Chapter 20
Linear, Second-Order Difference Equations

In this chapter, we will learn how to solve autonomous and non-autonomous linear second order difference equations.

Autonomous Equations

The general form of linear, autonomous, second order difference equation is

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = b. \tag{20.1}$$

In order to solve this we divide the equation in two parts: steady state part and homogeneous part. We solve these two parts separately and the complete solution is sum of these two solutions.

The steady state part is given by

$$\overline{y} + a_1 \overline{y} + a_2 \overline{y} = b \tag{20.2}$$

which implies that

$$\overline{y} = \frac{b}{1 + a_1 + a_2} \tag{20.3}$$

Notice that steady-state solution exists if and only if $1 + a_1 + a_2 \neq 0$. We will impose this condition through out this section.

The homogeneous part of the difference equation is given by

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0. \tag{20.4}$$

(20.4) has a trivial solution $y_t = 0$. In order to find non-trivial homogeneous solution, y_h, assume that the solution has following form

$$y_t = Ar^t \tag{20.5}$$

where A & $r \neq 0$ are two unknown constants. We would like to find the values of these two unknowns which satisfy (20.4). As we will see below r has the interpretation of **characteristic root or eigen value**.

Putting (20.5) in (20.4), we get

$$r^t(r^2 + a_1 r + a_2) = 0. \tag{20.6}$$

We are looking for the values of r which solve quadratic equation

$$r^2 + a_1 r + a_2 = 0. \tag{20.7}$$

(20.7) is known as **characteristic equation** associated with homogeneous difference equation (20.4). The values of r which satisfy (20.7) are known as **characteristic roots**.

The characteristic roots are given by

$$r_1,\ r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}. \tag{20.8}$$

Three Cases

Case I: $a_1^2 - 4a_2 > 0$; Real Distinct roots, $r_1 \neq r_2$ and

$$y_h = C_1 r_1^t + C_2 r_2^t \tag{20.9}$$

where C_1 and C_2 are two unknown coefficients to be determined by initial conditions.

Case II: $a_1^2 - 4a_2 + 0$; Real Repeated Roots $r_1 = r_2 = r = -\frac{a_1}{2}$ and

$$y_h = (C_1 + C_2 t) r^t. \tag{20.10}$$

Case III: $a_1^2 - 4a_2 < 0$; Complex Distinct roots, $r_1 \neq r_2$ and roots are given by conjugate complex numbers

$$r_1,\ r_2 = h \pm iv \tag{20.11}$$

where

$$h = -\frac{a_1}{2} \ \& \ v = \frac{\sqrt{4a_2 - a_1^2}}{2}. \qquad (20.11)$$

The solution to the homogeneous difference equation is

$$y_h = C_1(h + iv)^t + C_2(h - iv)^t. \qquad (20.12)$$

These complex numbers can be expressed in trigonometric forms as

$$h \pm iv = R(cos\theta \pm isin\theta) \qquad (20.13)$$

where R is the **absolute value** of the complex roots and given by

$$R = \sqrt{h^2 + v^2} = a_2^{1/2}. \qquad (20.14)$$

Using (20.14), we can rewrite (20.12) as

$$y_h =$$

$$C_1 R^t (cos\theta + isin\theta)^t + C_2 R^t (cos\theta - isin\theta)^t. \tag{20.15}$$

de Moivre's theorem implies that

$$[R(cos\theta + isin\theta)]^n = R^n [cos\theta n + isin\theta n]. \tag{20.16}$$

Putting (20.16) in (20.15), we have

$$y_h =$$

$$a_2^{t/2} (C_1 (cos\theta t + isin\theta t) + C_2 (cos\theta t - isin\theta t)$$

which simplifies to

$$y_h = a_2^{t/2}(B_1 cos\theta t + B_2 sin\theta t) \qquad (20.17)$$

where $B_1 = C_1 + C_2$ and $B_2 = (C_1 - C_2)i$.

The complete solution of the difference equation is given by

$$y_t = y_h + \overline{y} = y_h + \frac{b}{1 + a_1 + a_2}. \qquad (20.18)$$

Convergence

Steady state is stable if

$$\lim_{t \to \infty} y_t = \lim_{t \to \infty} y_h + \overline{y} = \overline{y}. \qquad (20.19)$$

This requires that $|r_1| \ \& |r_2| < 1$ if we have distinct real roots, $|r| < 1$ in the case of repeated roots and $a_2^{1/2} < 1$ in case of complex roots.

Whether steady-state is stable or not can also be determined by inspecting coefficients of the difference equation.

Theorem 20.6: The absolute value of the roots of the characteristic equation for the linear, autonomous, second-order difference equation are less than 1 if and only if the following three conditions are satisfied:

$$1 + a_1 + a_2 > 0 \tag{i}$$

$$1 - a_1 + a_2 > 0 \tag{ii}$$

$$a_2 < 1. \tag{iii}$$

Linear, Second-Order Difference Equation with a Variable Term

Suppose the difference equation is given by

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = b_t. \tag{20.20}$$

With variable b_t steady-state does not exist. We need to find out counter-part of the steady-state solution. We will call it particular solution and denote it by y_p. The homogeneous solution can be found by using the methods discussed earlier.

Under certain conditions we can find, y_p, using the **method of undetermined coefficients**.

Case I: b_t is a polynomial of n^{th} degree:

$$b_t = B_0 + B_1 t + ... + B_n t^n. \tag{20.21}$$

In this case, assume that

$$y_p = A_0 + A_1 t + ... + A_n t^n \tag{20.22}$$

where $A's$ are undetermined coefficients. After making this assumption, put (20.22) in (20.20) which would result in n linear equations in A. By solving this system of n linear equations we can find $A's$.

Case II:

$$b_t = Bk^t. \tag{20.23}$$

In this case assume that

$$y_p = Ak^t \tag{20.23}$$

and follow through as in case I.

Case III:

$$b_t = Bk^t(B_0 + B_1 t + ... + B_n t^n). \tag{20.24}$$

In this case, assume that

$$y_p = Ak^t(A_0 + A_1 t + ... + A_n t^n) \quad (20.25)$$

and follow through as in case I.

Note: An important caveat to above rules is that no term in the assumed solution should be equal to the homogeneous solution disregarding the multiplicative term. If this case arises, multiply the assumed solution by t^k, where k is the smallest positive integer such that the common terms are then eliminated.