Lecture 3

Dynamic General equilibrium Models

1. Introduction

In macroeconomics, we study behavior of economy-wide aggregates – e.g. GDP, savings, investment, employment and so on - and their interrelations. The behavior of aggregates and their interrelations are results of decisions and interactions of consumers and firms in different markets – goods market, labor market, and asset market. In addition, most of the issues in macroeconomics are inherently dynamic. In growth we are concerned with the behavior of output, investment, and consumption over long run. Business cycle relates to short-run movements in a number of variables *e.g.* GDP, employment, real wage, inflation, investment. Savings involve foregoing current consumption for the sake of higher future consumption. Investment decision requires comparison of current costs with expected future returns. Thus in macroeconomics, we are concerned with the behavior of agents across time and markets *i.e.* macroeconomics is about dynamics and general equilibrium.

In order to analyze macroeconomics issues, we need a framework which can handle both general equilibrium and dynamics. **Dynamic general equilibrium (DGE)** models provide one such framework. These models ensure that aggregate or economy-wide variables are consistent with the decisions and interactions of individual agents, and the decisions of individual agents are optimal given aggregate variables and other parameters.

In this lecture, we will develop a basic framework of these models and study some of its applications. First we will consider two-period models. Then we will analyze models with infinite horizon. DGE models generally have following building blocks:

1. **Description of the Economy/ Environment:** This section gives details about number and types of goods and agents, preferences (objective functions) of agents, their endowments, technology, structure of markets, trading processes, information structure, timing of events, time period, and sources of shocks. It is extremely important to clearly describe the economy. Basically this section lays out the structure of the economy and all the assumptions a modeler makes.

2. **Optimal Decisions of Agents:** This section analyzes optimal behavior of agents subject to given constraints *e.g.* consumers maximize utility subject to their budget constraints, firms maximize profit. This is partial equilibrium analysis. At this stage, it is very important to differentiate between what variables are choice variables of the agents and what variables they take as given.

   There are two types of variables - endogenous variables and exogenous variables. Endogenous variables are variables whose solution we are seeking. Exogenous variables are variables given from outside. Endogenous variables are also of two types : (i) individual choice variables, which individual agents choose; and (ii) aggregate or economy-wide endogenous variables, which are not chosen by individual agents, but are the outcomes of their interactions. For example, in the competitive market consumption by a consumer or production by a firm is individual choice variable, but the prices are aggregate endogenous variables. Individual agents while making decisions take exogenous and aggregate endogenous variables as given.

3. **Definition of Equilibrium:** This section tells us what constitutes an equilibrium. Usually equilibrium consists of a system of prices and allocation (quantity) which are consistent with the optimizing behavior of agents given the market structure and feasibility constraints (coming from endowments, technology etc.).

4. **Solution:** Ultimately, we are interested in the solution of the model, which involves expressing endogenous variables solely as function of exogenous variables. The optimal decisions of agents together with the definition of equilibrium allow us to find solution of the model.

To illustrate these elements, we will solve many examples. But first we are going to analyze some partial equilibrium models. We will assume that agents while making decisions take market prices as given.

1. Partial Equilibrium Models

Example 1

Suppose that the utility function of a consumer is $U(c_1, c_2)$, where c_1 and c_2 are consumption of good 1 and good 2 respectively. The utility function is an increasing and concave function of consumption of both goods. The budget constraint faced by the consumer is $p_1c_1 + p_2c_2 = Y$, where p_1, p_2, and Y are prices of good 1 and good 2 and income respectively. We want to find out the optimal choices of c_1 and c_2 (consumption bundle) given prices and income.

The consumer's problem is

$$\max_{c_1, c_2} U(c_1, c_2)$$

subject to

$$p_1c_1 + p_2c_2 = Y. \tag{1.1}$$

The easiest way to solve this problem is to put the budget constraint in the utility function (or the objective function). This way we convert the constrained optimization problem in an unconstrained optimization problem. Then, we have

$$\max_{c_1} U(c_1, \frac{Y}{p_2} - \frac{p_1}{p_2}c_1). \tag{1.2}$$

The first order condition is

$$U_1 = \frac{p_1}{p_2}U_2 \tag{1.3}$$

which can be rewritten as

$$\frac{U_1}{U_2} = \frac{p_1}{p_2}. \tag{1.4}$$

(1.4) equates the marginal rate substitution between the two goods to the ratio of their prices. Using equations (1.1) and (1.4) we can derive consumption functions $c_1(p_1, p_2, Y)$ and $c_2(p_1, p_2, Y)$.

Let us take an specific example. Suppose that $U(c_1, c_2) = \ln c_1 + \ln c_2$. Then from (1.4) we have

$$p_1 c_1 = p_2 c_2. \tag{1.5}$$

Putting (1.5) in the budget constraint we have $c_1 = \frac{1}{2}\frac{Y}{p_1}$. Then (1.5) implies that $c_2 = \frac{1}{2}\frac{Y}{p_2}$.

Example 2

Let us take another example with the labor-leisure choice. Suppose that the utility function of the consumer is $U(c, 1-l)$ where l is the amount of time worked ($1-l$ is the leisure). Assume that the utility function is an increasing and concave function of consumption and leisure. Let P and W be the price of the consumption good and wage respectively. What will be the optimal choices of consumption and leisure (labor supply) given prices?

The consumer's problem is

$$\max_{c,l} U(c, 1-l)$$

subject to

$$Pc = Wl. \tag{1.6}$$

Putting (1.6) in the objective function we have

$$\max_{l} U(Wl/P, 1-l). \tag{1.7}$$

The first order condition is

$$\frac{U_2}{U_1} = \frac{W}{P} \tag{1.8}$$

which equates the MRS between consumption and leisure to the real wage. Using (1.6) and (1.8) we can derive the individual demand function $c(W, P)$ and the labor supply function $l(W, p)$.

Example 3

Let us now take a two-period model where consumers face consumption-savings choi- ces. Suppose that the utility function of a consumer is $U(c_1, c_2)$, where c_1 and c_2 are consumption in time period 1 and 2 respectively. The consumer can save in terms of financial instrument at the net rate of interest

r. Let Y be the income in the first period. What will be the optimal choices of consumption and savings given the rate of interest?

The consumer's problem is

$$\max_{c_1, c_2, s} U(c_1, c_2)$$

subject to

$$c_1 + s = Y \ \& \tag{1.9}$$

$$c_2 = (1 + r)s \tag{1.10}$$

where s is the amount of saving in period 1.

By putting (1.9) and (1.10) in the objective function we have

$$\max_{s} U(Y - s, (1 + r)s).$$

The first order condition is

$$\frac{U_1}{U_2} = 1 + r \tag{1.11}$$

which equates the MRS between consumption in two periods to the gross rate of interest.

Using (1.11) and the budget constraints, we can derive consumption functions $c_1(r, Y)$ and $c_2(r, Y)$ and savings function $s(r, Y)$.

Example 4

Let us modify the previous problem as follows. Suppose that the consumer also has access to production technology which converts k units investment in period one to $f(k)$ units of goods in the second period. The production technology is an increasing and concave function of k. Now the consumer faces a **portfolio-choice problem**. It can enhance second period consumption by savings in the financial instrument or it can invest. What will be the optimal portfolio?

The consumer's problem is

$$\max_{c_1, c_2, s, k} U(c_1, c_2)$$

subject to

$$c_1 + s + k = Y \ \& \tag{1.12}$$

$$c_2 = (1+r)s + f(k). \tag{1.13}$$

By putting (1.12) and (1.13) in the objective function we have

$$\max_{s,k} U(Y - s - k, (1+r)s + f(k)). \tag{1.14}$$

The first order conditions are

$$s : \frac{U_1}{U_2} = 1 + r \ \& \tag{1.15}$$

$$k : \frac{U_1}{U_2} = f_k(k). \tag{1.16}$$

Combining (1.15) and (1.16), we have

$$f_k(k) = 1 + r \tag{1.17}$$

which equates the marginal product of capital to the gross rate of interest. Since the consumer now has two instruments of savings, at the optimum it must be indifferent between the two. Using (1.12)-(1.16) one can derive the consumption functions $c_1(r, Y)$ and $c_2(r, Y)$ and savings $s(r, Y)$ and investment functions $k(r, Y)$.

Example 5

Suppose that there is a firm. The production depends on investment, k, and labor input, l. More specifically, the production function, $f(k, l)$ is an increasing and concave function of investment and labor input. Let w, r, and δ be the real wage, the net rate of interest, and the rate of depreciation respectively. What would be the optimal choices of k and l?

The objective of the representative firms is to choose k and l in order to maximize the profit

$$PR \equiv f(k,l) + (1-\delta)k - (1+r)k - wl = f(k,l) - (\delta + r)k - wl. \tag{1.18}$$

The first order conditions are

$$MPK \equiv f_k(k,l) = r + \delta \ \& \tag{1.19}$$

$$MPL \equiv f_l(k,l) = w. \tag{1.20}$$

Using (1.19) and (1.20) we can derive the input demand functions $k(w,r)$ and $l(w,r)$. So far we have analyzed several partial equilibrium models. Let us now turn to general equilibrium models.

2. General Equilibrium Models

Let us begin with a two-period representative agent economy with production. By assumption, consumers in the economy have identical preferences and identical wealth levels.

Example 6

Environment

1. Economy lasts for two periods.
2. Single commodity.
3. Large number (unit measure) of identical consumers and identical firms.
4. Consumers own all firms equally.
5. In the first period each consumer is endowed with y units of good, which they can consume and save. They have zero endowment in the second period.
6. The preference of the 'representative consumer' is given by

$$U = \ln c_1 + \beta \ln c_2 \tag{2.1}$$

where c_i is the consumption at time i and $\beta \in (0,1)$ is the discount rate.

7. The ‘representative firm’ possesses a technology which converts k units investment in period one to k^{α} units of goods in the second period. Let δ be the rate of depreciation.

8. Tradings between consumers and firms take place in a competitive market.

Consumer Optimization

Let us first state the budget constraint of the representative consumer. The first period budget constraint is given by

$$c_1 + s = y \tag{2.2}$$

where s is the saving. The second period constraint is given by

$$c_2 = s(1+r) + PR \tag{2.3}$$

where r is the real rate of interest (taken as given by consumers) and PR is the profit repatriated by the representative firm to the representative consumer. We will define PR below. We can combine (2.2) and (2.3) and get intertemporal budget constraint of the consumer given by

$$c_1 + \frac{c_2}{1+r} = y + \frac{PR}{1+r}. \tag{2.4}$$

The consumer problem is

$$\max_{c_1, c_2} U = \ln c_1 + \beta \ln c_2 \tag{2.5}$$

subject to the inter-temporal budget constraint in (2.4). Let λ be the Langrangian multiplier associated with (2.4), then the first order conditions are

$$c_1 : \; MU_1 \equiv \frac{1}{c_1} = \lambda \tag{2.6}$$

$$c_2 : \; MU_2 \equiv \frac{\beta}{c_2} = \frac{\lambda}{1+r}. \tag{2.7}$$

Combining (2.6) and (2.7), we have

$$\frac{MU_1}{MU_2} \equiv MRS \equiv -\frac{dc_2}{dc_1} \equiv \frac{c_2}{\beta c_1} = 1 + r \tag{2.8}$$

where MRS is the marginal rate of substitution. (2.8) together with the budget constraint gives the consumption functions:

$$c_1 = \frac{1}{1+\beta}\left[y + \frac{PR}{1+r}\right] \tag{2.9}$$

$$c_2 = \frac{\beta(1+r)}{1+\beta}\left[y + \frac{PR}{1+r}\right]. \tag{2.10}$$

(2.9) and (2.10) show that the current consumption is a function of the life-time income and not only the current income. This illustrates the **permanent income hypothesis**. Savings/borrowings allow a consumer to consume more or less than the current income in a given period.

Firm Optimization

The objective of the representative firms is to choose k in order to maximize the profit

$$PR \equiv k^\alpha + (1-\delta)k - (1+r)k = k^\alpha - (\delta + r)k. \tag{2.11}$$

To simplify the problem, we will assume that $\delta = 1$ (100% depreciation). The first order condition yields

$$MPK \equiv \alpha k^{\alpha-1} = 1 + r \tag{2.12}$$

Definition of The Equilibrium

Competitive Equilibrium: A competitive equilibrium is the price (real rate of interest) r and allocation $\{c_1,\ c_2,\ k\}$ such that:

(a) the representative consumer maximizes its utility given prices and subject to its budget constraints;

(b) the representative firm maximizes profit given prices and technology; and

(c) supply equals demand for each good:

$$c_1 + k = y, \; c_2 = k^\alpha. \tag{2.13}$$

The last part of the definition pins down the equilibrium level of real rate of interest, r. In order to get equilibrium allocation and prices, use (2.8), (2.12), and market clearing condition (2.13). After some work, you can show that the equilibrium allocation and the real rate of interest satisfy:

$$k = \frac{\alpha\beta y}{1+\alpha\beta} \tag{2.14}$$

$$c_1 = \frac{1}{1+\alpha\beta} y \tag{2.15}$$

$$c_2 = \left[\frac{\alpha\beta y}{1+\alpha\beta}\right]^\alpha \tag{2.16}$$

$$r = \alpha \left[\frac{1+\alpha\beta}{\alpha\beta y}\right]^{1-\alpha} - 1. \tag{2.17}$$

Example 7
Heterogeneity: A Model of Private Debt/Credit

In the previous example, we assumed that all agents are alike. Let us now introduce heterogeneity. Agents can be heterogeneous in terms of their preferences, endowments, information etc. We will consider a simple model in which agents are heterogenous in terms of their endowment pattern. This will allow us to examine the issue of the circulation credit and debt.

Suppose that there are two types of individuals: borrowers, with no endowment in the first period and endowment y in the second period, and lenders with endowment y in the first period and no endowment in the second period. With this structure of endowment, borrowers would like to borrow in the first period while lenders would like to lend in order to finance their consumption in the second period.

Preferences and Constraints of Lenders

Let $c_{1,l}, c_{2,l}$, and l denote the first-period consumption, second-period consumption, and the amount of lending of a lender respectively. The first-period budget constraint for a lender is

$$c_{1,l} + l = y. \tag{2.18}$$

The second-period budget constraint is

$$c_{2,l} = (1 + r)l. \tag{2.19}$$

The life-time budget constraint is given by

$$c_{1,l} + \frac{c_{2,l}}{1 + r} = y. \tag{2.20}$$

The lender chooses $c_{1,l}, c_{2,l}, l$ in order to maximize

$$U(c_{1,l}, c_{2,l}) \tag{2.21}$$

subject to its budget constraints.

Optimal Choices of Lender

Putting (2.18) and (2.19) in (2.21), we have

$$\max_{l} U(y - l, (1 + r)l). \tag{2.22}$$

The first order condition is

$$\frac{U_1(c_{1,l}, c_{2,l})}{U_2(c_{1,l}, c_{2,l})} = 1 + r. \tag{2.23}$$

(2.23) equates the marginal rate of substitution between the current and the future consumption to the rate of interest. Using this equation, we can derive the amount lent, l, as a function of interest rate r, $l(r)$. Normally we assume that utility function is such that lending, l, is an increasing function of the real interest rate, r, *i.e.*, $l_1(r) > 0$. Using (2.18), (2.19), and (2.23), we can derive $c_{1,l}, c_{2,l}, l$ as a function of interest rate r.

Preferences and Constraints of Borrowers

Let $c_{1,b}, c_{2,b}$, and b denote the first-period consumption, second-period consumption, and the amount of borrowing of a borrower. Let r be the net rate of interest. The first-period budget constraint for a borrower is

$$c_{1,b} = b. \tag{2.24}$$

The second-period budget constraint is

$$c_{2,b} = y - (1+r)b. \tag{2.25}$$

The life-time budget constraint is given by

$$c_{1,b} + \frac{c_{2,b}}{r} = \frac{y}{1+r}. \tag{2.26}$$

The borrower chooses $c_{1,b}, c_{2,b}, b$ in order to maximize

$$U(c_{1,b}, c_{2,b}) \tag{2.27}$$

subject to its budget constraints.

Optimal Choices of Borrower

Putting (2.24) and (2.25) in (2.27), we have

$$\max_{l} U(b, y - (1+r)b). \tag{2.28}$$

The first order condition is

$$\frac{U_1(c_{1,b}, c_{2,b})}{U_2(c_{1,b}, c_{2,b})} = 1 + r. \tag{2.29}$$

(2.29) equates the marginal rate of substitution between the current and the future consumption to the real rate of interest. Using this equation, we can derive the amount borrowed, b, as a function of the real interest rate r, $b(r)$. Normally we assume that utility function is such that the borrowing, b, is a decreasing function of the interest rate, r, *i.e.*, $b_1(r) < 0$. Using (2.24), (2.25), and (2.29), we can derive $c_{1,b}, c_{2,b}, b$ as a function of interest rate r.

Note that (2.23) and (2.29) equates the MRS of lenders and borrowers:

$$\frac{U_1(c_{1,l}, c_{2,l})}{U_2(c_{1,l}, c_{2,l})} = \frac{U_1(c_{1,b}, c_{2,b})}{U_2(c_{1,b}, c_{2,b})} = 1 + r. \tag{2.30}$$

Definition of The Equilibrium

Competitive Equilibrium: A competitive equilibrium is the price (real rate of interest) r and allocation $\{c_{1,l},\ c_{2,l},\ c_{1,b},\ c_{2,b},\ b\ \&\ l\}$ such that:

(a) the representative lender maximizes its utility given prices and subject to its budget constraints;

(b) the representative borrower maximizes its utility given prices and subject to its budget constraints; and

(c) markets clear:

$$c_{1,l} + c_{1,b} = y, \ c_{2,l} + c_{2,b} = y, \ \& \ l(r) = b(r). \tag{2.31}$$

The condition that $l(r) = b(r)$ allows us to pin down the equilibrium rate of interest. Once we have determined the equilibrium real rate of interest, we can derive the allocations $c_{1,l}, c_{2,l}, l, c_{1,b}, c_{2,b}, b, l$.

3. The Social Planner Problem

For policy formulation, it is important to know whether the allocation made by the market is efficient and maximize social welfare. If market allocation is not efficient or social welfare maximizing, then what are the options available to policy makers/ government/ social planner?

In order to know whether a particular allocation is social welfare maximizing we need to have some kind of social preference which reflects preferences of individual agents. In general, ways to aggregate preferences of individual agents are subject to debate because of differing utilities. But in the case of representative agent economy, deriving social welfare maximizing allocation is particularly simple because every agent is identical. The socially optimal allocation maximizes the representative consumer's utility subject to the resource constraint.

$$\max_{c_1, c_2, k} U = U(c_1) + \beta U(c_2)$$

subject to resource constraints

$$c_1 + k = y \ \& \tag{3.1}$$

$$c_2 = f(k) + (1 - \delta)k. \tag{3.2}$$

The first order condition is

$$k : \ U_1(c_1) = \beta U_2(c_2)(f_k(k) + 1 - \delta). \tag{3.3}$$

Using (3.1)-(3.3) one can derive socially optimal allocation. This allocation satisfies the condition that

$$MRS = MRPT.$$

where MRPT is the marginal rate of product transformation given by $f'(k) + 1 - \delta$.

Example 8

Let us derive the socially optimal allocation in the economy considered in example 6.

$$\max_{c_1, c_2, k} U = \ln c_1 + \beta \ln c_2$$

subject to resource constraints (assuming $\delta = 1$)

$$c_1 + k = y \tag{3.4}$$

$$c_2 = k^\alpha \tag{3.5}$$

One can easily show that efficient allocation is given by:

$$k = \frac{\alpha\beta y}{1 + \alpha\beta} \tag{3.6}$$

$$c_1 = \frac{1}{1 + \alpha\beta} y \tag{3.7}$$

$$c_2 = \left[\frac{\alpha\beta y}{1 + \alpha\beta}\right]^\alpha \tag{3.8}$$

which coincides with market allocation (2.14, 2.15, and 2.16).

Why does social optimal allocation coincide with market allocation? From microeconomics, we know that socially optimal allocation is **Pareto optimal or efficient**. An allocation (in our case $\{c_1\ c_2\ k\}$) is *Pareto optimal* or Pareto efficient if production and distribution cannot be reorganized to increase the utility of one or more individuals without decreasing utility of others.

From the **first and second fundamental theorems of welfare economics** we know that competitive allocation is Pareto optimal (under certain conditions) and optimal allocation can be supported as competitive equilibria (under more restrictive conditions). Our example (actually all examples considered so far) satisfies conditions under which fundamental theorems apply and thus market allocation coincides with social optimal allocation.

In the competitive economies where the second fundamental theorem of welfare applies, usually it is easier to compute competitive equilibrium by solving the social planner problem, rather than going through the consumers and firms optimization problem and imposing the market clearing conditions. Steps involved in computing competitive equilibrium through this method are as follows:

1. Compute the socially optimal allocation.

2. Derive the real rate of interest by equating it to either MRS or MRPT and evaluating derivatives at the optimal allocation:

$$MRS \equiv \frac{MU_1}{MU_2} = MRPT \equiv MPK = 1 + r. \tag{3.9}$$

3. Other prices such as wages can be computed by evaluating the relevant MRS at the socially optimal allocation.

We can use the above method to compute the competitive equilibrium in economies which satisfy conditions of second fundamental theorem of welfare. However, there are many economies which do not satisfy these conditions. In such economies, the social planner allocation normally diverge from the market allocation. Such divergence raises interesting policy issues, e.g. whether policy interventions can improve market allocation. The examples of such economies are economies with distortionary taxes, imperfect competition (e.g. Keynesian models), increasing returns, externalities, OLG economies etc.

Example 9

Let us derive the socially optimal allocation in the economy considered in example 7. Since, there is heterogeneity we have think about how to aggregate the preferences of lenders and borrowers i.e. aggregate the individual preferences into one social preference. One reasonable way is to assume that the social preference is represented by the weighted average of individual preferences (**utilitarian social welfare function**). Let us suppose that the social planner puts weight $\lambda \in (0, 1)$ on the utility of the lender and $1 - \lambda$ on the utility of the borrowers. Thus, the social planner maximizes

$$\max_{c_{1,l}, c_{2,l}, c_{1,b}, c_{2,b}} \lambda U(c_{1,l}, c_{2,l}) + (1 - \lambda) U(c_{1,b}, c_{2,b})$$

subject to resource constraints:

$$c_{1,l} + c_{1,b} = y \ \& \tag{3.10}$$

$$c_{2,l} + c_{2,b} = y. \tag{3.11}$$

The first order conditions are

$$c_{1,l} : \ \lambda U_1(c_{1,l}, c_{2,l}) = (1 - \lambda) U_1(c_{1,b}, c_{2,b}) \ \& \tag{3.11}$$

$$c_{2,l} : \ \lambda U_2(c_{1,l}, c_{2,l}) = (1 - \lambda) U_2(c_{1,b}, c_{2,b}) \ \& \tag{3.12}$$

(3.11) and (3.12) imply that

$$\frac{U_1(c_{1,l}, c_{2,l})}{U_2(c_{1,l}, c_{2,l})} = \frac{U_1(c_{1,b}, c_{2,b})}{U_2(c_{1,b}, c_{2,b})} \tag{3.13}$$

just as in the market economy.

4. Uncertainty and Expectations

So far we have been dealing with economies without uncertainty. But the real world is full of uncertainty. In this section, we introduce uncertainty in two-period economies. We will assume that exogenous variables (technology, endowments, preferences, taxes, money supply etc.) can take more than one value in the second period. We will also assume that uncertainty about the values of exogenous variables can be expressed in terms of their probability distributions and all agents in the economy know these distributions. The question we are going to ask is: how allocations and prices are determined in economies in which agents face uncertainty about exogenous variables in the second period (no uncertainty in the first period)? The DGE model with uncertainty is known as **Dynamic Stochastic General Equilibrium (DSGE) model**.

Example 10

We begin with an example. Let us modify the environment example 6 by assuming that there is uncertainty about production function in the next period. Let the new production function be Ak^{α} where A is a random variable which can take values A^h and A^l with probabilities p^h and p^l respectively ($p^h + p^l = 1$). Consider h to be high state and l to be low state in the sense that $A^h > A^l$. We continue to assume that the depreciation rate $\delta = 1$. Now we want to find out allocations and prices in this economy. Before we proceed, let us define the expectation operator E. The expected value of A is given by

$$E(A) = p^h A^h + p^l A^l. \tag{4.1}$$

Notice that there are two states in the second period: high state and low state. Corresponding to these two states, there will be two consumption levels c_2^h and c_2^l in the second period. Thus, in this economy the objects of interest are $c_1,\ k,\ c_2^h,\ c_2^l,\ r$. In order to solve for these variables, let us setup the representative agent problem:

$$\max_{c_1, c_2^h, c_2^l, k} U = \ln c_1 + \beta[p^h \ln c_2^h + p^l \ln c_2^l] \equiv \ln c_1 + \beta E \ln c_2 \tag{4.2}$$

subject to

$$c_1 + k = y \tag{4.3}$$

$$c_2^h = A^h k^{\alpha} \tag{4.4}$$

$$c_2^l = A^l k^{\alpha}. \tag{4.5}$$

We have two constraints on the second period consumption corresponding to two states. Ultimately, only one of these will end up binding. Putting the budget constraints in the objective function, we have

$$\max_k U = \ln(y - k) + \beta[p^h \ln(A^h k^{\alpha}) + p^l \ln(A^l k^{\alpha})] \equiv \ln c_1 + \beta E \ln c_2 \tag{4.6}$$

The first order condition is given by

$$\frac{1}{y-k} = \frac{\alpha\beta}{k}. \tag{4.7}$$

(4.7) is an example of the **Euler equation**. The solution for optimal k is

$$k = \frac{\alpha\beta}{1+\alpha\beta} y \tag{4.8}$$

The optimal consumption plan is given by

$$c_1 = \frac{1}{1+\alpha\beta} y,\ c_2^i = A^i \left[\frac{\alpha\beta}{1+\alpha\beta} y\right]^\alpha \text{ for } i = h, l. \tag{4.9}$$

Now we have solved for optimal allocations. We can solve for the real rate of interests by using MPK. The real rate of interest will satisfy

$$r = E(A\alpha k^{\alpha-1}) - 1. \tag{4.10}$$

We have characterized allocations and prices for this particular example. Let us do it for a more general case.

Example 11

Suppose that the period utility is $u(c)$ with $u'(c) > 0$ and $u''(c) < 0$. The production function is $y = Af(k)$ with $f(0) = 0$, $f'(k) > 0$ and $f''(k) < 0$. Suppose that $f(k)$ satisfies Inada Conditions: $f'(0) = \infty$ and $f'(\infty) = 0$. As before $\delta = 1$.

The representative agent problem is

$$\max_{c_1, c_2^h, c_2^l, k} U = u(c_1) + \beta[p^h u(c_2^h) + p^l u(c_2^l)] \equiv u(c_1) + \beta E(u(c_2)) \tag{4.11}$$

subject to

$$c_1 + k = y \tag{4.12}$$

$$c_2^h = A^h f(k) \tag{4.13}$$

$$c_2^l = A^l f(k). \tag{4.14}$$

We can plug these constraints in the objective function (4.11) and get unconstrained maximization problem:

$$\max_k U = u(y-k)+\beta[p^h u(A^h f(k))+p^l u(A^l f(k))] \equiv u(y-k)+\beta E(u(Af(k))). \tag{4.15}$$

The first order condition satisfies

$$u'(c_1) = \beta \left[p^h u'(c_2^h)A^h f'(k) + p^l u'(c_2^l)A^l f'(k)\right]. \tag{4.16}$$

Using the expectation operator defined in (4.1), we can write (4.16) as

$$u'(c_1) = \beta E\left[u'(c_2)Af'(k)\right]. \tag{4.17}$$

Equations like (4.17) are known as **Euler equation**. It has straight forward interpretation. At the optimal level of k, the marginal cost of k (LHS) equals the expected marginal benefit from k (RHS). The marginal cost of investment is simply equal to the marginal utility of consumption forgone in the current period $u'(c)$. What is the gain from one unit of investment? One unit of investment produces $Af'(k)$ units of goods next period. In terms of utility this benefit is simply equal to $u'(c_2)Af'(k)$. Since, this utility occurs next period, we need to discount it in order to make it comparable to the current utility, and thus the expected marginal benefit from investment is given by the RHS of (4.17). Using this Euler equation together with the resource constraints we can derive optimal allocations. Once we get optimal allocations, using MPK we can get the real rates of interest.

Exercise: Show that the production function $y = Ak^\alpha$ satisfies Inada conditions. Let $u(c) = \ln c$. Using the Euler equation (4.17) and resource constraints show that optimal allocations and prices satisfy (4.8), (4.9), and (4.10).

5. Precautionary Saving

Effect of Income Risk and Capital Income Risk on Precautionary Saving

Example 12

The risk-averse consumer's problem is

$$\max_{c_1,c_2,s} U(c_1) + \beta U(c_2)$$

subject to

$$c_1 + s = y_1 \ \& \tag{5.1}$$

$$c_2 = \overline{R}s + \overline{y}_2 \tag{5.2}$$

where y_1, $\overline{y}_2$ & $\overline{R}$ are first period income, second period income, and the rate of interest respectively. Let $y_1 > \overline{y}_2$.

The optimal amount of saving is given by

$$U_c(y_1 - s) = \beta\overline{R}U_c(\overline{R}s + \overline{y}_2) \tag{5.3}$$

Denote the optimal amount of saving by $\overline{s}$.

Income Risk and Saving

Now suppose that the second period income, y_2, of the consumer is uncertain. The consumer has to save in the first period before knowing its second period income. Suppose that

$$Ey_2 = \overline{y}_2 \tag{5.4}$$

where E is the expectation operator. Denote the optimal amount of saving with uncertain income by s^*. The question is whether $s^* > \overline{s}$. Will uncertainty about future income will lead to higher saving (**precautionary saving**) in the first period?

Now the consumer's problem is

$$\max_{c_1,c_2,s} U(c_1) + \beta EU(c_2)$$

subject to

$$c_1 + s = y_1 \ \& \tag{5.5}$$

$$c_2 = \overline{R}s + y_2. \tag{5.6}$$

The optimal amount of saving, s^*, is given by

$$U_c(y_1 - s) = \beta\overline{R}EU_c(\overline{R}s + y_2). \tag{5.7}$$

Now compare (5.3) and (5.7). The LHS of both equations has identical expressions. However, the RHS has different expressions. Given the concavity of the utility function (or diminishing marginal utility), it is immediately clear that if

$$EU_c(\overline{R}s + y_2) > U_c(\overline{R}s + \overline{y}_2) \equiv U_c(E(\overline{R}s + y_2)) \tag{5.8}$$

$s^* > \overline{s}$.

It is the mathematical property (**Jensen's inequality**) that if a function $f(x)$ is a convex function of the random variable x then

$$Ef(x) > f(E(x)) \equiv f(\overline{x}) \tag{5.9}$$

where $\overline{x} = E(x)$.

Essentially we need to find out under what condition $U_c(\overline{R}s+y_2) \equiv U_c(c_2)$ is a convex function of y_2. Taking the second-derivative of $U_c(c_2)$ w.r.t. y_2 you can see that when the third derivative of the utility function is positive i.e. $U_{ccc}(c) > 0$, $U_c(c_2)$ will be a convex function of y_2. In this case, $s^* > \overline{s}$. The difference between $s^* - \overline{s}$ is known as *precautionary saving.* This is extra saving due to uncertainty in the future income.

Example : Suppose that $U(c) = \frac{c^{1-\mu}}{1-\mu}$. In this case, $U_{ccc}(c) = \mu(1 + \mu)c^{-(\mu+1)} > 0$. Thus, in case of income uncertainty there will be extra-saving.

Example : Suppose that we have quadratic utility function, $U(c) = ac - bc^2$. Since $U_{ccc}(c) = 0$, in this case, there will be no extra saving due to income uncertainty. Thus, the mere fact that an individual is **risk-averse** does not mean there will be precautionary saving. This is known as **certainty-equivalence** result.

Capital Income risk

Example 13

Now suppose that y_2 is certain, but there is uncertainty about the rate of interest R. Let R be the random variable with $E(R) = \overline{R}$. Now the optimization problem is

$$\max_{c_1, c_2, s} U(c_1) + \beta EU(c_2)$$

subject to

$$c_1 + s = y_1 \ \& \tag{5.10}$$

$$c_2 = Rs + \overline{y}_2 \tag{5.11}$$

The optimal amount of saving, s^{**}, is given by

$$U_c(y_1 - s) = \beta EU_c(Rs + \overline{y}_2)R. \tag{5.12}$$

We want to know under what condition $s^{**} > \overline{s}$. As before we need to derive the condition under which $U_c(Rs + \overline{y}_2)R \equiv U_c(c_2)R$ is a convex function of R. Taking the second derivative, you can show that $U_c(c_2)R$ is a convex function of R if

$$U_{ccc}(c_2)Rs + 2U_{cc}(c_2) > 0 \text{ or} \tag{5.13}$$

$$-\frac{U_{ccc}(c_2)}{U_{cc}(c_2)}Rs > 2. \tag{5.14}$$

As is evident $U_{ccc}(c_2) > 0$ is no longer sufficient (though necessary) to ensure extra-saving.

For example, with $U(c) = \frac{c^{1-\mu}}{1-\mu}$, (5.14) is satisfied only when $(\mu + 1)\frac{Rs}{c_2} > 2$. The reason is that risky capital income affects saving in two opposite ways. Firstly, the precautionary motive ($U_{ccc}(c_2) > 0$) has positive effect on saving. However, risky capital income also reduces the attractiveness of saving (negative substitution effect), which is captured by the term $U_{cc}(c_2)$. Only if the precautionary motive dominates the negative substitution effect, saving will be higher than the certainty case.

6. Infinite Horizon Modesl

Previously, we analyzed two-period models. Now, we extend our analysis to infinite periods. In particular, we develop infinitely-lived representative agent models. These models are commonly used to analyze macro issues. We first consider optimal consumption/savings problem in a model with perfect foresight. Then we extend the model to incorporate uncertainty. Introduction of uncertainty allows us to study business cycles. In the second part, we develop a basic version of Real Business Cycle (RBC) model, which is widely used to study business cycles. In lecture 4, we will examine more general versions of this model.

Example 14

Consider infinite horizon version of optimal consumption problem with production. The main difference from two-period models we considered is that now agents are infinitely lived. As before consumers own firms and all markets are competitive (Walrasian).

The optimization problem faced by the representative agent is

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln c_t \tag{6.1}$$

subject to (with $\delta = 1$)

$$k_{t+1} = k_t^\alpha - c_t, \ \forall t. \tag{6.2}$$

We can rewrite the problem as

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t [\ln c_t + \lambda_t [k_t^\alpha - c_t - k_{t+1}]] \tag{6.3}$$

where λ_t is the Langrangian multiplier associated with the resource constraint. The first order conditions are

$$c_t : \frac{1}{c_t} = \lambda_t \tag{6.4}$$

and

$$k_{t+1} : \beta^t \lambda_t = \beta^{t+1} \lambda_{t+1} \alpha k_{t+1}^{\alpha-1}. \tag{6.5}$$

Combining (6.4) and (6.5) we have

$$\frac{1}{c_t} = \frac{\alpha\beta k_{t+1}^{\alpha-1}}{c_{t+1}}. \tag{6.6}$$

(6.6) is the Euler equation linking consumptions in adjacent periods. In order to find the path of optimal consumption, we need to solve (6.6).

We will use *guess and verify method or method of undetermined coefficients* to solve this equation. As the name suggests, this method makes a guess about the optimal path of consumption, c_t, and then verifies whether the assumed path satisfies (6.6) or not.

Suppose that the optimal path of consumption has following form:

$$c_t = \mu k_t^{\alpha} \tag{6.7}$$

where μ is an unknown constant. Basically we are assuming that each period the decision maker consumes a constant fraction of output. The trick is to find an expression for μ which satisfies (6.6). Putting (6.7) in (6.6), we get

$$\frac{1}{\mu k_t^{\alpha}} = \frac{\alpha\beta k_{t+1}^{\alpha-1}}{\mu k_{t+1}^{\alpha}} \tag{6.8}$$

From (6.8) we have

$$\frac{1}{k_t^{\alpha}} = \frac{\alpha\beta}{k_{t+1}}. \tag{6.9}$$

This implies

$$k_{t+1} = \alpha\beta k_t^{\alpha} \tag{6.10}$$

Putting (6.7) and (6.10) in the budget constraint we have

$$k_t^{\alpha} = \mu k_t^{\alpha} + \alpha\beta k_t^{\alpha}. \tag{6.11}$$

(6.11) implies that $\mu = 1 - \alpha\beta$. The solution for optimal consumption is then

$$c_t = (1 - \alpha\beta) k_t^{\alpha}. \tag{6.12}$$

(6.10) and (6.12) show that investment and consumption are a constant proportion of output. This result is due to our assumptions of logarithmic utility function, Cobb-Douglas production function, and $100\%(\delta = 1)$ depreciation rate.

To solve this model, we used guess and verify method. However, there are limitations to this method. These methods work for only two classes of specifications of preferences and constraints, namely, variants of specification with linear constraints and quadratic preferences or Cobb-Douglas constraints and logarithmic preferences. Thus only in limited cases, dynamic general equilibrium models can be solved analytically. Generally, one uses approximation and/or numerical methods to solve these models. We will cover these methods in lecture 4.

Exercise: Derive steady state values of consumption, capital stock, real interest rate.

Example 15

Next we consider a model in which labor supply is endogenous. Suppose that labor market is competitive. Let n_t denote labor supplied by the representative agent at time t. Now the representative agent faces the following optimization problem

$$\max_{c_t, n_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t [\ln c_t + \ln(1 - n_t)] \tag{6.13}$$

subject to

$$k_t^{\alpha} n_t^{1-\alpha} = c_t + k_{t+1}, \ \forall t. \tag{6.14}$$

Again we can recast the problem as

$$\max_{c_t, n_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t [\ln c_t + \ln(1 - n_t) + \lambda_t [k_t^{\alpha} n_t^{1-\alpha} - c_t - k_{t+1}]]. \tag{6.15}$$

The first order conditions are:

$$c_t : \frac{1}{c_t} = \lambda_t; \tag{6.16}$$

$$n_t : \frac{1}{1 - n_t} = \lambda_t (1 - \alpha) k_t^{\alpha} n_t^{-\alpha}; \tag{6.17}$$

and

$$k_{t+1} : \lambda_t = \lambda_{t+1}\beta\alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha}. \tag{6.18}$$

(6.16) and (6.18) imply that

$$\frac{1}{c_t} = \frac{\beta\alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha}}{c_{t+1}} \tag{6.19}$$

which is the Euler equation linking consumption in adjacent periods. To derive the path of consumption, capital stock, and employment we can use guess and verify method. Again assume that

$$c_t = \mu k_t^{\alpha} n_t^{1-\alpha}. \tag{6.20}$$

Then using (6.14), (6.19) and (6.20) one can show that $\mu = (1 - \alpha\beta)$ and

$$c_t = (1 - \alpha\beta) k_t^{\alpha} n_t^{1-\alpha} \tag{6.21}$$

and

$$k_{t+1} = \alpha\beta k_t^{\alpha} n_t^{1-\alpha}. \tag{6.22}$$

(6.16), (6.17), (6.21) and (6.22) then imply that

$$n_t = \frac{1-\alpha}{2 - \alpha(1+\beta)}. \tag{6.23}$$

In this model, agents supply same amount of labor every period. This is the consequence of logarithmic and separable utility function.

Exercise: Derive steady state values of consumption, capital stock, and real wage.

7. Basic Real Business Cycle Model

Now we extend our analysis to consider models with uncertainty. Incorporation of uncertainty will allow us to study business cycle issues. In particular, we are going to study RBC model. This model has been very influential in studying business cycles.

Broadly, there are two approaches to study business cycles: (i) RBC models and (ii) Keynesian Models. There are two key differences between these two types of approaches. RBC models assume Walrasian markets and they attribute business cycles primarily to technology (productivity/real/ supply side) shocks. Keynesian models on the other hand assume imperfect markets with nominal rigidities and they attribute business cycles primarily to aggregate demand or nominal shocks. In this lecture, we will develop the basic version of RBC model.

The RBC model is a stochastic version of the optimal consumption problem analyzed above. In the basic RBC model, it is assumed that there is uncertainty with regard to technology or production function. This model brings out the effects of technology shock on consumption, output, employment etc.

Example 16

Consider a stochastic version of the optimal consumption problem analyzed above. Suppose that the production function is given by $A_t k_t^\alpha$ where $A_t = \exp^{\epsilon_t}$. ϵ_t is an **independently and identically distributed** (i.i.d) random variable with mean 0 and variance σ^2. The random variable A_t is called technology or productivity shock. It is assumed that technology shock is realized at the beginning of period t before consumption and investment decisions are made. Let E_t denote the expectation operator conditional on time t information set.

The optimization problem is

$$\max_{c_t, k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \tag{7.1}$$

subject to

$$k_{t+1} = A_t k_t^\alpha - c_t. \tag{7.2}$$

This optimization problem is known as the **real business cycle model**, which studies the effects of technology shocks on investment, consumption, output etc. We can recast the above optimization problem as

$$\max_{c_t, k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + \lambda_t [A_t k_t^\alpha - c_t - k_{t+1}]\right]. \tag{7.3}$$

The first order conditions are

$$c_t : \ \frac{1}{c_t} = \lambda_t \tag{7.4}$$

$$k_{t+1}: \ \lambda_t = \alpha\beta E_t \lambda_{t+1} A_{t+1} k_{t+1}^{\alpha-1}. \tag{7.5}$$

Combining (7.4) and (7.5) we get

$$\frac{1}{c_t} = \alpha\beta E_t \frac{A_{t+1} k_{t+1}^{\alpha-1}}{c_{t+1}} \tag{7.6}$$

which is again the Euler equation. In order to solve (7.6), we will use guess and verify method as before. Let

$$c_t = \mu A_t k_t^{\alpha}. \tag{7.7}$$

Putting (7.7) in (7.6) we have

$$\frac{1}{A_t k_t^{\alpha}} = \alpha\beta E_t \frac{1}{k_{t+1}}. \tag{7.8}$$

Combining (7.8) with the budget constraint we have

$$\frac{1}{A_t k_t^{\alpha}} = \frac{\alpha\beta}{A_t k_t^{\alpha} - c_t}. \tag{7.9}$$

From (7.9) we get

$$c_t = (1 - \alpha\beta) A_t k_t^{\alpha}. \tag{7.10}$$

(7.7) and (7.10) imply that $\mu = 1-\alpha\beta$. From (7.10) and the budget constraint we get

$$k_{t+1} = \alpha\beta A_t k_t^{\alpha}. \tag{7.11}$$

Taking log of (7.11) we get fundamental stochastic difference equation which tells us how capital stock evolves over time.

$$\ln k_{t+1} = \ln \alpha\beta + \alpha \ln k_t + \epsilon_t. \tag{7.12}$$

Using (7.12) we can trace out how capital accumulation evolves over time in response to a single shock, ϵ_t (**impulse response function**). One can also derive moments of the process of capital accumulation. Solving (7.12) backwards we have

$$\ln k_{t+1} = \ln \alpha\beta + \alpha[\ln \alpha\beta + \alpha \ln k_{t-1} + \epsilon_{t-1}] + \epsilon_t. \tag{7.13}$$

If we keep on repeating this process, we will get

$$\ln k_{t+1} = [1 + \alpha + \alpha^2 + ...] \ln \alpha\beta + [\epsilon_t + \alpha\epsilon_{t-1} + ...] \tag{7.14}$$

Then

$$E(\ln k_{t+1}) = \frac{\ln \alpha\beta}{1-\alpha} \tag{7.15}$$

$$V(\ln k_{t+1}) = \frac{\sigma^2}{1-\alpha^2}. \tag{7.16}$$

Similarly we can derive moments of other variables like consumption, income etc. Denote output by y_t. Then , we have

$$\ln y_t = \ln A_t + \alpha \ln k_t \equiv \alpha \ln k_t + \epsilon_t. \tag{7.17}$$

By combining (7.12) and (7.17) we have,

$$\ln k_t = \ln \alpha\beta + \ln y_{t-1}. \tag{7.18}$$

Then (7.17) and (7.18) imply that log of output follows a first order autoregressive process:

$$\ln y_{t+1} = \alpha \ln \alpha\beta + \alpha \ln y_t + \xi_{t+1}. \tag{7.19}$$

Let us now consider implications of technology shock on consumption, investment, and output. Using (7.19) one can derive the implications of technology shock on output. For this one needs to assume some value of α. α is estimated to be 0.33.

Now consider the effect of a one time positive technology shock. Let y_0 be the output at time $t-1$. Suppose that at time T technology shock is realized and let $\xi_T = 1$. Suppose that technology shock is entirely temporary, i.e. $\xi_t = 0,\ \forall t > T$. What would be the response of output. In period T log of output will be one unit higher than log of y_0. In $T+1$ it will be higher by 1/3 compared to log of y_0. In $T+2$ it will be higher by 1/9 and so on.

Example 17

The previous model does not say anything about employment and real wage. To consider the implications with regard to employment and real wage, let us introduce technology shock in example 15. Now the problem is

$$\max_{c_t, n_t, k_{t+1}} E_0 \left[\sum_{t=0}^{\infty} \beta^t [\ln c_t + \ln(1 - n_t) + \lambda_t [k_t^{\alpha} n_t^{1-\alpha} - c_t - k_{t+1}]] \right]. \qquad (7.20)$$

The first order conditions for consumption and employment continue to be given by (6.16) and (6.17). The first order condition for capital stock modifies to

$$k_{t+1} : \lambda_t = \beta \alpha E_t \lambda_{t+1} A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha}. \qquad (7.21)$$

(6.16) and (7.21) imply that

$$\frac{1}{c_t} = \alpha\beta \frac{E_t A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha}}{c_{t+1}}. \qquad (7.22)$$

which is the Euler equation linking consumption in adjacent periods. Using guess and verify method, one can show that

$$c_t = (1 - \alpha\beta) A_t k_t^{\alpha} n_t^{1-\alpha}; \qquad (7.23)$$

$$k_{t+1} = \alpha\beta A_t k_t^{\alpha} n_t^{1-\alpha} \qquad (7.24)$$

and

$$n_t = \frac{1 - \alpha}{2 - \alpha(1 + \beta)}. \qquad (7.25)$$

Since n_t is constant, the model implies that technology shock does not affect employment. Also since real wage is equal to the marginal product of labor ($= (1 - \alpha) y_t / n_t$), real wage is strongly pro-cyclical.