

BILEVEL POLYNOMIAL PROGRAMS AND SEMIDEFINITE RELAXATION METHODS*

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Abstract. A bilevel program is an optimization problem whose constraints involve the solution set to another optimization problem parameterized by upper level variables. This paper studies bilevel polynomial programs (BPPs), i.e., all the functions are polynomials. We reformulate BPPs equivalently as semi-infinite polynomial programs (SIPPs), using Fritz John conditions and Jacobian representations. Combining the exchange technique and Lasserre-type semidefinite relaxations, we propose a numerical method for solving BPPs. For simple BPPs, we prove the convergence to global optimal solutions. Numerical experiments are presented to show the efficiency of the proposed algorithm.

Key words. bilevel polynomial program, Lasserre relaxation, Fritz John condition, Jacobian representation, exchange method, semi-infinite programming

AMS subject classifications. 65K05, 90C22, 90C26, 90C34

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1. Introduction. We consider the following *bilevel polynomial program* (BPP):

$$(1.1) \quad (P) : \begin{cases} F^* := \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^p} F(x, y) \\ \text{s.t.} \quad G_i(x, y) \geq 0, \quad i = 1, \dots, m_1, \\ \quad \quad y \in S(x), \end{cases}$$

where F and all G_i are real polynomials in (x, y) , and $S(x)$ is the set of global minimizers of the following lower level program, which is parameterized by x :

$$(1.2) \quad \min_{z \in \mathbb{R}^p} f(x, z) \quad \text{s.t.} \quad g_j(x, z) \geq 0, \quad j = 1, \dots, m_2.$$

In (1.2), f and each g_j are polynomials in (x, z) . For convenience, denote by

$$Z(x) := \{z \in \mathbb{R}^p \mid g_j(x, z) \geq 0, \quad j = 1, \dots, m_2\}$$

the feasible set of (1.2). The inequalities $G_i(x, y) \geq 0$ are called upper (or outer) level constraints, while $g_j(x, z) \geq 0$ are called lower (or inner) level constraints. When $m_1 = 0$ (resp., $m_2 = 0$), there are no upper (resp., lower) level constraints. Similarly, $F(x, y)$ is the upper level (or outer) objective, and $f(x, z)$ is the lower level (or inner) objective. Denote the set

$$(1.3) \quad \mathcal{U} := \left\{ (x, y) \mid \begin{array}{l} G_i(x, y) \geq 0 \quad (i = 1, \dots, m_1), \\ g_j(x, y) \geq 0 \quad (j = 1, \dots, m_2) \end{array} \right\}.$$

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Then the feasible set of (P) is the intersection

$$(1.4) \quad \mathcal{U} \cap \{(x, y) : y \in S(x)\}.$$

Throughout the paper, we assume that for all $(x, y) \in \mathcal{U}$, $S(x) \neq \emptyset$, and consequently the feasible set of (P) is nonempty. When the lower level feasible set $Z(x) \equiv Z$ is independent of x , we call the problem (P) a *simple bilevel polynomial program* (SBPP). The SBPP is not mathematically simple but actually quite challenging. SBPPs have important applications in economics, e.g., the moral hazard model of the principal-agent problem [24]. When the feasible set of the lower level program $Z(x)$ depends on x , the problem (P) is called a *general bilevel polynomial program* (GBPP). GBPP is also an effective modeling tool for many applications in various fields; see, e.g., [9, 12] and the references therein.

1.1. Background. The bilevel program is a class of difficult optimization problems. Even for the case where all the functions are linear, the problem is NP-hard [4]. A general approach for solving bilevel programs is to transform them into single level optimization problems. A commonly used technique is to replace the lower level program by its Karush–Kuhn–Tucker (KKT) conditions. When the lower level program involves inequality constraints, the reduced problem becomes a so-called *mathematical program with equilibrium constraints* (MPEC) [23, 31]. If the lower level program is nonconvex, the optimal solution of a bilevel program may not even be a stationary point of the reduced single level optimization problem by using the KKT conditions. This was shown by a counterexample due to Mirrlees [24]. Moreover, even if the lower level program is convex, it was shown in [10] that a local solution to the MPEC obtained by replacing the lower level program by its KKT conditions may not be a local solution to the original bilevel program. Recently, [1] proposed replacing the lower level program with its Fritz John conditions instead of its KKT conditions. However, it was shown in [11] that the same difficulties remain; i.e., solutions to the MPEC obtained by replacing the lower level program by its Fritz John conditions may not be the solutions to the original bilevel program.

An alternative approach for solving BPPs is to use the value function [32, 42], which gives an equivalent reformulation. However, the optimal solution of the bilevel program may not be a stationary point of the value function reformulation. To overcome this difficulty, [43] proposed combining the KKT and the value function reformulations. Over the past two decades, many numerical algorithms were proposed for solving bilevel programs. However, most of them assume that the lower level program is convex, with few exceptions [21, 25, 27, 32, 39, 40, 41]. In [25, 27], an algorithm using the branch and bound in combination with the exchange technique was proposed to find approximate global optimal solutions. Recently, the smoothing techniques were used to find stationary points of the valued function or the combined reformulation of simple bilevel programs [21, 39, 40, 41].

In general, it is quite difficult to find global minimizers of nonconvex optimization problems. However, when the functions are polynomials, there exists much work on computing global optimizers, by using Lasserre-type semidefinite relaxations [18]. We refer the reader to [19, 20] for the recent work in this area. Recently, Jeyakumar et al. [17] worked on SBPPs. When the lower level program (1.2) is convex for each fixed x , they transformed (1.1) into a single level polynomial program, by using Fritz John conditions and the multipliers to replace the lower level program, and globally solving it by using Lasserre-type relaxations. When (1.2) is nonconvex for some x , by approximating the value function of lower level programs by a sequence of polynomi-

als, they propose reformulating (1.1) with approximate lower level programs by the value function approach, and globally solving the resulting sequence of polynomial programs by using Lasserre-type relaxations. The work [17] is very inspiring, because polynomial optimization techniques were proposed to solve BPPs. In this paper, we also use Lasserre-type semidefinite relaxations to solve BPPs, but we make different reformulations by using Jacobian representations and the exchange technique in semi-infinite programming.

1.2. From BPP to SIPP. A bilevel program can be reformulated as a *semi-infinite program* (SIP). Thus, the classical methods (e.g., the exchange method [7, 16, 38]) for SIPs can be applied to solve bilevel programs. For convenience of introduction, at the moment, we consider SBPPs; i.e., the feasible set $Z(x) \equiv Z$ in (1.2) is independent of x .

Before reformulating BPPs as SIPs, we show the fact

$$(1.5) \quad y \in S(x) \iff y \in Z, \quad H(x, y, z) \geq 0 \quad (\forall z \in Z),$$

where $H(x, y, z) := f(x, z) - f(x, y)$. Clearly, the “ \Rightarrow ” direction is true. Let us prove the reverse direction. Let $v(x)$ denote the value function:

$$(1.6) \quad v(x) := \inf_{z \in Z} f(x, z).$$

If (x, y) satisfies the right-hand side conditions in (1.5), then

$$\inf_{z \in Z} H(x, y, z) = v(x) - f(x, y) \geq 0.$$

Since $y \in Z$, we have $v(x) - f(x, y) \leq 0$. Combining these two inequalities, we get

$$v(x) = \inf_{z \in Z} f(x, z) = f(x, y)$$

and hence $y \in S(x)$.

By the fact (1.5), the problem (P) is equivalent to

$$(1.7) \quad (\tilde{P}) : \begin{cases} F^* := \min_{x \in \mathbb{R}^n, y \in Z} F(x, y) \\ \text{s.t.} \quad G_i(x, y) \geq 0, \quad i = 1, \dots, m_1, \\ H(x, y, z) \geq 0 \quad \forall z \in Z. \end{cases}$$

The problem (\tilde{P}) is a semi-infinite polynomial program (SIPP) if the set Z is infinite. Hence, the exchange method can be used to solve (\tilde{P}) . Suppose Z_k is a finite grid of Z . Replacing Z by Z_k in (\tilde{P}) , we get

$$(1.8) \quad (\tilde{P}_k) : \begin{cases} F_k^* := \min_{x \in \mathbb{R}^n, y \in Z} F(x, y) \\ \text{s.t.} \quad G_i(x, y) \geq 0, \quad i = 1, \dots, m_1, \\ H(x, y, z) \geq 0 \quad \forall z \in Z_k. \end{cases}$$

The feasible set of (\tilde{P}_k) contains that of (\tilde{P}) . Hence,

$$F_k^* \leq F^*.$$

Since Z_k is a finite set, (\tilde{P}_k) is a polynomial optimization problem. If, for some Z_k , we can get an optimizer (x^k, y^k) of (\tilde{P}_k) such that

$$(1.9) \quad v(x^k) - f(x^k, y^k) \geq 0,$$

then $y^k \in S(x^k)$ and (x^k, y^k) is feasible for (\tilde{P}) . In such a case, (x^k, y^k) must be a global optimizer of (\tilde{P}) . Otherwise, if (1.9) fails to hold, then there exists $z^k \in Z$ such that

$$f(x^k, z^k) - f(x^k, y^k) < 0.$$

For such a case, we can construct the new grid set as

$$Z_{k+1} := Z_k \cup \{z^k\}$$

and then solve the new problem (\tilde{P}_{k+1}) with the grid set Z_{k+1} . Repeating this process, we can get an algorithm for solving (\tilde{P}) approximately.

How does the above approach work in computational practice? Does it converge to global optimizers? Each subproblem (\tilde{P}_k) is a polynomial optimization problem, which is generally nonconvex. Theoretically, it is NP-hard to solve polynomial optimization globally. However, in practice, it can be solved successfully by Lasserre-type semidefinite relaxations (cf. [18, 19]). Recently, it was shown in [30] that Lasserre-type semidefinite relaxations are generally tight for solving polynomial optimization problems. About the convergence, we can see that $\{F_k^*\}$ is a sequence of monotonically increasing lower bounds for the global optimal value F^* , i.e.,

$$F_1^* \leq \dots \leq F_k^* \leq F_{k+1}^* \leq \dots \leq F^*.$$

By a standard analysis for SIP (cf. [16]), one can expect the convergence $F_k^* \rightarrow F^*$, under some conditions. However, we would like to point out that the above exchange process typically converges *very slowly* for solving BPPs. A major reason is that the feasible set of (\tilde{P}_k) is *much larger* than that of (\tilde{P}) . Indeed, the dimension of the feasible set of (\tilde{P}_k) is typically larger than that of (\tilde{P}) . This is because, for every feasible (x, y) in (\tilde{P}_k) , y must also satisfy optimality conditions for the lower level program (1.2). In the meanwhile, the y in (\tilde{P}_k) does not satisfy such optimality conditions. Typically, for (\tilde{P}_k) to approximate (\tilde{P}) reasonably well, the grid set Z_k should be *very big*. In practice, the above standard exchange method is not efficient for solving BPPs.

1.3. Contributions. In this paper, we propose a computational method for solving BPPs efficiently. First, we transform a BPP into an equivalent SIPP by using Fritz John conditions and Jacobian representations. Then we propose a new algorithm for solving BPPs by using the exchange technique and Lasserre-type semidefinite relaxations.

For each (x, y) that is feasible for (1.1), y is a minimizer for the lower level program (1.2) parameterized by x . If some constraint qualification conditions are satisfied, the KKT conditions hold. If such qualification conditions fail to hold, the KKT conditions might not be satisfied. However, the Fritz John conditions always hold for (1.2) (cf. [6, section 3.5] and [5] for optimality conditions for convex programs without constraint qualifications). So, we can add the Fritz John conditions to (\tilde{P}) , while the problem is not changed.

A disadvantage of using Fritz John conditions is the usage of multipliers, which need to be considered as new variables. Typically, using multipliers will make the polynomial program much harder to solve, because of new additional variables. To overcome this difficulty, the technique in [29, section 2] can be applied to avoid the usage of multipliers. This technique is known as Jacobian representations for optimality conditions.

The above observations motivate us to solve bilevel polynomial programs by combining Fritz John conditions, Jacobian representations, Lasserre relaxations, and the exchange technique. Our major results are as follows:

- Unlike some prior methods for solving BPPs, we do not assume the KKT conditions hold for the lower level program (1.2). Instead, we use the Fritz John conditions. This is because the KKT conditions may fail to hold for the lower level program (1.2), while the Fritz John conditions always hold. By using Jacobian representations, the usage of multipliers can be avoided. This greatly improves the computational efficiency.
- For SBPPs, we propose an algorithm using Jacobian representations, Lasserre relaxations, and the exchange technique. Its convergence to global minimizers is proved. The numerical experiments show that it is efficient for solving SBPPs.
- For GBPPs, we can apply the same algorithm, using Jacobian representations, Lasserre relaxations, and the exchange technique. The numerical experiments show that it works well for some GBPPs, while it is not theoretically guaranteed to get global optimizers. However, its convergence to global optimality can be proved under some assumptions.

The paper is organized as follows: In section 2, we review some preliminaries in polynomial optimization and Jacobian representations. In section 3, we propose a method for solving SBPPs and prove its convergence. In section 4, we consider GBPPs and show how the algorithm works. In section 5, we present numerical experiments to demonstrate the efficiency of the proposed methods. In section 6, we offer some conclusions and discussions about our method.

2. Preliminaries.

Notation. The symbol \mathbb{N} (resp., \mathbb{R} , \mathbb{C}) denotes the set of nonnegative integers (resp., real numbers, complex numbers). For an integer $n > 0$, $[n]$ denotes the set $\{1, \dots, n\}$. For $x := (x_1, \dots, x_n)$ and $\alpha := (\alpha_1, \dots, \alpha_n)$, denote the monomial

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

For a finite set T , $|T|$ denotes its cardinality. The symbol $\mathbb{R}[x] := \mathbb{R}[x_1, \dots, x_n]$ denotes the ring of polynomials in $x := (x_1, \dots, x_n)$ with real coefficients, whereas $\mathbb{R}[x]_k$ denotes its subspace of polynomials of degree at most k . For a polynomial $p \in \mathbb{R}[x]$, define the set product

$$p \cdot \mathbb{R}[x] := \{pq \mid q \in \mathbb{R}[x]\}.$$

It is the principal ideal generated by p . For a symmetric matrix W , $W \succeq 0$ (resp., $\succ 0$) means that W is positive semidefinite (resp., definite). For a vector $u \in \mathbb{R}^n$, $\|u\|$ denotes the standard Euclidean norm. The gradient of a function $f(x)$ is denoted by $\nabla f(x)$. If $f(x, z)$ is a function in both x and z , then $\nabla_z f(x, z)$ denotes the gradient with respect to z . For an optimization problem, argmin denotes the set of its optimizers.

2.1. Polynomial optimization. An ideal I in $\mathbb{R}[x]$ is a subset of $\mathbb{R}[x]$ such that $I \cdot \mathbb{R}[x] \subseteq I$ and $I + I \subseteq I$. For a tuple $p = (p_1, \dots, p_r)$ in $\mathbb{R}[x]$, $I(p)$ denotes the smallest ideal containing all p_i , i.e.,

$$I(p) = p_1 \cdot \mathbb{R}[x] + \cdots + p_r \cdot \mathbb{R}[x].$$

The k th truncation of the ideal $I(p)$, denoted by $I_k(p)$, is the set

$$p_1 \cdot \mathbb{R}[x]_{k-\deg(p_1)} + \cdots + p_r \cdot \mathbb{R}[x]_{k-\deg(p_r)}.$$

For the polynomial tuple p , denote its real zero set by

$$\mathcal{V}(p) := \{v \in \mathbb{R}^n \mid p(v) = 0\}.$$

A polynomial $\sigma \in \mathbb{R}[x]$ is said to be a sum of squares (SOS) if $\sigma = a_1^2 + \cdots + a_k^2$ for some $a_1, \dots, a_k \in \mathbb{R}[x]$. The set of all SOS polynomials in x is denoted by $\Sigma[x]$. For a degree m , denote the truncation

$$\Sigma[x]_m := \Sigma[x] \cap \mathbb{R}[x]_m.$$

For a tuple $q = (q_1, \dots, q_t)$, its quadratic module is the set

$$Q(q) := \Sigma[x] + q_1 \cdot \Sigma[x] + \cdots + q_t \cdot \Sigma[x].$$

The k th truncation of $Q(q)$ is the set

$$\Sigma[x]_{2k} + q_1 \cdot \Sigma[x]_{d_1} + \cdots + q_t \cdot \Sigma[x]_{d_t},$$

where each $d_i = 2k - \deg(q_i)$. For the tuple q , denote the basic semialgebraic set

$$\mathcal{S}(q) := \{v \in \mathbb{R}^n \mid q(v) \geq 0\}.$$

For the polynomial tuples p and q as above, if $f \in I(p) + Q(q)$, then clearly $f \geq 0$ on the set $\mathcal{V}(p) \cap \mathcal{S}(q)$. However, the reverse is not necessarily true. The sum $I(p) + Q(q)$ is said to be *archimedean* if there exists $b \in I(p) + Q(q)$ such that $S(b) = \{v \in \mathbb{R}^n : b(v) \geq 0\}$ is a compact set in \mathbb{R}^n . Putinar [33] proved that if a polynomial $f > 0$ on $\mathcal{V}(p) \cap \mathcal{S}(q)$ and if $I(p) + Q(q)$ is archimedean, then $f \in I(p) + Q(q)$. When f is only nonnegative (but not strictly positive) on $\mathcal{V}(p) \cap \mathcal{S}(q)$, we still have $f \in I(p) + Q(q)$, under some general conditions (cf. [30]).

Now, we review Lasserre-type semidefinite relaxations in polynomial optimization. More details can be found in [18, 19, 20]. Consider the general polynomial optimization problem

$$(2.1) \quad \begin{cases} f_{\min} := \min_{x \in \mathbb{R}^n} f(x) \\ \text{s.t. } p(x) = 0, q(x) \geq 0, \end{cases}$$

where $f \in \mathbb{R}[x]$ and p, q are tuples of polynomials. The feasible set of (2.1) is precisely the intersection $\mathcal{V}(p) \cap \mathcal{S}(q)$. The Lasserre's hierarchy of semidefinite relaxations for solving (2.1) is $(k = 1, 2, \dots)$

$$(2.2) \quad \begin{cases} f_k := \max \quad \gamma \\ \text{s.t. } f - \gamma \in I_{2k}(p) + Q_k(q). \end{cases}$$

When the set $I(p) + Q(q)$ is archimedean, Lasserre proved the convergence

$$f_k \rightarrow f_{\min} \quad \text{as } k \rightarrow \infty.$$

If there exist $k < \infty$ such that $f_k = f_{\min}$, the Lasserre's hierarchy is said to have finite convergence. Under the archimedeaness and some standard conditions in optimization known to be generic (i.e., linear independence constraint qualification, strict

complementarity and second order sufficiency conditions), the Lasserre's hierarchy has finite convergence. This was recently shown in [30]. On the other hand, there exist special polynomial optimization problems for which Lasserre's hierarchy fails to have finite convergence. However, such special problems belong to a set of measure zero in the space of input polynomials, as shown in [30]. Moreover, we can also get global minimizers of (2.1) by using the flat extension or flat truncation condition (cf. [28]). The optimization problem (2.2) can be solved as a semidefinite program, so it can be solved by semidefinite program packages (e.g., SeDuMi [36], SDPT3 [37]). A convenient and efficient software for using Lasserre relaxations is GloptiPoly 3 [15].

2.2. Jacobian representations. We consider the polynomial optimization problem that is similar to the lower level program (1.2):

$$(2.3) \quad \min_{z \in \mathbb{R}^p} f(z) \quad \text{s.t.} \quad g_1(z) \geq 0, \dots, g_m(z) \geq 0,$$

where $f, g_1, \dots, g_m \in \mathbb{R}[z] := \mathbb{R}[z_1, \dots, z_p]$. Let Z be the feasible set of (2.3). For $z \in Z$, let $J(z)$ denote the index set of active constraining functions at z .

Suppose z^* is an optimizer of (2.3). By the Fritz John condition (cf. [6, section 3.3.5]), there exists $(\mu_0, \mu_1, \dots, \mu_m) \neq 0$ such that

$$(2.4) \quad \mu_0 \nabla f(z^*) - \sum_{i=1}^m \mu_i \nabla g_i(z^*) = 0, \quad \mu_i g_i(z^*) = 0 \quad (i \in [m]).$$

A point like z^* satisfying (2.4) is called a Fritz John point. If we only consider active constraints, the above is then reduced to

$$(2.5) \quad \mu_0 \nabla f(z^*) - \sum_{i \in J(z^*)} \mu_i \nabla g_i(z^*) = 0.$$

The condition (2.4) uses multipliers μ_0, \dots, μ_m , which are often not known in advance. If we consider them as new variables, then it would increase the number of variables significantly. For the index set $J = \{i_1, \dots, i_k\}$, denote the matrix

$$B[J, z] := [\nabla f(z) \quad \nabla g_{i_1}(z) \quad \cdots \quad \nabla g_{i_k}(z)].$$

Then condition (2.5) means that the matrix $B[J(z^*), z^*]$ is rank deficient, i.e.,

$$\text{rank } B[J(z^*), z^*] \leq |J(z^*)|.$$

The matrix $B[J(z^*), z^*]$ depends on the active set $J(z^*)$, which is typically unknown in advance.

The technique in [29, section 2] can be applied to get explicit equations for Fritz John points, without using multipliers μ_i . For a subset $J = \{i_1, \dots, i_k\} \subseteq [m]$ with cardinality $|J| \leq \min\{m, p-1\}$, write its complement as $J^c := [m] \setminus J$. Then

$$B[J, z] \text{ is rank deficient} \iff \text{all } (k+1) \times (k+1) \text{ minors of } B[J, z] \text{ are zeros.}$$

There are totally $\binom{p}{k+1}$ equations defined by such minors. However, this number can be significantly reduced by using the method in [29, section 2]. The number of equations needed to characterize $B[J, z]$ as rank deficient can be reduced to

$$\ell(J) := p(k+1) - (k+1)^2 + 1.$$

It is much smaller than $\binom{p}{k+1}$. For cleanness of the paper, we do not repeat the construction of these minimum number defining polynomials. Interested readers are referred to [29, section 2] for details. List all the defining polynomials, which make $B[J, z]$ rank deficient, as

$$(2.6) \quad \eta_1^J, \dots, \eta_{\ell(J)}^J.$$

Consider the products of these polynomials with g_j 's:

$$(2.7) \quad \eta_1^J \cdot \left(\prod_{j \in J^c} g_j \right), \quad \dots, \quad \eta_{\ell(J)}^J \cdot \left(\prod_{j \in J^c} g_j \right).$$

They are all polynomials in z . The active set $J(z)$ is undetermined, unless z is known. We consider all possible polynomials as in (2.7), for all $J \subseteq [m]$, and collect them together. For convenience of notation, denote all such polynomials as

$$(2.8) \quad \psi_1, \quad \dots, \quad \psi_L,$$

where the number

$$\begin{aligned} L &= \sum_{J \subseteq [m], |J| \leq \min\{m, p-1\}} \ell(J) \\ &= \sum_{0 \leq k \leq \min\{m, p-1\}} \binom{m}{k} (p(k+1) - (k+1)^2 + 1). \end{aligned}$$

When m, k are big, the number L would be very large. This is an unfavorable feature of Jacobian representations.

We point out that the Fritz John points can be characterized by using the polynomials ψ_1, \dots, ψ_L . Define the set of all Fritz John points:

$$(2.9) \quad K_{FJ} := \left\{ z \in \mathbb{R}^p \mid \begin{array}{l} \exists (\mu_0, \mu_1, \dots, \mu_m) \neq 0, \mu_i g_i(z) = 0 \ (i \in [m]), \\ \mu_0 \nabla f(z) - \sum_{i=1}^m \mu_i \nabla g_i(z) = 0 \end{array} \right\}.$$

Let W be the set of real zeros of polynomials $\psi_j(z)$, i.e.,

$$(2.10) \quad W = \{z \in \mathbb{R}^p \mid \psi_1(z) = \dots = \psi_L(z) = 0\}.$$

It is interesting to note that the sets K_{FJ} and W are equal.

LEMMA 2.1. *For K_{FJ}, W as in (2.9)–(2.10), it holds that $K_{FJ} = W$.*

Proof. First, we prove that $W \subseteq K_{FJ}$. Choose an arbitrary $u \in W$, and let $J(u)$ be the active set at u . If $|J(u)| \geq p$, then the gradients $\nabla f(u)$ and $\nabla g_j(u)$ ($j \in J(u)$) must be linearly dependent, so $u \in K_{FJ}$. Next, we suppose $|J(u)| < p$. Note that $g_j(u) > 0$ for all $j \in J(u)^c$. By the construction, some of ψ_1, \dots, ψ_L are the polynomials as in (2.7),

$$\eta_t^{J(u)} \cdot \left(\prod_{j \in J(u)^c} g_j \right).$$

Thus, $\psi(u) = 0$ implies that all the polynomials $\eta_t^{J(u)}$ vanish at u . By their definition, we know the matrix $B[J(u), u]$ does not have full column rank. This means that $u \in K_{FJ}$.

Second, we show that $K_{FJ} \subseteq W$. Choose an arbitrary $u \in K_{FJ}$.

- Case I: $J(u) = \emptyset$. Then $\nabla f(u) = 0$. The first column of matrices $B[\emptyset, u]$ is zero, so all η_t^\emptyset and ψ_j vanishes at u and hence $u \in W$.
- Case II: $J(u) \neq \emptyset$. Let $I \subseteq [m]$ be an arbitrary index set with $|I| \leq \min\{m, p-1\}$. If $J(u) \not\subseteq I$, then at least one $j \in I^c$ belongs to $J(u)$. Thus, at least one $j \in I^c$ satisfies $g_j(u) = 0$, so all the polynomials

$$\eta_t^I \cdot \left(\prod_{j \in I^c} g_j \right)$$

vanish at u . If $J(u) \subseteq I$, then $\mu_i g_i(u) = 0$ implies that $\mu_i = 0$ for all $i \in I^c$. By definition of K_{FJ} , the matrix $B[I, u]$ does not have full column rank. So, the minors η_i^I of $B[I, u]$ vanish at u . By the construction of ψ_i , we know all ψ_i vanish at u , so $u \in W$.

The proof is completed by combining the above two cases. □

3. Simple bilevel polynomial programs. In this section, we study simple bilevel polynomial programs (SBPPs) and give an algorithm for computing global optimizers. For SBPPs as in (1.1), the feasible set $Z(x)$ for the lower level program (1.2) is independent of x . Assume that $Z(x)$ is constantly the semialgebraic set

$$(3.1) \quad Z := \{z \in \mathbb{R}^p \mid g_1(z) \geq 0, \dots, g_{m_2}(z) \geq 0\}$$

for given polynomials g_1, \dots, g_{m_2} in $z := (z_1, \dots, z_p)$. For each pair (x, y) that is feasible in (1.1), y is an optimizer for (1.2), which now becomes

$$(3.2) \quad \min_{z \in \mathbb{R}^p} f(x, z) \quad \text{s.t.} \quad g_1(z) \geq 0, \dots, g_{m_2}(z) \geq 0.$$

Note that the inner objective f still depends on x . So, y must be a Fritz John point of (3.2); i.e., there exists $(\mu_0, \mu_1, \dots, \mu_{m_2}) \neq 0$ satisfying

$$\mu_0 \nabla_z f(x, y) - \sum_{j \in [m_2]} \mu_j \nabla_z g_j(y) = 0, \quad \mu_j g_j(y) = 0 \quad (j \in [m_2]).$$

Let $K_{FJ}(x)$ denote the set of all Fritz John points of (3.2). The set $K_{FJ}(x)$ can be characterized by Jacobian representations. Let ψ_1, \dots, ψ_L be the polynomials constructed as in (2.8). Note that each ψ_j is now a polynomial in (x, z) , because the objective of (3.2) depends on x . Thus, each (x, y) feasible for (1.1) satisfies

$$\psi_1(x, y) = \dots = \psi_L(x, y) = 0.$$

For convenience of notation, denote the polynomial tuples

$$(3.3) \quad \xi := (G_1, \dots, G_{m_1}, g_1, \dots, g_{m_2}), \quad \psi := (\psi_1, \dots, \psi_L).$$

We call $\psi(x, y) = 0$ a *Jacobian equation*. Then the SBPP as in (1.1) is equivalent to the following SIPP:

$$(3.4) \quad \begin{cases} F^* := \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^p} F(x, y) \\ \text{s.t.} \quad \psi(x, y) = 0, \quad \xi(x, y) \geq 0, \\ \quad \quad H(x, y, z) \geq 0 \quad \forall z \in Z. \end{cases}$$

In the above, $H(x, y, z)$ is defined as in (1.5).

3.1. A semidefinite algorithm for SBPP. We have seen that the SBPP (1.1) is equivalent to (3.4), which is an SIPP. So, we can apply the exchange method to solve it. The basic idea of “exchange” is that we replace Z by a finite grid set Z_k in (3.4), and then solve it for a global minimizer (x^k, y^k) by Lasserre relaxations. If (x^k, y^k) is feasible for (1.1), we stop; otherwise, we compute global minimizers of $H(x^k, y^k, z)$ and add them to Z_k . Repeat this process until the convergence condition is met. We call (x^*, y^*) a global minimizer of (1.1), up to a tolerance parameter $\epsilon > 0$, if (x^*, y^*) is a global minimizer of the following approximate SIPP:

$$(3.5) \quad \begin{cases} F_\epsilon^* := \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^p} & F(x, y) \\ \text{s.t.} & \psi(x, y) = 0, \quad \xi(x, y) \geq 0, \\ & H(x, y, z) \geq -\epsilon \quad \forall z \in Z. \end{cases}$$

Summarizing the above, we get the following algorithm.

ALGORITHM 3.1 (a semidefinite relaxation algorithm for SBPP).

Input: Polynomials $F, f, G_1, \dots, G_{m_1}, g_1, \dots, g_{m_2}$ for the SBPP as in (1.1), a tolerance parameter $\epsilon \geq 0$, and a maximum number k_{\max} of iterations.

Output: The set \mathcal{X}^* of global minimizers of (1.1), up to the tolerance ϵ .

Step 1. Let $Z_0 = \emptyset, \mathcal{X}^* = \emptyset$, and $k = 0$.

Step 2. Apply Lasserre relaxations to solve

$$(3.6) \quad (P_k) : \begin{cases} F_k^* := \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^p} & F(x, y) \\ \text{s.t.} & \psi(x, y) = 0, \quad \xi(x, y) \geq 0, \\ & H(x, y, z) \geq 0 \quad (\forall z \in Z_k), \end{cases}$$

and get the set $S_k = \{(x_1^k, y_1^k), \dots, (x_{r_k}^k, y_{r_k}^k)\}$ of its global minimizers.

Step 3. For each $i = 1, \dots, r_k$, do the following:

(a) Apply Lasserre relaxations to solve

$$(3.7) \quad (Q_i^k) : \begin{cases} v_i^k := \min_{z \in \mathbb{R}^p} & H(x_i^k, y_i^k, z) \\ \text{s.t.} & \psi(x_i^k, z) = 0, \\ & g_1(z) \geq 0, \dots, g_{m_2}(z) \geq 0, \end{cases}$$

and get the set $T_i^k = \{z_{i,j}^k : j = 1, \dots, t_i^k\}$ of its global minimizers.

(b) If $v_i^k \geq -\epsilon$, then update $\mathcal{X}^* := \mathcal{X}^* \cup \{(x_i^k, y_i^k)\}$.

Step 4. If $\mathcal{X}^* \neq \emptyset$ or $k > k_{\max}$, stop; otherwise, update Z_k to Z_{k+1} as

$$(3.8) \quad Z_{k+1} := Z_k \cup T_1^k \cup \dots \cup T_{r_k}^k.$$

Let $k := k + 1$ and go to Step 2.

For the exchange method to solve the SIPP (3.4) successfully, the two subproblems (3.6) and (3.7) need to be solved globally at each iteration. This can be done by Lasserre’s hierarchy of semidefinite relaxations (cf. section 2.1).

(A) For solving (3.6) by Lasserre’s hierarchy, we get a sequence of monotonically increasing lower bounds for F_k^* , say, $\{\rho_\ell\}_{\ell=1}^\infty$, that is,

$$\rho_1 \leq \dots \leq \rho_\ell \leq \dots \leq F_k^*.$$

Here, ℓ is a relaxation order. If for some value of ℓ we get a feasible point (\hat{x}, \hat{y}) for (3.6) such that $F(\hat{x}, \hat{y}) = \rho_\ell$, then we must have

$$(3.9) \quad F(\hat{x}, \hat{y}) = F_k^* = \rho_\ell$$

and know (\hat{x}, \hat{y}) is a global minimizer. This certifies that Lasserre's relaxation of order ℓ is exact and (3.6) is solved globally; i.e., Lasserre's hierarchy has finite convergence. As recently shown in [30], Lasserre's hierarchy has finite convergence when the archimedeaness and some standard conditions well known in optimization to be generic (i.e., linear independence constraint qualification, strict complementarity, and second order sufficiency conditions) hold.

- (B) For a given polynomial optimization problem, there exist a sufficient (and almost necessary) condition for detecting whether or not Lasserre's hierarchy has finite convergence. The condition is *flat truncation*, proposed in [28]. It was proved in [28] that Lasserre's hierarchy has finite convergence if the flat truncation condition is satisfied. When the flat truncation condition holds, we can also get the point (\hat{x}, \hat{y}) in (3.9). In all of our numerical examples, the flat truncation condition is satisfied, so we know that Lasserre relaxations solved them exactly. There exist special optimization problems for which Lasserre relaxations are not exact (see, e.g., [19, Chapter 5]). Even for the worst-case scenario in which Lasserre's hierarchy fails to have finite convergence, flat truncation is still the right condition for checking asymptotic convergence. This is proved in [28, section 3].
- (C) In computational practice, semidefinite programs cannot be solved exactly, because round-off errors always exist in computers. Therefore, if $F(\hat{x}, \hat{y}) \approx \rho_\ell$, it is reasonable to claim that (3.6) is solved globally. This numerical issue is a common feature of most computational methods.
- (D) For the same reasons as above, the subproblem (3.7) can also be solved globally by Lasserre's relaxations. Moreover, (3.7) uses the equation $\psi(x_i^k, z) = 0$, obtained from Jacobian representation. As shown in [29], Lasserre's hierarchy of relaxations, in combination with Jacobian representations, always has finite convergence, under some nonsingularity conditions. This result has been improved in [13, Theorem 3.9] under weaker conditions. Flat truncation can be used to detect the convergence (cf. [28, section 4.2]).
- (E) For all $\epsilon_1 > \epsilon_2 > 0$, it is easy to see that $F_{\epsilon_1}^* \leq F_{\epsilon_2}^* \leq F^*$, and hence the feasible region and the optimal value of the bilevel problems are monotone. Indeed, we can prove $\lim_{\epsilon \rightarrow 0^+} F_\epsilon^* = F^*$ and the continuity of the optimal solutions; see [21, Theorem 4.1] for the result and a detailed proof. However, we should point out that if $\epsilon > 0$ is not small enough, then the solution of the approximate bilevel program may be very different from that for the original bilevel program. We refer the reader to [25, Example 4.1].
- (F) In Step 3 of Algorithm 3.1, the value of v_i^k is a measure for the feasibility of (x_i^k, y_i^k) in (3.4). This is because (x_i^k, y_i^k) is a feasible point for (3.4) if and only if $v_i^k \geq 0$. By using the exchange method, the subproblem (3.6) is only an approximation for (3.4), so typically we have $v_i^k < 0$ if (x_i^k, y_i^k) is infeasible for (3.4). The closer v_i^k is to zero, the better (3.6) approximates (3.4).

3.2. Two features of the algorithm. As in the introduction, we do not apply the exchange method directly to (1.7), but instead to (3.4). Both (1.7) and (3.4) are SIPPs that are equivalent to the SBPP (1.1). As the numerical experiments

will show, the SIPP (3.4) is much easier to solve by the exchange method. This is because, the Jacobian equation $\psi(x, y) = 0$ in (3.4) makes it much easier for (3.6) to approximate (3.4) accurately. Typically, for a finite grid set Z_k of Z , the feasible sets of (3.4) and (3.6) have the same dimension. However, the feasible set of (1.7) has smaller dimension than that of (1.8). Thus, it is usually very difficult for (1.8) to approximate (1.7) accurately by choosing a finite set Z_k . In contrast, it is often much easier for (3.6) to approximate (3.4) accurately. We illustrate this fact by the following example.

Example 3.2 (see [26, Example 3.19]). Consider the SBPP

$$(3.10) \quad \begin{cases} \min_{x \in \mathbb{R}, y \in \mathbb{R}} & F(x, y) := xy - y + \frac{1}{2}y^2 \\ \text{s.t.} & 1 - x^2 \geq 0, \quad 1 - y^2 \geq 0, \\ & y \in S(x) := \underset{1-z^2 \geq 0}{\operatorname{argmin}}_z f(x, z) := -xz^2 + \frac{1}{2}z^4. \end{cases}$$

Since $f(x, z) = \frac{1}{2}(z^2 - x)^2 - \frac{1}{2}x^2$, one can see that

$$S(x) = \begin{cases} 0, & x \in [-1, 0), \\ \pm \sqrt{x}, & x \in [0, 1]. \end{cases}$$

Therefore, the outer objective $F(x, y)$ can be expressed as

$$F(x, y) = \begin{cases} 0, & x \in [-1, 0), \\ \frac{1}{2}x \pm (x - 1)\sqrt{x}, & x \in [0, 1]. \end{cases}$$

So, the optimal solution and the optimal value of (3.10) are $(a = \frac{\sqrt{13}-1}{6})$

$$(x^*, y^*) = (a^2, a) \approx (0.1886, 0.4343), \quad F^* = \frac{1}{2}a^2 + a^3 - a \approx -0.2581.$$

If Algorithm 3.1 is applied without using the Jacobian equation $\psi(x, y) = 0$, we get the computational results shown in Table 1. Problem (3.10) cannot be solved reasonably well. In contrast, if we apply Algorithm 3.1 with the Jacobian equation $\psi(x, y) = 0$, then (3.10) is solved very well. The computational results are shown in Table 2. It takes only two iterations for the algorithm to converge.

TABLE 1
Computational results without $\psi(x, y) = 0$.

Iter k	(x_i^k, y_i^k)	$z_{i,j}^k$	F_k^*	v_i^k
0	$(-1, 1)$	4.098e-13	-1.5000	-1.5000
1	(0.1505, 0.5486)	± 0.3879	-0.3156	-0.0113
2	(0.0752, 0.3879)	± 0.2743	-0.2835	-0.0028
3	(0.2088, 0.5179)	± 0.4569	-0.2754	-0.0018
4	cannot be solved

For the lower level program (1.2), the KKT conditions may fail to hold. In such a case, the classical methods which replace (1.2) by the KKT conditions do not work at all. However, such problems can also be solved efficiently by Algorithm 3.1. The following are two such examples.

TABLE 2
Computational results with $\psi(x, y) = 0$

Iter k	(x_i^k, y_i^k)	$z_{i,j}^k$	F_k^*	v_i^k
0	(-1, 1)	3.283e-21	-1.5000	-1.5000
1	(0.1886, 0.4342)	± 0.4342	-0.2581	-3.625e-12

Example 3.3 (see [10, Example 2.4]). Consider the SBPP

$$(3.11) \quad F^* := \min_{x \in \mathbb{R}, y \in \mathbb{R}} (x-1)^2 + y^2 \quad \text{s.t.} \quad y \in S(x) := \underset{z \in Z := \{z \in \mathbb{R} \mid z^2 \leq 0\}}{\operatorname{argmin}} x^2 z.$$

It is easy to see that the global minimizer of this problem is $(x^*, y^*) = (1, 0)$. The set $Z = \{0\}$ is convex. By using the multiplier variable λ , we get a single level optimization problem:

$$\begin{cases} r^* := \min_{x \in \mathbb{R}, y \in \mathbb{R}, \lambda \in \mathbb{R}} (x-1)^2 + y^2 \\ \text{s.t. } x^2 + 2\lambda y = 0, \lambda \geq 0, y^2 \leq 0, \lambda y^2 = 0. \end{cases}$$

The feasible points of this problem are $(0, 0, \lambda)$ with $\lambda \geq 0$. We have $r^* = 1 > F^*$. The KKT reformulation approach fails in this example, since $y^* \in S(x^*)$ is not a KKT point. We solve the SBPP problem (3.11) by Algorithm 3.1. The Jacobian equation is $\psi(x, y) = x^2 y^2 = 0$, and we reformulate the problem as

$$\begin{cases} s^* := \min_{x \in \mathbb{R}, y \in \mathbb{R}} (x-1)^2 + y^2 \\ \text{s.t. } x^2(z-y) \geq 0 \quad \forall z \in Z, \\ \psi(x, y) = x^2 y^2 = 0. \end{cases}$$

This problem is not an SIPP actually, since the set Z only has one feasible point. At the initial step, we find its optimal solution $(x^*, y^*) = (1, 0)$, and it is easy to check that $\min_{z \in Z} H(x^*, y^*, z) = 0$, which certifies that it is the global minimizer of the SBPP problem (3.11).

Example 3.4. Consider the SBPP

$$(3.12) \quad \begin{cases} \min_{x \in \mathbb{R}, y \in \mathbb{R}^2} & F(x, y) := x + y_1 + y_2 \\ \text{s.t.} & x - 2 \geq 0, \quad 3 - x \geq 0, \\ & y \in S(x) := \underset{z \in Z}{\operatorname{argmin}} f(x, z) := x(z_1 + z_2), \end{cases}$$

where set Z is defined by the inequalities

$$g_1(z) := z_1^2 - z_2^2 - (z_1^2 + z_2^2)^2 \geq 0, \quad g_2(z) := z_1 \geq 0.$$

For all $x \in [2, 3]$, one can check that $S(x) = \{(0, 0)\}$. Clearly, the global minimizer of (3.12) is $(x^*, y^*) = (2, 0, 0)$, and the optimal value $F^* = 2$. At $z^* = (0, 0)$,

$$\nabla_z f(x, z^*) = \begin{bmatrix} x \\ x \end{bmatrix}, \quad \nabla_z g_1(z^*) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \nabla_z g_2(z^*) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The KKT condition does not hold for the lower level program, since $\nabla_z f(x, z^*)$ is not a linear combination of $\nabla_z g_1(z^*)$ and $\nabla_z g_2(z^*)$. By [30, Proposition 3.4], Lasserre

relaxations in (2.2) do not have finite convergence for solving the lower level program. One can check that

$$K_{FJ}(x) = \{(0, 0), (0.8990, 0.2409)\}^1$$

for all feasible x .

By Jacobian representation of $K_{FJ}(x)$, we get

$$\psi(x, z) = \left(xg_1(z)g_2(z), \quad -xz_1(z_1 + z_2 + 2(z_2 - z_1)(z_1^2 + z_2^2)), \quad -xg_1(z) \right).$$

Next, we apply Algorithm 3.1 to solve (3.12). Indeed, for $k = 0$, $Z_0 = \emptyset$, we get

$$(x_1^0, y_1^0) \approx (2.0000, 0.0000, 0.0000),$$

which is the true global minimizer. We also get

$$z_1^0 \approx (4.6320, -4.6330) \times 10^{-5}, \quad v_1^0 \approx -5.2510 \times 10^{-8}.$$

For a small value of ϵ (e.g., 10^{-6}), Algorithm 3.1 terminates successfully with the global minimizer of (3.12).

3.3. Convergence analysis. We study the convergence properties of Algorithm 3.1. For theoretical analysis, we are mostly interested in its performance when the tolerance parameter $\epsilon = 0$ or the maximum iteration number $k_{\max} = \infty$.

THEOREM 3.5. *For the SBPP as in (1.1), assume the lower level program is as in (3.2). Suppose the subproblems (P_k) and each (Q_i^k) are solved globally by Lasserre relaxations.*

- (i) *Assume $\epsilon = 0$. If Algorithm 3.1 stops for some $k < k_{\max}$, then each $(x^*, y^*) \in \mathcal{X}^*$ is a global minimizer of (1.1).*
- (ii) *Assume $\epsilon = 0$, $k_{\max} = \infty$, and the union $\cup_{k \geq 0} Z_k$ is bounded. Suppose Algorithm 3.1 does not stop and each $S_k \neq \emptyset$ is finite. Let (x^*, y^*) be an arbitrary accumulation point of the set $\cup_{k \geq 0} S_k$. If the value function $v(x)$, as in (1.6), is continuous at x^* , then (x^*, y^*) is a global minimizer of the SBPP problem (1.1).*
- (iii) *Assume $k_{\max} = \infty$, the union $\cup_{k \geq 0} Z_k$ is bounded, and the set $\Xi = \{(x, y) : \psi(x, y) = 0, \xi(x, y) \geq 0\}$ is compact. Let $\Xi_1 = \{x : \exists y, (x, y) \in \Xi\}$, which is the projection of Ξ onto the x -space. Suppose $v(x)$ is continuous on Ξ_1 . Then, for all $\epsilon > 0$, Algorithm 3.1 must terminate within finitely many steps, and each $(\bar{x}, \bar{y}) \in \mathcal{X}^*$ is a global minimizer of the approximate SIPP (3.5).*

Proof. (i) The SBPP (1.1) is equivalent to (3.4). Note that each optimal value $F_k^* \leq F^*$ and the sequence $\{F_k^*\}$ is monotonically increasing. If Algorithm 3.1 stops at the k th iteration, then each $(x^*, y^*) \in \mathcal{X}^*$ is feasible for (3.4), and also feasible for (1.1), so it holds that

$$F^* \geq F_k^* = F(x^*, y^*) \geq F^*.$$

This implies that (x^*, y^*) is a global optimizer of problem (1.1).

(ii) Suppose Algorithm 3.1 does not stop and each $S_k \neq \emptyset$ is finite. For each accumulation point (x^*, y^*) of the union $\cup_{k \geq 0} S_k$, there exists a sequence $\{k_\ell\}$ of integers such that $k_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$ and

$$(x^{k_\ell}, y^{k_\ell}) \rightarrow (x^*, y^*), \quad \text{where each } (x^{k_\ell}, y^{k_\ell}) \in S_{k_\ell}.$$

¹They are the solutions of the equations $g_1(z) = 0, z_1 + z_2 + 2(z_2 - z_1)(z_1^2 + z_2^2) = 0$.

Since the feasible set of problem (P_{k_ℓ}) contains the one for problem (1.1), we have $F_{k_\ell}^* = F(x^{k_\ell}, y^{k_\ell}) \leq F^*$, and hence $F(x^*, y^*) \leq F^*$ by the continuity of F . To show the opposite inequality it suffices to show that (x^*, y^*) is feasible for problem (1.1). Recall that the function ξ is defined as in (3.3). Since $\xi(x^{k_\ell}, y^{k_\ell}) \geq 0$ and $\psi(x^{k_\ell}, y^{k_\ell}) = 0$, by the continuity of the mappings ξ, ψ , we have $\xi(x^*, y^*) \geq 0$ and $\psi(x^*, y^*) = 0$. Define the function

$$(3.13) \quad \phi(x, y) := \inf_{z \in Z} H(x, y, z).$$

Clearly, $\phi(x, y) = v(x) - f(x, y)$, and $\phi(x^*, y^*) = 0$ if and only if (x^*, y^*) is a feasible point for (1.1). By the definition of $v(x)$ as in (1.6) and that $v(x)$ is continuous at x^* , we always have $\phi(x^*, y^*) \leq 0$. To prove $\phi(x^*, y^*) = 0$, it remains to show $\phi(x^*, y^*) \geq 0$. For all k' and for all $k_\ell \geq k'$, the point (x^{k_ℓ}, y^{k_ℓ}) is feasible for the subproblem $(P_{k'})$, so

$$H(x^{k_\ell}, y^{k_\ell}, z) \geq 0 \quad \forall z \in Z_{k'}.$$

Letting $\ell \rightarrow \infty$, we then get

$$(3.14) \quad H(x^*, y^*, z) \geq 0 \quad \forall z \in Z_{k'}.$$

The above is true for all k' . In Algorithm 3.1, for each k_ℓ , there exists $z^{k_\ell} \in T_i^{k_\ell}$, for some i , such that

$$\phi(x^{k_\ell}, y^{k_\ell}) = H(x^{k_\ell}, y^{k_\ell}, z^{k_\ell}).$$

Since $z^{k_\ell} \in Z_{k_{\ell+1}}$, by (3.14), we know

$$H(x^*, y^*, z^{k_\ell}) \geq 0.$$

Therefore, it holds that

$$(3.15) \quad \begin{aligned} \phi(x^*, y^*) &= \phi(x^{k_\ell}, y^{k_\ell}) + \phi(x^*, y^*) - \phi(x^{k_\ell}, y^{k_\ell}) \\ &\geq [H(x^{k_\ell}, y^{k_\ell}, z^{k_\ell}) - H(x^*, y^*, z^{k_\ell})] \\ &\quad + [\phi(x^*, y^*) - \phi(x^{k_\ell}, y^{k_\ell})]. \end{aligned}$$

Since z^{k_ℓ} belongs to the bounded set $\cup_{k \geq 0} Z_k$, there exists a subsequence $z^{k_\ell, j}$ such that $z^{k_\ell, j} \rightarrow z^* \in Z$. The polynomial $H(x, y, z)$ is continuous at (x^*, y^*, z^*) . Since $v(x)$ is continuous at x^* , $\phi(x, y) = v(x) - f(x, y)$ is also continuous at (x^*, y^*) . Letting $\ell \rightarrow \infty$, we get $\phi(x^*, y^*) \geq 0$. Thus, (x^*, y^*) is feasible for (3.4) and so $F(x^*, y^*) \geq F^*$. Earlier we proved $F(x^*, y^*) \leq F^*$, so (x^*, y^*) is a global optimizer of (3.4), i.e., (x^*, y^*) is a global minimizer of the SBPP problem (1.1).

(iii) Suppose otherwise, that the algorithm does not stop within finitely many steps. Then there exist a sequence $\{(x^k, y^k, z^k)\}$ such that $(x^k, y^k) \in S_k$, $z^k \in \cup_{i=1}^{r_k} T_i^k$,

$$H(x^k, y^k, z^k) < -\epsilon$$

for all k . Note that $(x^k, y^k) \in \Xi$ and $z^k \in Z_{k+1}$. By the assumption that Ξ is compact and $\cup_{k \geq 0} Z_k$ is bounded, the sequence $\{(x^k, y^k, z^k)\}$ has a convergent subsequence, say,

$$(x^{k_\ell}, y^{k_\ell}, z^{k_\ell}) \rightarrow (x^*, y^*, z^*) \quad \text{as } \ell \rightarrow \infty.$$

So, it holds that $(x^*, y^*) \in \Xi$, $z^* \in Z$, and $H(x^*, y^*, z^*) \leq -\epsilon$. Since Ξ is compact, the projection set Ξ_1 is also compact, and hence $x^* \in \Xi_1$. By the assumption, we

know $v(x)$ is continuous at x^* . Similar to the proof in (ii), we have $\phi(x^*, y^*) = 0$; then (x^*, y^*) is a feasible point for (1.1), and we will get

$$H(x^*, y^*, z^*) = f(x^*, z^*) - f(x^*, y^*) \geq 0.$$

However, this contradicts that $H(x^*, y^*, z^*) \leq -\epsilon$. Therefore, Algorithm 3.1 must terminate within finitely many steps.

Now suppose Algorithm 3.1 terminates within finitely many steps at $(\bar{x}, \bar{y}) \in \mathcal{X}^*$ with $\epsilon > 0$. Then (\bar{x}, \bar{y}) must be a feasible solution to the approximate SIPP (3.5). Hence it is obvious that (\bar{x}, \bar{y}) is a global minimizer of (3.5). \square

In Theorem 3.5, we assumed that the subproblems (P_k) and (Q_i^k) can be solved globally by Lasserre relaxations. This is a reasonable assumption. Please see the remarks (A)–(D) after Algorithm 3.1. In items (ii)–(iii), the value function $v(x)$ is assumed to be continuous at certain points. This can be satisfied under some conditions. The *restricted inf-compactness* (RIC) is such a condition. The value function $v(x)$ is said to have RIC at x^* if $v(x^*)$ is finite and there exist a compact set Ω and a positive number ϵ_0 , such that for all $\|x - x^*\| < \epsilon_0$ with $v(x) < v(x^*) + \epsilon_0$, there exists $z \in S(x) \cap \Omega$. For instance, if the set Z is compact, or the lower level objective $f(x^*, z)$ is weakly coercive in z with respect to set Z , i.e.,

$$\lim_{z \in Z, \|z\| \rightarrow \infty} f(x^*, z) = \infty,$$

then $v(x)$ has RIC at x^* ; see, e.g., [8, section 6.5.1]. Note that the union $\cup_{k \geq 0} Z_k$ is contained in Z . So, if Z is compact, then $\cup_{k \geq 0} Z_k$ is bounded.

PROPOSITION 3.6. *For the SBPP problem (1.1), assume the lower level program is as in (3.2). If the value function $v(x)$ has RIC at x^* , then $v(x)$ is continuous at x^* .*

Proof. On the one hand, since the lower level constraint is independent of x , the value function $v(x)$ is always upper semicontinuous [2, Theorem 4.22 (1)]. On the other hand, since the RIC holds, it follows from [8, p. 246] (or see the proof of [14, Theorem 3.9]) that $v(x)$ is lower semicontinuous. Therefore $v(x)$ is continuous at x^* . \square

4. General bilevel polynomial programs. In this section, we study GBPPs as in (1.1). For GBPPs, the feasible set $Z(x)$ of the lower level program (1.2) varies as x changes; i.e., the constraining polynomials $g_j(x, z)$ depends on x .

For each pair (x, y) that is feasible for (1.1), y is an optimizer for the lower level program (1.2) parameterized by x , so y must be a Fritz John point of (1.2); i.e., there exists $(\mu_0, \mu_1, \dots, \mu_{m_2}) \neq 0$ satisfying

$$\mu_0 \nabla_z f(x, y) - \sum_{j \in [m_2]} \mu_j \nabla_z g_j(x, y) = 0, \quad \mu_j g_j(x, y) = 0 \quad (j \in [m_2]).$$

For convenience, we still use $K_{FJ}(x)$ to denote the set of Fritz John points of (1.2) at x . The set $K_{FJ}(x)$ consists of common zeros of some polynomials. As in (2.3), choose the polynomials $(f(z), g_1(z), \dots, g_m(z))$ to be $(f(x, z), g_1(x, z), \dots, g_{m_2}(x, z))$, whose coefficients depend on x . Then construct ψ_1, \dots, ψ_L in the same way as in (2.8). Each ψ_j is also a polynomial in (x, z) . Thus, every (x, y) feasible in (1.1) satisfies $\psi_j(x, y) = 0$ for all j . For convenience of notation, we still denote the polynomial tuples ξ, ψ as in (3.3).

We have seen that (1.1) is equivalent to the generalized SIPP ($H(x, y, z)$ is as in (1.5))

$$(4.1) \quad \begin{cases} F^* := \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^p} F(x, y) \\ \text{s.t.} \quad \psi(x, y) = 0, \quad \xi(x, y) \geq 0, \\ H(x, y, z) \geq 0 \quad \forall z \in Z(x). \end{cases}$$

Note that the constraint $H(x, y, z) \geq 0$ in (4.1) is required for $z \in Z(x)$, which depends on x . Algorithm 3.1 can also be applied to solve (4.1). We first give an example to show how it works.

Example 4.1 (see [26, Example 3.23]). Consider the GBPP

$$(4.2) \quad \begin{cases} \min_{x, y \in [-1, 1]} x^2 \\ \text{s.t.} \quad 1 + x - 9x^2 - y \leq 0, \\ y \in \operatorname{argmin}_{z \in [-1, 1]} \{z \text{ s.t. } z^2(x - 0.5) \leq 0\}. \end{cases}$$

By simple calculations, one can show that

$$Z(x) = \begin{cases} \{0\}, & x \in (0.5, 1], \\ [-1, 1], & x \in [-1, 0.5], \end{cases} \quad S(x) = \begin{cases} \{0\}, & x \in (0.5, 1], \\ \{-1\}, & x \in [-1, 0.5]. \end{cases}$$

The set $\mathcal{U} = \{(x, y) \in [-1, 1]^2 : 1 + x - 9x^2 - y \leq 0, y^2(x - 0.5) \leq 0\}$. The feasible set of (4.2) is

$$\mathcal{F} := \left(\{(x, 0) : x \in (0.5, 1]\} \cup \{(x, -1) : x \in [-1, 0.5]\} \right) \cap \mathcal{U}.$$

One can show that the global minimizer and the optimal values are

$$(x^*, y^*) = \left(\frac{1 - \sqrt{73}}{18}, -1 \right) \approx (-0.4191, -1), \quad F^* = \left(\frac{1 - \sqrt{73}}{18} \right)^2 \approx 0.1757.$$

By the Jacobian representation of Fritz John points, we get the polynomial

$$\psi(x, y) = (x - 0.5)y^2(y^2 - 1).$$

We apply Algorithm 3.1 to solve (4.2). The computational results are reported in Table 3. As one can see, Algorithm 3.1 takes two iterations to solve (4.2) successfully.

TABLE 3
Results of Algorithm 3.1 for solving (4.2).

Iter k	(x_i^k, y_i^k)	$z_{i,j}^k$	F_k^*	v_i^k
0	(0.0000, 1.0000)	-1.0000	0.0000	-2.0000
1	(-0.4191, -1.0000)	-1.0000	0.1757	-2.4e-11

However, we would like to point out that Algorithm 3.1 might not solve GBPPs globally. The following is such an example.

Example 4.2 (see [26, Example 5.2]). Consider the GBPP

$$(4.3) \quad \begin{cases} \min_{x \in \mathbb{R}, y \in \mathbb{R}} & (x - 3)^2 + (y - 2)^2 \\ \text{s.t.} & 0 \leq x \leq 8, \quad y \in S(x), \end{cases}$$

where $S(x)$ is the set of minimizers of the optimization problem

$$\begin{cases} \min_{z \in \mathbb{R}} & (z - 5)^2 \\ \text{s.t.} & 0 \leq z \leq 6, \quad -2x + z - 1 \leq 0, \\ & x - 2z + 2 \leq 0, \quad x + 2z - 14 \leq 0. \end{cases}$$

It can be shown that

$$S(x) = \begin{cases} \{1 + 2x\}, & x \in [0, 2], \\ \{5\}, & x \in (2, 4], \\ \left\{7 - \frac{x}{2}\right\}, & x \in (4, 6], \\ \emptyset, & x \in (6, 8]. \end{cases}$$

The feasible set of (4.3) is thus the set

$$\mathcal{F} := \{(x, y) \mid x \in [0, 6], y \in S(x)\}.$$

It consists of three connected line segments. One can easily check that the global optimizer and the optimal values are

$$(x^*, y^*) = (1, 3), \quad F^* = 5.$$

The polynomial ψ in the Jacobian representation is

$$\psi(x, y) = (-2x + y - 1)(x - 2y + 2)(x + 2y - 14)y(y - 6)(y - 5).$$

We apply Algorithm 3.1 to solve (4.3). The computational results are reported in Table 4. For $\epsilon = 10^{-6}$, Algorithm 3.1 stops at $k = 1$ and returns the point (2.9972, 5.00), which is not a global minimizer.

TABLE 4
Results of Algorithm 3.1 for solving (4.3).

Iter k	(x_i^k, y_i^k)	$z_{i,j}^k$	F_k^*	v_i^k
0	(2.7996, 2.3998)	5.0021	0.2000	-6.7611
1	(2.9972, 5.0000)	5.0021	9.0001	4.41e-6

However, it is interesting to note that the computed solution (2.9972, 5.0000) \approx (3, 5), a local optimizer of problem (4.3).

Why does Algorithm 3.1 fail to find a global minimizer in Example 4.2? By adding z^0 to the discrete subset Z_1 , the feasible set of (P_1) becomes

$$\{x \in X, y \in Z(x)\} \cap \{\psi(x, y) = 0\} \cap \{|y - 5| \leq 0.0021\}.$$

It does not include the unique global optimizer $(x^*, y^*) = (1, 3)$. In other words, the reason is that $H(x^*, y^*, z^0) \geq 0$ fails to hold, and hence by adding z^0 , the true optimal solution (x^*, y^*) is not in the feasible region of problem (P_1) .

From the above example, we observe that the difficulty for solving GBPPs globally comes from the dependence of the lower level feasible set on x . For a global optimizer (x^*, y^*) , it is possible that $H(x^*, y^*, z_{i,j}^k) \not\geq 0$ for some $z_{i,j}^k$ at some step; i.e., (x^*, y^*) may fail to satisfy the newly added constraint in (P_{k+1}) : $H(x, y, z_{i,j}^k) \geq 0$. In other words, (x^*, y^*) may not be feasible for the subproblem (P_{k+1}) . Let \mathcal{X}_k be the feasible set of problem (P_k) . Since $Z_k \subseteq Z_{k+1}$, we have $\mathcal{X}_{k+1} \subseteq \mathcal{X}_k$, and (x^*, y^*) is not feasible for (P_ℓ) for all $\ell \geq k+1$. In such cases, Algorithm 3.1 will fail to find a global optimizer. However, this will not happen for SBPPs, since $Z(x) \equiv Z$ for all x . For all $z \in Z$, we have $H(x^*, y^*, z) \geq 0$; i.e., (x^*, y^*) is feasible for all subproblems (P_k) . This is why Algorithm 3.1 has convergence to global optimal solutions for solving SBPPs. However, under some further conditions, Algorithm 3.1 can solve GBPPs globally.

THEOREM 4.3. *For the GBPP as in (1.1), assume that the lower level program is as in (1.2) and the minimum value F^* is achievable at a point (\bar{x}, \bar{y}) such that $H(\bar{x}, \bar{y}, z) \geq 0$ for all $z \in Z_k$ and for all k . Suppose (P_k) and (Q_i^k) are solved globally by Lasserre relaxations.*

- (i) *Assume $\epsilon = 0$. If Algorithm 3.1 stops for some $k < k_{\max}$, then each $(x^*, y^*) \in \mathcal{X}^*$ is a global minimizer of the GBPP problem (1.1).*
- (ii) *Assume $\epsilon = 0$, $k_{\max} = \infty$, and the union $\cup_{k \geq 0} Z_k$ is bounded. Suppose Algorithm 3.1 does not stop and each $S_k \neq \emptyset$ is finite. Let (x^*, y^*) be an arbitrary accumulation point of the set $\cup_{k \geq 0} S_k$. If the value function $v(x)$, defined as in (1.6), is continuous at x^* , then (x^*, y^*) is a global minimizer of the GBPP problem (1.1).*
- (iii) *Assume $k_{\max} = \infty$, the union $\cup_{k \geq 0} Z_k$ is bounded, and the set $\Xi = \{(x, y) : \psi(x, y) = 0, \xi(x, y) \geq 0\}$ is compact. Let $\Xi_1 = \{x : \exists y, (x, y) \in \Xi\}$, the projection of Ξ onto the x -space. Suppose $v(x)$ is continuous on Ξ_1 . Then, for all $\epsilon > 0$, Algorithm 3.1 must terminate within finitely many steps.*

Proof. By the assumption, the point (\bar{x}, \bar{y}) is feasible for the subproblem (P_k) for all k . Hence, we have $F_k^* \leq F^*$. The rest of the proof is the same as the proof of Theorem 3.5. \square

In the above theorem, the existence of the point (\bar{x}, \bar{y}) satisfying the requirement may be hard to check. If $v(x)$ has restricted inf-compactness at x^* and the Mangasarian–Fromovitz constraint qualification (MFCQ) holds at all solutions of the lower level problem (1.2), then the value function $v(x)$ is Lipschitz continuous at x^* ; see [8, Corollary 1]. Recently, it was shown in [14, Corollary 4.8] that the MFCQ can be replaced by a weaker condition called quasi-normality in the above result.

5. Numerical experiments. In this section, we present numerical experiments for solving BPPs. In Algorithm 3.1, the polynomial optimization subproblems are solved by Lasserre semidefinite relaxations, implemented in software `Gloptipoly 3` [15] and the SDP solver `SeDuMi` [36]. The computation is implemented with MATLAB R2012a on a MacBook Pro 64-bit OS X (10.9.5) system with 16GB memory and 2.3 GHz Intel Core i7 CPU. In the algorithms, we set the parameters $k_{\max} = 20$ and $\epsilon = 10^{-5}$. In reporting computational results, we use (x^*, y^*) to denote the computed global optimizers, F^* to denote the value of the outer objective function F at (x^*, y^*) , v^* to denote $\inf_{z \in Z} H(x^*, y^*, z)$, *Iter* to denote the total of number of iterations for convergence, and *Time* to denote the CPU time taken to solve the problem (in seconds, unless stated otherwise). When $v^* \geq -\epsilon$, the computed point (x^*, y^*) is considered as a global minimizer of (P) , up to the tolerance ϵ . Mathematically, to solve BPPs exactly, we need to set $\epsilon = 0$. However, in computational practice, the round-off errors always exist, so we choose $\epsilon > 0$ to be a small number.

5.1. Examples of SBPPs.

Example 5.1 (see [26, Example 3.26]). Consider the SBPP

$$\left\{ \begin{array}{l} \min_{x \in \mathbb{R}^2, y \in \mathbb{R}^3} \quad x_1 y_1 + x_2 y_2^2 + x_1 x_2 y_3^3 \\ \text{s.t.} \quad x \in [-1, 1]^2, \quad 0.1 - x_1^2 \leq 0, \\ \quad \quad 1.5 - y_1^2 - y_2^2 - y_3^2 \leq 0, \\ \quad \quad -2.5 + y_1^2 + y_2^2 + y_3^2 \leq 0, \\ \quad \quad y \in S(x), \end{array} \right.$$

where $S(x)$ is the set of minimizers of

$$\min_{z \in [-1, 1]^3} \quad x_1 z_1^2 + x_2 z_2^2 + (x_1 - x_2) z_3^2.$$

It was shown in [26, Example 3.26] that the unique global optimal solution is

$$x^* = (-1, -1), \quad y^* = (1, \pm 1, -\sqrt{0.5}).$$

Algorithm 3.1 terminates after one iteration. It takes about 14.83 seconds. We get

$$\begin{aligned} x^* &\approx (-1, -1), & y^* &\approx (1, \pm 1, -0.7071), \\ F^* &\approx -2.3536, & v^* &\approx -5.71 \times 10^{-9}. \end{aligned}$$

Example 5.2. Consider the SBPP

$$(5.1) \quad \left\{ \begin{array}{l} \min_{x \in \mathbb{R}^2, y \in \mathbb{R}^3} \quad x_1 y_1 + x_2 y_2 + x_1 x_2 y_1 y_2 y_3 \\ \text{s.t.} \quad x \in [-1, 1]^2, \quad y_1 y_2 - x_1^2 \leq 0, \\ \quad \quad y \in S(x), \end{array} \right.$$

where $S(x)$ is the set of minimizers of

$$\left\{ \begin{array}{l} \min_{z \in \mathbb{R}^3} \quad x_1 z_1^2 + x_2^2 z_2 z_3 - z_1 z_3^2 \\ \text{s.t.} \quad 1 \leq z_1^2 + z_2^2 + z_3^2 \leq 2. \end{array} \right.$$

Algorithm 3.1 terminates after one iteration. It takes about 13.45 seconds. We get

$$\begin{aligned} x^* &\approx (-1, -1), & y^* &\approx (1.1097, 0.3143, -0.8184), \\ F^* &\approx -1.7095, & v^* &\approx -1.19 \times 10^{-9}. \end{aligned}$$

By Theorem 3.5, we know (x^*, y^*) is a global optimizer, up to a tolerance around 10^{-9} .

Example 5.3. We consider some test problems from [26]. For convenience of display, we choose problems that have common constraints $x \in [-1, 1]$ for the outer level program and $z \in [-1, 1]$ for the inner level program. When Algorithm 3.1 is applied, all these SBPPs are solved successfully. The outer objective $F(x, y)$, the inner objective $f(x, z)$, the global optimizers (x^*, y^*) , the number of consumed iterations *Iter*, the CPU time taken to solve the problem, the optimal value F^* , and the value v^* are reported in Table 5. In all problems, except Ex. 3.18 and Ex. 3.19, the optimal solutions we obtained coincide with those given in [26]. For Ex. 3.18, the global optimal solution for minimizing the upper level objective $-x^2 + y^2$ subject to constraints

TABLE 5

Results for some SBPP problems in [26]. They have the common constraints $x \in [-1, 1]$ and $z \in [-1, 1]$.

Problem	SBPP	(x^*, y^*)	Iter	Time	F^*	v^*
Ex. 3.14	$F = (x - 1/4)^2 + y^2$ $f = z^3/3 - xz$	(0.2500, 0.5000)	2	0.49	0.2500	-5.7e-10
Ex. 3.15	$F = x + y$ $f = xz^2/2 - z^3/3$	(-1.0000, 1.0000)	2	0.42	2.79e-8	-4.22e-8
Ex. 3.16	$F = 2x + y$ $f = -xz^2/2 - z^4/4$	(-0.5, -1), (-1, 0)	2	0.47	-2.0000	-6.0e-10
Ex. 3.17	$F = (x + 1/2)^2 + y^2/2$ $f = xz^2/2 + z^4/4$	(-0.2500, ± 0.5000)	4	1.12	0.1875	-8.3e-11
Ex. 3.18	$F = -x^2 + y^2$ $f = xz^2 - z^4/2$	(1.0000, 0.0000)	2	0.44	-1.0000	-3.1e-13
Ex. 3.19	$F = xy - y + y^2/2$ $f = -xz^2 + z^4/2$	(0.1886, 0.4343)	2	0.41	-0.2581	-3.6e-12
Ex. 3.20	$F = (x - 1/4)^2 + y^2$ $f = z^3/3 - x^2z$	(0.5000, 0.5000)	2	0.38	0.3125	-1.1e-10

$x, y \in [-1, 1]$ is $x^* = 1, y^* = 0$. It is easy to check that $y^* = 0$ is the optimal solution for the lower level problem parameterized by $x^* = 1$, and hence $x^* = 1, y^* = 0$ is also the unique global minimizer for the SBPP in Ex. 3.18. For Ex. 3.19, as shown in [26], the optimal solution must have $x^* \in (0, 1)$. For such x^* , $S(x^*) = \{\pm\sqrt{x^*}\}$. Plugging $y = \pm\sqrt{x}$ into the upper level objective, we have $F(x, y) = \pm x\sqrt{x} + \sqrt{x} + \frac{x}{2}$. It is obvious that the minimum over $0 < x < 1$ should occur when $y = \sqrt{x}$. So minimizing $F(x, y) = x\sqrt{x} - \sqrt{x} + \frac{x}{2}$ over $0 < x < 1$ gives $x^* = (\frac{\sqrt{13}-1}{6})^2 \approx 0.1886$, $y^* = \frac{\sqrt{13}-1}{6} \approx 0.4343$.

Example 5.4. Consider the SBPP

$$(5.2) \quad \begin{cases} \min_{x \in \mathbb{R}^4, y \in \mathbb{R}^4} & x_1^2 y_1 + x_2 y_2 + x_3 y_3^2 + x_4 y_4^2 \\ \text{s.t.} & \|x\|^2 \leq 1, y_1 y_2 - x_1 \leq 0, \\ & y_3 y_4 - x_3^2 \leq 0, y \in S(x), \end{cases}$$

where $S(x)$ is the set of minimizers of

$$\begin{cases} \min_{z \in \mathbb{R}^4} & z_1^2 - z_2(x_1 + x_2) - (z_3 + z_4)(x_3 + x_4) \\ \text{s.t.} & \|z\|^2 \leq 1, z_2^2 + z_3^2 + z_4^2 - z_1 \leq 0. \end{cases}$$

We apply Algorithm 3.1 to solve (5.2). The computational results are reported in Table 6. As one can see, Algorithm 3.1 stops when $k = 4$ and solves (5.2) successfully. It takes about 20 minutes to solve the problem. By Theorem 3.5, we know the point (x_i^k, y_i^k) obtained at $k = 4$ is a global optimizer for (5.2), up to a tolerance around 10^{-8} .

An interesting special case of SBPPs is that the inner level program has no constraints, i.e., $Z = \mathbb{R}^p$. In this case, the set $K_{F,J}(x)$ of Fritz John points is just the set of critical points of the inner objective $f(x, z)$. It is easy to see that the polynomial $\psi(x, z)$ is given as

$$\psi(x, z) = \left(\frac{\partial}{\partial z_1} f(x, z), \dots, \frac{\partial}{\partial z_p} f(x, z) \right).$$

TABLE 6
Results of Algorithm 3.1 for solving (5.2).

Iter k	(x_i^k, y_i^k)	F_k^*	v_i^k
0	(-0.0000, 1.0000, -0.0000, 0.0000, 0.6180, -0.7862, 0.0000, 0.0000)	-0.7862	-1.6406
1	(0.0000, -0.0000, 0.0000, -1.0000, 0.6180, -0.0000, 0.0000, -0.7862)	-0.6180	-0.3458
	(0.0003, -0.0002, -0.9999, 0.0000, 0.6180, 0.0001, -0.7861, -0.0000)	-0.6180	-0.3458
2	(0.0000, -0.0000, -0.8623, -0.5064, 0.6180, -0.0000, -0.6403, -0.4561)	-0.4589	-0.0211
3	(0.0000, -0.0000, -0.7098, -0.7042, 0.6180, -0.0000, -0.5570, -0.5548)	-0.4371	-6.37e-5
4	(0.0000, -0.0000, -0.7071, -0.7071, 0.6180, 0.0000, -0.5559, -0.5559)	-0.4370	-2.27e-8

Example 5.5 (SBPPs with $Z = \mathbb{R}^p$). Consider random SBPPs with ball conditions on x and no constraints on z :

$$(5.3) \quad \begin{cases} F^* := \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^p} F(x, y) \\ \text{s.t. } \|x\|^2 \leq 1, \quad y \in \operatorname{argmin}_{z \in \mathbb{R}^p} f(x, z), \end{cases}$$

where $F(x, y)$ and $f(x, z)$ are generated randomly as

$$F(x, y) := a_1^T [u]_{2d_1-1} + \|B_1 [u]^{d_1}\|^2, \\ f(x, z) := a_2^T [x]_{2d_2-1} + a_3^T [z]_{2d_2-1} + \left\| B_2 \begin{pmatrix} [x]^{d_2} \\ [z]^{d_2} \end{pmatrix} \right\|^2.$$

In the above, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_p)$, $z = (z_1, \dots, z_p)$, $u = (x, y)$, and $d_1, d_2 \in \mathbb{N}$. The symbol $[x]_d$ denotes the vector of monomials in x and of degrees $\leq d$, while $[x]^d$ denotes the vector of monomials in x and of degrees equal to d . The symbols $[y]_d, [y]^d, [u]^d$ are defined in the same way.

TABLE 7
Results for random SBPPs as in (5.3).

n	p	d_1	d_2	Iter			Time			v^*		
				Min	Avg	Max	Min	Avg	Max	Min	Avg	Max
2	3	3	2	1	1.9	5	00:01	00:02	00:06	-3.8e-6	-2.9e-7	-4.32e-8
3	3	2	2	1	1.6	2	00:04	00:07	00:09	-4.0e-6	-3.7e-7	-1.1e-10
3	3	3	2	1	1.7	2	00:04	00:07	00:10	-2.0e-6	-2.6e-7	-7.4e-11
4	2	2	2	1	1.4	3	00:04	00:06	00:09	-3.0e-6	-2.4e-7	-4.9e-12
4	3	2	2	1	2.3	5	00:15	00:41	01:36	-5.3e-6	-6.4e-7	-4.67e-9
5	2	2	2	1	1.9	4	00:14	00:33	01:13	-3.5e-6	-8.1e-7	-4.3e-11
5	3	2	2	1	1.8	3	06:30	10:04	11:56	-1.1e-6	-3.8e-7	-1.9e-10
6	2	2	2	1	2.0	4	04:02	09:56	17:39	-6.2e-6	-1.5e-6	-5.57e-7

We test the performance of Algorithm 3.1 for solving SBPPs in the form (5.3). The computational results are reported in Table 7. In the table, we randomly generated 20 instances for each case. *Avg Iter* denotes the average number of iterations taken by Algorithm 3.1, *Avg Time* denotes the average of consumed time, and *Avg(v*)* denotes the average of the values v^* . The consumed computational time is in the format mn:sc, with mn and sc standing for minutes and seconds, respectively. As we can see, these SBPPs were solved successfully. In Table 7, the computational time in the last two rows are much bigger than those in the previous rows. This is because the newly

added Jacobian equation $\psi(x, y) = 0$ has more polynomials and has higher degrees. Consequently, in order to solve (P_k) and (Q_k^k) globally by Lasserre relaxations, the relaxation orders need to be higher. This makes the semidefinite relaxations more difficult to solve.

Example 5.6 (random SBPPs with ball conditions). Consider the SBPP

$$(5.4) \quad \begin{cases} \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^p} & F(x, y) \\ \text{s.t.} & \|x\|^2 \leq 1, \quad y \in \underset{\|z\|^2 \leq 1}{\operatorname{argmin}} f(x, z). \end{cases}$$

The outer and inner objectives $F(x, y)$, $f(x, z)$ are generated as

$$F(x, y) = a^T [(x, y)]_{2d_1}, \quad f(x, z) = \begin{pmatrix} [x]_{d_2} \\ [z]_{d_2} \end{pmatrix}^T B \begin{pmatrix} [x]_{d_2} \\ [z]_{d_2} \end{pmatrix}.$$

The entries of the vector a and matrix B are generated randomly, obeying Gaussian distributions. The symbols like $[(x, y)]_{2d_1}$ are defined similarly as in Example 5.5. We apply Algorithm 3.1 to solve (5.4). The computational results are reported in Table 8. The meanings of *Avg Iter*, *Avg Time*, and *Avg(v^*)* are same as in Example 5.5. As we can see, the SBPPs as in (5.4) can be solved successfully by Algorithm 3.1.

TABLE 8
Results for random SBPPs in (5.4).

n	p	d_1	d_2	Iter			Time			v^*		
				Min	Avg	Max	Min	Avg	Max	Min	Avg	Max
3	2	2	2	1	2.6	6	00:01	00:03	00:06	-7.4e-7	-1.4e-7	2.0e-9
3	3	2	2	1	2.7	6	00:03	00:09	00:21	-2.6e-6	-6.5e-7	-1.5e-9
3	3	3	2	1	3.0	5	00:03	00:09	00:17	-2.9e-6	-3.6e-7	-1.1e-9
4	2	2	2	1	3.5	8	00:03	00:20	00:43	-1.8e-6	-5.0e-7	1.4e-9
4	3	2	2	1	2.6	5	00:12	00:31	01:01	-2.9e-6	-3.0e-7	1.8e-9
5	2	2	2	1	3.7	11	00:11	00:43	02:06	-3.9e-6	-1.7e-7	-3.4e-9
5	2	3	2	1	3.4	10	00:10	00:41	02:15	-3.6e-6	-5.4e-7	-1.5e-9
6	2	2	2	1	2.6	6	03:21	09:17	22:41	-4.3e-6	-5.7e-7	5.8e-10
6	2	3	2	1	2.4	5	03:15	08:23	17:42	-6.2e-7	-1.5e-7	2.7e-10

5.2. Examples of GBPPs.

Example 5.7. Consider the GBPP

$$(5.5) \quad \begin{cases} \min_{x \in \mathbb{R}^2, y \in \mathbb{R}^3} & \frac{1}{2}x_1^2y_1 + x_2y_2^2 - (x_1 + x_2)y_3 \\ \text{s.t.} & x \in [-1, 1]^2, \quad x_1 + x_2 - x_1^2 - y_1^2 - y_2^2 \geq 0, \\ & y \in S(x), \end{cases}$$

where $S(x)$ is the set of minimizers of

$$\begin{cases} \min_{z \in \mathbb{R}^3} & x_2(z_1z_2z_3 + z_2^2 - z_3^3) \\ \text{s.t.} & x_1 - z_1^2 - z_2^2 - z_3^2 \geq 0, \quad 1 - 2z_2z_3 \geq 0. \end{cases}$$

We apply Algorithm 3.1 to solve (5.5). Algorithm 3.1 terminates at the iteration $k = 0$. It takes about 10.18 seconds to solve the problem. We get

$$x^* \approx (1, 1), \quad y^* \approx (0, 0, 1), \quad F_0^* \approx -2, \quad v^* \approx -2.95 \times 10^{-8}.$$

Since $Z_0 = \emptyset$, we have $F_0^* \leq F^*$ (the global minimum value). Moreover, (x^*, y^*) is feasible for (5.5), so $F(x^*, y^*) \geq F^*$. Therefore, $F(x^*, y^*) = F^*$ and (x^*, y^*) is a global optimizer, up to a tolerance around 10^{-8} .

Example 5.8. Consider the GBPP

$$(5.6) \quad \begin{cases} \min_{x \in \mathbb{R}^4, y \in \mathbb{R}^4} & (x_1 + x_2 + x_3 + x_4)(y_1 + y_2 + y_3 + y_4) \\ \text{s.t.} & \|x\|^2 \leq 1, y_3^2 - x_4 \leq 0, \\ & y_2 y_4 - x_1 \leq 0, y \in S(x), \end{cases}$$

where $S(x)$ is the set of minimizers of

$$\begin{cases} \min_{z \in \mathbb{R}^4} & x_1 z_1 + x_2 z_2 + 0.1 z_3 + 0.5 z_4 - z_3 z_4 \\ \text{s.t.} & z_1^2 + 2z_2^2 + 3z_3^2 + 4z_4^2 \leq x_1^2 + x_3^2 + x_2 + x_4, \\ & z_2 z_3 - z_1 z_4 \geq 0. \end{cases}$$

We apply Algorithm 3.1 to solve (5.6). The computational results are reported in Table 9. Algorithm 3.1 stops with $k = 1$. It takes about 490.65 seconds to solve the problem. We are not sure whether the point (x_i^k, y_i^k) computed at $k = 1$ is a global optimizer or not.

TABLE 9
Results of Algorithm 3.1 for solving (5.6).

Iter k	(x_i^k, y_i^k)	F_k^*	v_i^k
0	(0.5442, 0.4682, 0.4904, 0.4942, -0.7792, -0.5034, -0.2871, -0.1855)	-3.5050	-0.0391
1	(0.5135, 0.5050, 0.4882, 0.4929, -0.8346, -0.4104, -0.2106, -0.2887)	-3.4880	3.29e-9

Example 5.9. In this example we consider some GBPP examples given in the literature. The problems and the computational results are displayed in Table 10. Problem 1 is [1, Example 3.1] and the optimal solution $(x^*, y^*) = (0, 0)$ is reported. Problem 2 is [43, Example 4.2] and the optimal solution $(x^*, y^*) = (1, 1)$ is reported. Problem 3 is [26, Example 3.22]. As shown in [26], the optimal solution should attain at a point satisfying $0 < x < 1$ and $y = -0.5 + 0.1x$. For (x, y) satisfying these conditions, the lower level constraint $0.01(1 + x^2) - y^2 \leq 0$ becomes inactive. Plugging $y = -0.5 + 0.1x$ into the upper level objective, the bilevel program becomes finding the minimum of the convex function $(x - 0.6)^2 + (-0.5 + 0.1x)^2$. Hence the optimal solution is $(x^*, y^*) = (\frac{65}{101}, \frac{44}{101})$. Problem 4 can be found in [26, Example 4.2] with the optimal solution $(x^*, y^*) = (1, 0, 1)$ reported. Problem 5 can be found in [26, Example 5.1], where the optimal solution $(x^*, y^*) = (5, 4, 2)$ is derived. Problem 6 is [10, Example 3.1]. As shown in [10], the optimal solution is $(x^*, y^*) = (\sqrt{0.5}, \sqrt{0.5})$. Problem 7 was originally given in [3, Example 3] and analyzed in [1]. It was reported in [1] that the optimal solution is $x^* = (0, 2)$, $y^* \approx (1.875, 0.9062)$. In fact we can show that the optimal solution is $x^* = (0, 2)$, $y^* = (\frac{15}{8}, \frac{29}{32})$ as follows. Since the upper objective is separable in x and y , it is easy to show that the optimal solution for the problem

$$\min_{(x_1, x_2) \geq 0} -x_1^2 - 3x_2 - 4y_1 + y_2^2 \quad \text{s.t.} \quad -x_1^2 - 2x_2 + 4 \geq 0$$

with y_1, y_2 fixed is $x_1^* = 0$, $x_2^* = 2$. Since $y^* = (\frac{15}{8}, \frac{29}{32})$ is the optimal solution to the lower level problem parameterized by $x^* = (0, 2)$, we conclude that the optimal solution is $x^* = (0, 2)$, $y^* = (\frac{15}{8}, \frac{29}{32})$. From Table 10, we can see that Algorithm 3.1 stops in very few steps with global optimal solutions for all problems.

TABLE 10
Results for some GBPPs.

No.	Small GBPPs	Results	
1	$\begin{cases} \min_{x \in \mathbb{R}, y \in \mathbb{R}} & -x - y \\ \text{s.t.} & y \in S(x) := \underset{z \in Z(x)}{\operatorname{argmin}} z \\ & Z(x) := \{z \in \mathbb{R} \mid -x + z \geq 0, -z \geq 0\}. \end{cases}$	F^*	-2.78e-13
		Iter	1
		x^*	3.82e-14
		y^*	2.40e-13
		v^*	-7.43e-13
		Time	0.19
2	$\begin{cases} \min_{x \in \mathbb{R}, y \in \mathbb{R}} & (x - 1)^2 + y^2 \\ \text{s.t.} & x \in [-3, 2], y \in S(x) := \underset{z \in Z(x)}{\operatorname{argmin}} z^3 - 3z \\ & Z(x) := \{z \in \mathbb{R} \mid z \geq x\}. \end{cases}$	F^*	0.9999
		Iter	2
		x^*	0.9996
		y^*	1.0000
		v^*	-4.24e-9
		Time	0.57
3	$\begin{cases} \min_{x \in \mathbb{R}, y \in \mathbb{R}} & (x - 0.6)^2 + y^2 \\ \text{s.t.} & x, y \in [-1, 1], y \in S(x) := \underset{z \in Z(x)}{\operatorname{argmin}} f(x, z) = z^4 + \frac{4}{30}(1 - x)z^3 \\ & \quad + (0.16x - 0.02x^2 - 0.4)z^2 + (0.004x^3 - 0.036x^2 + 0.08x)z, \\ & Z(x) := \{z \in \mathbb{R} \mid 0.01(1 + x^2) \leq z^2, z \in [-1, 1]\}. \end{cases}$	F^*	0.1917
		Iter	2
		x^*	0.6436
		y^*	-0.4356
		v^*	2.18e-10
		Time	0.52
4	$\begin{cases} \min_{x \in \mathbb{R}, y \in \mathbb{R}^2} & x^3 y_1 + y_2 \\ \text{s.t.} & x \in [0, 1], y \in [-1, 1] \times [0, 100], y \in S(x) := \underset{z \in Z(x)}{\operatorname{argmin}} -z_2 \\ & Z(x) := \{z \in \mathbb{R}^2 \mid xz_1 \leq 10, z_1^2 + xz_2 \leq 1, z \in [-1, 1] \times [0, 100]\}. \end{cases}$	F^*	1
		Iter	1
		x^*	1
		y^*	(0,1)
		v^*	3.45e-8
		Time	1.83
5	$\begin{cases} \min_{x \in \mathbb{R}, y \in \mathbb{R}^2} & -x - 3y_1 + 2y_2 \\ \text{s.t.} & x \in [0, 8], y \in [0, 4] \times [0, 6], y \in S(x) = \underset{z \in Z(x)}{\operatorname{argmin}} -z_1 \\ & Z(x) := \left\{ z \in \mathbb{R}^2 \mid \begin{array}{l} -2x + z_1 + 4z_2 \leq 16, 8x + 3z_1 - 2z_2 \leq 48 \\ 2x - z_1 + 3z_2 \geq 12, z \in [0, 4] \times [0, 6] \end{array} \right\}. \end{cases}$	F^*	-13
		Iter	1
		x^*	5
		y^*	(4,2)
		v^*	3.95e-6
		Time	0.38
6	$\begin{cases} \min_{x \in \mathbb{R}^2, y \in \mathbb{R}^2} & -y_2 \\ \text{s.t.} & y_1 y_2 = 0, x \geq 0, y \in S(x) := \underset{z \in Z(x)}{\operatorname{argmin}} z_1^2 + (z_2 + 1)^2 \\ & Z(x) := \left\{ z \in \mathbb{R}^2 \mid \begin{array}{l} (z_1 - x_1)^2 + (z_2 - 1 - x_1)^2 \leq 1, \\ (z_1 + x_2)^2 + (z_2 - 1 - x_2)^2 \leq 1 \end{array} \right\}. \end{cases}$	F^*	-1
		Iter	2
		x^*	(0.71, 0.71)
		y^*	(0,1)
		v^*	-3.77e-10
		Time	0.60
7	$\begin{cases} \min_{x \in \mathbb{R}^2, y \in \mathbb{R}^2} & -x_1^2 - 3x_2 - 4y_1 + y_2^2 \\ \text{s.t.} & (x, y) \geq 0, -x_1^2 - 2x_2 + 4 \geq 0, y \in S(x) := \underset{z \in Z(x)}{\operatorname{argmin}} z_1^2 - 5z_2 \\ & Z(x) := \left\{ z \in \mathbb{R}^2 \mid \begin{array}{l} x_1^2 - 2x_1 + x_2^2 - 2z_1 + z_2 + 3 \geq 0, \\ x_2 + 3z_1 - 4z_2 - 4 \geq 0 \end{array} \right\}. \end{cases}$	F^*	-12.6787
		Iter	2
		x^*	(0,2)
		y^*	(1.88, 0.91)
		v^*	2.40e-6
		Time	10.52

6. Conclusions and discussions. This paper studies how to solve both simple and general bilevel polynomial programs. We reformulate them as equivalent semi-infinite polynomial programs, using Fritz John conditions and Jacobian representations. Then we apply the exchange technique and Lasserre-type semidefinite relaxations to solve them. For solving SBPPs, we proposed Algorithm 3.1 and proved

its convergence to global optimal solutions. For solving GBPPs, Algorithm 3.1 can also be applied, but its convergence to global optimizers is not guaranteed. However, under some assumptions, GBPPs can also be solved globally by Algorithm 3.1. Extensive numerical experiments are provided to demonstrate the efficiency of the proposed method. To see the advantages of our method, we would like to make some comparisons with two existing methods for solving bilevel polynomial programs. The first one is the value function approximation approach proposed by Jeyakumar et al. [17]; the second one is the branch and bound approach proposed by Mitsos, Lemonidis, and Barton [27].

6.1. Comparison with the value function approximation approach. For solving SBPPs with convex lower level programs, a semidefinite relaxation method was proposed in [17, section 3], under the assumption that the lower level programs satisfy both the nondegeneracy condition and the Slater condition. It uses multipliers, appearing in the Fritz John conditions, as new variables in SOS-type representations. For SBPPs with nonconvex lower level programs, it was proposed in [17, section 4] to solve the following ϵ -approximation problem (for a tolerance parameter $\epsilon > 0$):

$$(6.1) \quad (P_\epsilon^k) : \begin{cases} F_\epsilon^k := \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^p} F(x, y) \\ \text{s.t.} \quad G_i(x, y) \geq 0, \quad i = 1, \dots, m_1, \\ \quad \quad g_j(y) \geq 0, \quad j = 1, \dots, m_2, \\ \quad \quad f(x, y) - J_k(x) \leq \epsilon. \end{cases}$$

In the above, $J_k(x) \in \mathbb{R}_{2k}[x]$ is a $\frac{1}{k}$ -solution for approximating the nonsmooth value function $v(x)$ [17, Algorithm 4.5]. For a given parameter $\epsilon > 0$, the method in [17, section 4] finds the approximating polynomial $J_k(x)$ first and then solves (P_ϵ^k) by Lasserre-type semidefinite relaxations. Theoretically, $\epsilon > 0$ can be chosen as small as possible. However, in computational practice, when $\epsilon > 0$ is very small, the degree $2k$ needs to be chosen very high, making it hard to compute $J_k(x)$. In the following, we give an example to compare our Algorithm 3.1 and the method in [17, section 4].

Example 6.1. Consider the SBPP

$$(6.2) \quad \begin{cases} F^* := \min_{x \in \mathbb{R}^2, y \in \mathbb{R}^2} y_1^3(x_1^2 - 3x_1x_2) - y_1^2y_2 + y_2x_2^3 \\ \text{s.t.} \quad x \in [-1, 1]^2, \quad y_2 + y_1(1 - x_1^2) \geq 0, \quad y \in S(x), \end{cases}$$

where $S(x)$ is the solution set of the optimization problem

$$v(x) := \min_{z \in \mathbb{R}^2} z_1z_2^2 - z_2^3 - z_1^2(x_2 - x_1^2) \quad \text{s.t.} \quad z_1^2 + z_2^2 \leq 1.$$

The computational results of applying Algorithm 3.1 is shown in Table 11. It took only two steps to solve the problem successfully. The set \mathcal{U} is compact. For each x , $S(x) \neq \emptyset$, since the lower level program is defined as a polynomial over a compact set. The value function $v(x)$ of lower level programs is continuous. The feasible set of problem (6.2) is nonempty and compact. At the iteration $k = 1$, the value v_1^k is almost zero, so the point $(0.5708, -1.0000, -0.1639, 0.9865)$ is a global optimizer of problem (6.2), up to a tolerance around 10^{-9} .

Next, we apply the method in [17, section 4]. We use the software Yalmip [22] to compute the approximating polynomial $J_k(x) \in \mathbb{R}_{2k}[x]$, as in [17, Algorithm 4.5].

TABLE 11
Computational results of Algorithm 3.1 for solving (6.2).

Iter k	(x_i^k, y_i^k)	$z_{i,j}^k$	F_k^*	v_i^k
0	(1.0000, -1.0000, -1.0000, 0.0000)	(-0.1355, 0.9908)	-4.0000	-3.0689
	(-1.0000, 1.0000, -1.0000, 0.0000)	(-0.2703, 0.9628)	-4.0000	-1.1430
1	(0.5708, -1.0000, -0.1639, 0.9865)	(-0.1638, 0.9865)	-1.0219	-4.76e-9

After that, we solve the problem (P_ϵ^k) by Lasserre-type semidefinite relaxations, for a parameter $\epsilon > 0$. Let F_ϵ^k denote the optimal value of (6.1). The computational results are shown in Table 12. As ϵ is close to 0, we can see that F_ϵ^k is close to the true optimal value $F^* \approx -1.0219$. Since the method in [17] depends on the choice of $\epsilon > 0$, we do not compare the computational time. In applications, the optimal value F^* is typically unknown. An interesting question for research is how to select a value of $\epsilon > 0$ that guarantees F_ϵ is close enough to F^* .

TABLE 12
Computational results of the method in [17, section 4].

ϵ	F_ϵ^2	F_ϵ^3	F_ϵ^4
1.0	-3.4372	-3.6423	-3.6439
0.5	-1.5506	-1.5909	-1.5912
0.25	-1.2718	-1.2746	-1.2750
0.125	-1.1746	-1.1775	-1.1779
0.05	-1.1193	-1.1224	-1.1228
0.01	-1.0897	-1.0930	-1.0934
0.005	-1.0858	-1.0892	-1.0897
0.001	-1.0827	-1.0862	-1.0867
0.0001	-1.0820	-1.0855	-1.0860

6.2. Comparison with the branch and bound approach. Mitsos, Lemonidis, and Barton [27] proposed a bounding algorithm for solving bilevel programs, in combination with the exchange technique. It works on finding a point that satisfies ϵ -optimality in the inner and outer programs. For the lower bounding algorithm, a relaxed program needs to be solved globally. The optional upper bounding problem is based on probing the solution obtained by the lower bounding procedure. The algorithm can be extended to use branching techniques. For cleanness of the paper, we do not repeat the details here. Interested readers are referred to [27]. We list some major differences between the method in our paper and the one in [27].

- The method in [27] is based on building a tree of nodes of subproblems, obtained by partitioning box constraints for the variables x, y . Our method does not need to build such a tree of nodes and does not require box constraints for partitioning.
- For each subproblem in the lower/upper bounding, a nonlinear nonconvex optimization, or a mixed integer nonlinear nonconvex optimization, needs to be solved globally or with ϵ -optimality. The software applications GMAS [34] and BARON [35] are applied to solve them. In contrast, our method does not solve these nonlinear nonconvex subproblems by BARON and GMAS. Instead, we solve them globally by Lasserre-type semidefinite relaxations, which are convex programs and can be solved efficiently by a standard SDP package like

SeDuMi. In our computational experiments, the subproblems are all solved globally by **GloptiPoly 3** [15] and **SeDuMi** [36].

In [27], the branch and bound method was implemented in **C++**, and the subproblems were solved by **BARON** and **GMAS**. In our paper, the method is implemented in **MATLAB**, and the subproblems are solved by **GloptiPoly 3** and **SeDuMi**. Their approaches and implementations are very different. It is hard to find a good way to compare them directly. However, for BPPs, the subproblems in [27] and in our paper are all polynomial optimization problems. To compare the two methods, it is reasonably well to compare the number of subproblems that are needed to be solved, although this may not be the best way.

We choose the seven SBPPs in Example 5.3, which were also in [27]. The numbers of subproblems are listed in Table 13. In the table, **B & B (I)** is the branch and bound method in [27] without branching; **B & B (II)** is the branch and bound method in [27] with branching; **#LBD** is the number of lower bounding subproblems; **#UBD** is the number of upper bounding subproblems; **#L-POP** is the number of subproblems (P_k) needs to be solved in Algorithm 3.1; **#U-POP** is the number of subproblems (Q_i^k) needs to be solved in Algorithm 3.1. The number of variables in lower bounding subproblems for branch and bound methods (I/II) and subproblem (P_k) for Algorithm 3.1 are the same, all equal to $n + p$; the number of variables in upper bounding subproblems for branch and bound methods (I/II) and subproblem (Q_i^k) for Algorithm 3.1 are the same, all equal to p . For problem Ex. 3.16, since the subproblem (P_k) has two optimal solutions, so we need to solve two subproblems (Q_i^k) to check if they are both global optimal solutions. From Table 13, one can see that Algorithm 3.1 has a smaller number of subproblems that need to be solved. If all the subproblems are solved by the same method, Algorithm 3.1 is expected to be more efficient.

TABLE 13

A comparison of the numbers of polynomial optimization subproblems in [27] and in Algorithm 3.1.

Problem	B & B (I)		B & B (II)		Alg. 3.1	
	#LBD	#UBD	#LBD	#UBD	#L-POP	#U-POP
Ex. 3.14	4	3	7	3	2	2
Ex. 3.15	2	1	3	1	2	2
Ex. 3.16	2	1	3	1	2	3
Ex. 3.17	19	18	37	18	4	4
Ex. 3.18	2	2	3	2	2	2
Ex. 3.19	13	12	27	14	2	2
Ex. 3.20	4	3	5	3	2	2

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