

# RIGOROUS POINTWISE APPROXIMATIONS FOR INVARIANT DENSITIES OF NONUNIFORMLY EXPANDING MAPS

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ABSTRACT. We use an Ulam-type discretization scheme to provide *pointwise* approximations for invariant densities of interval maps with a neutral fixed point. We prove that the approximate invariant density converges pointwise to the true density at a rate  $C^* \cdot \frac{\ln m}{m}$ , where  $C^*$  is a computable fixed constant and  $m^{-1}$  is the mesh size of the discretization.

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## 1. INTRODUCTION

Ulam-type discretization schemes provide rigorous approximations for dynamical invariants. Moreover, such discretizations are easily implementable on a computer. In [18] it was shown that the original Ulam method [23] is remarkably successful in approximating isolated spectrum of transfer operators associated with piecewise expanding maps of the interval. In particular, it was shown that this method provides rigorous approximations in the  $L^1$ -norm for invariant densities of Lasota-Yorke maps (see [18] and references therein). This method has been also successful when dealing with multi-dimensional piecewise expanding maps [20], and partially successful<sup>1</sup> in providing rigorous approximations for certain uniformly hyperbolic systems [10, 11]. Recently, Blank [5] and Murray [21] independently succeeded in applying the pure Ulam method in a non-uniformly hyperbolic setting. They obtained approximations in the  $L^1$ -norm for invariant densities of certain non-uniformly expanding maps of the interval<sup>2</sup>.

Although  $L^1$  approximations provide significant information about the long-term statistics of the underlying system, they are not helpful when dealing with rare events in dynamical systems. In fact, when studying *rare events* in dynamical

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*Date:* November 14, 2014.

*1991 Mathematics Subject Classification.* Primary 37A05, 37E05.

*Key words and phrases.* Interval maps with a neutral fixed point, Invariant densities, Ulam's method, Pointwise approximations.

<sup>1</sup>See [6] for examples where the pure Ulam method provides fake spectra for certain hyperbolic systems.

<sup>2</sup>In [21], in addition to proving convergence, Murray also obtained an upper bound on the rate of convergence.

systems [1, 15] one often obtains probabilistic laws that depend on pointwise information from the invariant density of the system. In particular, *extreme value laws* of interval maps with a neutral fixed point depend pointwise on the invariant density of the map [13].

Statistical properties of non-uniformly expanding maps were studied by Pianigiani in [22] who first proved existence of invariant densities of such maps. Later, it was independently proved in [14, 19, 24] that such maps exhibit polynomial decay of correlations. The slow mixing behaviour that such maps exhibit has made them good testing tools for real and difficult physical problems.

The difficulty in obtaining pointwise approximations for invariant densities of interval maps with a neutral fixed point is two fold. Firstly, the transfer operator associated with such maps does not have a spectral gap in a classical Banach space. Therefore, powerful perturbation results [16]<sup>3</sup> are not directly available in this setting. Secondly, invariant densities of such maps are not  $L^\infty$  functions. Consequently, to provide pointwise approximation of such densities, one should first measure the approximations in a ‘properly weighted’  $L^\infty$ -norm.

In this note we use a piecewise linear Ulam-type discretization scheme to provide pointwise approximations for invariant densities of nonuniformly expanding interval maps. Our main result is stated in Corollary 3.2. For  $x \in (0, 1]$  we prove that the approximate invariant density converges pointwise to the true density at a rate  $\frac{C^*}{x^{1+\alpha}} \cdot \frac{\ln m}{m}$ , where  $C^*$  is a computable fixed constant,  $\alpha \in (0, 1)$  is a fixed constant, and  $m^{-1}$  is the mesh size of the discretization. To overcome the spectral difficulties and the unboundedness of the densities which we discussed above, we first induce the map and obtain a uniformly piecewise, expanding and onto map. Then we perform our discretization on the induced space. After that we pull back, both the invariant density and the approximate one to the full space and measure their difference in a weighted  $L^\infty$ -norm. Full details of our strategy is given in subsection 3.2.

In section 2, we recall results on uniformly piecewise expanding and onto maps. Moreover, we introduce our discretization scheme and recall results about uniform approximations for invariant densities of uniformly piecewise expanding and onto maps. In section 3, we introduce our non-uniformly expanding system, set up our strategy, and state our main results, Theorem 3.1 and Corollary 3.2. Section 4 contains technical Lemmas and the proof of Theorem 3.1. Section 5 presents an algorithm based on the result of Corollary 3.2 and discusses its feasibility.

## 2. PRELIMINARIES

**2.1. A piecewise expanding system.** Let  $(\Delta, \mathfrak{B}, \hat{\lambda})$  denote the measure space where  $\Delta$  is a closed interval,  $\mathfrak{B}$  is Borel  $\sigma$ -algebra and  $\hat{\lambda}$  is normalized Lebesgue measure on  $\Delta$ . Let  $\hat{T} : \Delta \rightarrow \Delta$  be a measurable transformation. We assume that there exists a countable partition  $\mathcal{P}$  of  $\Delta$ , which consists of a sequence of intervals,  $\mathcal{P} = \{I_i\}_{i=0}^\infty$ , such that

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<sup>3</sup>See also [12] for another perturbation result.

- (1) for each  $i = 1, \dots, \infty$ ,  $\hat{T}_i := \hat{T}_i \circ_{|I_i}$  is monotone,  $C^2$  and it extends to a  $C^2$  function on  $\bar{I}_i$ ;
- (2)  $\hat{T}_i(I_i) = \Delta$ ; i.e., for each  $i = 1, \dots, \infty$ ,  $\hat{T}_i$  is onto;
- (3) there exists a constant  $D > 0$  such that  $\sup_i \sup_{x \in I_i} \frac{|\hat{T}_i''(x)|}{(\hat{T}_i'(x))^2} \leq D$ ;
- (4) there exists a number  $\gamma$  such that  $\frac{1}{|\hat{T}_i'|} \leq \gamma < 1$ .

Let  $\hat{\mathcal{L}} : L^1 \rightarrow L^1$  denote the transfer operator (Perron-Frobenius) [4, 8] associated to  $\hat{T}$ :

$$\hat{\mathcal{L}}f(x) = \sum_{y=\hat{T}^{-1}x} \frac{f(y)}{|\hat{T}'(y)|}.$$

Under the above assumptions, among other ergodic properties, it is well known (see for instance [7])  $\hat{T}$  admits a unique invariant density  $\hat{f}$ ; i.e.  $\hat{\mathcal{L}}\hat{f} = \hat{f}$ . Moreover,  $\hat{\mathcal{L}}$  admits a spectral gap when acting on the space of Lipschitz continuous functions over  $\Delta$  [2]<sup>4</sup>. We will denote by  $BV(\Delta)$  the space of functions of bounded variation defined on the interval  $\Delta$ . Set  $\|\cdot\|_{BV(\Delta)} := V_\Delta + \|\cdot\|_{1,\Delta}$ , where  $V_\Delta$  denotes the one-dimensional variation over  $\Delta$ . Then it is well known that  $(BV(\Delta), \|\cdot\|_{BV(\Delta)})$  is a Banach space and  $\hat{\mathcal{L}}$  satisfies the following inequality (see [22] for instance): there exists a constant  $C_{LY} > 0$  such that for any  $f \in BV(\Delta)$ , we have

$$(2.1) \quad V_\Delta \hat{\mathcal{L}}f \leq \gamma V_\Delta f + C_{LY} \|f\|_{1,\Delta}.$$

Inequality (2.1) is called the Lasota-Yorke inequality.

**2.2. Markov Discretization.** We now introduce a discretization scheme which enables us to obtain rigorous uniform approximation of  $\hat{f}$  the invariant density of  $\hat{T}$ . We use a piecewise linear approximations which was introduced by Ding and Li [9]. Let  $\eta = \{c_i\}_{i=0}^m$  be a partition of  $\Delta$  into intervals. Since uniform partitions are the first choice for numerical work, we set  $c_i - c_{i-1} = \frac{|\Delta|}{m}$ , where  $|\Delta|$  is the length of  $\Delta$ . Everything we do can be easily modified for non-uniform partitions with only minor notational changes. Let

$$\varphi_i = \chi_{[c_{i-1}, c_i]} \text{ and } \phi_i(x) = m \int_0^x \varphi_i d\lambda.$$

Let  $\psi_i$  denote a set of **hat** functions over  $\eta$ :

$$(2.2) \quad \psi_0 := (1 - \phi_1), \psi_m := \phi_m \text{ and for } i = 1, \dots, m-1, \psi_i := (\phi_i - \phi_{i+1}).$$

For  $f \in L^1$ , we set  $I_i := [c_{i-1}, c_i]$  and

$$f_i := \frac{m}{|\Delta|} \int_{I_i} f dx, \quad i = 1, 2, \dots, m,$$

the average of  $f$  over the associated partition cell. For  $f \in L^1$  we set

$$Q_m f := f_1 \psi_0 + \sum_{i=1}^{m-1} \frac{f_i + f_{i+1}}{2} \psi_i + f_m \psi_m$$

Obviously, the operator  $Q_m$  retains good stochastic properties; i.e.,

- for  $f \geq 0$ ,  $Q_m f \geq 0$ ;

<sup>4</sup>In [2], a Lasota-Yorke inequality was obtained for Markov interval maps with a finite partition. The proof carries over for piecewise onto maps with a countable number of branches satisfying assumptions of subsection 2.1.

- $\int Q_m f = \int f$ .

We now define a piecewise linear Markov discretization of  $\hat{\mathcal{L}}$  by

$$(2.3) \quad \mathbb{P}_m := Q_m \circ \hat{\mathcal{L}}.$$

Notice that  $\mathbb{P}_m$  is a finite-rank Markov operator whose range is contained in the space of continuous, piecewise linear functions with respect to  $\eta$ . The matrix representation of  $\mathbb{P}_m$  restricted to this finite-dimensional space and with respect to the basis  $\{\psi_i\}$  is a (row) stochastic matrix, with entries

$$p_{ij} := m \int_{I_j} \hat{\mathcal{L}}\psi_i \geq 0.$$

By the Perron-Frobenius Theorem for stochastic matrices [17],  $\mathbb{P}_m$  has a left invariant density  $\hat{f}_m$ ; i.e.,

$$\hat{f}_m = \hat{f}_m \mathbb{P}_m.$$

The following theorem was proved in [2]:

**Theorem 2.1.** *There exists a computable constant  $\hat{C}$  such that for any  $m \in \mathbb{N}$*

$$\|\hat{f} - \hat{f}_m\|_\infty \leq \hat{C} \frac{\ln m}{m}.$$

*Remark 2.2.* We recall that in [2] it was shown that the constant  $\hat{C}$ , which is independent of  $m$ , can be computed explicitly.

### 3. POINTWISE APPROXIMATIONS FOR INVARIANT DENSITIES OF MAPS WITH A NEUTRAL FIXED POINT

**3.1. The non-uniformly expanding system.** Let  $I = [0, 1]$  be the unit interval,  $\lambda$  be Lebesgue measure on  $[0, 1]$ . Let  $T : I \rightarrow I$  be a piecewise smooth map with two branches. We assume that

- $T(0) = 0$  and there is a  $x_0 \in (0, 1)$  such that  $T_1 = T|_{[0, x_0]}$ ,  $T_2 = T|_{[x_0, 1]}$  and  $T_1 : [0, x_0] \xrightarrow{\text{onto}} [0, 1]$ ,  $T_2 : [x_0, 1] \xrightarrow{\text{onto}} [0, 1]$ ;
- $T_1$  is  $C^1$  on  $[0, x_0]$ ,  $T_1$  is  $C^2$  on  $(0, x_0]$  and  $T_2$  is  $C^2$  on  $[x_0, 1]$ .
- $T'(0) = 1$  and  $T'(x) > 1$  for  $x \in (0, x_0)$ ;  $|T'(x)| \geq \beta > 1$  for  $x \in (x_0, 1)$ ;
- $T_1$  and  $T_1'$  have the form

$$T_1(x) = x + x^{1+\alpha} + x^{1+\alpha}\delta_0(x),$$

$$T_1'(x) = 1 + (1 + \alpha)x^\alpha + x^\alpha\delta_1(x),$$

where,  $0 < \alpha < 1$  and  $\delta_i(x) \rightarrow 0$  as  $x \rightarrow 0$  for  $i = 0, 1$  with  $\delta_0'(x) \geq 0$ .

It is well known that  $T$  admits a unique invariant density  $f^*$  [14, 19, 22, 24] and the system  $(I, T, f^* \cdot \lambda)$  exhibits a polynomial mixing rate [14, 19, 24]. Moreover, it is well known [14, 19, 24] that the  $T$ -invariant density,  $f^*$ , is not an  $L^\infty$ -function. In particular, near  $x = 0$ ,  $f^*(x)$  behaves like  $x^{-\alpha}$ . Despite this difficulty, we will show that, for any  $x \in (0, 1]$ , one can obtain rigorous pointwise approximation of  $f^*(x)$ .

**3.2. Strategy and the statement of the main result.** Recall that  $\alpha \in (0, 1)$ . We first define a suitable Banach space that contains  $f^*$ . More precisely, let  $\mathcal{B}$  denote the set of continuous functions on  $(0, 1]$  with the norm

$$\|f\|_{\mathcal{B}} = \sup_{x \in (0, 1]} |x^{1+\alpha} f(x)|.$$

When equipped with the norm  $\|\cdot\|_{\mathcal{B}}$ ,  $\mathcal{B}$  is a Banach space<sup>5</sup>. The fact that  $f^* \in \mathcal{B}$  follows from Lemma 3.3 of [14]. Our strategy for obtaining pointwise approximation  $f^*$  consists of the following steps:

- (1) We first induce  $T$  on  $\Delta \subset I$  and obtain a  $\hat{T}$  which satisfies the assumptions of subsection 2.1.
- (2) On  $\Delta$ , we use Theorem 2.1 to say that  $\hat{f}_m$ , the invariant density of the discretized operator  $\mathbb{P}_m := Q_m \circ \hat{\mathcal{L}}$ , defined in equation (2.3), provides a uniform approximation of  $\hat{f}$  the  $\hat{T}$ -invariant density.
- (3) Next we write  $f^*$  in terms of  $\hat{f}$ , and define a function  $f_m$  as the ‘pullback’ of  $\hat{f}_m$ .
- (4) Finally, we use steps (2) and (3) to prove that  $\|f^* - f_m\|_{\mathcal{B}} \leq C^* \frac{\ln m}{m}$ , and deduce a pointwise approximation of  $f^*$ .

**3.2.1. The induced system.** We induce  $T$  on  $\Delta := [x_0, 1]$ . For  $n \geq 0$  we define

$$x_{n+1} = T_1^{-1}(x_n).$$

Set

$$W_0 := (x_0, 1), \text{ and } W_n := (x_n, x_{n-1}), n \geq 1.$$

For  $n \geq 1$ , we define

$$Z_n := T_2^{-1}(W_{n-1}).$$

Then we define the induced map  $\hat{T} : \Delta \rightarrow \Delta$  by

$$(3.1) \quad \hat{T}(x) = T^n(x) \text{ for } x \in Z_n.$$

Observe that

$$T(Z_n) = W_{n-1} \text{ and } \tau_{Z_n} = n,$$

where  $\tau_{Z_n}$  is the first return time of  $Z_n$  to  $\Delta$ . An example of the map  $T$  and its induced counterpart  $\hat{T}$  are shown in Figures 1 and 2 respectively. It is well known (see for instance [24]) that the  $\hat{T}$  defined in (3.1) satisfies the assumptions of subsection 2.1, and, by Theorem 2.1, one can obtain a rigorous *uniform* approximation of its invariant density  $\hat{f}$ . Moreover, by Lemma 3.3 of [3],  $f^*$ , the invariant density of  $T$ , can be written in terms of  $\hat{f}$ :

$$(3.2) \quad f^*(x) = \begin{cases} c_\tau \hat{f}(x) & \text{for } x \in \Delta \\ c_\tau \sum_{n=1}^{\infty} \left( \frac{\hat{f}(T_2^{-1} T_1^{-(n-1)} x)}{|DT^{(n)}(T_2^{-1} T_1^{-(n-1)} x)|} \right) & \text{for } x \in I \setminus \Delta \end{cases},$$

where  $\hat{f}$  is the  $\hat{T}$ -invariant density,  $c_\tau^{-1} = \sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}(Z_k)$ , and  $\hat{\mu} = \hat{f} \cdot \hat{\lambda}$ .

<sup>5</sup>In what follows, we only use the metric properties of  $\mathcal{B}$ . In particular, the completeness of  $\mathcal{B}$  is not needed in our proofs.

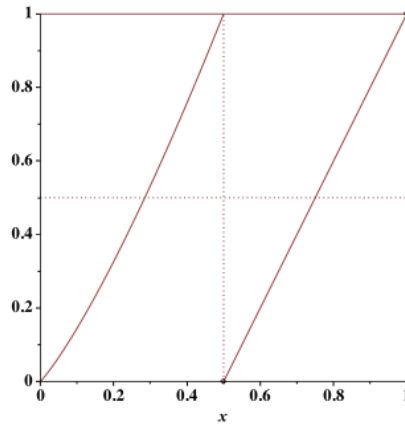


FIGURE 1. A typical example of a map  $T$  which belongs to the family defined in subsection 3.1.

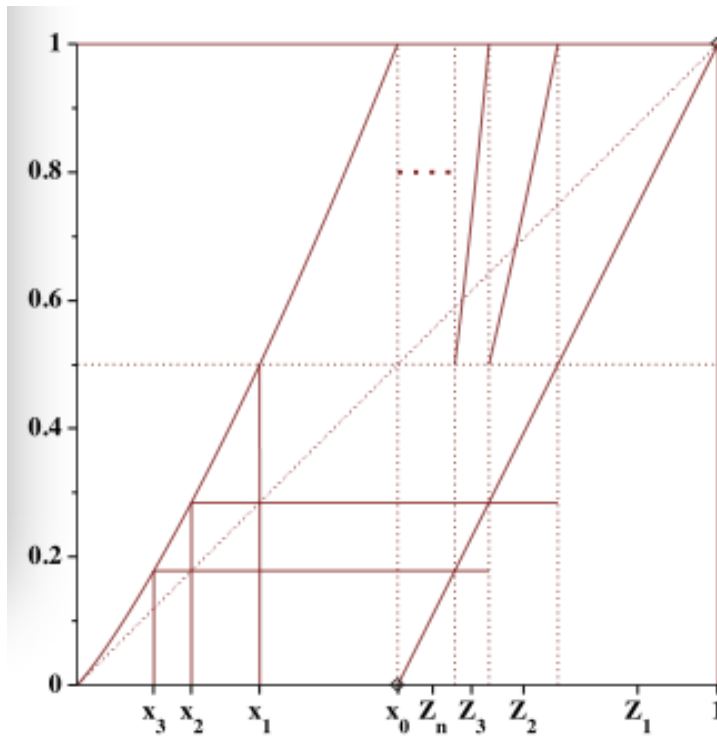


FIGURE 2. This figure shows the induced map  $\hat{T}$  corresponding to the map  $T$  of Figure 1.

3.2.2. *The approximate density and the statement of the main result.* Set

$$(3.3) \quad f_m(x) \stackrel{\text{def}}{=} \begin{cases} c_{\tau,m} \hat{f}_m(x) & \text{for } x \in \Delta \\ c_{\tau,m} \sum_{n=1}^{\infty} \left( \frac{c_{\tau,m} \hat{f}_m(T_2^{-1} T_1^{-(n-1)} x)}{|DT^{(n)}(T_2^{-1} T_1^{-(n-1)} x)|} \right) & \text{for } x \in I \setminus \Delta \end{cases},$$

where  $\hat{f}_m = \mathbb{P}_m f_m$ , and  $\mathbb{P}_m$  is the Markov discretization of  $\hat{\mathcal{L}}$  defined in (2.3),  $c_{\tau,m}^{-1} = \sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}_m(Z_k)$ , and  $\hat{\mu}_m = \hat{f}_m \cdot \hat{\lambda}$ . The next result shows that the function  $f_m$  defined in (3.3) provides a rigorous pointwise approximation of  $f^*$ .

**Theorem 3.1.** *For any  $m \in \mathbb{N}$  we have*

$$\|f^* - f_m\|_{\mathcal{B}} \leq C^* \frac{\ln m}{m},$$

where

$$C^* = \hat{C} \left( 1 + \frac{x_0^{1+\alpha}}{\beta} + M(1+\alpha) \right) C_4;$$

in particular,  $\hat{C}$  is the computable constant of Theorem 2.1,

$$M := \frac{C_1^{1+\alpha} e^{2C_0 C_1^{2\alpha}}}{\beta},$$

$$C_0 := \frac{\alpha(1+\alpha)}{2} [1 + 2\delta_0(x_0) + \delta_0^2(x_0)], \quad C_1 := (2[2^{\frac{1}{\alpha}} - 1])^{1/\alpha},$$

$$C_4 := 1 + C_3 \left( \frac{C_{LY}}{1-\gamma} + \frac{1}{|\Delta|} \right), \quad C_3 := \frac{1}{\beta} + \frac{C_2}{\beta|\Delta|} \left( \alpha + \frac{2-\alpha}{1-\alpha} \right),$$

and

$$C_2 = \frac{1-x_0}{x_0^{1+\alpha}} 2^{1+\frac{1}{\alpha}} [2^{\frac{1}{\alpha}} - 1]^{1+\frac{1}{\alpha}}.$$

As a direct consequence of the Theorem 3.1 we obtain the required pointwise approximation of  $f^*$ :

**Corollary 3.2.** *For any  $x \in (0, 1]$  we have*

$$|f^*(x) - f_m(x)| \leq \frac{C^*}{x^{1+\alpha}} \frac{\ln m}{m}.$$

*Proof.* For  $x \in (0, 1]$ , we have

$$|f^*(x) - f_m(x)| = \frac{1}{x^{1+\alpha}} |x^{1+\alpha}(f^*(x) - f_m(x))| \leq \frac{1}{x^{1+\alpha}} \|f^* - f_m\|_{\mathcal{B}} \leq \frac{1}{x^{1+\alpha}} C^* \frac{\ln m}{m}. \quad \square$$

## 4. PROOFS

4.1. **Technical lemmas.** We first introduce notation of certain functions which appear in the proof of Theorem 3.1. For  $x \in I \setminus \Delta$  set:

$$g(x) := \frac{\left(\frac{T_1 x}{x}\right)^{1+\alpha}}{T_1'(x)},$$

$$G_1(x) := \frac{x^{1+\alpha}}{|T_2'(T_2^{-1} x)|}; \quad \text{and for } n \geq 2, \quad G_n(x) := \frac{x^{1+\alpha}}{|DT^{(n)}(T_2^{-1} T_1^{-(n-1)} x)|}.$$

**Lemma 4.1.** *For  $x \in I \setminus \Delta$ , we have*

$$[1 + x^\alpha + x^\alpha \delta_0(x)]^{1+\alpha} \leq 1 + (1 + \alpha)[x^\alpha + x^\alpha \delta_0(x)] + \frac{\alpha(1 + \alpha)}{2}[x^\alpha + x^\alpha \delta_0(x)]^2.$$

*Proof.* Let

$$\phi_1(x) := [1 + x^\alpha + x^\alpha \delta_0(x)]^{1+\alpha}$$

and

$$\phi_2(x) := 1 + (1 + \alpha)[x^\alpha + x^\alpha \delta_0(x)] + \frac{\alpha(1 + \alpha)}{2}[x^\alpha + x^\alpha \delta_0(x)]^2.$$

Note that  $\phi_1(0) = \phi_2(0) = 1$ . Therefore, to prove the lemma, it is enough to prove that  $\phi_1'(x) \leq \phi_2'(x)$ . We have:

$$\begin{aligned} \phi_1'(x) &= (1 + \alpha)(1 + \xi(x))^\alpha \xi'(x) \\ \phi_2'(x) &= (1 + \alpha)(1 + \alpha \xi(x)) \xi'(x), \end{aligned}$$

where  $\xi(x) := x^\alpha + x^\alpha \delta_0(x) \geq 0^6$ . Notice that  $\xi'(x) \geq 0$ . Thus, we only need to show that

$$(4.1) \quad (1 + \xi(x))^\alpha \leq (1 + \alpha \xi(x)).$$

Indeed, (4.1) holds because  $(1 + \xi(0))^\alpha = (1 + \alpha \xi(0)) = 1$  and

$$[(1 + \xi(x))^\alpha]' = \frac{\alpha}{(1 + \xi(x))^{1-\alpha}} \xi'(x) \leq \alpha \xi'(x) = [1 + \alpha \xi(x)]'.$$

□

**Lemma 4.2.** *For  $x \in I \setminus \Delta$ , we have  $g(x) \leq 1 + C_0 x^{2\alpha}$ , where*

$$C_0 = \frac{\alpha(1 + \alpha)}{2}[1 + 2\delta_0(x_0) + \delta_0^2(x_0)].$$

*Proof.* Using Lemma 4.1, we have:

$$\begin{aligned} g(x) &= \frac{(\frac{T_1 x}{x})^{1+\alpha}}{T_1'(x)} = \frac{[1 + x^\alpha + x^\alpha \delta_0(x)]^{1+\alpha}}{1 + (1 + \alpha)x^\alpha + x^\alpha \delta_1(x)} \\ &\leq \frac{1 + (1 + \alpha)[x^\alpha + x^\alpha \delta_0(x)] + \frac{\alpha(1 + \alpha)}{2}[x^\alpha + x^\alpha \delta_0(x)]^2}{1 + (1 + \alpha)x^\alpha + x^\alpha \delta_1(x)} \\ &= \frac{1 + (1 + \alpha)[x^\alpha + x^\alpha \delta_0(x)]}{1 + (1 + \alpha)x^\alpha + x^\alpha \delta_1(x)} + \frac{\frac{\alpha(1 + \alpha)}{2}[x^\alpha + x^\alpha \delta_0(x)]^2}{1 + (1 + \alpha)x^\alpha + x^\alpha \delta_1(x)} \\ &\leq 1 + \frac{\alpha(1 + \alpha)}{2}[x^\alpha + x^\alpha \delta_0(x)]^2 \\ &= 1 + \frac{\alpha(1 + \alpha)}{2}(1 + 2\delta_0(x) + \delta_0^2(x))x^{2\alpha} \leq 1 + C_0 x^{2\alpha}. \end{aligned}$$

□

**Lemma 4.3.** *Let  $x_n = T_1^{-n} x_0$ . For  $n \geq 1$ ,  $x_n \leq C_1 n^{-\frac{1}{\alpha}}$ , where  $C_1 = (2[2^{\frac{1}{\alpha}} - 1])^{1/\alpha}$ .*

<sup>6</sup>It is obvious that  $\xi(0) = 0$  and for  $x > 0$ ,  $\xi(x) > 0$ .



*Proof.* Observe that  $C_1 > 1 \geq T_1^{-1}(x_0) = x_1$ . Therefore, the lemma is true for  $n = 1$ . Next, for  $n \geq 2$ , we suppose that  $x_{n-1} \leq C_1(n-1)^{-\frac{1}{\alpha}}$ , and prove that  $x_n \leq C_1 n^{-\frac{1}{\alpha}}$ . If it is false, that is  $x_n > C_1 n^{-\frac{1}{\alpha}}$ , then by our inductive statement on  $x_{n-1}$ , we have:

$$C_1(n-1)^{-\frac{1}{\alpha}} \geq x_{n-1} = T_1(x_n) > C_1 n^{-\frac{1}{\alpha}} [1 + C_1^\alpha n^{-1} + C_1^\alpha n^{-1} \delta_0(C_1 n^{-\frac{1}{\alpha}})].$$

This is equivalent to

$$n[(1 + \frac{1}{n-1})^{\frac{1}{\alpha}} - 1] > C_1^\alpha [1 + \delta_0(C_1 n^{-\frac{1}{\alpha}})].$$

By convexity of the function  $z^{\frac{1}{\alpha}}$ , it follows  $\frac{n}{n-1}[2^{\frac{1}{\alpha}} - 1] > C_1^\alpha [1 + \delta_0(C_1 n^{-\frac{1}{\alpha}})]$ , that is

$$C_1^\alpha < \frac{n}{n-1} [2^{\frac{1}{\alpha}} - 1] / [1 + \delta_0(C_1 n^{-\frac{1}{\alpha}})] < 2[2^{\frac{1}{\alpha}} - 1] = C_1^\alpha.$$

A contradiction. Therefore,  $x_n \leq C_1 n^{-\frac{1}{\alpha}}$ , and this completes the proof of the lemma.  $\square$

**Lemma 4.4.** For  $x \in I \setminus \Delta$ , we have

$$G_1(x) \leq \frac{x_0^{1+\alpha}}{\beta}.$$

and for  $n \geq 2$ ,

$$G_n(x) \leq M(n-1)^{-(1+\frac{1}{\alpha})},$$

where  $M = \frac{C_1^{1+\alpha} e^{2C_0} C_1^{2\alpha}}{\beta}$ .

*Proof.* For  $n = 1$ , it is easy to see that

$$G_1(x) \leq \frac{x_0^{1+\alpha}}{\beta}.$$

For  $n \geq 2$ , we have

$$\begin{aligned} G_n(x) &= \frac{x^{1+\alpha}}{|DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x)|} \\ &= \frac{x^{1+\alpha}}{|D(T_1 \circ T_1 \circ \dots \circ T_1 \circ T_2)(T_2^{-1}T_1^{-(n-1)}x)|} \\ &= \frac{x^{1+\alpha}}{T_1'(T_1^{-1}x) \cdot T_1'(T_1^{-2}x) \cdots T_1'(T_1^{-(n-1)}x) \cdot |T_2'(T_2^{-1}T_1^{-(n-1)}x)|} \\ (4.2) \quad &= \frac{(\frac{x}{T_1^{-1}x})^{1+\alpha}}{T_1'(T_1^{-1}x)} \cdot \frac{(\frac{T_1^{-1}x}{T_1^{-2}x})^{1+\alpha}}{T_1'(T_1^{-2}x)} \cdots \frac{(\frac{T_1^{-(n-2)}x}{T_1^{-(n-1)}x})^{1+\alpha}}{T_1'(T_1^{-(n-1)}x)} \cdot \frac{(T_1^{-(n-1)}x)^{1+\alpha}}{|T_2'(T_2^{-1}T_1^{-(n-1)}x)|} \\ &= g(T_1^{-1}x) \cdot g(T_1^{-2}x) \cdots g(T_1^{-(n-1)}x) \cdot \frac{(T_1^{-(n-1)}x)^{1+\alpha}}{|T_2'(T_2^{-1}T_1^{-(n-1)}x)|} \\ &\leq g(T_1^{-1}x) \cdot g(T_1^{-2}x) \cdots g(T_1^{-(n-1)}x) \cdot \frac{(T_1^{-(n-1)}x)^{1+\alpha}}{\beta}. \end{aligned}$$

By Lemmas 4.2 and 4.3, for any  $k \geq 1$ ,  $x \in [0, x_0)$ , we have

$$\begin{aligned} (4.3) \quad g(T_1^{-k}(x)) &\leq 1 + C_0(T_1^{-k}(x))^{2\alpha} \leq 1 + C_0(T_1^{-k}(x_0))^{2\alpha} \\ &= 1 + C_0(x_k)^{2\alpha} \leq 1 + C_0 C_1^{2\alpha} k^{-2}. \end{aligned}$$

Therefore, using (4.2) and (4.3), for  $n \geq 2$ , we obtain:

$$\begin{aligned}
G_n(x) &= \prod_{k=1}^{n-1} g(T_1^{-k}(x)) \cdot \frac{(T_1^{-(n-1)}(x))^{1+\alpha}}{\beta} \\
&\leq \prod_{k=1}^{n-1} (1 + C_0 C_1^{2\alpha} k^{-2}) \cdot \frac{C_1^{1+\alpha} (n-1)^{-(1+\frac{1}{\alpha})}}{\beta} \\
&= \exp\left\{\sum_{k=1}^{n-1} \ln(1 + C_0 C_1^{2\alpha} k^{-2})\right\} \cdot \frac{C_1^{1+\alpha} (n-1)^{-(1+\frac{1}{\alpha})}}{\beta} \\
&\leq \exp\left\{\sum_{k=1}^{n-1} C_0 C_1^{2\alpha} k^{-2}\right\} \cdot \frac{C_1^{1+\alpha} (n-1)^{-(1+\frac{1}{\alpha})}}{\beta} \\
&\leq \exp\left\{C_0 C_1^{2\alpha} \left(2 - \frac{1}{n-1}\right)\right\} \cdot \frac{C_1^{1+\alpha} (n-1)^{-(1+\frac{1}{\alpha})}}{\beta} \\
&\leq M (n-1)^{-(1+\frac{1}{\alpha})}.
\end{aligned}$$

□

**Lemma 4.5.**

$$\sum_{n=1}^{\infty} n \cdot \hat{\lambda}(Z_n) \leq C_3,$$

where  $C_3 = \frac{1}{\beta} + \frac{C_2}{\beta(1-x_0)}(\alpha + \frac{2-\alpha}{1-\alpha})$  and  $C_2 = \frac{1-x_0}{x_0^{1+\alpha}} 2^{1+\frac{1}{\alpha}} [2^{\frac{1}{\alpha}} - 1]^{1+\frac{1}{\alpha}}$ .

*Proof.* By Lemma 4.3, we have  $\lambda(W_n) = x_{n-1} - x_n = T_1(x_n) - x_n = \frac{1-x_0}{x_0^{1+\alpha}} x_n^{1+\alpha} \leq \frac{1-x_0}{x_0^{1+\alpha}} C_1^{1+\alpha} n^{-(1+\frac{1}{\alpha})} = C_2 n^{-(1+\frac{1}{\alpha})}$ . Since  $T_2(Z_n) = W_{n-1}$ , we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n \cdot \lambda(Z_n) &\leq \lambda(Z_1) + \sum_{n=2}^{\infty} n \cdot \frac{\lambda(W_{n-1})}{\beta} \\
&\leq \frac{1-x_0}{\beta} + \sum_{n=2}^{\infty} \frac{n(x_{n-2} - x_{n-1})}{\beta} \\
&= \frac{1-x_0}{\beta} + \sum_{n=1}^{\infty} \frac{(n+1)(x_{n-1} - x_n)}{\beta} \\
&= \frac{1-x_0}{\beta} + \sum_{n=1}^{\infty} \frac{n(x_{n-1} - x_n)}{\beta} + \sum_{n=1}^{\infty} \frac{(x_{n-1} - x_n)}{\beta} \\
&\leq \frac{1-x_0}{\beta} + \sum_{n=1}^{\infty} \frac{C_2}{\beta} n^{-\frac{1}{\alpha}} + \sum_{n=1}^{\infty} \frac{C_2}{\beta} n^{-(1+\frac{1}{\alpha})} \\
&\leq \frac{1-x_0}{\beta} + \frac{C_2}{\beta} \left(1 + \int_1^{\infty} x^{-\frac{1}{\alpha}} dx\right) + \frac{C_2}{\beta} \left(1 + \int_1^{\infty} x^{-(1+\frac{1}{\alpha})} dx\right) \\
&= \frac{1-x_0}{\beta} + \frac{C_2}{\beta} \left(\alpha + \frac{2-\alpha}{1-\alpha}\right) = (1-x_0) \cdot C_3.
\end{aligned}$$

This completes the proof of the lemma since  $\hat{\lambda}(\cdot) = \frac{\lambda(\cdot)}{1-x_0}$ . □

**Lemma 4.6.** *We have*

$$|c_{\tau,m} - c_\tau| \leq C_3 \cdot \hat{C} \frac{\ln m}{m}.$$

*Proof.* Using the fact that  $c_\tau \leq 1$ ,  $c_{\tau,m} \leq 1$  and Theorem 2.1, we have

$$\begin{aligned} |c_{\tau,m} - c_\tau| &\leq \left| \frac{1}{\sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}_m(Z_k)} - \frac{1}{\sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}(Z_k)} \right| \\ &= \left| \frac{\sum_{k=1}^{\infty} k [\hat{\mu}(Z_k) - \hat{\mu}_m(Z_k)]}{\sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}_m(Z_k) \cdot \sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}(Z_k)} \right| \\ &\leq \left( \sum_{k=1}^{\infty} k \int_{Z_k} |\hat{f} - \hat{f}_m| d\hat{\lambda} \right) \\ &\leq \|\hat{f} - \hat{f}_m\|_\infty \left( \sum_{k=1}^{\infty} k \hat{\lambda}(Z_k) \right) \\ &\leq \hat{C} \frac{\ln m}{m} \cdot C_3. \end{aligned}$$

In the last estimate, we have used Lemma 4.5. □

We now have all our tools ready to prove Theorem 3.1.

*Proof. (of Theorem 3.1)* Using (3.2) and (3.3), we have

$$\begin{aligned} (4.4) \quad &\|f^* - f_m\|_{\mathcal{B}} = \sup_{x \in (0,1]} |x^{1+\alpha} (f^*(x) - f_m(x))| \\ &\leq \sup_{x \in I \setminus \Delta} |x^{1+\alpha} (f^*(x) - f_m(x))| + \sup_{x \in \Delta} |x^{1+\alpha} (f^*(x) - f_m(x))| \\ &= \sup_{x \in I \setminus \Delta} \left| \sum_{n=1}^{\infty} \frac{x^{1+\alpha}}{DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x)} (c_\tau \hat{f}(T_2^{-1}T_1^{-(n-1)}x) - c_{\tau,m} \hat{f}_m(T_2^{-1}T_1^{-(n-1)}x)) \right| \\ &\quad + \sup_{x \in \Delta} |x^{1+\alpha} (c_\tau \hat{f}(x) - c_{\tau,m} \hat{f}_m(x))|. \end{aligned}$$

Notice that for  $x \in I \setminus \Delta$ , and  $n \geq 1$ ,  $z_n := T_2^{-1}T_1^{-(n-1)}x \in \Delta$ . Then using the fact that  $c_\tau \leq 1$ ,  $c_{\tau,m} \leq 1$ , Theorem 2.1, Lemma 4.6, and (4.4), we obtain:

$$\begin{aligned}
(4.5) \quad & \|f^* - f_m\|_{\mathcal{B}} \leq \sup_{x \in I \setminus \Delta} \left| \sum_{n=1}^{\infty} \frac{x^{1+\alpha}}{DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x)} \right| \cdot \sup_{z_n \in \Delta} |(c_\tau \hat{f}(z_n) - c_{\tau,m} \hat{f}_m(z_n))| \\
& + \sup_{x \in \Delta} |c_\tau \hat{f}(x) - c_{\tau,m} \hat{f}_m(x)| \\
& \leq \sup_{x \in I \setminus \Delta} \left| \sum_{n=1}^{\infty} \frac{x^{1+\alpha}}{DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x)} \right| \times \\
& \quad \left( \sup_{z_n \in \Delta} |\hat{f}(z_n) - \hat{f}_m(z_n)| + |c_\tau - c_{\tau,m}| \sup_{z_n \in \Delta} |\hat{f}(z_n)| \right) \\
& + \sup_{x \in \Delta} |\hat{f}(x) - \hat{f}_m(x)| + |c_\tau - c_{\tau,m}| \sup_{x \in \Delta} |\hat{f}(x)| \\
& \leq \hat{C} \frac{\ln m}{m} \left( \sup_{x \in I \setminus \Delta} \sum_{n=1}^{\infty} |G_n(x)| (1 + C_3 \sup_{z_n \in \Delta} |\hat{f}(z_n)|) + (1 + C_3 \sup_{x \in \Delta} |\hat{f}(x)|) \right).
\end{aligned}$$

Since  $\hat{f} \in BV(\Delta)$ , we have  $\sup_{x \in \Delta} |\hat{f}(x)| \leq V_\Delta \hat{f} + \frac{1}{1-x_0} \|\hat{f}\|_{1,\Delta}$ . Therefore, using the Lasota-Yorke inequality (2.1), we obtain  $\sup_{x \in \Delta} |\hat{f}(x)| \leq \frac{C_{LY}}{1-\gamma} + \frac{1}{1-x_0}$ . Using Lemma 4.4 and (4.5), we obtain:

$$\begin{aligned}
\|f^* - f_m\|_{\mathcal{B}} & \leq C_4 \hat{C} \frac{\ln m}{m} \left( 1 + \frac{x_0^{1+\alpha}}{\beta} + \sum_{n=2}^{\infty} M(n-1)^{-(1+\frac{1}{\alpha})} \right) \\
& = C_4 \hat{C} \frac{\ln m}{m} \left( 1 + \frac{x_0^{1+\alpha}}{\beta} + M \sum_{n=1}^{\infty} n^{-(1+\frac{1}{\alpha})} \right) \\
& \leq C_4 \hat{C} \left( 1 + \frac{x_0^{1+\alpha}}{\beta} + M(1+\alpha) \right) \cdot \frac{\ln m}{m}.
\end{aligned}$$

□

## 5. ALGORITHM AND FEASIBILITY

Given a map  $T$  satisfying the conditions of section 3.1, and  $x^* \in (0, 1]$ , we provide an algorithm based on corollary 3.2 that can be used to approximate  $f^*(x^*)$ , the  $T$ -invariant density at the point  $x^*$ , up to a pre-specified approximation error  $R$ .

### 5.1. Algorithm and output.

- (1) Compute the constants  $\beta, M, C_4$  which appear in Theorem 3.1.
- (2) Compute an upper bound on the constant  $\hat{C}$  which appears in Theorem 2.1.
- (3) Then use (1), (2) to compute  $C^*$  which appear in Theorem 3.1.
- (4) Find  $m^*$ , number of bins, such that

$$\frac{\ln m^*}{m^*} \leq \frac{x^{*1+\alpha}}{C^*} \cdot \frac{R}{3}.$$

- (5) Compute an approximate fixed point<sup>7</sup>.  $\tilde{f}_m > 0$  of  $\mathbb{P}_m$  so that  $\|\tilde{f}_m - \hat{f}_m\|_\infty \leq \varepsilon$  on  $\Delta$ , where  $\varepsilon$  is chosen such that

$$|\tilde{f}_m(x^*) - f_m(x^*)| \leq \frac{R}{3},$$

where

$$\tilde{f}_m(x) = \tilde{c}_{\tau,m} \sum_{n=1}^{\infty} \left( \frac{\tilde{f}_m(T_2^{-1}T_1^{-(n-1)}x)}{|DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x)|} \right),$$

$$\tilde{c}_{\tau,m}^{-1} := \sum_{k=1}^{\infty} k\tilde{\mu}_m(Z_k) \text{ and } \tilde{\mu} = \tilde{f}_m \cdot \hat{\lambda}.$$

- (6) Find  $N^*$  such that

$$|\tilde{c}_{\tau,m}(N^*) \sum_{n=1}^{N^*} \left( \frac{\tilde{f}_m(T_2^{-1}T_1^{-(n-1)}x^*)}{|DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x^*)|} \right) - \tilde{f}_m(x^*)| \leq \frac{R}{3},$$

$$\text{where } \tilde{c}_{\tau,m}^{-1}(N^*) := \sum_{k=1}^{N^*} k\tilde{\mu}_m(Z_k).$$

- (7) The approximate value of  $f^*(x^*)$  is given by<sup>8</sup>:

$$\tilde{f}_{m,N^*}(x^*) := \tilde{c}_{\tau,m}(N^*) \sum_{n=1}^{N^*} \left( \frac{\tilde{f}_m(T_2^{-1}T_1^{-(n-1)}x^*)}{|DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x^*)|} \right).$$

## 5.2. Feasibility.

- For (1), once the map  $T$  is given, the constants  $\beta, M, C_4$  can be computed analytically.
- For (2), the constant  $\hat{C}$  appears in the approximation done on the induced system (See Theorem 2.1). The induced system is a uniformly expanding map. The computation of  $\hat{C}$  can be done following the ideas of [2] which is based on the spectral stability result of [16].
- (3) is a consequence of (1) and (2).
- Once  $C^*$  is computed in (3), with  $m := m^*$  we know, from Corollary 3.2, that  $|f_m(x^*) - f^*(x^*)| \leq R/3$ .
- For (5) we should find out how small  $\varepsilon$  should be to ensure

$$|\tilde{f}_m(x^*) - f_m(x^*)| \leq \frac{R}{3}.$$

We propose the following method. To work out explicitly all the constants needed in verifying (5), we suppose  $x^* \in W_k$ , and  $T_1(x) = x + 2^\alpha x^{1+\alpha}$ . In the following estimates (5.2), (5.3) and (5.4), we prepare the ingredients to

<sup>7</sup>Recall that  $\hat{f}_m$  is the fixed point of the finite rank operator  $\mathbb{P}_m$  defined in (2.3). Here  $\tilde{f}_m$  is the computer approximation of  $\hat{f}_m$ ; i.e., (5) of the Algorithm takes care of the computer roundoff errors in computing the fixed point of  $\mathbb{P}_m$ . Since  $\hat{f}_m > 0$ , we also ask in this computation that  $\tilde{f}_m > 0$ . Note that the strict positivity of  $\hat{f}_m$  follows from the fact that the induced map  $\hat{T}$  is a piecewise onto map, which implies that the matrix representation of  $\mathbb{P}$  is irreducible, and consequently its Perron eigenvector is strictly positive (see Perron-Frobenius Theorem [17]).

<sup>8</sup>It is very important to notice that the approximation  $\tilde{f}_{m,N^*}(x^*)$  is a *finite* sum.

achieve our job. Firstly, following exactly the argument of Lemma 5.2 in [3], for  $x^* \in W_k$ ,

$$(5.1) \quad |DT^{(n)}(T_2^{-1}T_1^{n-1}x^*)| \geq \beta \left(\frac{n+k}{k+2}\right)^{\eta_k},$$

where

$$\eta_k = \frac{d(k+2)}{k+2+d}, \quad d = (1+\alpha)2^\alpha \left[ \frac{1}{2(1+\alpha)^{\frac{1}{\alpha}}} + d_1 \right]^\alpha > 1,$$

and

$$d_1 = \frac{1}{2(\alpha)^{\frac{1}{\alpha}}} - \frac{1}{2(1+\alpha)^{\frac{1}{\alpha}}}.$$

Consequently,  $\eta_k > 1$ , and for  $x^* \in W_k$ ,

$$(5.2) \quad \sum_{n=1}^{\infty} \frac{1}{|DT^{(n)}(T_2^{-1}T_1^{n-1}x^*)|} \leq \frac{1}{\beta} \left(\frac{k+2}{k}\right)^{\eta_k} \frac{k}{\eta_k - 1}.$$

Secondly, using the same argument as the one in the proof of Lemma 4.6, we have

$$(5.3) \quad |\tilde{c}_{\tau,m} - c_{\tau,m}| \leq C_3 \cdot \|\tilde{f}_m - \hat{f}_m\|_\infty.$$

Thirdly, it is well known<sup>9</sup> that for  $g \in BV(\Delta)$  we have  $V_\Delta Q_m g \leq g$ , where  $Q_m$  is the discretization defined in subsection 2.2. Therefore,  $\mathbb{P}_m$  satisfies the same Lasota-Yorke inequality (2.1) as  $\hat{\mathcal{L}}$ . In particular, this implies that  $\hat{f}_m$ , the  $\mathbb{P}_m$  fixed point, satisfies:

$$(5.4) \quad \|\hat{f}_m\|_\infty \leq \frac{C_{LY}}{1-\gamma} + \frac{1}{|\Delta|}.$$

Consequently, using (5.2), (5.3), (5.4) and that  $\tilde{c}_{\tau,m} \leq 1$ , we obtain

$$(5.5) \quad |\tilde{f}_m(x^*) - f_m(x^*)| \leq \|\tilde{f}_m - \hat{f}_m\|_\infty \cdot \frac{1}{\beta} \left(\frac{k+2}{k}\right)^{\eta_k} \frac{k}{\eta_k - 1} \left[1 + C_3 \left(\frac{C_{LY}}{1-\gamma} + \frac{1}{|\Delta|}\right)\right].$$

Thus, to ensure  $|\tilde{f}_m(x^*) - f_m(x^*)| \leq \frac{R}{3}$ ,  $\tilde{f}_m$  should be computed to a precision that satisfies

$$\|\tilde{f}_m - \hat{f}_m\|_\infty \leq \frac{1}{\frac{1}{\beta} \left(\frac{k+2}{k}\right)^{\eta_k} \frac{k}{\eta_k - 1} \left[1 + C_3 \left(\frac{C_{LY}}{1-\gamma} + \frac{1}{|\Delta|}\right)\right]} \cdot \frac{R}{3}.$$

- For (6), as in (5), we also suppose  $x^* \in W_k$ , and  $T_1(x) = x + 2^\alpha x^{1+\alpha}$ . Firstly, by (5.1), each term in the sum

$$(5.6) \quad \sum_{n=1}^{\infty} \left( \frac{\tilde{f}_m(T_2^{-1}T_1^{-(n-1)}x^*)}{|DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x^*)|} \right)$$

is bounded above by

$$\sup_{y \in \Delta} \tilde{f}_m(y) \cdot \frac{1}{\beta} \left(\frac{k+2}{n+k}\right)^{\eta_k}.$$

Since  $\eta_k > 1$ , the tail of the sum, starting from  $N_1^* + 1$ , in (5.6) can be approximated as follows:

<sup>9</sup>See for instance [9] Lemma 2.3.

Choose  $N_1^*$  such that

$$(5.7) \quad \sup_{y \in \Delta} \tilde{f}_m(y) \cdot \frac{(N_1^* + k)^{-\eta_k + 1}}{\eta_k - 1} \cdot (k + 2)^{\eta_k} \leq \frac{R}{6}.$$

Secondly, using  $\sum_{k=1}^{\infty} k \tilde{\mu}_m(Z_k) \geq 1$ , and recalling that  $\alpha \in (0, 1)$  and  $\inf_{y \in \Delta} \tilde{f}_m(y) > 0$ , we have

$$(5.8) \quad |\tilde{c}_{\tau, m} - \tilde{c}_{\tau, m}(N_2^*)| = \frac{\sum_{N_2^*+1}^{\infty} k \tilde{\mu}_m(Z_k)}{\sum_{k=1}^{\infty} k \tilde{\mu}_m(Z_k) \cdot \sum_{k=1}^{N_2^*} k \tilde{\mu}_m(Z_k)} \leq \sup_{y \in \Delta} \tilde{f}_m \frac{\sum_{N_2^*+1}^{\infty} k \hat{\lambda}(Z_k)}{\sum_{k=1}^{N_2^*} k \tilde{\mu}_m(Z_k)} \\ \leq \sup_{y \in \Delta} \tilde{f}_m \frac{\sum_{N_2^*+1}^{\infty} \frac{C_2}{\beta} \cdot k \cdot k^{-1-\frac{1}{\alpha}}}{\inf_{y \in \Delta} \tilde{f}_m \cdot \hat{\lambda}(Z_1)} \leq \frac{\sup_{y \in \Delta} \tilde{f}_m}{\inf_{y \in \Delta} \tilde{f}_m \cdot \hat{\lambda}(Z_1)} \frac{C_2}{\beta} \cdot \frac{N_2^{*\frac{-1}{\alpha}+1}}{\frac{1}{\alpha} - 1}.$$

Thirdly, using (5.2) and (5.8), we have

$$(5.9) \quad |\tilde{c}_{\tau, m} - \tilde{c}_{\tau, m}(N_2^*)| \tilde{f}_{m, N_2^*}(x^*) \leq \left( \frac{\sup_{y \in \Delta} \tilde{f}_m}{\inf_{y \in \Delta} \tilde{f}_m \cdot \hat{\lambda}(Z_1)} \frac{C_2}{\beta} \cdot \frac{N_2^{*\frac{-1}{\alpha}+1}}{\frac{1}{\alpha} - 1} \right) \tilde{f}_m(x^*) \\ \leq \frac{\sup_{y \in \Delta} \tilde{f}_m^2}{\inf_{y \in \Delta} \tilde{f}_m \cdot \hat{\lambda}(Z_1)} \frac{C_2}{\beta^2} \cdot \frac{N_2^{*\frac{-1}{\alpha}+1}}{\frac{1}{\alpha} - 1} \left( \frac{k+2}{k} \right)^{\eta_k} \frac{k}{\eta_k - 1}.$$

Choose<sup>10</sup>  $N_2^*$  such that the right hand side of (5.9) is not greater than  $\frac{R}{6}$ . Finally, choose  $N^* = \max\{N_1^*, N_2^*\}$ . This will lead to the desired estimate  $|\tilde{f}_m(x^*) - \tilde{f}_{m, N^*}(x^*)| \leq \frac{R}{3}$ .

- for (7) using the above six steps, we obtain

$$|f^*(x^*) - \tilde{f}_{m, N^*}(x^*)| \leq |f^*(x^*) - f_m(x^*)| + |f_m(x^*) - \tilde{f}_m(x^*)| + |\tilde{f}_m(x^*) - \tilde{f}_{m, N^*}(x^*)| \leq R;$$

i.e., the **finite sum**  $\tilde{f}_{m, N^*}(x^*) := \tilde{c}_{\tau, m}(N^*) \sum_{n=1}^{N^*} \left( \frac{\tilde{f}_m(T_2^{-1} T_1^{-(n-1)} x^*)}{|DT^{(n)}(T_2^{-1} T_1^{-(n-1)} x^*)|} \right)$  is a rigorous approximation of  $f^*(x^*)$  up to the pre-specified error  $R$ .

**Acknowledgment:** The authors would like to thank an anonymous referee for comments that improved both the presentation and the content of the paper.

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<sup>10</sup>In our analysis we verified many items of the Algorithm analytically. Many of these steps maybe verified iteratively using the computer, For instance,  $N_2^*$  maybe found iteratively using a computer. Simply keep  $\sum_{k=1}^{N_2^*} k \tilde{\mu}_m(Z_k)$  in the denominator on the right hand side of (5.9) instead of replacing it by  $\inf_{y \in \Delta} \tilde{f}_m \cdot \hat{\lambda}(Z_1)$ . The iterative method may find a smaller value of  $N_2^*$  that achieves the job.

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