If z^* is an M-,C-stationary point of vector MPCC with $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^J \times \mathbb{R}^J$, then z^* is an M-,C-stationary point of the associated SOCMPCC with $(\lambda^g, \lambda^h, \tilde{\lambda}^G, \tilde{\lambda}^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^\tau \times \mathbb{R}^\tau$, where $\tilde{\lambda}_i^G = (\lambda_i^G, 0, \dots, 0) \in \mathbb{R}^{m_i}$ and $\tilde{\lambda}_i^H = (\lambda_i^H, 0, \dots, 0) \in \mathbb{R}^{m_i}$ for $i = 1, \dots, J$.*

Proof. Part (a). Recall that a point z^* is said to be an S-stationary point of the MPCC if there exists $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^J \times \mathbb{R}^J$ such that

$$(26) \quad \begin{cases} \nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^J \nabla G_i(z^*)\lambda_i^G + \sum_{i=1}^J \nabla H_i(z^*)\lambda_i^H = 0, \\ \lambda^g \geq 0, \ g(z^*)^T\lambda^g = 0, \\ \lambda_i^H = 0 \ \text{if} \ i \in I_G(z^*) \cap I_H^+(z^*), \\ \lambda_i^G = 0 \ \text{if} \ i \in I_G^+(z^*) \cap I_H(z^*), \\ \lambda_i^G \leq 0, \ \lambda_i^H \leq 0 \ \text{if} \ i \in I_G(z^*) \cap I_H(z^*). \end{cases}$$

Note that

$$I_{\widetilde{G}}(z^*) = I_G(z^*), \ I_{\widetilde{G}}^+(z^*) = I_G^+(z^*), \ I_{\widetilde{H}}(z^*) = I_H(z^*), \ I_{\widetilde{H}}^+(z^*) = I_H^+(z^*).$$

Let

$$(27) \quad \begin{aligned} \tilde{\lambda}^G &= (\tilde{\lambda}_1^G, \dots, \tilde{\lambda}_J^G) \quad \text{with} \quad \tilde{\lambda}_i^G = (\lambda_i^G, 0, \dots, 0) \in \mathbb{R}^{m_i}, \\ \tilde{\lambda}^H &= (\tilde{\lambda}_1^H, \dots, \tilde{\lambda}_J^H) \quad \text{with} \quad \tilde{\lambda}_i^H = (\lambda_i^H, 0, \dots, 0) \in \mathbb{R}^{m_i}. \end{aligned}$$

Then

$$\nabla \widetilde{G}_i(z^*)\tilde{\lambda}_i^G = \nabla G_i(z^*)\lambda_i^G \quad \text{and} \quad \nabla \widetilde{H}_i(z^*)\tilde{\lambda}_i^H = \nabla H_i(z^*)\lambda_i^H.$$

From $\lambda_i^G \leq 0, \lambda_i^H \leq 0$, we have $\tilde{\lambda}_i^G \in -\mathcal{K}_i, \tilde{\lambda}_i^H \in -\mathcal{K}_i$. Thus (26) implies that z^* is an S-stationary point of a-SOCMPCC.

Conversely, assume that z^* is an S-stationary point of the associated SOCMPCC reformulation. Notice that

$$\nabla \widetilde{G}_i(z^*)\tilde{\lambda}_i^G = (\tilde{\lambda}_i^G)_1 \nabla G_i(z^*) \quad \text{and} \quad \nabla \widetilde{H}_i(z^*)\tilde{\lambda}_i^H = (\tilde{\lambda}_i^H)_1 \nabla H_i(z^*).$$

In addition, $\tilde{\lambda}_i^G \in -\mathcal{K}_i, \ \tilde{\lambda}_i^H \in -\mathcal{K}_i$ imply $(\tilde{\lambda}_i^G)_1 \leq 0, (\tilde{\lambda}_i^H)_1 \leq 0$. Hence z^* is an S-stationary point of MPCC with $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^J \times \mathbb{R}^J$ satisfying (26), where $\lambda_i^G = (\tilde{\lambda}_i^G)_1$ and $\lambda_i^H = (\tilde{\lambda}_i^H)_1$ for $j = 1, \dots, J$.

Part (b). Recall that a point z^* is said to be an M-stationary point of the vector MPCC if there exists $(\lambda, \mu, u, v) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^J \times \mathbb{R}^J$ such that

$$\begin{cases} \nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^J \nabla G_i(z^*)\lambda_i^G + \sum_{i=1}^J \nabla H_i(z^*)\lambda_i^H = 0, \\ \lambda^g \geq 0, \ g(z^*)^T\lambda^g = 0, \\ \lambda_i^H = 0 \ \text{if} \ i \in I_G(z^*) \cap I_H^+(z^*), \\ \lambda_i^G = 0 \ \text{if} \ i \in I_G^+(z^*) \cap I_H(z^*), \\ \lambda_i^G < 0, \ \lambda_i^H < 0, \ \text{or} \ \lambda_i^G\lambda_i^H = 0 \ \text{if} \ i \in I_G(z^*) \cap I_H(z^*). \end{cases}$$

For $\tilde{\lambda}^G$ and $\tilde{\lambda}^H$ given as in (27), we have

$$\begin{cases} \nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^J \nabla \widetilde{G}_i(z^*)\tilde{\lambda}_i^G + \sum_{i=1}^J \nabla \widetilde{H}_i(z^*)\tilde{\lambda}_i^H = 0, \\ \lambda^g \geq 0, \ g(z^*)^T\lambda^g = 0, \\ \tilde{\lambda}_i^H = 0 \ \text{if} \ i \in I_{\widetilde{G}}(z^*) \cap I_{\widetilde{H}}^+(z^*), \\ \tilde{\lambda}_i^G = 0 \ \text{if} \ i \in I_{\widetilde{G}}^+(z^*) \cap I_{\widetilde{H}}(z^*), \\ \tilde{\lambda}_i^G, \tilde{\lambda}_i^H \in -\mathcal{K}_i \ \text{or} \ \tilde{\lambda}_i^G = 0, \tilde{\lambda}_i^H \in \mathbb{R}^{m_i}, \ \text{or} \ \tilde{\lambda}_i^H = 0, \tilde{\lambda}_i^G \in \mathbb{R}^{m_i} \\ \hspace{4cm} \text{if} \ i \in I_{\widetilde{G}}(z^*) \cap I_{\widetilde{H}}(z^*). \end{cases}$$

Hence z^* is an M-stationary point for the associated SOCMPCC. The proof for the C-stationary condition is similar and is omitted. □

In general the converse statement of part (b) in Theorem 8.1 does not hold. In fact it is easy to see that the M- and C-stationary conditions for the associated SOCMPCC are weaker than the one for the original MPCC since by taking $\lambda_i^G := (\tilde{\lambda}_i^G)_1$ and $\lambda_i^H := (\tilde{\lambda}_i^H)_1$ we may not be able to obtain the M- and C-stationary conditions for the original MPCC. This is illustrated by the following example, where z^* is an M-stationary (or C-stationary) point of the SOCMPCC reformulation, but not an M-stationary (or C-stationary) point of the original MPCC.

Example 5. Consider an example of MPCC given in [6]:

$$\min \quad z_1 - \frac{25}{8}z_2 - z_3 - \frac{1}{2}z_4$$
$$\text{s.t.} \quad z_4^2 \leq 0,$$
$$0 \leq G_i(z) \perp H_i(z) \geq 0, \quad i = 1, 2,$$

where $G_1(x) = 6z_1 - z_3 - z_4$, $G_2(x) = z_1$, $H_1(x) = -6z_2 - z_3$, and $H_2(x) = -z_2$.

It is easy to see that $z^* = (0,0,0,0)$ is the unique optimal solution. The only nonempty index set is $I_G(z^*) \cap I_H(z^*) = \{1,2\}$. Consider the W-stationary system for MPCC:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{25}{8} \\ -1 \\ -\frac{1}{2} \end{pmatrix} + \lambda^g \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_1^G \begin{pmatrix} 6 \\ 0 \\ -1 \\ -1 \end{pmatrix} + \lambda_2^G \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$+ \lambda_1^H \begin{pmatrix} 0 \\ -6 \\ -1 \\ 0 \end{pmatrix} + \lambda_2^H \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

where $\lambda^g \geq 0$. The solutions are $\lambda^g \geq 0, \lambda_1^G = -\frac{1}{2}, \lambda_2^G = 2, \lambda_1^H = -\frac{1}{2}, \lambda_2^H = -\frac{1}{8}$, and hence $z^* = (0,0,0,0)$ is an W-stationary point. But since $\lambda_2^G \lambda_2^H < 0$, z^* is not a C-stationary point and hence not an M-stationary point. Now we reformulate the problem as an SOCMPCC:

$$\min \quad z_1 - \frac{25}{8}z_2 - z_3 - \frac{1}{2}z_4$$
$$\text{s.t.} \quad z_4^2 \leq 0,$$
$$\mathcal{K}_i \ni \widetilde{G}_i(z) \perp \widetilde{H}_i(z) \in \mathcal{K}_i, \quad i = 1, 2,$$

where $\mathcal{K}_i$ is the 2-dimensional second-order cone, $\widetilde{G}_1(x) = (6z_1 - z_3 - z_4, 0)$, $\widetilde{G}_2(x) = (z_1, 0)$, $\widetilde{H}_1(x) = (-6z_2 - z_3, 0)$, and $\widetilde{H}_2(x) = (-z_2, 0)$. The only nonempty index set is $I_{\widetilde{G}}(z^*) \cap I_{\widetilde{H}}(z^*) = \{1,2\}$. We now increase the dimensions of the multipliers from 1 to 2 with the first components kept the same. Let

$$\tilde{\lambda}_1^G := \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad \tilde{\lambda}_2^G := \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad \tilde{\lambda}_1^H := \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad \tilde{\lambda}_2^H := \begin{pmatrix} -\frac{1}{8} \\ -\frac{1}{8} \end{pmatrix}.$$

Then $\tilde{\lambda}_1^G, \tilde{\lambda}_1^H \in -\mathcal{K}_1$. Let $\xi = (1,1)$. Then $\tilde{\lambda}_2^G \perp \xi, \tilde{\lambda}_2^H \perp \hat{\xi}$. Hence z^* is an M-stationary (also a C-stationary) point for the corresponding SOCMPCC.

From this example, it is inspiring to see that by increasing the dimension of the second-order cone, we can obtain new and weaker M- or C-stationary conditions which can be used to identify candidates for optimality when the M- or C-stationary conditions of the original MPCC do not hold.

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