Second-order Optimality Conditions for Mathematical Programs with Equilibrium Constraints

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Abstract. We study second-order optimality conditions for mathematical programs with equilibrium constraints (MPEC). Firstly, we improve some second-order optimality conditions for standard nonlinear programming problems using some newly discovered constraint qualifications in the literature, and apply them to MPEC. Then, we introduce some MPEC variants of these new constraint qualifications, which are all weaker than the MPEC linear independence constraint qualification, and derive several second-order optimality conditions for MPEC under the new MPEC constraint qualifications. Finally, we discuss the isolatedness of local minimizers for MPEC under very weak conditions.

Key Words. Mathematical program with equilibrium constraints, second-order optimality condition, constraint qualification, isolatedness.

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1 Introduction

MPEC is a constrained optimization problem in which the essential constraints are defined by some parametric variational inequalities or parametric complementarity systems. It plays a very important role in many fields, such as engineering design, economic equilibria, transportation science, multilevel game, and mathematical programming itself. However, this kind of problems is generally difficult to deal with because its constraints fail to satisfy the standard Mangasarian-Fromovitz constraint qualification (MFCQ) at any feasible point [1].

A lot of research has been done during the last two decades to study the first-order optimality

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conditions for MPEC, such as Clarke (C-), Mordukhovich (M-), Strong (S-), Bouligrand (B-) stationarity conditions; see, e.g., [1–8]. At the same time, algorithms for solving MPEC have been proposed by using a number of approaches, such as sequential quadratic programming approach, penalty function approach, relaxation approach, active set identification approach, etc.; see, e.g., [9–11] and the references therein for more details.

In this paper, we focus on second-order optimality conditions for MPEC. First-order optimality conditions tell us how the first derivatives of the functions involved are related to each other at locally optimal solutions. However, for some feasible directions in the tangent cone such as the so-called critical directions, we cannot determine from the first derivative information alone whether the objective function increases or decreases in this direction. Second-order optimality conditions examine the second derivative terms in the Taylor series expansions of the functions involved to see whether these extra information resolves the issue of increase or decrease in the objective. Essentially, the second-order optimality conditions are concerned with the curvature of the so-called MPEC Lagrangian function in the critical directions. Moreover, second-order optimality conditions play important roles in convergence analysis for numerical algorithms and the stability anslysis for MPEC; see, e.g., [12–18].

Compared with the first-order optimality conditions, there are very little research done with the second-order optimality conditions for MPEC. Scheel and Scholtes [2] showed that S-stationary points satisfying the refined second-order sufficient optimality conditions are strictly and locally optimal and they derived a strong second-order necessary optimality condition under the MPEC strict MFCQ (MPEC-SMFCQ). Izmailov [19] investigated second-order optimality conditions under the MPEC linear independence constraint qualification (MPEC-LICQ). In this paper, we study second-order optimality conditions for MPEC systematically. Note that, recently, several new constraint qualifications weaker than the LICQ and MFCQ have been introduced for standard nonlinear programming problems. We use these new constraint qualifications to derive some second-order optimality conditions for standard nonlinear programming problems in Section 2 and apply the obtained results to MPEC. We further introduce some MPEC variants of these new constraint qualifications, which are weaker than the MPEC-LICQ, and derive some second-order optimality conditions for MPEC in terms of S- and C-multipliers under these new MPEC constraint qualifications. Moreover, we establish some relationships between various second-order optimality conditions for MPEC in terms of the classical NLP multipliers and S-multipliers respectively. It is interesting to see that not all second-order optimality conditions in terms of the classical NLP multipliers and S-multipliers are equivalent. In addition, unlike the first-order conditions, the second-order conditions in terms of singular multipliers provide some information for optimality. Finally, we discuss the isolatedness of local minimizers under some weak conditions. Since MPEC includes standard nonlinear programming problems as special cases, some of our results are new even for nonlinear programming problems.

2 Second-order Optimality Conditions for Standard Nonlinear Programming Problems

In this section, we review and improve various second-order optimality conditions for the standard nonlinear programming problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq 0,\ h(x) = 0, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^p$, and $h : \mathbb{R}^n \to \mathbb{R}^q$ are all twice differentiable functions. Given $r \geq 0$, we let L^r be the *generalized Lagrangian function* for (1) defined by

$$L^r(x, y) := rf(x) + g(x)^T\lambda + h(x)^T\mu,$$

where $y := (\lambda, \mu)$. For a mapping $\psi : \mathbb{R}^n \to \mathbb{R}^l$ and a vector $x \in \mathbb{R}^n$, we denote by $\nabla\psi(x)$ the transposed Jacobian of ψ at x. According to the terminology used by Clarke [20], we call y an *index r multiplier* for (1) at a feasible point x iff $(r, y) \neq 0$ and

$$\nabla_x L^r(x, y) = 0, \quad \lambda \geq 0, \quad g(x)^T\lambda = 0.$$

An index 0 multiplier is usually referred to as a singular multiplier (see, e.g., [21]) or an abnormal multiplier (see, e.g., [20]). Denote by $\mathcal{F}$ the feasible region of (1). For $x^* \in \mathcal{F}$, let $\mathcal{M}^r(x^*)$ denote the set of all index r multipliers for (1) at x^* and $I_g^* := \{i \mid g_i(x^*) = 0\}$. Moreover, the *linearized cone* of $\mathcal{F}$ at

x^* is defined by

$$\mathcal{L}(x^*) := \{d \in \mathbb{R}^n \mid \nabla g_i(x^*)^T d \leq 0 \text{ for } i \in I_g^*,\ \nabla h_j(x^*)^T d = 0 \text{ for } j = 1, \ldots, q\},$$

and the *critical cone* at x^* is defined by

$$\mathcal{C}(x^*) := \{d \in \mathbb{R}^n \mid \nabla f(x^*)^T d \leq 0\} \cap \mathcal{L}(x^*),$$

respectively.

There are three types of second-order optimality conditions for (1). The definitions of strong second-order optimality conditions are classical; see, e.g., [22,23]. The refined second-order optimality condition was introduced in [24], and subsequently studied in [21]. The weak second-order optimality condition was studied from theoretical and practical points of view in [25–27].

Definition 2.1 (Second-order optimality conditions for NLP) Let $x^* \in \mathcal{F}$.

(i) We say that the *strong second-order necessary optimality condition* (SSONC) holds at x^* iff $\mathcal{M}^1(x^*) \neq \emptyset$ and, for every $y^* \in \mathcal{M}^1(x^*)$, there holds

$$d^T \nabla_x^2 L^1(x^*, y^*) d \geq 0, \qquad \forall d \in \mathcal{C}(x^*).$$

We say that the *strong second-order sufficient optimality condition* (SSOSC) holds at x^* iff, for every $y^* \in \mathcal{M}^1(x^*)$, there holds

$$d^T \nabla_x^2 L^1(x^*, y^*) d > 0, \qquad \forall d \in \mathcal{C}(x^*) \backslash \{0\}.$$

(ii) We say that the *refined second-order necessary optimality condition* (RSONC) holds at x^* iff, for every $d \in \mathcal{C}(x^*)$, there exists $y^* \in \mathcal{M}^r(x^*)$ such that

$$d^T \nabla_x^2 L^r(x^*, y^*) d \geq 0.$$

We say that the *refined second-order sufficient optimality condition* (RSOSC) holds at x^* iff, for every $d \in \mathcal{C}(x^*) \backslash \{0\}$, there exists $y^* \in \mathcal{M}^r(x^*)$ such that

$$d^T \nabla_x^2 L^r(x^*, y^*) d > 0.$$

(iii) We say that the *weak second-order necessary optimality condition* (WSONC) holds at x^* iff there exists $y^* \in \mathcal{M}^1(x^*)$ such that

$$d^T \nabla_x^2 L^1(x^*, y^*) d \geq 0, \qquad \forall d \in \bar{\mathcal{C}}(x^*),$$

where

$$\bar{\mathcal{C}}(x^*) := \{d \mid \nabla g_i(x^*)^T d = 0 \text{ for } i \in I_g^*, \ \nabla h_j(x^*)^T d = 0 \text{ for } j = 1, \ldots, q\}.$$

It is obvious that the following relationships hold:

$$\begin{array}{ccc} \text{SSOSC} & & \text{SSONC} \\ \Downarrow & \swarrow\!\!\Downarrow & \Downarrow \\ \text{RSOSC} & \text{RSONC} & \text{WSONC} \end{array}$$

Note that, for the sufficient optimality conditions to hold, no constraint qualification is required.

Proposition 2.1 (RSOSC for NLP) ([21, Proposition 5.48]) *Let $x^* \in \mathcal{F}$. If the RSOSC holds at x^*, then x^* is a locally optimal solution with quadratic growth, that is, there are constants $c > 0$ and $\delta > 0$ such that*

$$f(x) \geq f(x^*) + c\|x - x^*\|^2, \qquad \forall x \in \mathcal{F} \cap \mathcal{B}_\delta(x^*),$$

where $\mathcal{B}_\delta(x^*) := \{x \mid \|x - x^*\| < \delta\}$ with Euclidean vector norm $\|\cdot\|$.

However, for a necessary optimality condition to hold, a constraint qualification is required usually. We next recall the definition of LICQ and its various relaxations. The CRCQ was introduced by Janin in [28], and its relaxation RCRCQ was introduced by Minchenko and Stakhovski in [29]. The even weaker WCR condition was introduced by Andreani et al in [30]. Note that the WCR condition is not a constraint qualification [30].

Definition 2.2 (LICQ and its relaxations) Let $x^* \in \mathcal{F}$.

(i) We say that the *linear independence constraint qualification (LICQ)* holds at x^* iff the family of gradients $\{\nabla g_i(x^*), \nabla h_j(x^*) \mid i \in I_g^*, j = 1, \cdots, q\}$ is linearly independent.

(ii) We say that the *constant rank constraint qualification* (CRCQ) holds at x^* iff there exists $\delta > 0$ such that, for each $\mathcal{I} \subseteq I_g^*$ and $\mathcal{J} \subseteq \{1, \ldots, q\}$, the family of gradients $\{\nabla g_i(x), \nabla h_j(x) \mid i \in \mathcal{I}, j \in \mathcal{J}\}$ has the same rank for every $x \in \mathcal{B}_\delta(x^*)$.

(iii) We say that the *relaxed constant rank constraint qualification* (RCRCQ) holds at x^* iff there exists $\delta > 0$ such that, for each $\mathcal{I} \subseteq I_g^*$, the family of gradients $\{\nabla g_i(x),\ \nabla h_j(x) \mid i \in \mathcal{I}, j = 1, \cdots, q\}$ has the same rank for every $x \in \mathcal{B}_\delta(x^*)$.

(iv) We say that the *weak constant rank* condition (WCR) holds at x^* iff there exists $\delta > 0$ such that the family of gradients $\{\nabla g_i(x), \nabla h_j(x) \mid i \in I_g^*, j = 1, \cdots, q\}$ has the same rank for every $x \in \mathcal{B}_\delta(x^*)$.

Recall that, for $A := \{a^1, \cdots, a^l\}$ and $B := \{b^1, \cdots, b^s\}$, (A, B) is said to be *positively linearly dependent* iff there exist α and β such that $\alpha \geq 0$, $(\alpha, \beta) \neq 0$, and

$$\sum_{i=1}^{l} \alpha_i a^i + \sum_{j=1}^{s} \beta_j b^j = 0.$$

Otherwise, (A, B) is said to be *positively linearly independent*.

We now recall the positively linear independence constraint qualification (PLICQ) and its relaxations. It is well-known that the MFCQ is equivalent to the PLICQ. The PLICQ is also called *no nonzero abnormal multiplier constraint qualification* (NNAMCQ) or *basic constraint qualification* (Basic CQ). The CPLD was introduced by Qi and Wei in [31] and was shown to be a constraint qualification by Andreani et al. in [32]. The relaxed CPLD was introduced by Andreani et al. in [33]. The CRSC was introduced by Andreani et al. in [34], and it is weaker than the RCPLD (see [34, Theorem 4.2]).

Definition 2.3 (MFCQ and its relaxations) Let $x^* \in \mathcal{F}$.

(i) We say that the *positive linear independence constraint qualification* (PLICQ) holds at x^* iff the family of gradients $(\{\nabla g_i(x^*) \mid i \in I_g^*\}, \{\nabla h_j(x^*) \mid j = 1, \cdots, q\})$ is positively linearly independent.

(ii) We say that the *constant positive linear dependence* condition (CPLD) holds at x^* iff, for each $\mathcal{I} \subseteq I_g^*$ and $\mathcal{J} \subseteq \{1, \ldots, q\}$, whenever $(\{\nabla g_i(x^*) \mid i \in \mathcal{I}\}, \{\nabla h_j(x^*) \mid j \in \mathcal{J}\})$ is positively linearly dependent, there exists $\delta > 0$ such that, for every $x \in \mathcal{B}_\delta(x^*)$, $\{\nabla g_i(x), \nabla h_j(x) \mid i \in \mathcal{I}, j \in \mathcal{J}\}$ is linearly dependent.

(iii) Let $\mathcal{J} \subseteq \{1, \cdots, q\}$ be such that $\{\nabla h_j(x^*)\}_{j\in\mathcal{J}}$ is a basis for span $\{\nabla h_j(x^*)\}_{j=1}^q$. We say that the *relaxed constant positive linear dependence* condition (RCPLD) holds at x^* iff there exists $\delta > 0$ such that

- $\{\nabla h_j(x)\}_{j=1}^q$ has the same rank for each $x \in \mathcal{B}_\delta(x^*)$;
- for each $\mathcal{I} \subseteq I_g^*$, if $(\{\nabla g_i(x^*) \mid i \in \mathcal{I}\}, \{\nabla h_j(x^*) \mid j \in \mathcal{J}\})$ is positively linearly dependent, then $\{\nabla g_i(x), \nabla h_j(x) \mid i \in \mathcal{I}, j \in \mathcal{J}\}$ is linearly dependent for each $x \in \mathcal{B}_\delta(x^*)$.

(iv) Let $\mathcal{J}_- := \{i \in I_g^* \mid -\nabla g_i(x^*) \in \mathcal{L}(x^*)^o\}$, where o stands for the *polar* operation. We say that the *constant rank of the subspace component* condition (CRSC) holds at x^* iff there exists $\delta > 0$ such that the family of gradients $\{\nabla g_i(x), \nabla h_j(x) \mid i \in \mathcal{J}_-, j \in \{1, \cdots, q\}\}$ has the same rank for every $x \in \mathcal{B}_\delta(x^*)$.

Note that the CPLD can be regarded as a complement for the PLICQ. In fact, it is easy to see that the CPLD holds at x^* if the PLICQ holds or, for each $\mathcal{I} \subseteq I_g^*$ and $\mathcal{J} \subseteq \{1, \ldots, q\}$, whenever $(\{\nabla g_i(x^*) \mid i \in \mathcal{I}\}, \{\nabla h_j(x^*) \mid j \in \mathcal{J}\})$ is positively linearly dependent, there exists $\delta > 0$ such that $\{\nabla g_i(x), \nabla h_j(x) \mid i \in \mathcal{I}, j \in \mathcal{J}\}$ is linearly dependent for every $x \in \mathcal{B}_\delta(x^*)$. It is similar for the RCPLD and CRSC. See [33, 34] for the relations among the above constraint qualifications.

In the literature, the SSONC requires the LICQ. A counterexample due to Arutyunov [35] (see also [36]) shows that one cannot relax LICQ to MFCQ. Recently, Minchenko and Stakhovski [29] showed that the SSONC holds under the RCRCQ, which is weaker than the LICQ.

Proposition 2.2 (SSONC for NLP) [29, Theorem 6] *Let x^* be a locally optimal solution of* (1)*. If the RCRCQ holds at x^*, then x^* satisfies the SSONC with $r = 1$.*

In the following, we improve the standard results for the RSONC. We first give an equivalent form of the Clarke calmness condition [20]. See [37] for the equivalence between the Clarke calmness and exact penalization.

Definition 2.4 (Calmness) Let x^* be a locally optimal solution of (1). We say that problem (1) is (Clarke) *calm* at x^* iff x^* is also a locally optimal solution of the penalized problem

$$\min_x \; f(x) + \kappa(\max\{g_1(x), \cdots, g_p(x), 0\} + \|h(x)\|) \tag{2}$$

for some positive constant κ.

Note that, since the concept involves the objective function, the calmness condition is not a constraint qualification in the classical sense. It is just a sufficient condition under which the first-order conditions and the RSONC with $r = 1$ hold at a local minimizer. The calmness condition may hold even when Guignard constraint qualification [38], which is the weakest constraint qualification, does not hold. For example, it is easy to verify that the optimization problem

$$\min\ x_1 + x_2 \quad \text{s.t. } (x_1 - x_2)^2 = 0,\ x_1 + 1 \geq 0,\ x_2 + 1 \geq 0$$

is calm at the optimal solution $(-1, -1)$, but the Guignard constraint qualification fails at the point. The standard constraint qualification for the RSONC with $r = 1$ to hold at a local minimizer is the MFCQ; see, e.g., [21]. The following theorem shows that the RSONC with $r = 1$ actually holds at a local minimizer under the calmness condition.

Theorem 2.1 (RSONC for NLP) *Let x^* be a locally optimal solution of* (1)*. Then the RSONC holds at x^*. Moreover, if* (1) *is calm at x^*, then x^* satisfies the RSONC with $r = 1$.*

Proof. The fact that the RSONC holds at any local solution x^* can be found in [21, Proposition 5.48]. We now suppose that (1) is calm at x^*. By the calmness condition, there exists $\kappa > 0$ such that x^* is a locally optimal solution of (2). It is easy to see that $(x, t, s) = (x^*, 0, 0)$ is a locally optimal solution of the problem

$$\begin{array}{ll} \min & f(x) + \kappa(t + \sum_{j=1}^{q} s_j) \\ \text{s.t.} & t \geq g_i(x), \quad i = 1, \cdots, p, \\ & t \geq 0, \\ & s_j \geq h_j(x),\ s_j \geq -h_j(x), \quad j = 1, \cdots, q. \end{array}$$

It is straightforward to verify that the set of nonzero singular multipliers for the above problem at $(x^*, 0, 0)$ is empty, and hence the set of index 1 multipliers must be nonempty by virtue of the Fritz John necessary optimality condition. It follows immediately that the MFCQ holds, and then the RSONC holds for the above problem at $(x^*, 0, 0)$ with $r = 1$. By direct calculation, the critical cone of the above problem at

$(x^*,0,0)$ is

$$\hat{\mathcal{C}}(x^*,0,0)=\left\{(d_1,d_2,d_3)\;\middle|\;\begin{array}{l}\nabla f(x^*)^T d_1+\kappa d_2+\kappa e^T d_3\le 0,\ -d_2\le 0\\ \nabla g_i(x^*)^T d_1-d_2\le 0\ i\in I_g^*\\ \nabla h_j(x^*)^T d_1-d_{3j}\le 0,\ -\nabla h_j(x^*)^T d_1-d_{3j}\le 0\ j=1,\cdots,q\end{array}\right\}.$$

For every $d\in\mathcal{C}(x^*)$, $(d,0,0)\in\hat{\mathcal{C}}(x^*,0,0)$ holds clearly. On the other hand, the fact that $(\lambda^g,\mu,\lambda^h,\lambda^{-h})$ is a multiplier for the above problem implies that $(\lambda^g,\lambda^h-\lambda^{-h})$ is a multiplier for (1). It follows immediately that the RSONC holds for (1) with $r=1$. □

Recently, Andreani et al. showed that the CRSC implies the existence of local error bound in [34, Theorem 5.4]. It follows from the Clarke's exact penalty principle [20, Proposition 2.4.3] that the existence of local error bound implies the calmness condition; see, e.g., [3, Proposition 4.2]. Consequently, from Theorem 2.1, we get the following result immediately. Note that, since the CRSC is much weaker than the MFCQ, our result improves the classical result.

Corollary 2.1 (RSONC under CRSC) *Let x^* be a locally optimal solution of* (1). *If the CRSC condition holds at x^*, then the RSONC holds at x^* with $r=1$.*

Robinson [39] showed the isolatedness of KKT points under the condition that the MFCQ and SSOSC hold. Qi and Wei [31] showed that a KKT point satisfying the CPLD and SSOSC in the sense of Robinson [40], which is stronger than the one in this paper, is isolated. Since the RCPLD is weaker than both CPLD and MFCQ, the following theorem improves the above results. Since the proof is a simplified version of Theorem 4.1, we omit it here.

Theorem 2.2 (Isolatedness under RCPLD) *Suppose that $x^*\in\mathcal{F}$ is a KKT point, i.e., $\mathcal{M}^1(x^*)\neq\emptyset$. If the RCPLD and SSOSC hold at x^*, then x^* is an isolated KKT point of* (1).

Recall that the *tangent cone* of a set X at $x^*\in X$ is a closed cone defined by

$$\mathcal{T}_X(x^*):=\{\,d\,|\,\exists t_k\ge 0\text{ and }x^k\to_X x^*\text{ s.t. }t_k(x^k-x^*)\to d\},$$

where $x^k\to_X x^*$ means that $x^k\to x^*$ with $x^k\in X$ for each k.

Lemma 2.1 [33, Theorem 3] *If the RCPLD holds at $x^*\in\mathcal{F}$, then the Abadie CQ holds at x^*, i.e.,*

$\mathcal{T}_{\mathcal{F}}(x^*) = \mathcal{L}_{\mathcal{F}}(x^*)$.

The following result improves Theorem 3.1 of [30] in that the MFCQ assumption is replaced by the weaker condition $\mathcal{M}^1(x^*) \neq \emptyset$.

Theorem 2.3 (WSONC for NLP) *Let x^* be a locally optimal solution of* (1)*. If the WCR condition holds at x^* and $\mathcal{M}^1(x^*) \neq \emptyset$, then x^* satisfies the WSONC.*

Proof. For any given $y^* \in \mathcal{M}^1(x^*)$, we define the set

$$\hat{\mathcal{F}} := \{x \mid g_i(x) = 0 \text{ for } i \in I_g^*,\ h_j(x) = 0 \text{ for } j = 1, \cdots, q\}.$$

Its linearized cone at x^* is given by

$$\mathcal{L}_{\hat{\mathcal{F}}}(x^*) = \{d \mid \nabla g_i(x^*)^T d = 0 \text{ for } i \in I_g^*,\ \nabla h_j(x^*)^T d = 0 \text{ for } j = 1, \cdots, q\}.$$

By the WCR assumption, the RCPLD holds at $x^* \in \hat{\mathcal{F}}$. It follows from Lemma 2.1 that $\mathcal{L}_{\hat{\mathcal{F}}}(x^*) = \mathcal{T}_{\hat{\mathcal{F}}}(x^*)$. For any given $d \in \mathcal{L}_{\hat{\mathcal{F}}}(x^*)$, we have from the definition of tangent cone that there exist $t_k \geq 0$ and $x^k \to_{\hat{\mathcal{F}}} x^*$ such that $\lim_{k\to\infty} t_k(x^k - x^*) = d$. By the Taylor series expansion, we have

$$\begin{aligned} L^1(x^k, y^*) = {} & L^1(x^*, y^*) + \nabla_x L^1(x^*, y^*)^T (x^k - x^*) + \frac{1}{2}(x^k - x^*)^T \nabla_x^2 L^1(x^*, y^*)(x^k - x^*) \\ & + o(\|x^k - x^*\|^2). \end{aligned} \tag{3}$$

Since $x^k \in \hat{\mathcal{F}}$ and $y^* \in \mathcal{M}^1(x^*)$, we have $L^1(x^k, y^*) = f(x^k)$, $L^1(x^*, y^*) = f(x^*)$, and $\nabla_x L^1(x^*, y^*) = 0$. Thus, from (3), we have

$$f(x^k) = f(x^*) + \frac{1}{2}(x^k - x^*)^T \nabla_x^2 L^1(x^*, y^*)(x^k - x^*) + o(\|x^k - x^*\|^2). \tag{4}$$

Noting that $x^k \in \mathcal{F}$ for every k sufficiently large and x^* is locally optimal, we have $f(x^k) \geq f(x^*)$ when k is sufficiently large. It follows from (4) that

$$(x^k - x^*)^T \nabla_x^2 L^1(x^*, y^*)(x^k - x^*) + o(\|x^k - x^*\|^2) \geq 0.$$

Multiplying it by t_k^2 and taking a limit, we get the desired result. $\square$

Consider the mathematical program with equilibrium constraints (MPEC)

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g(x) \le 0,\ h(x) = 0, \\
& 0 \le G(x) \ \perp\ H(x) \ge 0,
\end{aligned}
\tag{5}
$$

where $\{f, g, h\}$ are assumed as above, $G, H : \mathbb{R}^n \to \mathbb{R}^m$ are all twice differentiable functions, and $a \perp b$ means that the vector a is perpendicular to the vector b. We can treat the MPEC (5) as the following nonlinear programming problem with equality and inequality constraints:

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g(x) \le 0,\ h(x) = 0, \\
& G(x) \ge 0,\ H(x) \ge 0,\ G(x)^T H(x) \le 0.
\end{aligned}
\tag{6}
$$

In order to facilitate the notation, for a given feasible point x^* of (6), we let I_g^* be the same as above and

$$
\begin{cases}
\mathcal{I}^* := \{\, i \mid G_i(x^*) = 0 < H_i(x^*)\}, \\
\mathcal{J}^* := \{\, i \mid G_i(x^*) = 0 = H_i(x^*)\}, \\
\mathcal{K}^* := \{\, i \mid G_i(x^*) > 0 = H_i(x^*)\}.
\end{cases}
$$

Obviously, $\{\mathcal{I}^*, \mathcal{J}^*, \mathcal{K}^*\}$ is a partition of $\{1, 2, \cdots, m\}$. For simplicity, we also denote by L^r, $\mathcal{C}$, and $\mathcal{M}^r$ the generalized Lagrangian, the critical cone, and the set of all r index multipliers of (6), respectively. It is easy to distinguish them from the context. In particular, the generalized Lagrangian of (6) is

$$
L^r(x, \lambda, \mu, \alpha, \beta, \xi) := rf(x) + g(x)^T\lambda + h(x)^T\mu - G(x)^T\alpha - H(x)^T\beta + \xi G(x)^T H(x),
$$

the critical cone associated with a feasible point x^* of (6) is

$$\mathcal{C}(x^*) := \left\{ d \left| \begin{array}{l} \nabla f(x^*)^T d \leq 0 \\ \nabla g_i(x^*)^T d \leq 0 \text{ if } i \in I_g^*,\ \nabla h_i(x^*)^T d = 0 \text{ for } i = 1, 2, \cdots, q \\ \nabla G_i(x^*)^T d = 0 \text{ if } i \in \mathcal{I}^*,\ \nabla H_i(x^*)^T d = 0 \text{ if } i \in \mathcal{K}^* \\ \nabla G_i(x^*)^T d \geq 0 \text{ if } i \in \mathcal{J}^*,\ \nabla H_i(x^*)^T d \geq 0 \text{ if } i \in \mathcal{J}^* \end{array} \right. \right\},$$

and $\mathcal{M}^r(x^*)$ is the set of all index r multipliers (λ, μ, u, v) such that $0 \neq (x^*, \lambda, \mu, u, v)$ satisfies the following Fritz-John optimality condition of (6):

$$\begin{cases} \nabla_x L_1^r(x, \lambda, \mu, \alpha, \beta, \xi) = 0, \\ \min(\lambda, -g(x)) = 0,\ h(x) = 0,\ \xi \geq 0, \\ \min(\alpha, G(x)) = 0,\ \min(\beta, H(x)) = 0,\ G(x)^T H(x) = 0. \end{cases} \tag{7}$$

For (6), although the MFCQ is not satisfied, some weaker constraint qualifications may hold; see e.g. [41, 42]. The following example illustrates that the RCRCQ may hold and, by Proposition 2.2, the SSONC holds.

Example 2.1 Consider the problem

$$\begin{array}{ll} \min & (x_1 - 1)^2 + (x_2 + 1)^2 \\ \text{s.t.} & x_1 - 1 = 0, \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_1 x_2 \leq 0. \end{array}$$

The only feasible point of the problem is $x^* = (1, 0)$ and the LICQ fails at x^*. Set $h(x) := x_1 - 1$, $g_1(x) := -x_1$, $g_2(x) := -x_2$, and $g_3(x) := x_1 x_2$. Then the active index set of inequality constraints at x^* is $\{2, 3\}$, which contains $\emptyset$, $\{2\}$, $\{3\}$, $\{2, 3\}$ as subsets. It is not hard to see that the families of gradients

$$\{\nabla h(x^*)\}, \{\nabla h(x), \nabla g_2(x^*)\}, \{\nabla h(x), \nabla g_3(x^*)\}, \{\nabla h(x), \nabla g_2(x^*), \nabla g_3(x^*)\}$$

have rank one, two, two, two near x^*, respectively. Thus, the RCRCQ holds at x^*.

Note that there always exist singular NLP multipliers for the MPEC (5) since the MFCQ does not hold at every feasible point. It is interesting that, although a singular NLP multiplier may not be useful for first-order optimality, the second-order optimality conditions in terms of singular NLP multipliers may provide useful information. We illustrate this point by using the following example.

Example 2.2 Consider the problem

$$\begin{array}{ll} \min & -x_2 \\ \text{s.t.} & x_1 - x_2 = 0, \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_1 x_2 \leq 0. \end{array}$$

The only feasible solution, which is certainly the only optimal solution, is $x^* = (0,0)$. The critical cone is

$$\begin{aligned} \mathcal{C}(x^*) &= \{d \in \mathbb{R}^2 \mid -d_2 \leq 0, d_1 - d_2 = 0, -d_1 \leq 0, -d_2 \leq 0\} \\ &= \{d \in \mathbb{R}^2 \mid d_1 = d_2, d_1 \geq 0, d_2 \geq 0\}. \end{aligned}$$

It is obvious that $(\mu, \alpha, \beta, \xi)$ with $\mu = \alpha = \beta = 0, \xi = 1$ is a singular multiplier. For such a singular multiplier, we have

$$\nabla_x^2 L^0(x^*, \mu, \alpha, \beta, \xi) = \nabla^2 \varphi(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $\varphi(x_1 x_2) := x_1 x_2$. Hence, for each $d \in \mathcal{C}(x^*) \setminus \{0\}$, we have $d = (h, h)$ with $h > 0$. Therefore,

$$d^T \nabla_x^2 L^0(x^*, \mu, \alpha, \beta, \xi) d = 2h^2 > 0$$

holds for each $d \in \mathcal{C}(x^*) \setminus \{0\}$. That is, the RSOSC with $r = 0$ is satisfied.

The above two examples illustrate that, in some cases, we can apply the standard results for NLP to the MPEC (5) directly. However, since the Abadie CQ for (6) is only satisfied in fairly restrictive circumstances (see, e.g., [42]) and the above constraint qualifications imply the Abadie CQ, it is more reasonable to investigate the properties of (5) from the MPEC structure and specificity. In the next section, we investigate the MPEC second-order optimality conditions.

3 MPEC Second-order Optimality Conditions

First, we review the first-order optimality conditions for MPEC. It is easy to see that the MPEC (5) can be rewritten as the following optimization problem with a geometric constraint:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & F(x) \in \Lambda, \end{aligned} \tag{8}$$

where

$$F(x) := (g(x), h(x), \Psi(x))^T, \quad \Psi(x) := (-G_1(x), -H_1(x), \cdots, -G_m(x), -H_m(x)),$$

and

$$\Lambda :=]-\infty, 0]^p \times \{0\}^q \times C^m, \quad C := \{\, (a,b) \in \mathbb{R}^2 \mid 0 \le -a \perp -b \ge 0\}.$$

In the rest of the paper, we denote by X the feasible region of the MPEC (5) and define the *generalized MPEC-Lagrangian function* of the MPEC (5) as

$$\begin{aligned} L^r_{MPEC}(x, \lambda, \mu, u, v) \quad &:= \quad rf(x) + g(x)^T\lambda + h(x)^T\mu - G(x)^T u - H(x)^T v \\ &= \quad rf(x) + F(x)^T y, \quad y := (\lambda, \mu, u, v). \end{aligned}$$

We need the following normal cones.

Definition 3.1 [43, 44] The *regular normal cone* of a set $\Omega \subset \mathbb{R}^n$ at $x^* \in \Omega$ is a closed convex cone defined by

$$\hat{\mathcal{N}}_\Omega(x^*) := \{d \mid d^T(x - x^*) \le o(\|x - x^*\|) \text{ for each } x \in \Omega\},$$

and the *limiting normal cone* of Ω at $x^* \in \Omega$ is a closed cone defined by

$$\mathcal{N}_\Omega(x^*) := \{d \mid d = \lim_{k\to\infty} d^k \text{ with } d^k \in \hat{\mathcal{N}}_\Omega(x^k) \text{ and } x^k \to_\Omega x^*\}.$$

Definition 3.2 [2,4–6] Let $x^* \in X$. We say that x^* is *Clarke stationary* (C-stationary) to (5) iff there exists (λ, μ, u, v) satisfying

$$\begin{cases} \nabla_x L^1_{MPEC}(x^*, \lambda, \mu, u, v) = 0, \\ \lambda \geq 0,\ g(x^*)^T\lambda = 0, \\ u_i = 0, \qquad i \in \mathcal{K}^*, \\ v_i = 0, \qquad i \in \mathcal{I}^*, \\ \text{either } u_i v_i \geq 0, \qquad i \in \mathcal{J}^*. \end{cases} \tag{9}$$

We say that x^* is *Mordukhovich stationary* (M-stationary) to (5) iff there exists (λ, μ, u, v) satisfying

$$\begin{cases} \nabla_x L^1_{MPEC}(x^*, \lambda, \mu, u, v) = 0, \\ \lambda \geq 0,\ g(x^*)^T\lambda = 0, \\ u_i = 0, \qquad i \in \mathcal{K}^*, \\ v_i = 0, \qquad i \in \mathcal{I}^*, \\ \text{either } u_i v_i = 0 \text{ or } u_i > 0, v_i > 0, \qquad i \in \mathcal{J}^*. \end{cases} \tag{10}$$

We say that x^* is *strongly stationary* (S-stationary) to (5) iff there exists (λ, μ, u, v) satisfying

$$\begin{cases} \nabla_x L^1_{MPEC}(x^*, \lambda, \mu, u, v) = 0, \\ \lambda \geq 0,\ g(x^*)^T\lambda = 0, \\ u_i = 0, \qquad i \in \mathcal{K}^*, \\ v_i = 0, \qquad i \in \mathcal{I}^*, \\ u_i \geq 0,\ v_i \geq 0, \qquad i \in \mathcal{J}^*. \end{cases} \tag{11}$$

By straightforward calculation, we can get the following result (see, e.g., [3,8]).

Proposition 3.1 *For any* $(a,b) \in C$*, we have*

$$\begin{aligned}
\mathcal{T}_C(a,b) &= \left\{ (d_1,d_2) \;\middle|\; \begin{array}{r} d_1 = 0, d_2 \in \mathbb{R} \text{ if } a = 0 > b \\ d_1 \in \mathbb{R}, d_2 = 0 \text{ if } a < 0 = b \\ (d_1,d_2) \in C \text{ if } a = b = 0 \end{array} \right\}, \\
\hat{\mathcal{N}}_C(a,b) &= \left\{ (d_1,d_2) \;\middle|\; \begin{array}{r} d_1 \in \mathbb{R}, d_2 = 0 \text{ if } a = 0 > b \\ d_1 = 0, d_2 \in \mathbb{R} \text{ if } a < 0 = b \\ d_1 \geq 0, d_2 \geq 0 \text{ if } a = b = 0 \end{array} \right\}, \\
\mathcal{N}_C(a,b) &= \left\{ (d_1,d_2) \;\middle|\; \begin{array}{r} d_1 \in \mathbb{R}, d_2 = 0 \text{ if } a = 0 > b \\ d_1 = 0, d_2 \in \mathbb{R} \text{ if } a < 0 = b \\ \text{either } d_1 > 0, d_2 > 0 \text{ or } d_1 d_2 = 0 \text{ if } a = b = 0 \end{array} \right\}.
\end{aligned}$$

Based on the above result, we can easily obtain the following result.

Proposition 3.2 *Let* $x^* \in X$*. The following statements are true:*

(a) *The MPEC linearized cone*

$$\begin{aligned}
\mathcal{L}_{MPEC}(x^*) &:= \{d \mid \nabla F(x^*)^T d \in \mathcal{T}_\Lambda(F(x^*))\} \\
&= \left\{ d \;\middle|\; \begin{array}{l} \nabla g_i(x^*)^T d \leq 0, \quad i \in I_g^* \\ \nabla h_i(x^*)^T d = 0, \quad i = 1, \cdots, q \\ \nabla G_i(x^*)^T d = 0, \quad i \in \mathcal{I}^* \\ \nabla H_i(x^*)^T d = 0, \quad i \in \mathcal{K}^* \\ 0 \leq \nabla G_i(x^*)^T d \perp \nabla H_i(x^*)^T d \geq 0, \quad i \in \mathcal{J}^* \end{array} \right\}.
\end{aligned}$$

(b) *The M-stationarity condition is equivalent to*

$$\nabla f(x^*) + \nabla F(x^*) y = 0, \quad y \in \mathcal{N}_\Lambda(F(x^*)).$$

(c) *The S-stationarity condition is equivalent to*

$$\nabla f(x^*) + \nabla F(x^*) y = 0, \quad y \in \hat{\mathcal{N}}_\Lambda(F(x^*)).$$

We now study the second-order optimality conditions for the MPEC (5) by C-/M-/S-multipliers. To this end, we define the *MPEC critical cone* at $x^* \in X$ as

$$\mathcal{C}_{MPEC}(x^*) := \mathcal{L}_{MPEC}(x^*) \cap \{d \mid \nabla f(x^*)^T d \leq 0\}. \tag{12}$$

For a given $x^* \in X$, we denote by $\mathcal{M}_S^r(x^*)$ the set of all index r S-multipliers (λ, μ, u, v) such that $(r, \lambda, \mu, u, v) \neq 0$ and (11) is satisfied with L^1_{MPEC} replaced by L^r_{MPEC}, denote by $\mathcal{M}_M^r(x^*)$ the set of all index r M-multipliers (λ, μ, u, v) such that $(r, \lambda, \mu, u, v) \neq 0$ and (10) is satisfied with L^1_{MPEC} replaced by L^r_{MPEC}, and denote by $\mathcal{M}_C^r(x^*)$ the set of all index r C-multipliers (λ, μ, u, v) such that $(r, \lambda, \mu, u, v) \neq 0$ and (9) is satisfied with L^1_{MPEC} replaced by L^r_{MPEC}, respectively.

Definition 3.3 Let $x^* \in X$.

(i) We say that the *S-multiplier strong second-order necessary condition* (S-SSONC) holds at x^* iff $\mathcal{M}_S^1(x^*) \neq \emptyset$ and, for every $y^* \in \mathcal{M}_S^1(x^*)$, there holds

$$d^T \nabla_x^2 L^1_{MPEC}(x^*, y^*) d \geq 0, \qquad \forall d \in \mathcal{C}_{MPEC}(x^*).$$

We say that the *S-multiplier strong second-order sufficient condition* (S-SSOSC) holds at x^* iff, for every $y^* \in \mathcal{M}_S^1(x^*)$, there holds

$$d^T \nabla_x^2 L^1_{MPEC}(x^*, y^*) d > 0, \qquad \forall d \in \mathcal{C}_{MPEC}(x^*) \backslash \{0\}.$$

(ii) We say that the *S-multiplier refined second-order necessary condition* (S-RSONC) holds at x^* iff, for every $d \in \mathcal{C}_{MPEC}(x^*)$, there exists $y^* \in \mathcal{M}_S^r(x^*)$ such that

$$d^T \nabla_x^2 L^r_{MPEC}(x^*, y^*) d \geq 0.$$

We say that the *S-multiplier refined second-order sufficient condition* (S-RSOSC) holds at x^* iff, for every $d \in \mathcal{C}_{MPEC}(x^*) \backslash \{0\}$, there exists $y^* \in \mathcal{M}_S^r(x^*)$ such that

$$d^T \nabla_x^2 L^r_{MPEC}(x^*, y^*) d > 0.$$

(iii) We say that the *S-multiplier weak second-order necessary condition* (S-WSONC) holds at x^* iff there exists $y^* \in \mathcal{M}_S^1(x^*)$, there holds

$$d^T \nabla_x^2 L_{MPEC}^1(x^*, y^*) d \geq 0, \qquad \forall d \in \bar{\mathcal{C}}_{MPEC}(x^*),$$

where

$$\bar{\mathcal{C}}_{MPEC}(x^*) := \left\{ d \;\middle|\; \begin{array}{l} \nabla g_i(x^*)^T d = 0 \text{ if } i \in I_g^*, \quad \nabla h_i(x^*)^T d = 0 \text{ for } i = 1, \cdots, q \\ \nabla G_i(x^*)^T d = 0 \text{ if } i \in \mathcal{I}^* \cup \mathcal{J}^*, \quad \nabla H_i(x^*)^T d = 0 \text{ if } i \in \mathcal{K}^* \cup \mathcal{J}^* \end{array} \right\}.$$

It is well-known that there is a connection between the NLP multipliers and the S-multipliers; see, e.g., [12,17]. We next investigate the connection between the classical second-order optimality condition and the S-multiplier second-order optimality condition. Although the equation (13) in Theorem 3.1 is given in [19, Proposition 1], we give a proof for completeness.

Theorem 3.1 (a) *Let $d \in \mathcal{C}_{MPEC}(x^*)$ and $(\lambda, \mu, \alpha, \beta, \xi) \in \mathcal{M}^r(x^*)$. If*

$$d^T \nabla_x^2 L^r(x^*, \lambda, \mu, \alpha, \beta, \xi) d \;\geq (>)\; 0,$$

then there exists (u, v) such that $(\lambda, \mu, u, v) \in \mathcal{M}_S^r(x^*)$ and

$$d^T \nabla_x^2 L_{MPEC}^r(x^*, \lambda, \mu, u, v) d \;\geq (>)\; 0.$$

(b) *Let $d \in \mathcal{C}(x^*)$ and $(\lambda, \mu, u, v) \in \mathcal{M}_S^r(x^*)$. If*

$$d^T \nabla_x^2 L_{MPEC}^r(x^*, \lambda, \mu, u, v) d \;\geq (>)\; 0,$$

then there exists (α, β, ξ) such that $(\lambda, \mu, \alpha, \beta, \xi) \in \mathcal{M}^r(x^*)$ and

$$d^T \nabla_x^2 L^r(x^*, \lambda, \mu, \alpha, \beta, \xi) d \;\geq (>)\; 0.$$

Proof. Set $u := \alpha - \xi H(x^*)$ and $v := \beta - \xi G(x^*)$ with

$$\xi \geq \max\left(0, \max_{i\in\mathcal{I}^*}(-\frac{u_i}{H_i(x^*)}), \max_{i\in\mathcal{K}^*}(-\frac{v_i}{G_i(x^*)})\right).$$

Then $(\lambda,\mu,u,v) \in \mathcal{M}_S^r(x^*)$ if and only if $(\lambda,\mu,\alpha,\beta,\xi) \in \mathcal{M}^r(x^*)$; see, e.g., [12,17]. By direct calculation, we get

$$\begin{aligned}\nabla_x^2 L^r(x^*,\lambda,\mu,\alpha,\beta,\xi) \;\; &= \;\; \nabla_x^2 L^r_{MPEC}(x^*,\lambda,\mu,u,v)\\ &\quad +\xi\sum_{i=1}^m \left(\nabla G_i(x^*)\nabla H_i(x^*)^T + \nabla H_i(x^*)\nabla G_i(x^*)^T\right),\end{aligned}$$

and hence

$$d^T\nabla_x^2 L^r(x^*,\lambda,\mu,\alpha,\beta,\xi)d = d^T\nabla_x^2 L^r_{MPEC}(x^*,\lambda,\mu,u,v)d + 2\xi\sum_{i=1}^m \nabla G_i(x^*)^T d\;\nabla H_i(x^*)^T d. \qquad (13)$$

Therefore, if $d \in \mathcal{C}_{MPEC}(x^*)$, we have

$$d^T\nabla_x^2 L^r(x^*,\lambda,\mu,\alpha,\beta,\xi)d = d^T\nabla_x^2 L^r_{MPEC}(x^*,\lambda,\mu,u,v)d,$$

which means that (a) is true.

We next show (b). Let $d \in \mathcal{C}(x^*)$. If $d \in \mathcal{C}_{MPEC}(x^*)$, then the result follows from (13) immediately. If $d \in \mathcal{C}(x^*) \setminus \mathcal{C}_{MPEC}(x^*)$, there must exist a $j_0 \in \mathcal{J}^*$ such that $G_{j_0}(x^*)^T d > 0$ and $H_{j_0}(x^*)^T d > 0$. Moreover, by $d \in \mathcal{C}(x^*)$, we have $\nabla G_i(x^*)^T d \nabla H_i(x^*)^T d \geq 0$ for each i. Thus, the result must hold if ξ is sufficiently large. □

Corollary 3.1 *The RSONC (or RSOSC/WSONC) holds at x^* for* (6) *if and only if the S-RSONC (or S-RSOSC/S-WSONC) holds at x^*. Moreover, if the SSOSC (or SSONC) holds at x^* for* (6)*, then the S-SSOSC (or S-SSONC) holds at x^*.*

Proof. The first part of the corollary is obvious from Theorem 3.1. We next show that the SSOSC implies the S-SSOSC. For any given $(\lambda,\mu,u,v) \in \mathcal{M}_S^r(x^*)$ and $d \in \mathcal{C}_{MPEC}(x^*)$, there exists (α,β,γ) such that $(\lambda,\mu,\alpha,\beta,\gamma) \in \mathcal{M}^r(x^*)$. Since the SSOSC holds at x^* and $\mathcal{C}_{MPEC}(x^*) \subseteq \mathcal{C}(x^*)$, we have from (1) of Theorem 3.1 that the S-SSOSC holds. In a similar way, we can show that the SSONC implies the

S-SSONC. $\square$

The following example shows that the S-SSOSC does not imply the SSOSC.

Example 3.1 Consider the problem

$$\begin{array}{ll} \min & x_1 x_2 \\ \text{s.t.} & x_2 \geq 0, \\ & 0 \leq x_1 - x_2 \perp x_1 + x_2 \geq 0. \end{array}$$

It is obvious that $x^* = (0,0)$ is an optimal solution. Since, for any S-multiplier (λ, u, v),

$$\nabla_x^2 L^1_{MPEC}(x^*, \lambda, u, v) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the MPEC critical cone is $\mathcal{C}_{MPEC}(x^*) = \{(d_1, d_2) \mid d_1 = d_2 \geq 0\}$, it is not hard to know that the S-SSOSC holds at x^*. On the other hand, it is obvious that $(\lambda, \alpha, \beta, \xi)$ with $\lambda = \alpha = \beta = \xi = 0$ is an NLP-multiplier. The critical cone is $\mathcal{C}(x^*) = \{(d_1, d_2) \mid d_1 - d_2 \geq 0, d_2 \geq 0\}$. Picking $\bar{d} = (1, 0)$, we have

$$\bar{d}^T \nabla_x^2 L^1(x^*, \lambda, \alpha, \beta, \xi)\bar{d} = 0 \not> 0,$$

which implies that the SSOSC does not hold at x^*.

The following example shows that the S-SSONC does not imply the SSONC.

Example 3.2 Consider the problem

$$\begin{array}{ll} \min & -x_1^2 - x_2^2 \\ \text{s.t.} & x_1 - x_2 = 0, \\ & 0 \leq x_1 \perp x_2 \geq 0. \end{array}$$

It is obvious that the only feasible point $x^* = (0,0)$ is an optimal solution. Since the MPEC critical cone at x^* is $\mathcal{C}_{MPEC}(x^*) = \{(d_1, d_2) \mid d_1 = d_2 = 0\}$, the S-SSONC holds at x^* for any S-multiplier. On the other hand, it is obvious that $(\lambda, \alpha, \beta, \xi) = (0,0,0,0)$ is an NLP-multiplier. The critical cone is

$\mathcal{C}(x^*) = \{(d_1, d_2) \mid d_1 = d_2 \geq 0\}$. Picking $\bar{d} = (1,1)$, we have

$$\bar{d}^T \nabla_x^2 L^1(x^*, \lambda, \alpha, \beta, \xi)\bar{d} = -4 < 0,$$

which implies that the SSONC does not hold at x^*.

Scheel and Scholtes [2, Theorem 7 (2)] showed that the S-RSOSC at a feasible point with $r = 1$ implies that the point is a strict local minimizer. In what follows, we show that the second-order sufficient condition in terms of either the S-multiplier or the singular S-multiplier provides a useful information for local optimality. This is very interesting for MPEC since singular S-multipliers always exist.

Theorem 3.2 *Let $x^* \in X$. If the S-RSOSC holds at x^*, then there exist $\delta > 0$ and $c > 0$ such that*

$$f(x) \geq f(x^*) + c\|x - x^*\|^2, \qquad \forall x \in X \cap \mathcal{B}_\delta(x^*).$$

Proof. By Corollary 3.1, the S-RSOSC holds at x^* if and only if the RSOSC holds at x^*. The conclusion follows from Proposition 2.1 immediately. □

In order to study second-order necessary optimality conditions for MPEC, we next extend various relaxations of LICQ for (1) in Definition 2.2 to the MPEC (5). All these conditions are weaker than the MPEC-LICQ and MPEC Linear CQ. The latter means that all constraint functions are affine [4]. The MPEC-CRCQ has been studied to analyze convergence of relaxation methods for solving MPEC in [45, 46]. The other weaker constraint qualifications may be also useful in convergence analysis.

Definition 3.4 Let $x^* \in X$.

(i) We say that the *MPEC constant rank constraint qualification* (MPEC-CRCQ) is satisfied at x^* iff there exists $\delta > 0$ such that, for any $\mathcal{I}_1 \subseteq I_g^*$, $\mathcal{I}_2 \subseteq \{1, \cdots, q\}$, $\mathcal{I}_3 \subseteq \mathcal{I}^* \cup \mathcal{J}^*$, and $\mathcal{I}_4 \subseteq \mathcal{K}^* \cup \mathcal{J}^*$, the family of gradients

$$\left\{\nabla g_i(x), \nabla h_j(x), \nabla G_\imath(x), \nabla H_\jmath(x) \,\middle|\, i \in \mathcal{I}_1, j \in \mathcal{I}_2, \imath \in \mathcal{I}_3, \jmath \in \mathcal{I}_4\right\}$$

has the same rank for each $x \in \mathcal{B}_\delta(x^*)$.

(ii) We say that the *MPEC relaxed constant rank constraint qualification* (MPEC-RCRCQ) is satisfied at x^* iff there exists $\delta > 0$ such that, for any $\mathcal{I}_1 \subseteq I_g^*$ and $\mathcal{I}_2, \mathcal{I}_3 \subseteq \mathcal{J}^*$, the family of gradients

$$\left\{ \nabla g_i(x), \nabla h_j(x), \nabla G_\imath(x), \nabla H_\jmath(x) \,\middle|\, i \in \mathcal{I}_1, j = 1, \cdots, q, \imath \in \mathcal{I}^* \cup \mathcal{I}_2, \jmath \in \mathcal{K}^* \cup \mathcal{I}_3 \right\}$$

has the same rank for each $x \in \mathcal{B}_\delta(x^*)$.

(iii) We say that the *MPEC weak relaxed constant rank* condition (MPEC-WRCR) is satisfied at x^* iff there exists $\delta > 0$ such that, for any $\mathcal{I} \subseteq I_g^*$, the family of gradients

$$\left\{ \nabla g_i(x), \nabla h_j(x), \nabla G_\imath(x), \nabla H_\jmath(x) \,\middle|\, i \in \mathcal{I}, j = 1, \cdots, q, \imath \in \mathcal{I}^* \cup \mathcal{J}^*, \jmath \in \mathcal{K}^* \cup \mathcal{J}^* \right\}$$

has the same rank for each $x \in \mathcal{B}_\delta(x^*)$.

(iv) We say that the *MPEC weak constant rank* condition (MPEC-WCR) is satisfied at x^* iff there exists $\delta > 0$ such that the family of gradients

$$\left\{ \nabla g_i(x), \nabla h_j(x), \nabla G_\imath(x), \nabla H_\jmath(x) \,\middle|\, i \in I_g^*, j = 1, \cdots, q, \imath \in \mathcal{I}^* \cup \mathcal{J}^*, \jmath \in \mathcal{K}^* \cup \mathcal{J}^* \right\}$$

has the same rank for each $x \in \mathcal{B}_\delta(x^*)$.

We are ready to study the second-order necessary conditions for MPEC. To this end, we introduce two sets. For any $y^* \in \mathcal{M}_C^1(x^*)$, define

$$\tilde{X} := \left\{ x \,\middle|\, \begin{array}{l} g_i(x) = 0 \text{ if } i \in I_g^* \text{ and } \lambda_i^* > 0, \quad g_i(x) \leq 0 \text{ if } i \in I_g^* \text{ and } \lambda_i^* = 0 \\ h_i(x) = 0 \text{ for } i = 1, \cdots, q, \quad G_i(x) = 0 \text{ if } i \in \mathcal{I}^*, \ H_i(x) = 0 \text{ if } i \in \mathcal{K}^* \\ G_i(x) = 0 \text{ if } i \in \mathcal{J}^* \text{ and } u_i^* \neq 0, \quad H_i(x) = 0 \text{ if } i \in \mathcal{J}^* \text{ and } v_i^* \neq 0, \\ 0 \leq G_i(x) \perp H_i(x) \geq 0 \text{ if } i \in \mathcal{J}^* \end{array} \right\}.$$

Its linearized cone at x^* is given by

$$\tilde{\mathcal{C}}(x^*) = \left\{ d \;\middle|\; \begin{array}{l} \nabla g_i(x^*)^T d = 0 \text{ if } i \in I_g^* \text{ and } \lambda_i^* > 0, \quad \nabla g_i(x^*)^T d \leq 0 \text{ if } i \in I_g^* \text{ and } \lambda_i^* = 0 \\ \nabla h_i(x^*)^T d = 0 \text{ for } i = 1, 2, \cdots, q \\ \nabla G_i(x^*)^T d = 0 \text{ if } i \in \mathcal{I}^*, \quad \nabla H_i(x^*)^T d = 0 \text{ if } i \in \mathcal{K}^* \\ \nabla G_i(x^*)^T d = 0 \text{ if } i \in \mathcal{J}^* \text{ and } u_i^* \neq 0, \quad \nabla H_i(x^*)^T d = 0 \text{ if } i \in \mathcal{J}^* \text{ and } v_i^* \neq 0, \\ 0 \leq \nabla G_i(x^*)^T d \perp \nabla H_i(x^*)^T d \geq 0 \text{ if } i \in \mathcal{J}^* \end{array} \right\}.$$

It is easy to see that $\tilde{X}$ is a subset of the feasible region X locally around x^*. In the case of nonlinear programs, $\tilde{\mathcal{C}}(x^*)$ is equal to the classical critical cone. The following second-order condition holds without any constraint qualification.

Theorem 3.3 *Let x^* be a locally optimal solution of* (5) *and $y^* \in \mathcal{M}_C^1(x^*)$. Then*

$$d^T \nabla_x^2 L_{MPEC}^1(x^*, y^*) d \geq 0, \quad \forall d \in \mathcal{T}_{\tilde{X}}(x^*).$$

Proof. Let $d \in \mathcal{T}_{\tilde{X}}(x^*)$. By definition, there exist $t_k \geq 0$ and $x^k \to_{\tilde{X}} x^*$ such that $\lim\limits_{k\to\infty} t_k(x^k - x^*) = d$. By the Taylor series expansion, we have

$$\begin{aligned} L_{MPEC}^1(x^k, y^*) \;\; = \;\; & L_{MPEC}^1(x^*, y^*) + \nabla_x L_{MPEC}^1(x^*, y^*)^T (x^k - x^*) \\ & + \frac{1}{2}(x^k - x^*)^T \nabla_x^2 L_{MPEC}^1(x^*, y^*)(x^k - x^*) + o(\|x^k - x^*\|^2). \end{aligned} \tag{14}$$

Since $x^k \in \tilde{X}$ and $y^* \in \mathcal{M}_C^1(x^*)$, we have $L_{MPEC}^1(x^k, y^*) = f(x^k)$, $\nabla_x L_{MPEC}^1(x^*, y^*) = 0$, and $L_{MPEC}^1(x^*, y^*) = f(x^*)$. Thus, we have from (14) that

$$f(x^k) = f(x^*) + \frac{1}{2}(x^k - x^*)^T \nabla_x^2 L_{MPEC}^1(x^*, y^*)(x^k - x^*) + o(\|x^k - x^*\|^2). \tag{15}$$

Since $x^k \in X$ for each k sufficiently large and x^* is locally optimal, we have $f(x^k) \geq f(x^*)$ when k is sufficiently large. Thus, it follows from (15) that

$$(x^k - x^*)^T \nabla_x^2 L_{MPEC}^1(x^*, y^*)(x^k - x^*) + o(\|x^k - x^*\|^2) \geq 0.$$

Multiplying it by t_k^2 and taking a limit, we get the desired result. $\square$

As a consequence, we have the following strong second-order necessary optimality condition.

Corollary 3.2 *Let x^* be a locally optimal solution of (5) and $y^* \in \mathcal{M}_C^1(x^*)$. If the Abadie CQ for $\tilde{X}$ holds at x^*, i.e., $\tilde{\mathcal{C}}(x^*) \subseteq \mathcal{T}_{\tilde{X}}(x^*)$, then*

$$d^T \nabla_x^2 L_{MPEC}^1(x^*, y^*) d \geq 0, \quad \forall d \in \tilde{\mathcal{C}}(x^*).$$

We now investigate the conditions under which $\tilde{\mathcal{C}}(x^*) \subseteq \mathcal{T}_{\tilde{X}}(x^*)$ holds.

The following result shows that the MPEC-RCRCQ is a constraint qualification for M-stationarity.

Lemma 3.1 *Suppose that the MPEC-RCRCQ holds at $x^* \in X$. Then the MPEC Abadie CQ holds at x^*, i.e., $\mathcal{T}_X(x^*) = \mathcal{L}_{MPEC}(x^*)$.*

Proof. Let $\mathcal{P}(\mathcal{J}^*) := \{(\mathcal{J}_1^*, \mathcal{J}_2^*) \mid \mathcal{J}_1^* \cup \mathcal{J}_2^* = \mathcal{J}^*, \mathcal{J}_1^* \cap \mathcal{J}_2^* = \emptyset\}$. For each partition $(\mathcal{J}_1^*, \mathcal{J}_2^*) \in \mathcal{P}(\mathcal{J}^*)$, we consider the following restricted problem associated with (5):

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq 0, \ h(x) = 0, \\ & G_{\mathcal{I}^* \cup \mathcal{J}_1^*}(x) = 0, \ G_{\mathcal{J}_2^*}(x) \geq 0, \\ & H_{\mathcal{K}^* \cup \mathcal{J}_2^*}(x) = 0, \ G_{\mathcal{J}_1^*}(x) \geq 0. \end{aligned} \tag{16}$$

Denote by $X(\mathcal{J}_1^*, \mathcal{J}_2^*)$ the feasible region of (16). It is not difficult to see that

$$\mathcal{T}_X(x^*) = \bigcup_{(\mathcal{J}_1^*, \mathcal{J}_2^*) \in \mathcal{P}(\mathcal{J}^*)} \mathcal{T}_{X(\mathcal{J}_1^*, \mathcal{J}_2^*)}(x^*), \quad \mathcal{L}_{MPEC}(x^*) = \bigcup_{(\mathcal{J}_1^*, \mathcal{J}_2^*) \in \mathcal{P}(\mathcal{J}^*)} \mathcal{L}_{X(\mathcal{J}_1^*, \mathcal{J}_2^*)}(x^*), \tag{17}$$

where $\mathcal{L}_{X(\mathcal{J}_1^*, \mathcal{J}_2^*)}(x^*)$ is the linearized tangent cone of $X(\mathcal{J}_1^*, \mathcal{J}_2^*)$ at x^*.

Since x^* satisfies the MPEC-RCRCQ, the RCRCQ holds at $x^* \in X(\mathcal{J}_1^*, \mathcal{J}_2^*)$ for each partition $(\mathcal{J}_1^*, \mathcal{J}_2^*) \in \mathcal{P}(\mathcal{J}^*)$. Since the RCRCQ implies the RCPLD, it follows from Lemma 2.1 that

$$\mathcal{T}_{X(\mathcal{J}_1^*, \mathcal{J}_2^*)}(x^*) = \mathcal{L}_{X(\mathcal{J}_1^*, \mathcal{J}_2^*)}(x^*), \qquad \forall (\mathcal{J}_1^*, \mathcal{J}_2^*) \in \mathcal{P}(\mathcal{J}^*).$$

This, together with (17), implies $\mathcal{T}_X(x^*) = \mathcal{L}_{MPEC}(x^*)$. $\square$

Theorem 3.4 *If the MPEC-RCRCQ holds at $x^* \in X$ and $y^* \in \mathcal{M}^1_C(x^*)$, then $\tilde{\mathcal{C}}(x^*) \subseteq \mathcal{T}_{\tilde{X}}(x^*)$.*

Proof. Since the MPEC-RCRCQ holds at $x^* \in X$, the MPEC-RCRCQ holds at $x^* \in \tilde{X}$. It follows from Lemma 3.1 that $\tilde{\mathcal{C}}(x^*) \subseteq \mathcal{T}_{\tilde{X}}(x^*)$. $\square$

Now we are ready to show that, under the MPEC-RCRCQ, any locally optimal solution satisfies the S-SSONC. This result improves the result of Scheel and Scholtes [2, Theorem 7 (1)], who proved the S-RSONC (weaker than the S-SSONC) under the MPEC SMFCQ (stronger than the MPEC-RCRCQ).

Theorem 3.5 *Suppose that x^* is a locally optimal solution of (5) and the MPEC-RCRCQ holds at x^*. Then $\mathcal{M}^1_M(x^*) \neq \emptyset$ and, for any $y^* \in \mathcal{M}^1_S(x^*)$,*

$$d^T \nabla^2_x L^1_{MPEC}(x^*, y^*) d \geq 0, \qquad \forall d \in \mathcal{C}_{MPEC}(x^*).$$

Proof. By Lemma 3.1, we have $\mathcal{T}_X(x^*) = \mathcal{L}_{MPEC}(x^*)$ and then, by [4, Theorem 3.1], $\mathcal{M}^1_M(x^*) \neq \emptyset$. Let $y^* \in \mathcal{M}^1_S(x^*)$. By Theorem 3.4, we have $\tilde{\mathcal{C}}(x^*) = \mathcal{T}_{\tilde{X}}(x^*)$. By virtue of Corollary 3.2, it suffices to show that $\mathcal{C}_{MPEC}(x^*) \subseteq \tilde{\mathcal{C}}(x^*)$. In fact, for any $d \in \mathcal{C}_{MPEC}(x^*)$, by definition of MPEC critical cone, we have $\nabla f(x^*)^T d \leq 0$ and $d \in \mathcal{L}_{MPEC}(x^*)$. On the other hand, by the local optimality of x^*, we have

$$\nabla f(x^*)^T d \geq 0, \qquad \forall d \in \mathcal{T}_X(x^*) = \mathcal{L}_{MPEC}(x^*).$$

It follows that $\nabla f(x^*)^T d = 0$. Thus, we have from (11) that $d \in \tilde{\mathcal{C}}(x^*)$. $\square$

Theorem 3.6 *Let x^* be a locally optimal solution of the MPEC (5). If the MPEC-WRCR condition holds at x^*, then, for any $y^* \in \mathcal{M}^1_C(x^*)$, we have*

$$d^T \nabla^2_x L^1_{MPEC}(x^*, y^*) d \geq 0, \qquad \forall d \in \hat{\mathcal{C}}(x^*),$$

where

$$\hat{\mathcal{C}}(x^*) := \left\{ d \;\middle|\; \begin{array}{l} \nabla g_i(x^*)^T d = 0 \text{ if } i \in I^*_g \text{ and } \lambda^*_i > 0 \\ \nabla g_i(x^*)^T d \leq 0 \text{ if } i \in I^*_g \text{ and } \lambda^*_i = 0 \\ \nabla h_i(x^*)^T d = 0 \text{ for } i = 1, \cdots, q \\ \nabla G_i(x^*)^T d = 0 \text{ if } i \in \mathcal{I}^* \cup \mathcal{J}^*, \quad \nabla H_i(x^*)^T d = 0 \text{ if } i \in \mathcal{K}^* \cup \mathcal{J}^* \end{array} \right\}.$$

Proof. Consider the set

$$\hat{X} := \left\{ x \;\middle|\; \begin{array}{l} g_i(x) = 0 \text{ if } i \in I_g^* \text{ and } \lambda_i^* > 0, \quad g_i(x) \le 0 \text{ if } i \in I_g^* \text{ and } \lambda_i^* = 0 \\ h_i(x) = 0 \text{ for } i = 1, \cdots, q \\ G_i(x) = 0 \text{ if } i \in \mathcal{I}^* \cup \mathcal{J}^*, \quad H_i(x) = 0 \text{ if } i \in \mathcal{K}^* \cup \mathcal{J}^* \end{array} \right\}.$$

Since the MPEC-WRCR condition holds at x^*, the RCRCQ holds at x^* for $\hat{X}$. Thus, we have from Lemma 2.1 that

$$\mathcal{T}_{\hat{X}}(x^*) = \hat{\mathcal{C}}(x^*).$$

Therefore, the desired result can be obtained from Theorem 3.3 and $\mathcal{T}_{\hat{X}}(x^*) \subseteq \mathcal{T}_{\tilde{X}}(x^*)$. □

The second-order necessary condition given in Theorem 3.6 is reasonable because the cone $\hat{\mathcal{C}}(x^*)$ is actually the critical cone of the following tightened problem associated with (5):

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g(x) \le 0, \; h(x) = 0, \\ & G_{\mathcal{I}^* \cup \mathcal{J}^*}(x) = 0, \; H_{\mathcal{K}^* \cup \mathcal{J}^*}(x) = 0. \end{array} \tag{18}$$

Theorem 3.7 *Let x^* be a locally optimal solution of the MPEC* (5)*. If the MPEC-WCR condition holds at x^*, then, for any $y^* \in \mathcal{M}_C^1(x^*)$, we have*

$$d^T \nabla_x^2 L_{MPEC}^1(x^*, y^*) d \ge 0, \qquad \forall d \in \bar{\mathcal{C}}_{MPEC}(x^*).$$

Proof. Consider the set

$$\bar{X} := \left\{ x \;\middle|\; \begin{array}{l} g_i(x) = 0, \text{ if } i \in I_g^*, \quad h_i(x) = 0 \text{ for } i = 1, \cdots, q \\ G_i(x) = 0 \text{ if } i \in \mathcal{I}^* \cup \mathcal{J}^*, \quad H_i(x) = 0 \text{ if } i \in \mathcal{K}^* \cup \mathcal{J}^* \end{array} \right\}.$$

Since the MPEC-WCR condition holds at x^*, the RCRCQ holds at x^* for $\hat{X}$. Thus, we have from Lemma 2.1 that

$$\mathcal{T}_{\bar{X}}(x^*) = \bar{\mathcal{C}}_{MPEC}(x^*).$$

Therefore, the desired result can be obtained from Theorem 3.3 and $\mathcal{T}_{\bar{X}}(x^*) \subseteq \mathcal{T}_{\tilde{X}}(x^*)$. □

Note that the cone $\bar{\mathcal{C}}_{MPEC}(x^*)$ defined in Theorem 3.7 is independent of the objective function of (5).

We close this section by illustrating that both $\tilde{\mathcal{C}}(x^*) \subseteq \mathcal{T}_{\tilde{X}}(x^*)$ and the MPEC Abadie CQ are strictly weaker than the MPEC-RCRCQ and, moreover, the MPEC-WRCR and the MPEC-WCR imply neither $\tilde{\mathcal{C}}(x^*) \subseteq \mathcal{T}_{\tilde{X}}(x^*)$ nor the MPEC Abadie CQ, and neither of them is a constraint qualification for the M-stationarity.

The following example shows that $\tilde{\mathcal{C}}(\bar{x}) \subseteq \mathcal{T}_{\tilde{X}}(\bar{x})$ and the MPEC Abadie CQ hold at some point $\bar{x}$, but the MPEC-RCRCQ does not hold at $\bar{x}$.

Example 3.3 Consider the problem

$$\begin{array}{ll} \min & f(x_1, x_2) \equiv 1 \\ \text{s.t.} & x_2 \leq 0, \ -x_1 + x_2^2 \leq 0, \\ & 0 \leq x_1 \perp x_2 \geq 0. \end{array}$$

It is obvious that every feasible point is a globally optimal solution. Take $\bar{x} := (0,0)$. Since $\left\{\binom{1}{0}, \binom{-1}{2x_2}\right\}$ has rank one when $x = \bar{x}$ and rank two when $x_2 \neq \bar{x}_2$, the MPEC-RCRCQ does not hold at $\bar{x}$. Denote by X the feasible region of the problem. It is not hard to show that $(0,0;0,0,0,0)$ is an S-stationary pair for the problem and $\tilde{X} = X$. It is easy to verify that $\mathcal{T}_{\tilde{X}}(\bar{x}) = \tilde{\mathcal{C}}(\bar{x}) = \{(d_1, d_2) \mid d_1 \geq 0, \ d_2 = 0\}$ and $\mathcal{T}_X(\bar{x}) = \mathcal{L}_{MPEC}(\bar{x})$.

The following example shows that both the MPEC-WRCR and MPEC-WCR hold at some point $\bar{x}$, but $\tilde{\mathcal{C}}(\bar{x}) \subseteq \mathcal{T}_{\tilde{X}}(\bar{x})$ and the MPEC Abadie CQ do not hold at $\bar{x}$.

Example 3.4 Consider the problem

$$\begin{array}{ll} \min & f(x_1, x_2) \equiv 1 \\ \text{s.t.} & x_2 - x_1^2 = 0, \\ & 0 \leq -x_1 \perp x_2 \geq 0. \end{array}$$

It is obvious that the only feasible point is $\bar{x} := (0,0)$. Since $\left\{\binom{-2x_1}{1}, \binom{-1}{0}, \binom{0}{1}\right\}$ has rank two for any x, both the MPEC-WRCR and the MPEC-WCR hold at $\bar{x}$. Denote by X the feasible region of the problem. It is not hard to see that $(0,0;1,0,1)$ is an S-stationary pair of the problem and $\tilde{X} = X$. By direct calculation, we can get $\mathcal{T}_{\tilde{X}}(\bar{x}) = \{0\}$ and $\tilde{\mathcal{C}}(\bar{x}) = \{(d_1, d_2) \mid d_1 \leq 0,\ d_2 = 0\}$. Thus, $\tilde{\mathcal{C}}(\bar{x}) \nsubseteq \mathcal{T}_{\tilde{X}}(\bar{x})$. Moreover, $\mathcal{T}_X(\bar{x}) = \{0\}$ and $\mathcal{L}_{MPEC}(\bar{x}) = \{(d_1, d_2) \mid d_1 \leq 0,\ d_2 = 0\}$, which means that the MPEC Abadie CQ does not holds at $\bar{x}$.

The following example, which originates from [17, Example 2.2], shows that both the MPEC-WRCR and MPEC-WCR are not constraint qualifications for M-stationarity.

Example 3.5 Consider the problem

$$\begin{array}{ll} \min & x_1 + x_2 - x_3 - \dfrac{1}{2}x_4 \\ \text{s.t.} & x_4^2 = 0, \\ & -6x_1 + x_3 + x_4 = 0,\ -6x_2 + x_3 = 0, \\ & 0 \leq x_1 \perp x_2 \geq 0. \end{array} \tag{19}$$

It is obvious that the unique feasible point $\bar{x} = (0, 0, 0, 0)$ is a global solution. Moreover, it is not hard to see that the MPEC-WRCR and MPEC-WCR hold at $\bar{x}$. By solving the M-stationarity conditions, we can obtain that the multipliers corresponding to $\mathcal{J}^*$ is $u = v = -2$. Since $u < 0$ and $v < 0$, the unique minimizer $(0, 0, 0, 0)$ is not an M-stationary point.

4 Isolatedness of Local Minimizers

In this section, we investigate conditions under which an M- or S-stationary point of the MPEC (5) must be an isolated local minimizer. Theorem 3.2 shows that any feasible point satisfying the S-RSOSC is locally optimal. However, the following example, which originates from [39], shows that an S-stationary point satisfying the S-SSOSC (stronger than the S-RSOSC) may not be an isolated local minimizer. To ensure a stationary point to be isolated, some constraint qualification is required.

Example 4.1 Consider the problem

$$\begin{array}{ll} \min & \dfrac{1}{2}(x^2 + y^2) \\ \text{s.t.} & h(x) = 0, \\ & 0 \leq x \perp y \geq 0, \end{array}$$

where

$$h(x) := \begin{cases} x^5 \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The feasible region is $\{(0,y) \mid y \geq 0\} \cup \{(\frac{1}{k\pi}, 0) \mid k = 1, 2, \cdots\}$. It is not difficult to verify that every point of the set $\{(\frac{1}{k\pi}, 0) \mid k = 0, 1, 2, \cdots\}$ is both locally optimal and S-stationary, and the S-RSOSC holds at $(0,0)$. However, $(0,0)$ is neither an isolated locally optimal solution nor an isolated S-stationary point.

Definition 4.1 We say that the *M-multiplier strong second-order sufficient condition* (M-SSOSC) holds at $x^* \in X$ iff, for every $y^* \in \mathcal{M}_M^1(x^*)$,

$$d^T \nabla_x^2 L_{MPEC}^1(x^*, y^*) d > 0, \qquad \forall d \in \mathcal{C}_{MPEC}(x^*) \backslash \{0\}.$$

Recently, the CPLD for nonlinear programs in Definition 2.3 was extended to MPEC in [45] and [7], respectively. Note that the constraint qualification in the sense of [7] is weaker than the one in the sense of [45].

Definition 4.2 [7] We say that the *MPEC constant positive linear dependent* condition (MPEC-CPLD) holds at $x^* \in X$ iff, for any $\mathcal{I}_1 \subseteq I_g^*$, $\mathcal{I}_2 \subseteq \{1, \cdots, q\}$, $\mathcal{I}_3 \subseteq \mathcal{I}^* \cup \mathcal{J}^*$ and $\mathcal{I}_4 \subseteq \mathcal{K}^* \cup \mathcal{J}^*$, whenever there exist multipliers, not all zero, $\{\lambda, \mu, u, v\}$ with $\lambda_i \geq 0$ for each $i \in \mathcal{I}_1$, either $u_l v_l = 0$ or $u_l > 0, v_l > 0$ for each $l \in \mathcal{J}^*$, such that

$$\sum_{i \in \mathcal{I}_1} \lambda_i \nabla g_i(x^*) + \sum_{j \in \mathcal{I}_2} \mu_j \nabla h_j(x^*) - \sum_{l \in \mathcal{I}_3} u_l \nabla G_l(x^*) - \sum_{l \in \mathcal{I}_4} v_l \nabla H_l(x^*) = 0,$$

there exists a neighborhood $B(x^*)$ of x^* such that, for any $x \in B(x^*)$, the vectors

$$\{\nabla h_i(x) |\ i \in \mathcal{I}_1\}, \quad \{\nabla g_j(x) |\ j \in \mathcal{I}_2\}, \quad \{\nabla G_l(x) |\ l \in \mathcal{I}_3\}, \quad \{\nabla H_l(x) |\ l \in \mathcal{I}_4\}$$

are linearly dependent.

The MPEC-CPLD is obviously weaker than the MPEC NNAMCQ and MPEC-CRCQ. Motivated by the RCPLD in [33], we define an MPEC-type RCPLD, which is weaker than the MPEC-CPLD and

MPEC-RCRCQ. Given a point x and three index sets $\mathcal{I}_1 \subseteq \{1, \cdots, q\}$ and $\mathcal{I}_2, \mathcal{I}_3 \subseteq \{1, \cdots, m\}$, we denote

$$\mathcal{G}(x; \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3) := \{\nabla h_j(x), \nabla G_\imath(x), \nabla H_\jmath(x) \mid j \in \mathcal{I}_1, \imath \in \mathcal{I}_2, \jmath \in \mathcal{I}_3\}.$$

Definition 4.3 Let $\mathcal{I}_1 \subseteq \{1, \cdots, q\}$, $\mathcal{I}_2 \subseteq \mathcal{I}^*$, and $\mathcal{I}_3 \subseteq \mathcal{K}^*$ be such that $\mathcal{G}(x^*; \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ is a basis for the generated space span $\mathcal{G}(x^*; \{1, \cdots, q\}, \mathcal{I}^*, \mathcal{K}^*)$. We say that the *MPEC-relaxed constant positive linear dependence* (MPEC-RCPLD) holds at x^* iff there exists $\delta > 0$ such that

– $\mathcal{G}(x; \{1, \cdots, q\}, \mathcal{I}^*, \mathcal{K}^*)$ has the same rank for each $x \in \mathcal{B}_\delta(x^*)$;

– for each $\mathcal{I}_4 \subseteq I_g^*$ and $\mathcal{I}_5, \mathcal{I}_6 \subseteq \mathcal{J}^*$, whenever there exist multipliers, not all zero, $\{\lambda, \mu, u, v\}$ with $\lambda_i \geq 0$ for each $i \in \mathcal{I}_4$, either $u_l v_l = 0$ or $u_l > 0, v_l > 0$ for each $l \in \mathcal{J}^*$, such that

$$\sum_{i \in \mathcal{I}_4} \lambda_i \nabla g_i(x^*) + \sum_{j \in \mathcal{I}_1} \mu_j \nabla h_j(x^*) - \sum_{\imath \in \mathcal{I}_2 \cup \mathcal{I}_5} u_\imath \nabla G_\imath(x^*) - \sum_{\jmath \in \mathcal{I}_3 \cup \mathcal{I}_6} v_\jmath \nabla H_\jmath(x^*) = 0,$$

then, for any $x \in \mathcal{B}_\delta(x^*)$, the vectors

$$\{\nabla g_i(x) |\ i \in \mathcal{I}_4\}, \quad \{\nabla h_j(x) |\ j \in \mathcal{I}_1\}, \quad \{\nabla G_\imath(x) |\ \imath \in \mathcal{I}_2 \cup \mathcal{I}_5\}, \quad \{\nabla H_\jmath(x) |\ \jmath \in \mathcal{I}_3 \cup \mathcal{I}_6\}$$

are linearly dependent.

The MPEC-RCPLD has been shown to be a constraint qualification for M-stationarity in [47]. In the case where there is no complementarity constraint, the MPEC-RCPLD reduces to the relaxed constant positive linear dependence (RCPLD) condition introduced recently in [33] for standard nonlinear programs.

The following result can be seen as a corollary of Carathéodory's lemma [48].

Lemma 4.1 [33] *If $x = \sum_{i=1}^{m+p} \alpha_i v_i$ with $\{v_i \mid i = 1, \cdots, m\}$ to be linearly independent and $\alpha_i \neq 0$ for $i = m+1, \cdots, m+p$, then there exist $\mathcal{J} \subseteq \{m+1, \cdots, m+p\}$ and $\bar{\alpha}_i$ for $i \in \{1, \cdots, m\} \cup \mathcal{J}$ such that $x = \sum_{i \in \{1, \cdots, m\} \cup \mathcal{J}} \bar{\alpha}_i v_i$ with $\alpha_i \bar{\alpha}_i > 0$ for every $i \in \mathcal{J}$ and $\{v_i \,|\imath \in \{1, \cdots, m\} \cup \mathcal{J}\}$ is linearly independent.*

Theorem 4.1 *Let x^* be an M-stationary point of the MPEC* (5). *Suppose that both the MPEC-RCPLD and M-SSOSC hold at x^*. Then there exists a constant $\delta > 0$ such that, if $x \in \mathcal{B}_\delta(x^*) \cap X$ and (x, y)*

satisfies (10) *for some* y*, there must hold* $x = x^*$*.*

Proof. Suppose to the contrary that there exist a sequence $\{x^k\} \subseteq X \backslash \{x^*\}$ converging to x^* and an associated multiplier sequence $\{y^k\}$ satisfying

$$\begin{cases} 0 = \nabla f(x^k) + \nabla F(x^k) y^k, \\ y^k \in \mathcal{N}_\Lambda(F(x^k)). \end{cases} \tag{20}$$

We first show that there exists a bounded subsequence $\{\bar{y}^k\}_{k\in K}$ satisfying (20). Assume that $\nabla f(x^k) \neq 0$ for each k (otherwise, we can choose a subsequence and redefine $y^k = 0$ for each $k \in K$, which is obviously bounded and satisfies (20)). For sake of convenience, denote

$$\begin{cases} I_g^k := \{\, i \mid g_i(x^k) = 0\}, \\ \mathcal{I}^k := \{\, i \mid G_i(x^k) = 0 < H_i(x^k)\}, \\ \mathcal{J}^k := \{\, i \mid G_i(x^k) = 0 = H_i(x^k)\}, \\ \mathcal{K}^k := \{\, i \mid G_i(x^k) > 0 = H_i(x^k)\}. \end{cases}$$

Since $\mathcal{I}^* \subseteq \mathcal{I}^k$ and $\mathcal{K}^* \subseteq \mathcal{K}^k$ for each k sufficiently large, it follows from the first equality in (20) that

$$\begin{aligned} 0 \;=\; & \nabla f(x^k) + \sum_{i=1}^{q} \mu_i^k \nabla h_i(x^k) - \sum_{\imath\in\mathcal{I}^*} u_\imath^k \nabla G_\imath(x^k) - \sum_{\jmath\in\mathcal{K}^*} v_\jmath^k \nabla H_\jmath(x^k) - \\ & \sum_{\imath\in\mathcal{I}^k\backslash\mathcal{I}^*\cup\mathcal{J}^k\cap\mathrm{supp}(u^k)} u_\imath^k \nabla G_\imath(x^k) - \sum_{\jmath\in\mathcal{K}^k\backslash\mathcal{K}^*\cup\mathcal{J}^k\cap\mathrm{supp}(v^k)} v_\jmath^k \nabla H_\jmath(x^k) + \sum_{j\in\mathrm{supp}(\lambda^k)} \lambda_j^k \nabla g_j(x^k), \end{aligned} \tag{21}$$

where $\mathrm{supp}(a) := \{i \mid a_i \neq 0\}$. Let $\mathcal{I}_1 \subseteq \{1, \cdots, q\}$, $\mathcal{I}_2 \subseteq \mathcal{I}^*$, and $\mathcal{I}_3 \subseteq \mathcal{K}^*$ be such that $\mathcal{G}(x^*; \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ is a basis for $\mathrm{span}\,\mathcal{G}(x^*; \{1, \cdots, q\}, \mathcal{I}^*, \mathcal{K}^*)$. Then $\mathcal{G}(x^k; \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ is linearly independent for k sufficiently large. Since there is a constant $\delta > 0$ such that the rank of $\mathcal{G}(x; \{1, \cdots, q\}, \mathcal{I}^*, \mathcal{K}^*)$ is constant for each $x \in \mathcal{B}_\delta(x^*)$, we have that $\mathcal{G}(x^k; \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ is a basis for

$$\mathrm{span}\,\mathcal{G}(x^k; \{1, \cdots, q\}, \mathcal{I}^*, \mathcal{K}^*)$$

for k sufficiently large. Thus, it follows from (21) and Lemma 4.1 that there exist $\mathcal{I}_4^k \subseteq \mathrm{supp}(\lambda^k)$, $\mathcal{I}_5^k \subseteq \mathcal{I}^k \setminus \mathcal{I}^* \cup \mathcal{J}^k \cap \mathrm{supp}(u^k)$, $\mathcal{I}_6^k \subseteq \mathcal{K}^k \setminus \mathcal{K}^* \cup \mathcal{J}^k \cap \mathrm{supp}(v^k)$, and $\bar{\mu}^k$, $\bar{\lambda}^k$, $\bar{u}^k$, and $\bar{v}^k$ such that

$$\begin{aligned} 0 &= \nabla f(x^k) + \sum_{i\in\mathcal{I}_1} \bar{\mu}_i^k \nabla h_i(x^k) - \sum_{\imath\in\mathcal{I}_2} \bar{u}_\imath^k \nabla G_\imath(x^k) - \sum_{\jmath\in\mathcal{I}_3} \bar{v}_\jmath^k \nabla H_\jmath(x^k) \\ &\quad + \sum_{j\in\mathcal{I}_4^k} \bar{\lambda}_j^k \nabla g_j(x^k) - \sum_{\imath\in\mathcal{I}_5^k} \bar{u}_\imath^k \nabla G_\imath(x^k) - \sum_{\jmath\in\mathcal{I}_6^k} \bar{v}_\jmath^k \nabla H_\imath(x^k), \end{aligned}$$

and

$$\{\nabla g_j(x^k) \mid j \in \mathcal{I}_4^k\}, \{\nabla h_i(x^k) \mid i \in \mathcal{I}_1\}, \{\nabla G_\imath(x^k) \mid \imath \in \mathcal{I}_2 \cup \mathcal{I}_5^k\}, \{\nabla H_\jmath(x^k) \mid \jmath \in \mathcal{I}_3 \cup \mathcal{I}_6^k\}$$

are linearly independent for k sufficiently large. Set $\bar{\lambda}_i^k = 0$ for $i \notin \mathcal{I}_4^k$, $\bar{\mu}_i^k = 0$ for $i \notin \mathcal{I}_1$, and $\bar{u}_\imath^k = 0$ for $\imath \notin \mathcal{I}_2 \cup \mathcal{I}_5^k$, $\bar{v}_\jmath^k = 0$ for $\jmath \notin \mathcal{I}_3 \cup \mathcal{I}_6^k$. It is not hard from the above process and Lemma 4.1 that $(\bar{\lambda}^k, \bar{\mu}^k, \bar{u}^k, \bar{v}^k) \in \mathcal{M}_M^1(x^k)$. Without any loss of generality, we assume that $\mathcal{I}_4^k \equiv \mathcal{I}_4$, $\mathcal{I}_5^k \equiv \mathcal{I}_5$, and $\mathcal{I}_6^k \equiv \mathcal{I}_6$. Then

$$\{\nabla g_j(x^k) \mid j \in \mathcal{I}_4\}, \{\nabla h_i(x^k) \mid i \in \mathcal{I}_1\}, \{\nabla G_\imath(x^k) \mid \imath \in \mathcal{I}_2 \cup \mathcal{I}_5\}, \{\nabla H_\jmath(x^k) \mid \jmath \in \mathcal{I}_3 \cup \mathcal{I}_6\} \tag{22}$$

are linearly independent. It is not hard to get that $\mathcal{I}_4 \subseteq I_g^*$, $\mathcal{I}_1 \subseteq \{1, \cdots, q\}$, and $\mathcal{I}_5, \mathcal{I}_6 \in \mathcal{J}^*$ by $\mathcal{I}^k \cup \mathcal{J}^k \cup \mathcal{K}^k = \mathcal{I}^* \cup \mathcal{J}^* \cup \mathcal{K}^*$. Assume that $\bar{y}^k = (\bar{\lambda}^k, \bar{\mu}^k, \bar{u}^k, \bar{v}^k)$ is unbounded and, without any loss of generality, $\bar{y}^k/\|\bar{y}^k\| \to \bar{y}^*$ with $\|\bar{y}^*\| = 1$. Since $\bar{y}^k = (\bar{\lambda}^k, \bar{\mu}^k, \bar{u}^k, \bar{v}^k) \in \mathcal{M}_M^1(x^*)$, we have

$$\begin{cases} 0 = \nabla f(x^k) + \nabla F(x^k)\bar{y}^k, \\ \bar{y}^k \in \mathcal{N}_\Lambda(F(x^k)). \end{cases}$$

Thus, we have from the outer semicontinuity of $\mathcal{N}_\Lambda$ (see, e.g., [44, Proposition 6.6]) that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \Big(\frac{\nabla_x f(x^k)}{\|\bar{y}^k\|} + \frac{\nabla_x F(x^k)\bar{y}^k}{\|\bar{y}^k\|} \Big) = \nabla_x F(x^*)\bar{y}^*, \\ \bar{y}^* &= \lim_{k\to\infty} \frac{\bar{y}^k}{\|\bar{y}^k\|} \in \limsup_{k\to\infty} \mathcal{N}_\Lambda(F(x^k)) = \mathcal{N}_\Lambda(F(x^*)). \end{aligned}$$

Thus, we have $0 \neq \bar{y}^* = (\bar{\lambda}^*, \bar{\mu}^*, \bar{u}^*, \bar{v}^*) \in \mathcal{M}_M^1(x^*)$ and

$$\sum_{i\in\mathcal{I}_4} \bar{\lambda}_i^* \nabla g_i(x^*) + \sum_{j\in\mathcal{I}_1} \bar{\mu}_j^* \nabla h_j(x^*) - \sum_{\imath\in\mathcal{I}_2\cup\mathcal{I}_5} \bar{u}_\imath^* \nabla G_\imath(x^*) - \sum_{\jmath\in\mathcal{I}_3\cup\mathcal{I}_6} \bar{v}_\jmath^* \nabla H_\jmath(x^*) = 0.$$

Clearly, $\bar{\lambda}_i^* \geq 0$ for $i \in \mathcal{I}_4$ and either $\bar{u}_l^* \bar{v}_l^* = 0$ or $\bar{u}_l^* > 0$, $\bar{v}_l^* > 0$ for $l \in \mathcal{J}^*$. This, together with (22), contradicts the MPEC-RCPLD assumption at x^*. Thus, $\{\bar{y}^k\}_{k\in K}$ is bounded. Without any loss of generality, we may assume that $\{y^k\}$ is a bounded sequence satisfying (20). We may further assume that $y^k \to y^*$. Clearly, from the outer semicontinuity of $\mathcal{N}_\Lambda$, $y^* \in \mathcal{N}_\Lambda(F(x^*))$ and hence $y^* \in \mathcal{M}_M^1(x^*)$. Assume further that $(x^k - x^*)/\|x^k - x^*\| \to d^0$ with $\|d^0\| = 1$. Note that, for each k,

$$\Lambda \ni F(x^k) = F(x^*) + \nabla F(x^*)^T (x^k - x^*) + o(\|x^k - x^*\|),$$

and hence

$$\nabla F(x^*)^T d^0 = \lim_{k\to\infty} \frac{F(x^k) - F(x^*)}{\|x^k - x^*\|} \in \mathcal{T}_\Lambda(F(x^*)). \tag{23}$$

It follows from (20) and [44, Proposition 6.41] that

$$y^k \in \prod_{i=1}^{l} \mathcal{N}_{(-\infty,0]}(g_i(x^k)) \times \prod_{i=1}^{q} \mathcal{N}_{\{0\}}(h_i(x^k)) \times \prod_{i=1}^{m} \mathcal{N}_C(-G_i(x^k), -H_i(x^k)). \tag{24}$$

It follows from (24) and Proposition 3.1 that, for each j

$$y_j^k F_j(x^k) = 0. \tag{25}$$

We next show that $(y^k)^T F(x^*) = (y^*)^T F(x^k) = 0$ for each k sufficiently large. In fact, if $F_j(x^*) \neq 0$ for some j, then $F_j(x^k) \neq 0$ when k is sufficiently large and hence, by (25), $y_j^k = 0$. As a result, we always have $(y^k)^T F(x^*) = 0$. We can show $(y^*)^T F(x^k) = 0$ when k is sufficiently large in a similar way. Therefore, we have that, for each k sufficiently large,

$$(y^k)^T F(x^*) = (y^*)^T F(x^k) = (y^k)^T F(x^k) = 0. \tag{26}$$

Noting that $\{y^k\}$ is bounded, we have that, for each k sufficiently large,

$$\begin{aligned} 0 &= (y^k)^T F(x^k) \\ &= (y^k)^T F(x^*) + (y^k)^T \nabla F(x^*)^T (x^k - x^*) + o(\|x^k - x^*\|) \\ &= (y^k)^T \nabla F(x^*)^T (x^k - x^*) + o(\|x^k - x^*\|). \end{aligned}$$

Dividing it by $\|x^k - x^*\|$ and taking a limit, we obtain

$$(y^*)^T \nabla F(x^*)^T d^0 = 0.$$

It follows from (20) that

$$0 = \nabla f(x^k)^T (x^k - x^*) + (y^k)^T \nabla F(x^k)^T (x^k - x^*). \tag{27}$$

Dividing it by $\|x^k - x^*\|$ and taking a limit, we have $\nabla f(x^*)^T d^0 = 0$. This, together with (23), indicates $d^0 \in \mathcal{C}_{MPEC}(x^*)$ and then, by the M-SSOSC assumption,

$$(d^0)^T \nabla_x^2 L_{MPEC}^1(x^*, y^*) d^0 > 0. \tag{28}$$

Let k be large enough. Consider the smooth function s_k defined by

$$s_k(t) := (\nabla f(x_t) + \nabla F(x_t) y_t)^T (x^k - x^*) - F(x_t)^T (y^k - y^*),$$

where $(x_t, y_t) := (1-t)(x^*, y^*) + t(x^k, y^k)$. It is not difficult to see from (10) and (20) that the first term vanishes when $t = 0$ or $t = 1$. On the other hand,

- if $t = 0$, it follows from (26) and $y^* \in \mathcal{N}_\Lambda(F(x^*))$ that

$$F(x_t)^T (y^k - y^*) = F(x^*)^T (y^k - y^*) = 0;$$

- if $t = 1$, it follows from (26) that

$$F(x_t)^T(y^k - y^*) = F(x^k)^T y^k - F(x^k)^T y^* = 0.$$

As a result, we have $s_k(0) = 0 = s_k(1)$. By the mean value theorem, there exists $t_k \in (0, 1)$ such that

$$0 = s'_k(t_k) = (x^k - x^*)^T \nabla_x^2 L^1_{MPEC}(x_{t_k}, y_{t_k})(x^k - x^*).$$

Dividing it by $\|x^k - x^*\|^2$ and taking a limit, we obtain a contradiction to (28). This completes the proof.

□

We can get the following isolatedness result of S-stationary point immediately. Due to the fact that a limit point of S-stationary points may not be S-stationary, we do not know whether the M-SSOSC can be weakened to the S-SSOSC.

Corollary 4.1 *Let x^* be an S-stationary point of the MPEC* (5)*. Suppose that both the MPEC-RCPLD and M-SSOSC hold at x^*. Then there exists a neighborhood V of x^* containing no other S-stationary point.*

Theorem 4.2 *If the MPEC-CPLD holds at $x^* \in X$, there exists $\delta > 0$ such that, for each $x \in \mathcal{B}_\delta(x^*) \cap X$, the MPEC-CPLD holds at x.*

Proof. Assume to the contrary that there exists a sequence $\{x^k\} \to_X x^*$ such that the MPEC-CPLD does not hold at x^k for each k. That is, for each k, there exist index sets $\mathcal{I}_1^k \subseteq I_g^k$, $\mathcal{I}_2^k \subseteq \{1, \cdots, q\}$, $\mathcal{I}_3^k \subseteq \mathcal{I}^k \cup \mathcal{J}^k$, and $\mathcal{I}_4^k \subseteq \mathcal{K}^k \cup \mathcal{J}^k$, where $\{I_g^k, \mathcal{I}^k, \mathcal{J}^k, \mathcal{K}^k\}$ are the same as in the proof of Theorem 20, and multipliers, not all zero, $\{\lambda^k, \mu^k, u^k, v^k\}$ with $\lambda_i^k \geq 0$ for each $i \in \mathcal{I}_1^k$, either $u_l^k v_l^k = 0$ or $u_l^k > 0, v_l^k > 0$ for each $l \in \mathcal{J}^k$ such that

$$\sum_{i \in \mathcal{I}_1^k} \lambda_i^k \nabla g_i(x^k) + \sum_{j \in \mathcal{I}_2^k} \mu_j^k \nabla h_j(x^k) - \sum_{l \in \mathcal{I}_3^k} u_l^k \nabla G_l(x^k) - \sum_{l \in \mathcal{I}_4^k} v_l^k \nabla H_l(x^k) = 0, \tag{29}$$

and there exists sequence $\{y^{k,\nu}\}$ converging to x^k as $\nu \to \infty$ such that

$$\left\{ \nabla g_i(y^{k,\nu}), \nabla h_j(y^{k,\nu}), \nabla G_l(y^{k,\nu}), \nabla H_\jmath(y^{k,\nu}) \;\middle|\; i \in \mathcal{I}_1^k, j \in \mathcal{I}_2^k, l \in \mathcal{I}_3^k, \jmath \in \mathcal{I}_4^k \right\} \tag{30}$$

is linearly independent. Without any loss of generality, we may assume that $\mathcal{I}_1^k \equiv \mathcal{I}_1$, $\mathcal{I}_2^k \equiv \mathcal{I}_2$, $\mathcal{I}_3^k \equiv \mathcal{I}_3$, and $\mathcal{I}_4^k \equiv \mathcal{I}_4$. It is not difficult to get that $\mathcal{I}_1 \subseteq I_g^*$, $\mathcal{I}_2 \subseteq \{1, \cdots, q\}$, $\mathcal{I}_3 \subseteq \mathcal{I}^* \cup \mathcal{J}^*$, and $\mathcal{I}_4 \subseteq \mathcal{K}^* \cup \mathcal{J}^*$. Set $\lambda_i^k = 0$ for $i \notin \mathcal{I}_1$, $u_i^k = 0$ for $i \notin \mathcal{I}_3$, and $v_i^k = 0$ for $i \notin \mathcal{I}_4$. We may further assume that $(\lambda^k, \mu^k, u^k, v^k)/\|(\lambda^k, \mu^k, u^k, v^k)\| \to (\lambda^*, \mu^*, u^*, v^*)$ with $(\lambda^*, \mu^*, u^*, v^*) = 1$. Dividing (29) by $\|(\lambda^k, \mu^k, u^k, v^k)\|$ and taking a limit in (29), we have

$$\sum_{i \in \mathcal{I}_1} \lambda_i^* \nabla g_i(x^*) + \sum_{j \in \mathcal{I}_2} \mu_j^* \nabla h_j(x^*) - \sum_{l \in \mathcal{I}_3} u_l^* \nabla G_l(x^*) - \sum_{l \in \mathcal{I}_4} v_l^* \nabla H_l(x^*) = 0. \tag{31}$$

It is not hard to get from (29) and Proposition 3.1 that

$$y^k = (\lambda^k, \mu^k, u^k, v^k)/\|(\lambda^k, \mu^k, u^k, v^k)\| \in \mathcal{N}_\Lambda(F(x^k)).$$

Therefore, we have from the outer semicontinuity of $\mathcal{N}_\Lambda$ that $y^* \in \mathcal{N}_\Lambda(F(x^*))$. Thus, we have from Proposition 3.1 that

$$\text{either } \ u_l^* v_l^* = 0 \ \text{ or } \ u_l^* > 0, v_l^* > 0 \ \text{ for each } \ l \in \mathcal{J}^*. \tag{32}$$

By the diagonalization law, we have from (30) that there exists $\{z^k\} \to x^*$ such that

$$\Big\{ \nabla g_i(z^k), \nabla h_j(z^k), \nabla G_l(z^k), \nabla H_\jmath(z^k) \;\Big|\; i \in \mathcal{I}_1, j \in \mathcal{I}_2, l \in \mathcal{I}_3, \jmath \in \mathcal{I}_4 \Big\}$$

is linearly independent, which together with (31)–(32) gives a contradiction with the assumption that the MPEC-CPLD holds at x^*. This completes the proof. $\square$

Corollary 4.2 *Let x^* be an M-stationary point of the MPEC* (5)*. Suppose that both MPEC-CPLD and M-SSOSC hold at x^*. Then there exists a neighborhood V of x^* such that x^* is the only possibly local minimizer of the MPEC* (5) *in $V \cap X$. If, in addition, x^* is an S-stationary point of* (5)*, then x^* is the only local minimizer of the MPEC* (5)*in $V \cap X$.*

Proof. Let V_1 be the open neighborhood in Theorem 4.1 such that there is no other M-stationary point in $V_1 \cap X$ and V_2 be the open neighborhood in Theorem 4.2 such that the MPEC-CPLD holds at each element in $V_2 \cap X$. Set $V := V_1 \cap V_2$. Since any locally optimal solution satisfying the MPEC-CPLD must be M-stationary [7], the set V is the required one. The rest of the theorem follows from Theorem 3.2. □

We further have the following result from Theorem 3.2 and Corollary 4.2. Note that the M-SSOSC for the MPEC (5) with linear constraints is independent of M-multipliers.

Corollary 4.3 *Suppose that F is affine and x^* is an S-stationary point of* (5)*. Let the M-SSOSC hold at x^*. Then there exists a neighborhood V of x^* containing no other local minimizer of* (5)*.*

Finally, we summarize the relationships among the constraint qualifications involved in this paper and [2, 8, 47] in Figure 1. Especially, the MPEC pseudonormality and MPEC quasinormality were introduced in [8], in which the relations "the MPEC pseudonormality ⇒ local error bound and the MPEC Abadie constraint qualification" and "the MPEC quasinormality ⇒ the M-stationarity" were shown. Moreover, Guo and Lin [47] showed that the MPEC RCPLD ⇒ the M-stationarity. In addition, the MPEC SMFCQ and MPEC MFCQ, which are stronger than the MPEC NNAMCQ, were introduced in [2], and it was showed that local minimizers of the MPEC (5) are S-stationary and the C-stationary, respectively.

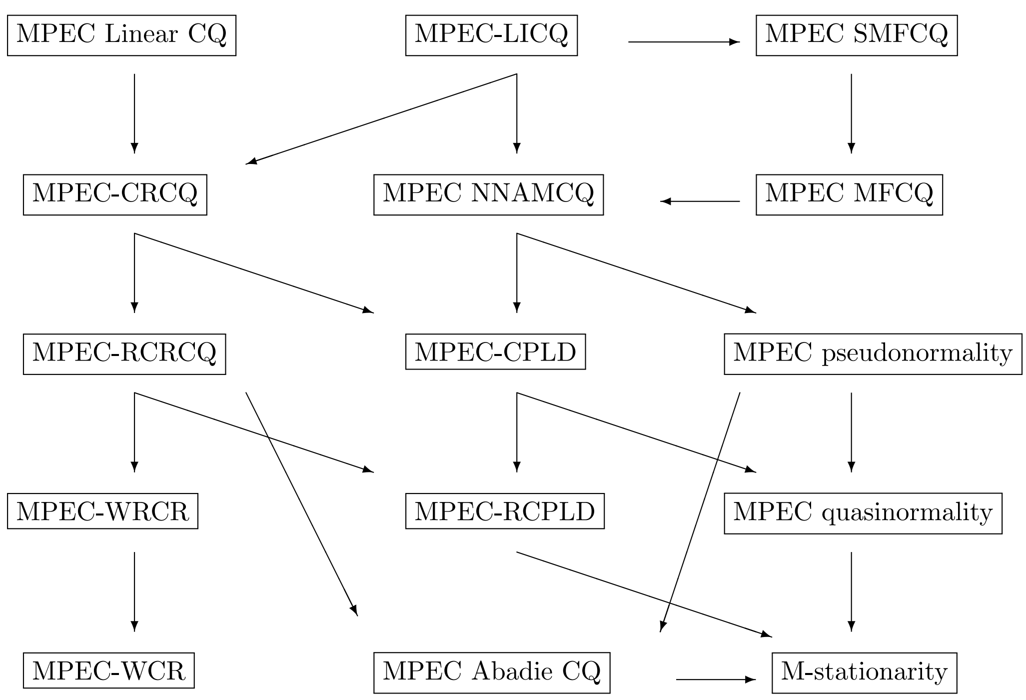

Figure 1: Relations among various CQs

5 Conclusions

We have improved various second-order optimality conditions for standard nonlinear programs by using some newly discovered constraint qualifications. We have also presented some new constraint qualifications for MPEC and, based on these new constraint qualifications, we obtained several second-order sufficient and necessary optimality conditions for MPEC. In addition, we have shown the isolatedness of M-/S-stationary points and local minimizers of MPEC under very weak conditions.

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