POLYNOMIAL DECAY OF CORRELATIONS IN THE GENERALIZED BAKER'S TRANSFORMATION

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ABSTRACT. We introduce a family of area preserving generalized baker's transformations acting on the unit square and having sharp polynomial rates of mixing for Hölder data. The construction is geometric, relying on the graph of a single variable "cut function". Each baker's map B is non-uniformly hyperbolic and while the exact mixing rate depends on B, all polynomial rates can be attained. The analysis of mixing rates depends on building a suitable Young tower for an expanding factor. The mechanisms leading to a slow rate of correlation decay are especially transparent in our examples due to the simple geometry in the construction. For this reason we propose this class of maps as an excellent testing ground for new techniques for the analysis of decay of correlations in non-uniformly hyperbolic systems. Finally, some of our examples can be seen to be extensions of certain 1-D non-uniformly expanding maps that have appeared in the literature over the last twenty years thereby providing a unified treatment of these interesting and well-studied examples.

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1. INTRODUCTION

The analysis of non-uniformly hyperbolic systems has undergone an explosion of activity in the last decade with a range of new techniques becoming available; notably Young towers [33, 34], hyperbolic times [3, 4, 5, 6, 7] and, earlier, Pesin theory for maps with singularities [23]. The application of this machinery on 'real life' examples is often highly technical, with substantial effort being required, for example, to isolate and analyze lower-dimensional expanding factors whose mixing properties drive statistics for the higher-dimensional hyperbolic map. In this paper we consider a class of two-dimensional *Generalized Baker's Transformations* (GBTs) whose simple geometry allows, via Young towers, the extraction of sharp polynomial rates of correlation decay on Hölder observables. The constructions and proofs are explicit and geometrically natural. The relevant one-dimensional expanding factors are certain piecewise C^2, Markov, non-uniformly expanding maps of the unit interval, the most familiar and well-understood examples we know of for analysis of the connection between hyperbolicity and mixing rates.

The extension from baker's to generalized baker's is easy to describe. Specifically, a two-dimensional map B on the unit square $S = [0,1]^2$ is determined by a *cut*
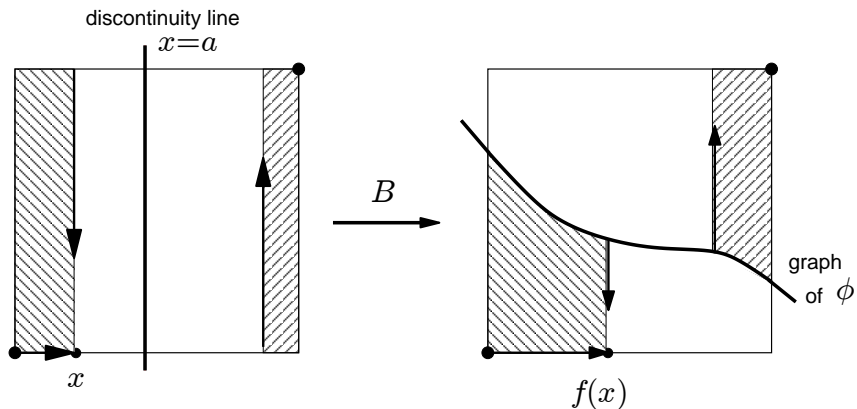

FIGURE 1. Generalized Baker's Transformation

function ϕ whose graph $y = \phi(x)$ partitions S into lower and upper pieces. The cut function is assumed to be measurable and to satisfy $0 \leq \phi(x) \leq 1$; these are the only constraints in the construction. The two-dimensional dynamics are depicted in Figure 1 and defined by mapping the vertical lines $\{x = 0\}, \{x = 1\}$, into themselves, and sending vertical fibres into vertical fibres (the fibre over x goes to part of the fibre over $f(x)$) in such a way that areas are preserved; if we define $a = \int_0^1 \phi(t)\,dt$ then the rectangle $[0, a] \times [0, 1]$ maps to the lower part of the square under the graph of ϕ and $[a, 1] \times [0, 1]$ maps to the upper part. The resulting map B preserves Lebesgue measure $m \times m$ on the square S. B necessarily has a discontinuity along the vertical line $\{x = a\}$. Clearly B is hyperbolic: through each point on the square passes a contracting leaf (vertical line) and an expanding leaf (the graph of a measurable function). B is uniformly hyperbolic if and only if the cut function ϕ is bounded away from zero and one as depicted in the Figure 1.

When $\phi \equiv 1/2$ the map is the classical baker's transformation.

The construction was introduced in [13] where many basic dynamical properties were established. For example, regularity conditions on the cut allow one to conclude that B is ergodic, or even Bernoulli. Perhaps more surprisingly, it was shown that every measure preserving transformation T on a (nonatomic, standard, Borel) probability space with entropy satisfying[2] $0 < h(T) < \log 2$ is measurably isomorphic to some GBT on the square S, so in some sense, these are universal examples of measure preserving systems.

A recent literature search uncovered more than 80 articles describing generalized baker's maps, of which the construction above represents only one possible direction. Some investigations consider only locally affine, measure preserving transformations, a minor variant of the classical example and a subcase of the present construction. There are also fat baker's tranformations – noninvertible maps where the expansion in the unstable direction dominates contraction on vertical fibres (for example, see [1, 30, 31]). Generically such maps admit an absolutely continuous

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[2]log will always mean the natural logarithm. The upper bound in this inequality is not essential; any finite entropy map may be represented by a gbt, provided you allow multiple cut functions on the square.

invariant SRB measure. The recent article [22] studies baker's transformations on non-square domains whose expanding factors are certain $\beta-$transformations.

Our main goal in this paper is to establish sharp polynomial decay of correlation estimates for some non-uniformly hyperbolic examples of the map B above acting on 2-D Hölder observables. Estimating correlation decay rates for multidimensional non-uniformly hyperbolic systems has proved to be rather difficult in general and the majority of early results of this type gave upper bounds in the exponential or stretched exponential class (see, for example [33] (Billiards), [14] (periodic Lorentz gas), [12] (Henon maps) for the former, and [19] (following [32, 6]) for the latter). Subexponential decay can arise in the *flow* associated to the corresponding billiard map whereby slow decay for a planar billiard map can be carried to the flow. [26] establishes, in a reasonably general setting, this sort of phenomenon, capturing earlier results such as [15, 16] for certain semi-dispersing planar billiards. [10] does a similar thing for dispersing billiards with cusps, Bunimovich flowers and Bunimovich stadia. [24] studies a family of measure preserving maps on the 2-Torus (with a neutral fixed point), establishing $O(n^{-2}(\log n)^4)$ upper bounds on correlation decay, with the observation that the logarithmic term is almost certainly a technical artifact. To the best of our knowledge, only in the case of the Bunimovich stadium has a polynomial upper bound on the rate (in this case $O(1/n)$) been rigorously proved to be sharp (see [25, 17] for upper bounds and [9], Corollary 1.3 for the corresponding lower bound).

Although the simple geometry of our class could be viewed as artificial, it is extremely effective for illustrating some of the obstacles (and techniques used to overcome them) that have been central to the subject in recent years.

In order to carry out our analysis, we first establish the corresponding rate-of-mixing result on an appropriate 1-D expanding factor f. This map arises naturally from the action of B on the invariant family of 'vertical fibres'; f will be a piecewise monotone and continuous map of the unit interval having neutral fixed points[3] at $x = 0$ and $x = 1$.

Non-uniformly expanding $1-$D interval maps such as our f are currently much better understood than the corresponding multidimensional transformations. Analysis of maps of the interval having neutral fixed points was carried out in [27] and references cited there. This early work also anticipates one of the most fruitful modern approaches: the construction of Markov Extensions or Young towers (see [33, 34]). Indeed, we also begin by constructing a suitable Young tower for f after which, upper bounds on the rate of decay of correlation against Hölder data are routine to obtain. In our case these rates are polynomial (the exact rate depending on which map f from the family is being considered; all polynomial mixing rates may be attained simply by the choice of parameters leading to f (and B)).

Recently, [18] investigated a parameterized family of 1-D circle maps on $[-1, 1]$ proving they have polynomial mixing rates. It turns out these maps are conjugate to certain f given by our construction (see Example 2.3). On the other hand, our examples need not be symmetric, and the $2-$D connections we are motivated by in this paper are not investigated.

Analysis of the mixing properties of B proceeds by lifting the corresponding estimates for f back to the square along stable fibres. In this case, because of the

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[3]Meaning, points $x*$ such that $f(x*) = x*$ and $f'(x*) = 1$. Such fixed points can be stable, unstable or neither, in general. In our case they will be *unstable* fixed points.

simple geometry, this step is relatively simple compared to previous studies in the literature, including the ones cited above.

Another approach to the study of non-uniformly hyperbolic maps depends on the analysis of *hyperbolic times*. In [11] we show that, while all our examples f have positive density of hyperbolic times, the first hyperbolic time function may or may not be integrable, depending on the order of tangency of the cut function to the boundary of the square at $(0,1)$ and $(1,0)$. Indeed, it is possible to obtain sharp estimates on $m\{h_{\sigma,\delta} > n\}$ where m is Lebesgue measure and $h_{\sigma,\delta}(x)$ denotes the first (σ,δ)-hyperbolic time for the orbit at x (see [2, 3, 4, 5] for definitions and related computations). Analysis of hyperbolic times for our maps f will not be used in this paper.

In the next section we set up the notation used throughout the paper and define our family of baker's maps B. In Section 3 we begin with a brief review of the Young tower construction in a form that best suits our application. In Sections 4–7 we build towers for the 1-D maps f induced by B acting on the stable leaves and establish sharp rates of correlation decay for these systems with respect to 1-D Hölder observables. We complete the work in Section 8 by lifting the 1-D results in a natural way to identical decay estimates on the 2-D maps B.

Some elementary computations (essentially calculus exercises) are gathered in Appendix 1.

2. Generalized baker's maps

With respect to notation from the previous section, the relevant equations are easy to derive:
$$(x,y) \mapsto (f(x), g(x,y)) = B(x,y)$$
where

$$g(x,y) = \begin{cases} \phi(f(x))\, y & \text{if } x \le a, \\[2mm] y + \phi(f(x))(1-y) & \text{if } x > a, \end{cases}$$

(1)   and

$$x = \begin{cases} \int_0^{f(x)} \phi(t)\, dt & \text{if } x \le a, \\[2mm] 1 - \int_{f(x)}^1 [1 - \phi(t)]\, dt & \text{if } x > a. \end{cases}$$

Note that the function f appears implicitly in Equations (1). Provided the set of t where ϕ takes on the value 0 or 1 is of measure zero, it is easy to see that f (and hence g) is uniquely defined for every $x \in [0,1]$ (respectively, for $(x,y) \in S$). This will be the case for all examples in this paper.

Even without this restriction, by construction, $B(x,y)$ is defined by Equation (1) for Lebesgue almost every point (x,y) in the square S, is invertible[4] and preserves two-dimensional Lebesgue measure. For details, and a formula for B^{-1} see [13]. The sub-sigma-algebra of vertical fibres $\{x = x_0\}$ on S is invariant under[5] B and the associated (non-invertible) factor is naturally identified with the map f, a two branched, piecewise increasing map on $[0,1]$.

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[4]In the usual sense of being invertible off a set of measure zero on the square.

[5]But not for B^{-1}, since B maps fibres into partial fibres, in general

Define $\pi : S \to [0,1]$ by $\pi(x,y) = x$. Then $\pi \circ B = f \circ \pi$ encodes the factor relationship between f and B and if $m \times m$ denotes Lebesgue measure on S, then[6] $\pi_*(m \times m) = m$ is Lebesgue measure on $[0,1]$, so f is also Lebesgue measure preserving (but now on the unit interval).

From the definition of $g(x,y)$, $g(x,y) \lessgtr \phi(f(x))$ according to whether $x \lessgtr a$; thus, the position of a point (x,y) on a vertical fibre $\pi^{-1}x$ determines the inverse history of possible f-orbits, while the position x specifies the future trajectory under f. In this way, B represents an *inverse limit* or invertible cover of the endomorphism f; in fact, in many cases, B proves to be the *natural extension* of f (for a precise treatment and conditions under which this will hold, see Section 4 of [13]). In all our examples, B will be the natural extension of f.

For each $n \geq 0$ the action of B^n is affine on each vertical fibre, and the skew-product character of B is emphasized through the formula:

$$(2) \qquad\qquad B^n(x,y) = (f^n(x), g_n(x,y))$$

where $g_0(x,y) = y$ and

$$g_n(x,y) = \begin{cases} \phi(f^n(x))\, g_{n-1}(x,y) & \text{if } f^{n-1}(x) \leq a \\ \phi(f^n(x)) + (1 - \phi(f^n(x)))\, g_{n-1}(x,y) & \text{otherwise.} \end{cases}$$

The geometry is illustrated in Figure 2 for the case $n = 2$. Sometimes we'll use the notation $\tilde{\phi} = \partial_y g_1$ for the contractive factor on the fibres. Then $\tilde{\phi}$ depends only on x, and indeed

$$(3) \qquad\qquad \partial_y g_n = \Pi_{k=0}^{n-1} \tilde{\phi}(f^k(x)).$$

Provided the cut function is smooth, at each point (x,y) in the interior of S minus the vertical line $\{x = a\}$ we can compute the Jacobian matrix of B using the expressions in Equation (1)and the fact that $f'(x) = [\phi(f(x))]^{-1}$ for $0 < x < a$ (with a similar expression for $a < x < 1$).

$$(4) \qquad DB(x,y) = \begin{cases} \begin{bmatrix} \frac{1}{\phi(f(x))} & 0 \\[2mm] \frac{\phi'(f(x))}{\phi(f(x))}y & \phi(f(x)) \end{bmatrix} & \text{if } 0 < x < a \\[8mm] \begin{bmatrix} \frac{1}{1-\phi(f(x))} & 0 \\[2mm] \frac{\phi'(f(x))}{1-\phi(f(x))}(1-y) & 1 - \phi(f(x)) \end{bmatrix} & \text{if } a < x < 1 \end{cases}$$

Observe that the measure-preserving property for B is again confirmed since clearly $\det DB(x,y) = 1$.

2.1. The baker's family $B_{\alpha,\alpha'}$. We consider a family of generalized baker's maps indexed by two *hyperbolicity parameters* $0 < \alpha, \alpha' < \infty$ through the definition of the cut function $\phi = \phi_{\alpha,\alpha'}$. Assume:

(1) ϕ is decreasing and $0 \leq \phi \leq 1$ on $[0,1]$
(2) $\phi(0) = 1$ and there is a smooth function g_0 on $(0,1)$ such that

$$\phi(t) = 1 - c_0 t^\alpha + g_0(t)$$

with $c_0 > 0$ and $g_0' = o(t^{\alpha-1})$ for t near 0.

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[6]We adopt the standard notation $T_*\nu = \nu \circ T^{-1}$ for a map T and measure ν.
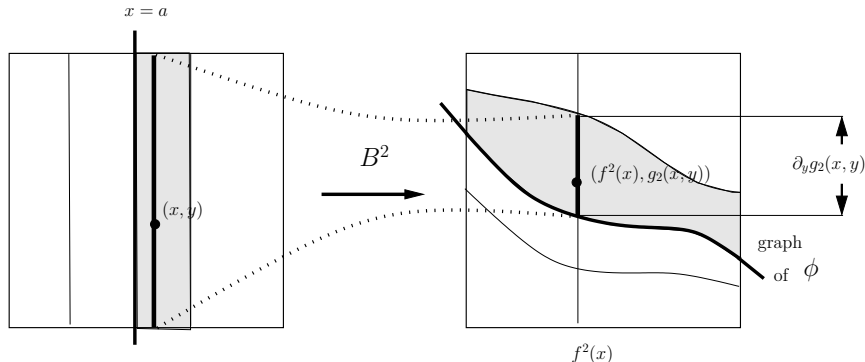
FIGURE 2. Second iterate of a generalized baker's map acting on a vertical cylinder.

(3) $\phi(1) = 0$ and there exists a smooth function g_1 on $(0,1)$ such that

$$\phi(1-t) = c_1 t^{\alpha'} + g_1(t)$$

where $c_1 > 0$ and $g_1' = o(t^{\alpha'-1})$ for t near 0.

These conditions imply that the cut function $\phi = \phi_{\alpha,\alpha'}$ is smooth[7] on $(0,1)$ with continuous extension to $[0,1]$ and that $0 < \phi(t) < 1$ for all $0 < t < 1$. It follows that the map f defined by Equation 1 is piecewise strictly increasing and expanding ($f' \geq 1$) with respect to the intervals $[0,a]$ and $[a,1]$. Each branch is surjective and C^2 when restricted to the interior of its domain ($(0,a)$ or $(a,1)$ respectively).

2.2. **Example.** Set $\alpha = \alpha' = 1$, $c_0 = c_1 = 1$ and $g_i \equiv 0$. Then $\phi(x) = 1 - x$ and $a = 1/2$. The map B is non-uniformly hyperbolic, with lines of fixed points along $\{x = 0\}$ and $\{x = 1\}$

The integrals defining f in (1) are easily computed, yielding

$$f(x) = \begin{cases} 1 - \sqrt{1 - 2x} & \text{if } x < \frac{1}{2}, \\ \sqrt{2x - 1} & \text{if } x > \frac{1}{2}. \end{cases}$$

We emphasize again that f is a measure-preserving circle endomorphism on $[0,1)$ with a discontinuity in f' at the single point $a = \frac{1}{2}$, and a neutral fixed point at $x = 0$, but in this case, with quadratic order of contact. Thus the example does not fit into the usual picture for maps with neutral fixed points (eg: [27, 34] or the AFN maps of [35]) . In fact, the branches of f do not have bounded distortion in the usual sense, since $f'(x) \to \infty$ as $x \to \frac{1}{2}$. Observe, however, that the slow escape of mass in the neighbourhood of the neutral point $x = 0$ is perfectly balanced by a very small rate of arrival in these intervals (for example, $f^{-1}([0,\epsilon)) \setminus [0,\epsilon) = [\frac{1}{2}, \frac{1}{2} + O(\epsilon^2)))$. It is this mechanism which allows all maps in our family to have a finite invariant measure, despite the fixed points being only weakly repelling.

This example has been studied previously in the literature. It is described in [35] where it is attributed to M. Thaler. [29] studied the baker's map B associated to this ϕ, proving that it is isomorphic to a Bernoulli shift by showing that the partition into regions above and below the cut function was weakly-Bernoulli (i.e.

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[7]If $\alpha, \alpha' > 1$ both the cut function ϕ and its derivative extend continuously to $[0,1]$ with $\phi'(0) = \phi'(1) = 0$.

satisfying a certain mixing rate on cylinders; see Section 8 of [29]). The map f also appears in Alves-Araújo [4] as an example having a non-integrable first hyperbolic time.

2.3. **Example.** Set $\alpha' = \alpha \in (0, \infty)$, $c_0 = c_1 = 2^{\alpha-1}$ and $g_i \equiv 0$. Let $\phi = \phi_\alpha$ denote the cut function. Then an easy computation shows that $a = \int \phi_\alpha = 1/2$ and

$$(5) \qquad \phi = \phi_\alpha(x) = \begin{cases} 1 - 2^{\alpha-1}x^\alpha & \text{if } x \leq \frac{1}{2}, \\ 2^{\alpha-1}(1-x)^\alpha & \text{if } x \geq \frac{1}{2}, \end{cases}$$

Let f_α be the resulting 1-D expanding map. It is straightforward to check that this map is conjugate[8] to the family of examples studied in [28], in turn motivated by earlier investigations of [20] and [21] and recently treated rigorously in [8] and [18]. The results in the next theorem recover decay rates from these studies as a special case.

2.4. **Statement of the main results.**

Theorem 1. [Decay of Correlations for f and B] *Let ϕ, f and B be as prescribed above and set $\gamma = \max\{\alpha, \alpha'\}$.*

  (1) *If φ is essentially bounded and measurable and ψ is Hölder continuous on $[0, 1]$ then*

$$\left| \int_0^1 \varphi \circ f^n \, \psi \, dm - \int_0^1 \varphi \, dm \int_0^1 \psi \, dm \right| = O(n^{-1/\gamma}).$$

  (2) *If φ and ψ are both Hölder continuous on S then*

$$\left| \int_S \varphi \circ B^n \, \psi \, dm \times m - \int_S \varphi \, dm \times m \int_S \psi \, dm \times m \right| = O(n^{-1/\gamma}).$$

If ϕ is symmetric (i.e.: $\phi(1-t) = 1 - \phi(t)$) then in both cases the rates above are sharp, even for Lipschitz continuous data.

Precise versions of the first part are given in Theorems 4 and 6, while Theorem 7 handles the second part.

3. Young Towers

In order to proceed, we outline the machinery developed in [33, 34] for analysis of non-uniformly hyperbolic dynamics using an abstract tower extension.

The construction begins with a set Δ_0, along with a σ-algebra $\mathcal{B}_0$ of subsets of Δ_0 and a finite measure μ_0 on $\mathcal{B}_0$. A ($\mathcal{B}_0$-measurable) *return time function* $R : \Delta_0 \to \mathbb{Z}^+$ defines a *tower*

$$\Delta := \{(z, l) \; : \; z \in \Delta_0, \; l \in \mathbb{Z}, \; 0 \leq l < R(z)\}.$$

Regarding Δ as a subset of $\Delta_0 \times \mathbb{Z}^+$, denote

$$\Delta_l = \Delta \cap (\Delta_0 \times \{l\})$$

—the lth level of the tower (when there is no ambiguity, we allow the identification $\Delta_0 \equiv (\Delta_0 \times \{0\})$). The measure μ_0 is extended to the tower Δ by defining $A \times \{l\} \subseteq \Delta_l$ to be measurable if $A \in \mathcal{B}_0$ and setting $\mu(A \times \{l\}) := \mu_0(A)$. Naturally, $\mu|_{\Delta_0} = \mu_0$. μ is called the *reference measure* on the tower Δ.

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[8]Via the affine conjugacy $x \to \frac{1+x}{2}$; the parameters satisfy $\gamma = \alpha + 1$.

Let $\{\Delta_{0,i}\}$ be a measurable and countable partition of Δ_0 such that R is constant on each atom of the partition.

Remark 1. *We emphasize at this point that the tower construction is carried out in the measurable category, so for example, the term* partition *above refers to a collection of measurable subsets which are disjoint mod zero and whose union is Δ_0 mod zero with respect to μ. Similarly, R is understood to be $\mathcal{B}-$measurable and constant $\mu-$a.e. on each $\Delta_{0,i}$.*

A map $F : \Delta \to \Delta$ is provided on the tower such that $F(z,l) = (z, l+1)$ if $l < R(z) - 1$ and $F(z, R(z) - 1) \in \Delta_0$. We denote by $F^R : \Delta_0 \to \Delta_0$ the first return map to Δ_0, that is $F^R(z,0) := F(z, R(z) - 1) \in \Delta_0$. The return time function R can be extended from Δ_0 to a function $\hat{R}$ on Δ as the first passage time to Δ_0 $(\hat{R}(z,l) = R(z) - l)$. F carries the partition of Δ_0 into a partition η of the tower: $\Delta_{l,i} = \{(z,l) \in \Delta \ : \ z \in \Delta_{0,i}\}$ and one assumes that the partition generates, in the sense that $\bigvee_{j=0}^{\infty} F^{-j}\eta$ separates the points of Δ. For our purposes, suppose also that $F^R : \Delta_{0,i} \to \Delta_0$ is bijective (μ-a.e.) for each i, and both $F^R|_{\Delta_{0,i}}$ and its inverse are nonsingular with respect to μ. The Jacobian of this return map with respect to μ will be denoted by JF^R and on each $\Delta_{0,i}$, $JF^R > 0$, again by assumption.

Regularity of functions on Δ is measured with respect to a *separation time* on the tower. Roughly speaking, a Hölder function will give similar values to x and y if the first n terms of the orbits of x and y visit the same sequence of atoms of η as one another[9]. The measure of separation s is defined as follows:

Definition 1. *In the notation established above:*
  (1) *if x,y are in different atoms of η, $s(x,y) = 0$;*
  (2) *if $x, y \in \Delta_{0,i_0}$, set $s(x,y)$ to be the minimum $n > 0$ such that $(F^R)^n(x), (F^R)^n(y)$ lie in different atoms η;*
  (3) *if $x, y \in \Delta_{l,i}$ put $s(x,y) := s(F^{\hat{R}}(x), F^{\hat{R}}(y)) - 1 = s(x', y')$ where $x', y' \in \Delta_{0,i}$ are the first unique preimages of x, y in Δ_0 under iteration by F^{-1}.*

Clearly $s < \infty$ since $\bigvee_{j=0}^{\infty} F^{-j}\eta$ separates points. In fact, s distinguishes two classes of Hölder functions: for $0 < \beta < 1$

$$C_\beta(\Delta) = \{\psi : \Delta \to \mathbb{R} : \exists c_\psi \text{ s.t. } \forall x, y \in \Delta, |\psi(x) - \psi(y)| \le c_\psi \beta^{s(x,y)}\}$$

and

$$C_\beta^+(\Delta) \ = \{\psi : \Delta \to [0, \infty) : \exists c_\psi \text{ s.t. for each } l, i \text{ either } \psi \equiv 0 \text{ on } \Delta_{l,i}$$
$$\text{or } \psi > 0 \text{ on } \Delta_{l,i} \text{ and } |\tfrac{\psi(x)}{\psi(y)} - 1| \le c_\phi \beta^{s(x,y)} \ \forall x, y, \in \Delta_{l,i}\}.$$

The regularity of F is described by a Hölder condition on the Jacobian of the maps $(F^R|_{\Delta_{0,i}})^{-1} : \Delta_0 \mapsto \Delta_{0,i}$ (anticipating their appearance in the transfer operator for F^R): we suppose there exist $0 < \beta < 1$ and C such that

$$(6) \qquad \left| \frac{JF^R(x)}{JF^R(y)} - 1 \right| \le C\beta^{s(F^R(x), F^R(y))}, \ \forall \ i, \ \forall \ x, y \in \Delta_{0,i}.$$

We adopt the conventional notation for asymptotics of sequences: $x_n = \mathrm{O}(y_n)$ means there exists a constant $C < \infty$ such that for all large n, $x_n \le Cy_n$ and $x_n \approx y_n$ if both $x_n = \mathrm{O}(y_n)$ and $y_n = \mathrm{O}(x_n)$.

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[9]From this point on we simplify notation and write x instead of (z,l) for points in the tower.

Theorem 2. [Young's Theorem (part of Theorems 1-3) in [34]] *Assume the set-ting and notation above (including the regularity condition (6)). Assume also that $\int_{\Delta_0} R \, d\mu < \infty$ and that $gcd\{R_i\} = 1$ where $R_i := R|_{\Delta_{0,i}}$. Then,*

(1) *F admits an absolutely continuous (w.r.t. μ) invariant probability measure ν on Δ with $\frac{d\nu}{d\mu} > 0$. Moreover, the system (F, ν) is exact.*

Furthermore, if there is a constant $\gamma > 0$ such that $\mu\{\hat{R} > n\} = \mathrm{O}(n^{-\gamma})$ then:

(2) *for a probability measure λ with $\frac{d\lambda}{dm} \in C_\beta^+(\Delta)$ we have*

$$|F_*^n \lambda - \nu| = \mathrm{O}(n^{-\gamma});$$

(3) *for each $\varphi \in L^\infty$ and $\psi \in C_\beta(\Delta)$ we have*

$$\left| \int_\Delta (\varphi \circ F^n) \psi \, d\nu - \int_\Delta \varphi \, d\nu \int_\Delta \psi \, d\nu \right| \leq |\varphi|_\infty C_\psi \, n^{-\gamma}$$

where $C_\psi < \infty$ depends on ψ and the tower.

Observe that $\mu\{\hat{R} > n\} = \sum_{l>n} \mu(\Delta_l)$ so the asymptotics above are exactly the decay rate of the mass in the top of the tower. The theorem shows that these rates simultaneously control (i) the relaxation rates of non-invariant measures (with suitable Hölder densities) under iteration by F to the invariant measure, and (ii) the rate of correlation decay with respect to the invariant measure over a large class of regular functions. (The decay of correlation statement is slightly different to [34, Theorem 3], and follows immediately from the speed of convergence to equilibrium for measures—see [34, Section 5.1].)

4. Towers for f

For the rest of this article we will assume that the values $\alpha, \alpha' \in (0, \infty)$, constants $c_i > 0$ and functions g_i defining ϕ have been chosen subject to the conditions in Section 2.1, and the baker's map B and interval map f are therefore determined. We now show how the abstract tower construction applies to our map f.

Note that f admits a period–2 orbit $\{x_0, x_0'\}$ since f^2 is a four-branched, piece-wise continuous and onto map. We may assume that[10] $x_0 < a$ and $x_0' > a$. To illustrate using Example 2.2, we have $x_0 = \sqrt{2} - 1$ and $x_0' = 2 - \sqrt{2}$.

Let $\Delta_0 = [x_0, x_0')$. Let $\{x_n\}$ be defined under the left branch of f (recursively) by $f(x_n) = x_{n-1}$. Put $J_n = [x_{n+1}, x_n)$. A parallel construction under the right branch yields a sequence x_n' and intervals $J_n' = [x_n', x_{n+1}')$ in $[x_0', 1]$. Finally, put $I_{n+1} = f^{-1}(J_n) \setminus J_{n+1}$ (and similarly for $\{I_n'\}$). Observe that the half open subintervals $I_n \subseteq (a, x_0')$ while $I_n' \subseteq [x_0, a)$. Let R denote the *first return time* function to Δ_0. Under the map f, we have

(7) $$I_k \to J_{k-1} \to J_{k-2} \to \cdots \to J_0 \to \Delta_0,$$

and similarly for the I_n' and J_n' intervals. Note that each application in the compo-sition is injective and onto. Thus, $R(x) = k + 1$ when $x \in I_k^{(')}$; moreover, f^R maps bijectively to Δ_0 from each $I_k^{(')}$. To summarize, in the terminology of the previous

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[10]Let x_0 be the fixed point for f^2 on the second branch.

section, the base of the tower is taken to be Δ_0, with Borel sets and Lebesgue measure m; Δ_0 is partitioned by two infinite sets of half-open intervals $\Delta_{0,i} = I_i \times \{0\}$ and $\Delta'_{0,i} = I'_i \times \{0\}$. Then, $R|_{\Delta^{(\prime)}_{0,i}} = i + 1$ $(i \geq 1)$ and the tower is

$$\Delta = \cup_{i=1}^{\infty} \cup_{l=0}^{i} (\Delta_{l,i} \cup \Delta'_{l,i}),$$

where $\Delta^{(\prime)}_{l,i} := \Delta^{(\prime)}_{0,i} \times \{l\}$, embedding the tower in $\Delta_0 \times \mathbb{Z}^+$.

The tower map is

$$F(x,l) = \begin{cases} (x, l+1) & \text{if } l < R(x) - 1, \\ (f^R(x), 0) & \text{if } l = R(x) - 1 \text{ and } R = R(x). \end{cases}$$

To establish the regularity condition (6) and estimate the distribution of the tail of R, we use the following asymptotics on x_n and intervals I_n and J_n.

Lemma 1.   (i) $x_n \approx \left(\frac{1}{n}\right)^{1/\alpha}$; $1 - x'_n \approx \left(\frac{1}{n}\right)^{1/\alpha'}$
   (ii) $m(J_k) \approx \left(\frac{1}{k}\right)^{1+1/\alpha}$; $m(J'_k) \approx \left(\frac{1}{k}\right)^{1+1/\alpha'}$
   (iii) for $x \in I_k, I'_k$, $f'(x) \approx k$
   (iv) $m(I_k) \approx \left(\frac{1}{k}\right)^{2+1/\alpha}$; $m(I'_k) \approx \left(\frac{1}{k}\right)^{2+1/\alpha'}$
   (v) if $\rho > 0$ then $x_k - x_{k+n} \approx x_k \frac{n}{k}$ when $n \leq \rho k$.

Proof: See Appendix 1.

The separation function s is given by Definition 1 with respect to the partition η of Δ, although we emphasize that $\Delta'_{l,i}$ and $\Delta_{l,i} \neq \Delta'_{l,i}$ are understood to be different atoms in η even though the value of the return time function R is the same on both intervals.

Lemma 2. *There exists a constant $\beta = \beta(f) < 1$ such that if $x, y \in \Delta_0$ and $s(x,y) = n$ then $|x - y| \leq \beta^n$*

Proof: Set $\beta := \min\left\{[f'(x'_0)]^{-1}, [f'(x_0)]^{-1}\right\} < 1$ and observe that on the set Δ_0, $f' \geq \beta^{-1} > 1$, and hence $(f^R)' \geq \beta^{-1}$ (recall $f' \geq 1$ everywhere). Therefore, if x, y lie in a common atom $\Delta^{(\prime)}_{0,i} \subseteq (f^R)^{-1}[x_0, x'_0]$ with $x = (f^R)^{-1}(x')$, $y = (f^R)^{-1}(y')$ then $|x - y| \leq \beta$. The result follows by induction on $i \leq n$.

Lemma 3 (Uniform distortion)**.** *Let $y, z \in \Delta_0$ and suppose that $s(y,z) \geq 1$. Then there is a constant $D > 1$ (depending on f but not y, z) such that*

$$\left| \frac{f^{R'}(y)}{f^{R'}(z)} - 1 \right| \leq \frac{D(D-1)}{m(\Delta_0)} |f^R(y) - f^R(z)|.$$

Proof: See Appendix 1.

Remark 2. *The ambient measure μ_0 from the abstract tower construction is chosen to be Lebesgue measure $m|_{[x_0, x'_0]}$. Its lift to the tower Δ under F is the product of Lebesgue measure with counting measure restricted to Δ, which we will denote by m_Δ. Note, however, that since m is invariant for f, $m|_{\Delta_0}$ is f^R–invariant on Δ_0. Since $F^R(x) = f^R(x) \; \forall \; x \in \Delta_0$, m_Δ is $F-$ invariant on the tower. Therefore F^R and its inverse satisfy the required nonsingularity assumption as maps between $\Delta^{(\prime)}_{0,i}$ and Δ_0.*

5. Mixing rates I – upper bounds for the tower map (F, Δ)

Recall that m_Δ denotes the product of Lebesgue measure with counting measure on the tower Δ.

Theorem 3. *Fix f be as in the previous section and any $\beta \geq \beta(f)$ as in Lemma 2. Set $\gamma = \max\{\alpha, \alpha'\}$. Then*

(1) *$m_\Delta(\Delta) = 1$ and m_Δ is the unique absolutely continuous $F-$invariant probability measure on Δ. Moreover, the system (F, m_Δ) is exact, hence ergodic and mixing.*

(2) *For each absolutely continuous probability measure λ such that $\frac{d\lambda}{dm_\Delta} \in C_\beta^+$ we have*

$$|F_*^n \lambda - m_\Delta| = O(n^{-\frac{1}{\gamma}})$$

(3) *For every $\varphi \in L^\infty(\Delta)$ and $\psi \in C_\beta(\Delta)$ we have*

$$\left| \int \varphi \circ F^n \; \psi \, dm_\Delta - \int \varphi \, dm_\Delta \int \psi \, dm_\Delta \right| \leq |\varphi|_\infty \, C_\psi \, n^{-\frac{1}{\gamma}}$$

where $C_\psi < \infty$ depends only on ψ and f.

Proof: (1) Since F is non-singular with respect to m_Δ (see Remark 2), Lemmas 2 and 3 give the regularity estimate (6) on the tower map F with $\beta := \beta(f)$, $D := D(f)$ and $C := \frac{D(D-1)}{m(\Delta_0)}$ (one simply observes that $|f^R(y) - f^R(z)| \leq \beta^{s(f^R(y), f^R(z))}$ and that $F^R = f^R$). It follows that (6) is satisfied for every $\beta \geq \beta(f)$. Next, using Lemma 1 we can estimate

$$\int_{\Delta_0} R(x) \, dm(x) = \sum_{k=1}^\infty (k+1) m(I_k \cup I_k') \leq K \sum_{k=1}^\infty (k+1) \left(\frac{1}{k}\right)^{2+\frac{1}{\gamma}} < \infty$$

for some constant K. Moreover, this shows

$$\int_{\Delta_0} R(x) \, dm(x) = O\left(\sum_{k=1}^\infty \left(\frac{1}{k}\right)^{1+\frac{1}{\gamma}} \right)$$

Finally, we note that the values taken by the return time function are $R = 2, 3, \ldots$ so the gcd condition in Theorem 2 also holds. Applying the theorem to our tower yields an invariant measure ν on Δ equivalent (i.e. mutually absolutely continuous) to m_Δ. Since the latter is already $F-$invariant, we claim $m_\Delta = \nu$.

To confirm this, note that since ν is ergodic we can decompose $m_\Delta = p \nu + (1 - p) \nu_\perp$ where ν and $\nu_\perp$ are mutually singular. If there is a set A such that $\nu_\perp(A) > 0$ but $\nu(A) = 0$ then $m_\Delta(A) = 0$ since m_Δ and ν are equivalent measures. Hence $(1 - p) = 0$, establishing the claim.

Conclusions (2)-(3) of Theorem 2 also apply since

$$m_\Delta(\hat{R} > n) = \sum_{l>n} m_\Delta(\Delta_l) = \sum_{l>n} (l - n) \, m(I_l \cup I_l') \approx \left(\frac{1}{n}\right)^{\frac{1}{\gamma}}$$

(by Lemma 1).

6. Mixing rates II – upper bounds for the factor map $(f, [0, 1])$

The tower (F, Δ) provides a representation for the dynamics of f oriented around the *induced transformation* f^R of first returns to Δ_0. In order to interpret the mixing results of Theorem 3 in terms of the original map f we first extract f as a factor of F. For $(x, l) \in \Delta$ define

$$\Phi(x, l) = f^l(x)$$

(For convenience set $f(a) = 0$ which is consistent with viewing f as a continuous circle endomorphism). Now:

(1) $\Phi|_{\Delta_0} \equiv \mathrm{id}_{[x_0, x_0')}$
(2) For $l > 0$, Φ maps Δ_l injectively onto $[0, x_0) \cup [x_0', 1)$
(3) $\Phi^{-1}(J_k) = \bigcup_{l=1}^{\infty} I_{l+k} \times \{l\}$ (with a similar equality for $\cdot'$)
(4) There exists a D' such that for all $l < i$, if $A \subseteq I_{0,i} \times \{l\} = \Delta_{l,i}$ then

$$(8) \qquad\qquad D'^{-1} \leq \frac{m(A)}{m(I_i)} \frac{m(J_{i-l})}{m(\Phi(A))} \leq D'$$

(with a similar inequality for $\cdot'$).

(5) The semi-conjugacy property:

$$\Phi \circ F(x, l) \;=\; \begin{cases} \Phi(f^{l+1}(x), 0) & \text{if } x \in \Delta_{0,l}, \\ \Phi(x, l+1) & \text{if } x \in \Delta_{0,k}, \; k > l \end{cases}$$

$$= f^{l+1}(x) = f(f^l(x)) = f \circ \Phi(x, l).$$

(6) That $\Phi_* m_\Delta = m_{[0,1]}$. This computation can be done by bare hands, or one can use the F–invariance of m_Δ as follows: From Theorem 3 we know that $f_* \Phi_* m_\Delta = \Phi_* F_* m_\Delta = \Phi_* m_\Delta$, and since (F, m_Δ) is ergodic, $(f, \Phi_* m_\Delta)$ is ergodic. Moreover, $m_{[0,1]} \ll \Phi_* m_\Delta$ by the distortion relation (8), so equality of the two measures follows by the same argument as in the proof of Theorem 3(1).

Now suppose ψ is ζ–Hölder continuous[11] as a function on $[0, 1]$, and denote $\hat{\psi} := \psi \circ \Phi$ (the natural lift to Δ).

Lemma 4. *Let $\beta = \beta(f)$ from Lemma 2. If ψ is a γ–Hölder then $\hat{\psi} \in C_{\beta_0}(\Delta)$, where $\beta_0 = \beta^\gamma$.*

Proof: We need to check the regularity condition on $\hat{\psi}$. First, if $(x, l), (y, k)$ are not on the same level of the tower, then $s((x, l), (y, k)) = 0$ and we estimate (for any choice of β)

$$|\hat{\psi}(x, l) - \hat{\psi}(y, k)| \leq 2 \, |\psi|_\infty \beta^0$$

In fact, the same inequality holds also whenever $s((x, l), (y, l)) = 0$ on the same level of the tower in which case $c_\psi = 2\|\psi\|_\infty$ will do the job. Now suppose $s((x, l), (y, l)) = n > 0$. Then, with C and $\zeta > 0$ from the Hölder condition on ψ and applying Lemma 2 we obtain

$$\begin{aligned} |\hat{\psi}(x, l) - \hat{\psi}(y, l)| \;&= |\psi(f^l(x)) - \psi(f^l(y))| \\ &\leq C|f^l(x) - f^l(y)|^\zeta \\ &\leq C|(F^R(x)) - (F^R(y))|^\zeta \\ &\leq C\beta^{(n-1)\zeta} = C\beta^{-\zeta}(\beta^\zeta)^n \end{aligned}$$

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[11]Meaning, $|\psi(x) - \psi(y)| \leq C|x - y|^\zeta$, for some C, $\zeta > 0$ and all x, y.

where we have used $s(F^R(x), F^R(y)) = n - 1$. Therefore it suffices to take $c_{\hat\psi} = \max\{C\beta^{-\zeta}, 2\,|\psi|_\infty\}$ and $\beta_0 = \beta^\zeta$ in the definition of $C_{\beta_0}(\Delta)$.

Theorem 4. *Let $\gamma = \max\{\alpha, \alpha'\}$.*

(1) *The system (f, m) is exact and hence B acting on S is a K-automorphism.*
(2) *If $d\lambda = \psi\, dm$ is any absolutely continuous probability measure with ψ Hölder continuous, then*

$$|f_*^n \lambda - m| = O(n^{-\frac{1}{\gamma}}).$$

(3) *If $\varphi \in L^\infty[0, 1]$ and $\psi : [0, 1] \to \mathbb{R}$ is Hölder continuous, then*

$$\left| \int_0^1 \varphi \circ f^n\, \psi\, dm - \int_0^1 \varphi\, dm \int_0^1 \psi\, dm \right| \le |\varphi|_\infty\, C_\psi\, n^{-\frac{1}{\gamma}}$$

where $C_\psi < \infty$ depends only on ψ and f.

Proof: Denote again by m_Δ Lebesgue measure on the tower. Since (f, m) is a factor of the exact system (F, m_Δ), it is also exact, and hence its natural extension B on S is a $K-$automorphism. Next we may assume $\zeta \le 1$ in the Hölder condition, so $\beta^\zeta \ge \beta$. Finally, observe the elementary identity

$$\int_{[0,1]} q(x)dm(x) = \int_{[0,1]} q(x)d\Phi_* m_\Delta = \int_\Delta \hat{q}\, dm_\Delta$$

Now an application of Lemma 4, combined with the decay of correlations result in Theorem 3, using the value of $\beta^\zeta \ge \beta(f)$ yields the result.

7. Mixing rates III – lower bounds for the factor map $(f, [0, 1])$

The upper bounds on speed of convergence to equilibrium and correlation decay obtained in Theorem 4 in parts (2) and (3) are in fact sharp in many situations.

We first treat the measure decay result, where lower bounds on the decay rate are effectively determined by the behaviour of initial densities in the neighbourhoods of the neutral fixed points at 0 and 1. The argument is quite intuitive.

We say a probability measure λ *is separated from m at x* if either

$$\limsup_{\epsilon \to 0^+} \frac{\lambda(x-\epsilon, x+\epsilon)}{m(x-\epsilon, x+\epsilon)} < 1 \text{ or } \liminf_{\epsilon \to 0^+} \frac{\lambda(x-\epsilon, x+\epsilon)}{m(x-\epsilon, x+\epsilon)} > 1.$$

Theorem 5. [Sharp decay rates for measures] *Let $\lambda \ll m$ be a probability measure on $[0, 1]$ such that $\varphi := \frac{d\lambda}{dm} \in L^\infty$. If λ is separated from m at 0 then for $n \in \mathbb{N}$, $|f_*^n \lambda - m| \ge c\, n^{-1/\alpha}$ ($c > 0$ is a constant depending on λ and α). If λ is separated from m at 1, the same result holds with α replaced by α'.*

While it is possible for correlations to decay faster than the rate specified in Theorem 4, L^∞ initial densities which differ slightly from their equilibrium value at the neutral fixed points *must decay slowly*.

Proof: We consider the case of a measure λ separated from m at zero. The proof of the second part of the theorem is identical.

Suppose first that $\limsup_{x\to 0} \frac{\lambda[0,x]}{x} < 1$. Let $\epsilon, \delta > 0$ be such that $\lambda[0, u) < (1 - \delta)\, u$ for all $u \in (0, \epsilon)$. Write $f^{-n}[0, u) = [0, v) \cup A_n$ where $f^n(v) = u$ and A_n is a union of $2^n - 1$ subintervals of $(v, 1]$. Then, $f_*^n \lambda[0, u) \le \lambda[0, v) + |\frac{d\lambda}{dm}|_\infty\, m(A_n)$.

Since m is f invariant, $u = m[0, u] = m \circ f^{-n}[0, u] = v + m(A_n)$. Now let $u = x_k$, where k is large enough that $x_k < \epsilon$ and $k \geq n$. Then, $v = x_{k+n}$ and

$$f_*{}^n \lambda[0, u] \leq (1 - \delta) v + \left| \tfrac{d\lambda}{dm} \right|_\infty (u - v) \leq (1 - \delta) x_k + \left| \tfrac{d\lambda}{dm} \right|_\infty c_2 \, x_k \, \tfrac{n}{k}$$

(where the finite c_2 is chosen corresponding to $\rho = 1$ in Lemma 1 (v)). Now, choose $N \in \mathbb{N}$ such that $\frac{\left| \frac{d\lambda}{dm} \right|_\infty c_2}{N} < \frac{\delta}{2}$ and $x_N < \epsilon$. Using $u = x_k = x_{nN}$,

$$f_*{}^n \lambda[0, x_{nN}) \leq (1 - \delta) x_{nN} + \tfrac{\delta}{2} x_{nN}.$$

Consequently, $|f_*{}^n \lambda - m| \geq |f_*{}^n \lambda[0, x_{nN}) - m[0, x_{nN})| \geq \tfrac{\delta}{2} x_{nN} \geq \tfrac{\delta}{2} c_1 \left(\tfrac{1}{nN} \right)^{1/\alpha}$, by Lemma 1 (i).

Now suppose $\liminf_{x \to 0} \frac{\lambda[0,x]}{x} > 1$ and let $\psi = \frac{d\lambda}{dm}$. Let $\lambda' = \left(1 - \frac{\psi - 1}{|\psi - 1|_\infty} \right) m$. Then the proof of the first part of the lemma applies to λ' and $|f_*^n \lambda' - m| = |f_*^n \lambda - m| / |\psi - 1|_\infty$.

It is more delicate to obtain lower bounds on the decay rates of regular (ie: Hölder) functions. One approach is to exploit symmetry of the cut function, when this is available.

We say that the cut function ϕ is symmetric if

(9) $$1 - \phi(t) = \phi(1 - t) \text{ for all } t \in [0, 1]$$

Equivalently, $\alpha = \alpha'$, $c_0 = c_1$ and $g_0 = g_1$.

It follows that $a = \int_0^1 \phi(t)\, dt = 1/2$ and $x_n' = 1 - x_n$ for every n. Note that Examples 2.2 and 2.3 satisfy this condition.

Theorem 6. [Sharp decay rates for Hölder data] *Suppose the cut function ϕ satisfies symmetry equation (9). Then there are Lipschitz functions φ, ψ and a constant c_α such that*

$$\left| \int_0^1 \varphi \circ f^n \, \psi \, dm - \int_0^1 \varphi \, dm \int_0^1 \psi \, dm \right| \geq c_\alpha \, n^{-1/\alpha}.$$

The proof is in Appendix 2.

8. Mixing rates IV – polynomial decay of correlations for $(B_\alpha, m \times m)$

Suppose φ, ψ are two bounded measurable functions on a Borel probability space (X, p) and T is a measure preserving map on X. We write

$$\mathrm{Cor}_n(\varphi, \psi) = \left| \int_X \varphi \circ T^n \, \psi \, dp - \int_X \varphi \, dp \int_X \psi \, dp \right|.$$

Theorem 7. *Let ϕ be a cut function as detailed in Section 2.1, let B be the associated baker's transformation and set $\gamma = \max\{\alpha, \alpha'\}$. If φ and ψ are Hölder continuous on S then with respect to the measure $m \times m$ we have*

$$\mathrm{Cor}_n(\varphi, \psi) = O(n^{-1/\gamma}).$$

The constant in the order notation depends on φ, ψ and γ. If ϕ satisfies the symmetry condition (9), there are φ, ψ for which this rate is sharp.

The proof proceeds in the expected fashion: by applying the 1-dimensional decay result for f to suitably chosen φ_0 that depend only on the "future" (that is, are φ_0 that are constant on vertical fibres). If $\varphi_0(x,y)$ depends only on x then $\overline{\varphi} = \varphi_0 \circ \pi^{-1}$ has an unambiguous definition (recall $\pi(x,y) = x$), and hence

$$(10) \qquad \mathrm{Cor}_n(\varphi_0, \psi) = \left| \int_0^1 \overline{\varphi} \circ f^n \, \overline{\psi}(x) \, dm - \int_0^1 \overline{\varphi} \, dm \int_0^1 \overline{\psi}(x) \, dm \right|$$

where $\overline{\psi}(x) = \int_0^1 \psi(x,y) \, dm(y)$.

Proof of (10): Since $\varphi_0(x',y') = \varphi_0(x',0)$ for each (x',y')

$$\varphi_0 \circ B^n(x,y) = \varphi_0(f^n(x), g_n(x,y)) = \varphi_0 \circ \pi^{-1}(f^n(x)) = \overline{\varphi} \circ f^n(x)$$

(see (2)). Hence, by Fubini's theorem,

$$\int_S \varphi_0 \circ B^n \, \psi \, dm \times m = \int_0^1 \overline{\varphi}(f^n(x)) \int_0^1 \psi(x,y) \, dm(y) \, dm(x) = \int_0^1 \overline{\varphi} \circ f^n \overline{\psi} \, dm.$$

Since,

$$\int_S \varphi_0 \, dm \times m = \int_S \varphi_0 d(\pi_* m) = \int_0^1 \varphi_0 \circ \pi^{-1} \, dm \qquad \text{and} \qquad \int_S \psi \, dm \times m = \int_0^1 \overline{\psi} \, dm$$

the proof is complete. $\square$

It is evident that the lower bounds on the rate of correlation decay obtained for f in Theorem 6 carry over to B: simply extend the one-dimensional functions to vertical fibres by translation. Lifting the upper bounds requires more work, and exploits the fact that for a Hölder continuous φ, $\varphi \circ B^n$ is very nearly constant on "most" fibres when n is large.

Lemma 5. *Let φ be Hölder continuous on S. Let B and γ be as defined in Theorem 7. Then there is a constant C such that for each sufficiently large k there are functions $\varphi_0, \varphi_1, \varphi_2$ such that*

$$\varphi \circ B^k = \varphi_0 + \varphi_1 + \varphi_2$$

where

(1) *φ_0 is constant on vertical fibres and $|\varphi_0|_\infty \leq |\varphi|_\infty$,*
(2) *$|\varphi_1|_\infty \leq k^{-1/\gamma}$ and*
(3) *$|\varphi_2|_{L^1} \leq C \, |\varphi|_\infty \, k^{-1/\gamma}$.*

Proof: Let k be fixed. We begin with some notation: let $\hat{\Delta}_0 = [x_0, x_0') \times [0,1] \subset S$ (where $\{x_0, x_0'\}$ is the period 2 orbit of f from Section 4) and let

$$\beta = \sup_{x \in [x_0, x_0')} \max\{\phi(x), 1 - \phi(x)\}.$$

Then, when $B(x,y) \in \hat{\Delta}_0$, $\tilde{\phi}(x) \leq \beta$ (see equation (3)), so vertical fibres are contracted by at least β every time the orbit visits $\hat{\Delta}_0$. If an orbit segment $\{B^n(x,y) : 0 \leq n < k\}$ has made at least N visits to $\hat{\Delta}_0$ then

$$(11) \qquad |B^k(x,y) - B^k(x,y')| = |g_k(x,y) - g_k(x,y')| = \partial_y g_k \, |y - y'| \leq \beta^N$$

(again, see (3) and note that $0 \leq \tilde{\phi} \leq 1$). If φ is ζ–Hölder then there is a constant C_φ such that $|\varphi(x,y) - \varphi(x',y')| \leq C_\varphi |(x,y) - (x',y')|^\zeta$. Choose N such that $C_\varphi(\beta^N)^\zeta \leq k^{-1/\gamma}$. Clearly $N \approx \log k \ll k$. Next, define a "good set"

$$G_k = \left\{ (x,y) \in S \ : \ B^{n_j}(x,y) \in \hat{\Delta}_0 \text{ for } n_1 < \cdots < n_N < k \right\}$$

and put $\varphi_0(x,y) = (\varphi \circ B^k)(x,0)\mathbf{1}_{G_k}(x,y)$, $\varphi_1 = (\varphi \circ B^k)\mathbf{1}_{G_k} - \varphi_0$ and $\varphi_2 = (\varphi \circ B^k)\mathbf{1}_{S \setminus G_k}$.

Since φ_0 takes only values of φ (and 0 outside G_k), $|\varphi_0|_\infty \leq |\varphi|_\infty$. Moreover, since $B^n(x,y) \in \hat{\Delta}_0$ if and only if $f^n(x) \in [x_0, x_0')$, G_k is a union of vertical fibres, so $\mathbf{1}_{G_k}(x,y)$ depends only on x. This establishes the claimed properties of φ_0.

For φ_1, if $(x,y) \in G_k$ then $\{B^n(x,y)\}_{0 \leq n \leq k}$ has made at least N visits to $\hat{\Delta}_0$, so

$$|\varphi(B^k(x,y)) - \varphi(B^k(x,y'))| \leq C_\phi \, (\beta^N)^\zeta \leq k^{-1/\gamma}$$

by the Hölder property, (11) and the choice of N.

Claim: There are constants c_1 and c_2 (independent of k) such that for all large enough k

$$m \times m\{S \setminus G_k\} \leq c_1 \, k^{-1/\gamma} + c_2 \, N^{2+1/\gamma} \, k^{-1-1/\gamma}.$$

Proof of the lemma, given the claim: All that remains is to control φ_2. Since N grows like $\log k$, taking $C = c_1 + 1$ gives $m \times m\{S \setminus G_k\} \leq C \, k^{-1/\gamma}$ for all large enough k. The bound on $|\varphi_2|_{L^1}$ follows.

Proof of claim: Let

$$\tau_1(x) = \min\{n \geq 0 \ : \ B^n(x,y) \in \hat{\Delta}_0\} = \min\{n \geq 0 \ : \ f^n(x) \in \Delta_0\}$$

and $\tau_{i+1}(x) = \tau_i(x) + R(f^{\tau_i(x)}(x))$ where R is the usual return time function to the "base of the tower" Δ_0. Note that $f^{\tau_i} = (f^R)^{i-1} \circ f^{\tau_1}$. Let

$$H_k = \{x \ : \ \tau_1(x) \leq k/2 \text{ and } \tau_{i+1}(x) - \tau_i(x) \leq k/2N, \quad i = 1, \ldots, N-1\}.$$

Clearly, $H_k \times [0,1] \subset G_k$ so

$$m \times m\{S \setminus G_k\} \leq m\{[0,1] \setminus H_k\} \quad \leq \quad m\{\tau_1 > k/2\} + \sum_{i=1}^{N-1} m\{\tau_{i+1} - \tau_i > k/2N\}$$

$$(12) \qquad\qquad\qquad = \sum_{j+1 > k/2} m(J_j \cup J_j')$$

$$+ \sum_{i=1}^{N-1} m \circ (f^R)^{-(i-1)}\{R \circ f^{\tau_1} > k/2N\}$$

using $\tau_1|_{J_j^{(\prime)}} = j+1$ and the definition of τ_{i+1}. Next, $m|_{\Delta_0}$ is invariant under f^R, so

$$\sum_{i=1}^{N-1} m \circ (f^R)^{-(i-1)}\{R \circ f^{\tau_1} > k/2N\} \quad = \quad (N-1)\, m\{R \circ f^{\tau_1} > k/2N\}$$

$$(13) \qquad\qquad\qquad = \quad (N-1)\, m \circ (f^{\tau_1})^{-1}\{D_k\}.$$

where $D_k = \{R > k/2N\} = \cup_{j+1 > k/2N}(I_j \cup I_j')$. Note that $m(D_k) \approx (k/2N)^{-1-1/\gamma}$ (Lemma 1). Since $f^{\tau_1} = id|_{[x_0, x_0']} + \sum_{j=0}^\infty f^{j+1}|_{J_j \cup J_j'}$ and each branch of f^{τ_1} has uniformly bounded distortion (see proof of Lemma 3), there is a constant $c \geq 1$ such that

$$m \circ (f^{\tau_1})^{-1}\{D_k\} \quad \leq \quad m(D_k) + c \sum_{j=0}^\infty \frac{m(D_k)}{m(\Delta_0)}(m(J_j) + m(J_j'))$$

$$(14) \qquad\qquad\qquad \leq \quad c\,\frac{m(D_k)}{m(\Delta_0)} \leq c'(k/2N)^{-1-1/\gamma}.$$

Combining (12), (13), (14) and the estimate $\sum_{j+1>k/2} m(J_j \cup J_j') \approx (k/2)^{-1/\gamma}$ from Lemma 1 completes the proof.

Proof of Theorem 7: First, $\overline{\psi}$ inherits the Hölder property from ψ. Put $n' = \lfloor n/3 \rfloor$, $k = n - n'$ and decompose

$$\varphi \circ B^k = \varphi_0 + \varphi_1 + \varphi_2$$

as in Lemma 5. Then,

$$\mathrm{Cor}_n(\varphi, \psi) = \mathrm{Cor}_{n'}(\varphi \circ B^k, \psi) \leq \mathrm{Cor}_{n'}(\varphi_0, \psi) + \sum_{i=1}^{2} \mathrm{Cor}_{n'}(\varphi_i, \psi).$$

The latter two terms are bounded above by $C\,n^{-1/\gamma}$ for some constant C independent of n and the first term is $O((n')^{-1/\gamma}) = O(n^{-1/\gamma})$ by (10) and Theorem 4 part 3. $\qquad \square$

APPENDIX 1: PRECISE DISTORTION AND DECAY ESTIMATES

Assume that α, α', c_0, c_1, g_0 and g_1 are given, defining ϕ as in Section 2.1, the generalized baker's transformation B and two branched expanding map f. As noted in Equation 4 we compute

$$f'(x) = \begin{cases} \frac{1}{\phi(f(x))} & x < a, \\ \frac{1}{1-\phi(f(x))} & x > a. \end{cases}$$

From the expression for ϕ, estimates on g_0 and the expression

$$x - f^{-1}(x) = \int_0^x (1 - \phi(t))\, dt,$$

valid under the left branch of f, we obtain constants C_0, $\delta_0 > 0$ such that for all $0 \leq x \leq \delta_0$ we have

(15) $$C_0^{-1} x^{1+\alpha} \leq x - f^{-1}(x) \leq C_0 x^{1+\alpha}.$$

A similar estimate holds for x near 1 using the right branch of f: There exists a constant C_1 and $\delta_1 > 0$ such that for all $1 - \delta_1 \leq x \leq 1$

(16) $$C_1^{-1}(1-x)^{1+\alpha'} \leq f^{-1}(x) - x \leq C_1(1-x)^{1+\alpha'}$$

Continue with the notation x_0 the left most period–2 point, $x_k = f^{-1}(x_{k-1}) \cap [0, x_k)$ and similarly for x_k'.

Proof of Lemma 1 on asymptotics of the x_n, x_n'. (i) We first establish the estimates on x_n. First, for any $y \geq \delta^{-1}$, $z \geq 0$, the mean value theorem and (15) give

$$\frac{[\frac{1}{y}]^{1/\alpha} - [\frac{1}{y+z}]^{1/\alpha}}{[\frac{1}{y}]^{1/\alpha} - f^{-1}([\frac{1}{y}]^{1/\alpha})} \;\leq\; \frac{C_0}{\alpha} \left[\frac{1}{y+\theta z}\right]^{1/\alpha-1} \left(\frac{1}{y} - \frac{1}{y+z}\right) y^{1+1/\alpha}$$

(17) $$= \frac{C_0}{\alpha} \left[\frac{y}{y+\theta z}\right]^{1/\alpha} \left[\frac{y+\theta z}{y+z}\right] z$$

(where $\theta \in [0,1]$). The upper and lower bounds are obtained by distinct applications of (17). First, fix n such that $x_n^{-\alpha} < \delta_0$ and set $y = x_n^{-\alpha}$ and $z = \left[\frac{C_0}{\alpha}\right]^{-1}$.

Then the RHS of Equation 17 is bounded above by 1, so that

$$[\tfrac{1}{y}]^{1/\alpha} - [\tfrac{1}{y+z}]^{1/\alpha} \leq [\tfrac{1}{y}]^{1/\alpha} - f^{-1}([\tfrac{1}{y}]^{1/\alpha}).$$

In particular, $f^{-1}([\frac{1}{y}]^{1/\alpha}) \le [\frac{1}{y+z}]^{1/\alpha}$, so that by using $y = x_n{}^{-\alpha}$ and induction, for all $k \ge 0$,

$$x_{n+k} = f^{-k}(x_n) = f^{-k}([\frac{1}{y}]^{1/\alpha}) \le \left[\frac{1}{y+k\,z}\right]^{1/\alpha} \le \frac{1}{z^{1/\alpha}}\left[\frac{1}{k}\right]^{1/\alpha} \approx \left[\frac{1}{n+k}\right]^{1/\alpha}.$$

On the other hand, whenever $y \ge z$ then the RHS of (17) is bounded below by $\frac{1}{\alpha\,C_0}\frac{1}{2^{1+1/\alpha}}z$. Pick $z = C_0\alpha 2^{1+1/\alpha}$ and set $y = \max\{z, x_n{}^{-\alpha}\}$. Then

$$x_{n+k} = f^{-k}(x_n) \ge f^{-k}([\frac{1}{y}]^{1/\alpha}) \ge \left[\frac{1}{y+k\,z}\right]^{1/\alpha} \ge \frac{1}{(2\,y)^{1/\alpha}}\left[\frac{1}{k}\right]^{1/\alpha} \approx \left[\frac{1}{n+k}\right]^{1/\alpha}.$$

This establishes the asymptotics for the x_k. The estimates on x_k' are similar, using α' instead of α and Equation (16) instead of Equation (15).

(ii) Since $J_k = [x_{k+1}, x_k)$, we have $m(J_k) = x_k - x_{k+1} \approx x_k{}^{1+\alpha} \approx \left[\frac{1}{k}\right]^{1+1/\alpha}$ by (15) and part (i) of the lemma. The estimate on J_k' using x_k' is similar.

(iii) Observe that on (a, x_0'), $f' > 1$ is decreasing so for $x \in I_k := [t_{k+1}, t_k]$ we have $f'(t_{k+1}) \ge f'(x) \ge f'(t_k)$. But, by part (i), for all sufficiently large k,

$$f'(t_k) = (1 - \phi(x_{k-1}))^{-1} \approx \left((k-1)^{\frac{1}{\alpha}}\right)^{\alpha} \approx k$$

The argument for intervals I_k' in $[x_0, a)$ is similar.

(iv) Since $f : I_k \to J_{k-1}$ bijectively, there is an $x \in I_k$ such that

$$m(I_k) = \frac{m(J_{k-1})}{f_\alpha'(x)} \approx \left[\frac{1}{k-1}\right]^{1+1/\alpha}\frac{1}{k} \approx \left[\frac{1}{k}\right]^{2+\frac{1}{\alpha}}$$

using (ii) and (iii). The argument for the I_k' is similar.

(v) When $n \le \rho\,k$, $[\frac{1}{k+n}] \approx [\frac{1}{k}]$ so the estimate follows from parts (i) and (ii) and the fact that $x_k - x_{k+n} = \sum_{k \le i < k+n} m(J_i)$. $\qquad\square$

Proof of Lemma 3 on uniform distortion. We assume that $y, z \in I_i \subset \Delta_{0,i} \subseteq (a, x_0')$. The case where $y, z \in I_i{}'$ is similar. For each $1 \le k < i + 1 = R$ let $y_k = f^{R-k}(y)$ and $z_k = f^{R-k}(z)$. Thus $y_k, z_k \in J_{k-1}$. Now,

$$[\log(f')]'|_{J_k} = \frac{f''}{f'}\Big|_{J_k} = \left[\frac{-\phi'}{\phi^2}\right] \circ f|_{J_k} \approx \left(\left[\frac{1}{k+1}\right]^{1/\alpha}\right)^{\alpha-1}$$

The final estimate in this expression follows from two observations. First note that $\phi|_{f(J_j)} \ge \phi(x_0') > 0$, providing a uniform lower bound on the denominator for all $j = 0, 1, \dots$ and second, $-\phi' \circ f(x) = \alpha c_0[f(x)]^{\alpha-1} + g_0'(f(x)) \approx [f(x)]^{\alpha-1} \approx x^{\alpha-1}$ whenever $x \in [0, x_0]$ since $x \le f(x) \le 2x$. Thus,

$$\left|\log\frac{f'(y_k)}{f'(z_k)}\right| \le c\left[\frac{1}{k}\right]^{1-1/\alpha}|y_k - z_k| \quad = \quad c\left[\frac{1}{k}\right]^{1-1/\alpha} m(J_{k-1})\frac{|y_k-z_k|}{m(J_{k-1})}$$

$$(18) \hspace{4cm} \le \quad c'\left[\frac{1}{k}\right]^2\frac{|y_k-z_k|}{m(J_{k-1})} \le c'\left[\frac{1}{k}\right]^2$$

since $|y_k - z_k| \le m(J_{k-1}) \approx m(J_k)$, where the latter estimate uses Lemma 1 (ii).

A slightly different computation is required for the first iterate.

$$[\log(f')]'|_{I_i} = \frac{f''}{f'}\Big|_{I_i} = \left[\frac{\phi'}{[1-\phi]^2}\right] \circ f|_{I_i}$$

Therefore, for some t in I_i between y and z we have

$$(19) \hspace{3cm} \left|\log\frac{f'(y)}{f'(z)}\right| = \frac{|\phi'(f(t))|}{[1-\phi(f(t))]^2}|y - z| \approx \frac{m(I_i)}{m(J_{i-1})}\frac{|y-z|}{m(I_i)}$$

Here we have used $1 - \phi \approx x^\alpha$, for $x \approx 0$ $|\phi'(x)| \approx x^{\alpha-1}$, $f(t) \in J_{i-1}$, (hence $f(t) \approx \left(\frac{1}{i-1}\right)^{\frac{1}{\alpha}}$) and estimate (ii) from Lemma 1. Next, observe that for some $t_0 \in J_{i-1}$

$$(20) \qquad \frac{m(I_i)}{m(J_{i-1})} = \frac{1}{m(J_{i-1})} \int_{J_{i-1}} 1 - \phi = 1 - \phi(t_0) \approx \frac{1}{i-1} \approx \frac{1}{i}$$

since then $t_0 \approx \left(\frac{1}{i-1}\right)^{\frac{1}{\alpha}}$. Therefore

$$(21) \qquad \left|\log \frac{f'(y)}{f'(z)}\right| \leq \frac{c''}{i} \frac{|y-z|}{m(I_i)} \leq \frac{c''}{i}$$

for some c'' independent of y, z, i (but possibly depending on α).

Now, since $(f^R)'(y) = f'(y) f'(y_{R-1}) \cdots f'(y_1)$ (and similarly for z),
$$(22)$$
$$\left|\log \frac{(f^R)'(y)}{(f^R)'(z)}\right| = \left|\log \frac{f'(y)}{f'(z)}\right| + \sum_{k=1}^{i} \left|\log \frac{f'(y_k)}{f'(z_k)}\right| < \frac{c''}{i} + c' \sum_{k=1}^{\infty} \frac{1}{k^2} \leq c'' + c' \sum_{k=1}^{\infty} \frac{1}{k^2} \stackrel{\text{def}}{=} C.$$

Now put $D = e^C$. Since the inequality in (22) holds uniformly for any choice of $y, z \in I_i$ and the map $f^R : I_i \to \Delta_0$ is bijective, we have

$$\frac{|y-z|}{m(I_i)} \leq D \frac{|f^R(y) - f^R(z)|}{m(\Delta_0)}.$$

Similarly, $\frac{(f^k)'(y_k)}{(f^k)'(z_k)} \leq D$ and since $f^k(y_k) = f^R(y)$ and $f^k(z_k) = f^R(z)$,

$$\frac{|y_k - z_k|}{m(J_{k-1})} \leq D \frac{|f^R(y) - f^R(z)|}{m(f^k(J_{k-1}))} = D \frac{|f^R(y) - f^R(z)|}{m(\Delta_0)}.$$

The last two displayed expressions can now be used to refine (21) and (18), yielding

$$\left|\log \frac{f'(y)}{f'(z)}\right| \leq \frac{c''}{i} D \frac{|f^R(y) - f^R(z)|}{m(\Delta_0)} \text{ and } \left|\log \frac{f'(y_k)}{f'(z_k)}\right| \leq c' \left[\frac{1}{k}\right]^2 D \frac{|f^R(y) - f^R(z)|}{m(\Delta_0)}$$

from which:

$$\left|\log \frac{(f^R)'(y)}{(f^R)'(z)}\right| \leq C D \frac{|f^R(y) - f^R(z)|}{m(\Delta_0)}.$$

Finally, if $|\log x| < C$ then $|\log x| > \frac{C}{e^C - 1} |x - 1|$ by an elementary convexity estimate. In view of (22),

$$\left|\frac{(f^R)'(y)}{(f^R)'(z)} - 1\right| \leq \frac{D-1}{C} \left|\log \frac{(f^R)'(y)}{(f^R)'(z)}\right| \leq \frac{D(D-1)}{m(\Delta_0)} |f^R(y) - f^R(z)|. \qquad \square$$

APPENDIX 2: LOWER BOUNDS FOR HÖLDER OBSERVABLES

A function $\psi : [0,1] \to \mathbb{R}$ will be called *anti-symmetric* if $\psi(1-x) = -\psi(x)$ for each $x \in [0,1]$.

Lemma 6. *Let ϕ be a cut function satisfying symmetry condition (9) and let f denote the expanding 1-D expanding map determined by ϕ via (1). Suppose that ψ is decreasing and anti-symmetric. Then $\frac{d}{dm} f_*^n(\psi m)$ is decreasing and anti-symmetric for each $n > 0$.*

Proof: First, let L be the Frobenius–Perron (transfer) operator for f, so $\frac{d}{dm} f_*^n(\psi m) = L^n \psi$. By induction, it suffices to show that $L\psi$ has the required properties. Next, since the cut-function ϕ satisfies Equation (9) for each $t \in [0,1]$, the transformation f satisfies $f(1-x) = 1 - f(x)$ for each $x \neq 1/2$. Let L_- be the Frobenius-Perron operator for $x \mapsto (1-x)$, so $LL_- = L_-L$ and $L_-\psi = -\psi$. Then

$$L\psi(1-x) = L_-L\psi(x) = LL_-\psi(x) = L(-\psi)(x) = -L\psi(x).$$

Next, since $\psi(1/2) = -\psi(1/2)$, $\psi(1/2) = 0$ and therefore $\psi \mathbf{1}_{(0,1/2)} \geq 0 \geq \psi \mathbf{1}_{(1/2,1)}$ (and also $L\psi(1/2) = 0$). Since ϕ is a decreasing function, $1/f' = \phi \circ f$ is decreasing on $(0,1/2)$, so $\psi_1 := L(\psi \mathbf{1}_{(0,1/2)})$ is decreasing. A similar argument shows that $\psi_2 := L(\psi \mathbf{1}_{(1/2,1)})$ is decreasing, so $L\psi = \psi_1 + \psi_2$ is decreasing.

Proof of Theorem 6: Let $\varphi(x) = \psi(x) = x$ and put $\lambda = m + (\psi - 1/2)\, m$. Then λ is a probability measure and since $\int \varphi \, dm = 1/2$,

$$\int (\varphi - 1/2) d(f_*^n \lambda) = \int (\varphi - 1/2) \circ f^n \, d\lambda = \int \varphi \circ f^n \, \psi \, dm - \int \varphi \, dm \int \psi \, dm.$$

Now, $f_*^n \lambda = m - (L^n(1/2 - \psi))\, m$ where L is the Frobenius–Perron operator for f, so the previous equation can be rewritten as

$$(23) \qquad \int (1/2 - \varphi)\, L^n(1/2 - \psi)\, dm = \int \varphi \circ f^n \, \psi \, dm - \int \varphi \, dm \int \psi \, dm.$$

By Lemma 6, $L^n(1/2 - \psi)$ is decreasing and antisymmetric (and in particular is non-negative on $(0,1/2)$, non-positive on $(1/2,1)$). Hence, $(1/2 - \varphi)\, L^n(1/2 - \psi) \geq 0$ and so

$$(24) \quad
\begin{aligned}
\int_0^1 (1/2 - \varphi)\, L^n(1/2 - \psi)\, dm \;&\geq\; \int_0^{1/4} (1/2 - \varphi)\, L^n(1/2 - \psi)\, dm \\
&\geq\; \tfrac{1}{4} \int_0^{1/4} L^n(1/2 - \psi)\, dm \\
&\geq\; \tfrac{1}{4}\tfrac{1}{2} \int_0^{1/2} L^n(1/2 - \psi)\, dm \\
&=\; \tfrac{1}{4}\tfrac{1}{4} \int_0^1 |L^n(1/2 - \psi)|\, dm = \tfrac{1}{16}|f_*^n \lambda - m|
\end{aligned}$$

(the last equality follows by the definition of λ). Clearly, λ is separated from m at 0, so the theorem follows from equations (23), (24) and Theorem 5. $\square$

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