First hyperbolic times for intermittent maps with unbounded derivative*†

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Abstract

We establish some statistical properties of the *hyperbolic times* for a class of nonuniformly expanding dynamical systems. The maps arise as factors of area preserving maps of the unit square via a geometric baker's map type construction, exhibit intermittent dynamics, and have unbounded derivatives. The geometric approach captures various examples from the literature over the last thirty years. The statistics of these maps are controlled by the order of tangency (linked to a single parameter α where $0 < \alpha < \infty$) that a certain "cut function" makes with the boundary of the square. Previously, a direct Young tower construction has been used to obtain optimal correlation decay rates of $O(n^{-1/\alpha})$ for Hölder observables and all values of the parameter α. A CLT is obtained when $0 < \alpha < 1$.

The asymptotics of a natural hyperbolic time for this family of maps are analsyed via the same Young tower. By using a large deviations result of Melbourne and Nicol, we prove that the first hyperbolic time is integrable if and only if the parameter satisfies $0 < \alpha < 1$. Furthermore, within this restricted range of parameters, concentration inequalities recently established by Chazottes and Gouëzel imply sharp $O(n^{-1/\alpha})$ bounds on the tail distribution of first hyperbolic times. As shown by Alves, Viana and others, knowledge of the tail distribution of the hyperbolic times leads to upper bounds on the rate of decay of correlations and derivation of a CLT. Comparing to the results obtained directly for this family of maps, the latter estimates via hyperbolic times are suboptimal, even over the restricted range of parameters $0 < \alpha < 1$.

Let $f : X \to X$ be a dynamical system which is expanding on average, but not necessarily uniformly with every time-step. Amongst the important questions to ask about f

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are: is there an invariant SRB-probability measure? how quickly do correlations between observables decay under iteration by f? does a central limit theorem hold? are these properties stable to perturbations of f? When f is *uniformly expanding* the answers to these questions are well understood (see eg [8, 20]), but the situation for non-uniformly expanding f is more delicate. Difficulties arise from the fact that orbits of f may experience periods of local contraction as well as expansion (for example, quadratic maps [9]), rapidly varying derivatives near singularities leading to unbounded distortion (eg [6]), or indifferent fixed points [25].

The theory of *hyperbolic times* has proved useful for analysing the statistical properties of non-uniformly expanding maps [2, 3, 4, 6, 24]. The idea was introduced in [1] to handle specific non-uniformly expanding families (Alves-Viana maps [24] and certain quadratic maps [15]), and has since been developed for various non-uniformly expanding and partially hyperbolic classes [4]. Gouëzel [17] has used hyperbolic times to show that the Alves-Viana maps exhibit stretched exponential decay of correlations. Alves, Luzzatto and Pinheiro [6] prove polynomial decay of correlations (and a CLT) for a class of non-uniformly expanding maps by using hyperbolic times and a Young tower construction. Further results are obtained in [5] for one-dimensional families. A survey paper discussing many of these ideas is found in [2].

Roughly speaking, hyperbolic times are defined as follows[1]: Given an orbit $\{f^k x\}$ of a point $x \in X$, an integer $n > 0$ is a *hyperbolic time* for x if for all $1 \leq l \leq n$ the cumulative derivative $\Pi_{k=n-l}^{k=n-1} |Df(f^l(x))|$ grows exponentially in l. In addition, if the map has a nonempty set $\mathcal{S}$ of singular points we require the distance from $f^{n-l}(x)$ to $\mathcal{S}$ to be bounded below by an exponential in l, essentially an exponential escape condition. These exponential rates are to be chosen uniformly for $x \in X$. Note that for uniformly expanding maps with bounded distortion both conditions automatically hold and every n is a hyperbolic time.

Therefore, the idea is to choose certain times at which the accumulated expansion and escape from the singular set mimic the uniformly expanding case even though there may have been times along the way where these properties failed. In this way, many good statistical properties can be recovered.

There are at least two important statistics associated with hyperbolic times: their long-run frequency of occurrence, and the distribution of *first hyperbolic times*. Obtaining precise quantitative control of the distribution of hyperbolic times can be an important step [6, 5] in further analysis of statistical properties of the map, including the above-mentioned rates of decay of correlation and CLT.

In [3] a map F on the interval $[-1, 1]$ is introduced that has positive density of hyperbolic times, but for which the first hyperbolic time fails to be integrable. F has a number of special properties (symmetry, preservation of Lebesgue measure, and a pair of indifferent fixed points with quadratic tangencies). In this paper we present a class of interval maps $\mathcal{C}_\alpha$ (parametrised by[2] $\alpha \in (0, \infty)$) which arise as nonuniformly expanding one-dimensional factors of geometrically derived *generalized baker's transformations* (GBTs) [10]. Each map in $\mathcal{C}_\alpha$ has an indifferent fixed point (IFP) at 0, and in fact the map F in [3] is conjugate

[1]The exact definition is detailed in equation (7) in Section 2.
[2]And certain continuous functions on $[0, 1]$.

to a certain map $f_1 \in \mathcal{C}_1$. Each $f \in \mathcal{C}_\alpha$ has a positive long-run frequency of hyperbolic times (by an argument from [4]), and integrability of first hyperbolic times holds when $\alpha \in (0, 1)$. However, as α increases through 1 this integrability is lost (Theorem 1). In this way, $\alpha = 1$ appears as a transition point for our families $\mathcal{C}_\alpha$.

As becomes clear in our proof of Theorem 1, the non-integrability is entirely due to lower bounds on the first hyperbolic time which are determined by escape statistics from the neighbourhood of the IFP(s). These same escape statistics are then used to provide upper bounds on the first hyperbolic times in Theorem 3, completing the analysis and providing sharp estimate on tail asymptotics for hyperbolic times for the range $0 < \alpha < 1$. While precise statements are given in Theorems 1 and 3, if m denotes Lebesgue measure and $h(x)$ denotes the first hyperbolic time on an orbit beginning at x then for some $C < \infty$, $\frac{1}{C} n^{-1/\alpha} \leq m\{x \, : \, h(x) \geq n\} \leq C\, n^{-1/\alpha}$.

In [5, 6] hyperbolic times asymptotics are used to estimate correlation decay rates and to establish CLT's for nonuniformly hyperbolic systems. When applied to our family $\mathcal{C}_\alpha$, these results imply upper bounds on correlation decay rates of $O(n^{-1/\alpha+1})$; these fail to be sharp: a direct computation via Young towers yields $O(n^{-1/\alpha})$ rates, as detailed in [11]. The range of parameters in our family leading to a CLT is similarly underestimated by the hyperbolic times analysis ($\alpha < 1/2$ compared to $\alpha < 1$ for the direct computation). Remark 3 at the end of Section 1 provides a comparative analysis of these two approaches.

In summary, we present a geometric family of maps $\mathcal{C}_\alpha$ where the distribution of the tail of the first hyperbolic time can be determined (sharply) by a single parameter α. The general purpose machinery of hyperbolic times enables analysis of statistical properties for our maps (such as CLTs and bounds on correlation decay rates), and we compare these with optimal results.

The class $\mathcal{C}_\alpha$ is presented in Section 1, hyperbolic times are reviewed and lower bounds are derived in Section 2 and sharp upper bounds are established (via a large deviations result of Chazottes and Gouëzel [12] on a suitable Young tower [25]) in Section 3. In Section 4 we discuss our results in the context of the existing literature on hyperbolic times. Some technical estimates are placed in an appendix (Section 5).

Notation: We write $f(n) = O(g(n))$ to mean there is a constant $C < \infty$ such that $f(n) \leq C\, g(n)$ and $f(n) \asymp g(n)$ to mean $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

1 Generalized baker's transformations and $\mathcal{C}_\alpha$

The generalized baker's construction [10] defines a large class of invertible, Lebesgue-measure-preserving maps of the unit square $S = [0, 1] \times [0, 1]$. Specifically, a two-dimensional map B on S is determined by a measurable *cut function* ϕ on $[0, 1]$ satisfying $0 \leq \phi \leq 1$. The graph $y = \phi(x)$ partitions the square S into upper and lower pieces and the line $\{x = a\}$, where $a = \int_0^1 \phi(t)\, dt$, partitions the square into a 'left half' $[0, a] \times [0, 1]$ and a 'right half' $[a, 1] \times [0, 1]$. The *generalized baker's transformation* (GBT) B maps the left half into the lower piece and the right half into the upper piece in such a way that:

- Vertical lines in the left (right) half are mapped affinely into vertical 'half lines' under (over) the graph of the cut function ϕ.

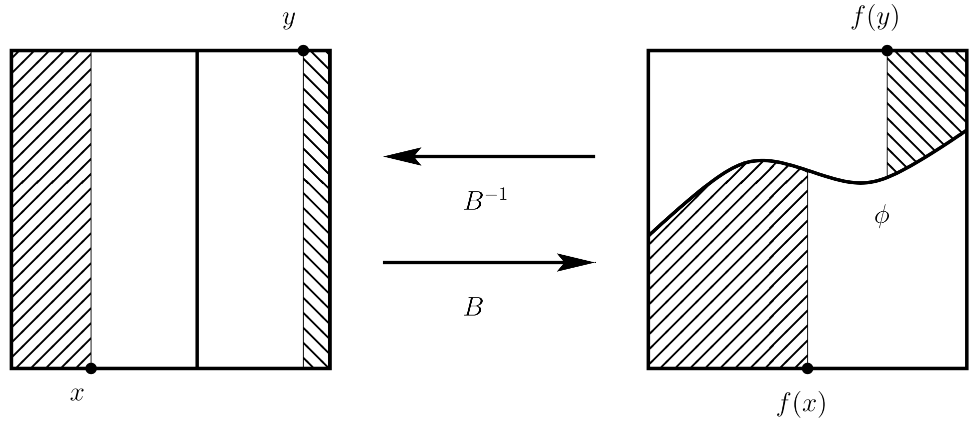

Figure 1: The GBT

- B preserves two-dimensional Lebesgue measure.
- The factor action f of B restricted to vertical lines is (conjugate to) a piecewise monotone increasing, Lebesgue-measure-preserving interval map on $[0,1]$ with two monotonicity pieces $[0,a]$ and $[a,1]$.

The action of a typical GBT is presented in Figure 1.

When $\phi \equiv 1/2$ the map is the classical baker's transformation where the action on vertical lines is an affine contraction and the map f is $x \to 2x \pmod 1$. On the other hand, every measure-preserving transformation T on a (nonatomic, standard, Borel) probability space with entropy satisfying $0 < h(T) < \log 2$ is measurably isomorphic to some generalized baker's transformation on the square S (see [10]).

In order to proceed, we establish some notation. Each GBT B has a skew-product form $B(x,y) = (f(x), g(x,y))$ where

$$g(x,y) = \begin{cases} \phi(f(x))\, y & x \le a, \\ y + \phi(f(x))\,(1-y) & x > a, \end{cases} \text{ and } \begin{cases} x = \int_0^{f(x)} \phi(t)\, dt & x \le a, \\ 1 - x = \int_{f(x)}^1 (1-\phi(t))\, dt & x > a, \end{cases} \qquad (1)$$

defines f implicitly.

Furthermore, by construction the vertical lines $\{x = 0\}$, $\{x = 1\}$ are mapped into themselves by B so that $0, 1$ are fixed points of f. If ϕ is continuous then f is differentiable on both $(0,a)$ and $(a,1)$ and

$$\frac{df}{dx} = \begin{cases} \frac{1}{\phi(f(x))} & x < a \\ \frac{1}{1-\phi(f(x))} & x > a. \end{cases} \qquad (2)$$

Since $\phi(t) \in [0,1]$ for each $t \in [0,1]$, $\frac{df}{dx} \ge 1$, so f is expanding, each branch of f is increasing, and may have infinite derivative at preimages of places where $\phi(t) \in \{0,1\}$. We will call f the **expanding factor** of B. Note also that if ϕ is a *decreasing* function then $\frac{df}{dx}$ is increasing on $(0,a)$ and decreasing on $(a,1)$.

Lemma 1 (Properties of GBTs) *Let f be defined by (1). Each $x \in [0,1]$ has two preimages under f: $x_l < a$ and $x_r > a$ and moreover*

(i) For every $x \in [0,1]$,

$$x_l = \int_0^x \phi(t)\, dt \text{ and } x_r - a = \int_0^x 1 - \phi(t)\, dt = x - x_l; \tag{3}$$

(ii) $\frac{dx_l}{dx} = \phi$ and $\frac{dx_r}{dx} = 1 - \phi$ Lebesgue almost everywhere;

(iii) Lebesgue measure m is f–invariant.

Proof: See appendix. □

A class $\mathcal{C}_\alpha$ of expanding factors of GBTs

Let $\alpha \in (0, \infty)$. Maps $f \in \mathcal{C}_\alpha$ arise as expanding factors of GBTs whose cut functions ϕ satisfy

- ϕ is continuous and decreasing function on $[0,1]$ with $0 \le \phi \le 1$.
- there is a constant c_0 and a C^1 function g_0 on $(0,1)$ such that near $t = 0$

  $$\phi(t) = 1 - c_0\, t^\alpha + g_0(t)$$

  where $\frac{dg_0}{dt} = o(t^{\alpha-1})$;
- either $\phi(1) > 0$ or there are constants $c_1 \in (0,\infty)$, $\alpha' \le \alpha$ and a C^1 function g_1 on $(0,1)$ such that near $t = 1$

  $$\phi(t) = c_1\,(1-t)^{\alpha'} + g_1(1-t)$$

  where $\frac{dg_1}{dt} = o(t^{\alpha'-1})$.

It follows from these conditions that ϕ is C^1 on $(0,1)$ and therefore each $f \in \mathcal{C}_\alpha$ has two piecewise increasing C^2 branches f_l, f_r with respect to the partition into intervals $(0,a)$ and $(a,1)$ (where $a = a(\phi)$). The branch f_l has continuous extension to $[0,a]$ (similarly for f_r and $[a,1]$) and $f_r'(x) \to \infty$ as $x \to a^+$.

Near $x = 0$ each $f \in \mathcal{C}_\alpha$ has the formula

$$f(x) = x + \frac{c_0}{1+\alpha}\, x^{1+\alpha} + o(x^{1+\alpha}),$$

giving an *indifferent fixed point* (IFP) at 0. If $\phi(1) = 0$ then f also has an IFP at $x = 1$, and the order of tangency of the graph of f near 1 is $O((1-t)^{1+\alpha'})$. For maps with IFPs at both 0 and 1 where the order of tangency is higher at 1 than 0, the conjugacy $\phi(t) \mapsto 1 - \phi(1-t)$ will put the higher order tangency at 0. There is thus no loss of generality in assuming that the "most indifferent" point is at $x = 0$. In case $\phi(1) > 0$, equation (2) shows that the fixed point at 1 is hyperbolic.

Example 1 [Alves-Araújo map F]. See [3]. Let $\phi(t) = 1-t$. Then $\alpha = \alpha' = 1 = c_0 = c_1$ and $g_0 = g_1 = 1$. Then

$$f_1(x) := \begin{cases} 1 - \sqrt{1-2\,x} & x < 1/2, \\ \sqrt{2x-1} & x > 1/2. \end{cases}$$

In [23], Rahe established that the map B is Bernoulli (using techniques from [16, 14]). Moreover, $f_1 \in \mathcal{C}_1$ is conjugate (by an affine scaling $[0,1] \to [-1,1]$) to a map presented in [2, 3] which has non-integrable first hyperbolic time. Despite this, f_1 exhibits polynomial decay of correlations for Hölder observables with rate $O(1/n)$ [11]. $\square$

Example 2 [Symmetric case]. Let $\phi(t) = 1 - (2t)^\alpha/2$ for $t \in [0, 1/2]$ and $\phi(t) = 1 - \phi(1-t)$ for $t \in [1/2, 1]$. Then ϕ is symmetric (and $\alpha = \alpha'$); let the expanding factor of the corresponding GBT be denoted by f_α. Then f_α has indifferent fixed points at 0 and 1 with tangency of order $(1+\alpha)$; moreover $\frac{df}{dx} \to \infty$ as $x \to \frac{1}{2}$. In [11] it is shown that Hölder continuous functions have correlation decay rate $O(n^{-1/\alpha})$ under f_α for all $\alpha \in (0,\infty)$. The paper [13] obtains similar results for a conjugate class of maps on $[-1,1]$. Theorem 1 and Theorem 3 below imply that the *first hyperbolic time* is integrable if and only if $\alpha < 1$, showing that the map f_1 emerges as an interesting transition point in the class $\mathcal{C}_\alpha$.

Remark 1 *The class of examples discussed here actually have a rather long history in the mathematical physics literature. For other examples see* [19, 18, 22, 26, 7].

A useful dynamical partition

To analyse $f \in \mathcal{C}_\alpha$ we make a convenient partition of $[0,1]$. First, observe that since f^2 has four 1–1 and onto branches, f admits a period-2 orbit $\{x_0, y_0\}$ where $0 < x_0 < a < y_0 < 1$. Next, for each $n > 0$ let $x_n = f^{-1}(x_{n-1}) \cap (0,a)$ and $x'_n = f^{-1}(x_{n-1}) \cap (a, 1)$. Then

$$0 \cdots < x_{n+1} < x_n < \cdots < x_0 < a \qquad \text{and} \qquad a < \cdots < x'_n < \cdots < x'_2 < y_0 < 1.$$

Defining $y_n = f^{-1}(y_{n-1}) \cap (a,1)$ and $y'_n = f^{-1}(y_{n-1}) \cap (0,a)$ allows a similar partitioning of (x_0, a) and $(y_0, 1)$ (note that $x'_1 = y_0$ and $y'_1 = x_0$). Put

$$J_n = (x_{n+1}, x_n), I_n = (x'_{n+1}, x'_n), J'_n = (y_n, y_{n+1}), I'_n = (y'_n, y'_{n+1}).$$

These intervals partition $[0,1]$ from left to right as

$$0 \cdots J_n \cdots J_0\, x_0\, I'_1 \cdots I'_n \cdots a \cdots I_n \cdots I_1\, y_0\, J'_0 \cdots J'_n \cdots 1$$

with

$$I_k \xrightarrow{f} J_{k-1} \xrightarrow{f} \cdots J_0 \xrightarrow{f} (\cup_l I_l \cup I'_l)$$

(and similarly for the $J'_\cdot, I'_\cdot$ intervals). See Figure 2.

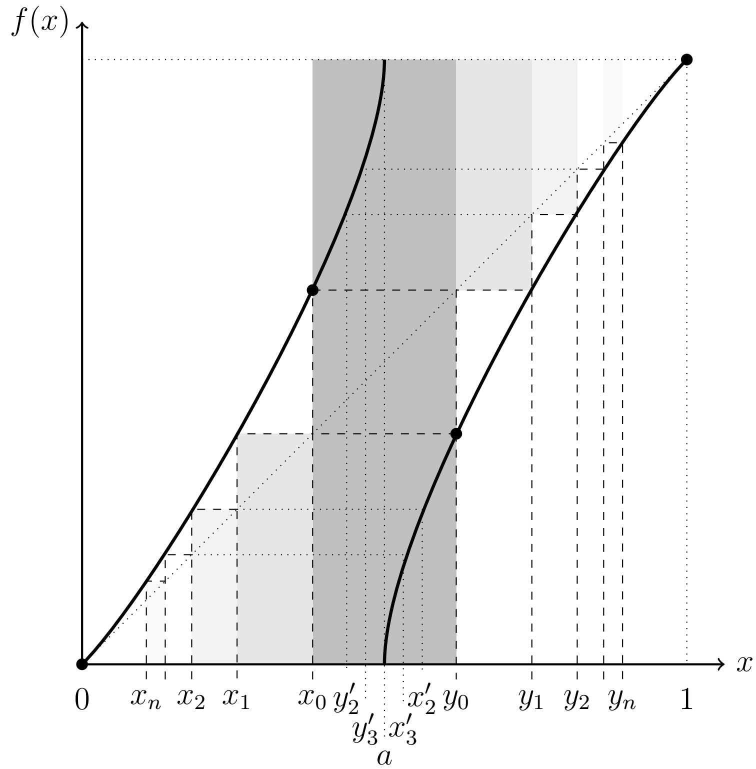

Figure 2: A map f with indifferent fixed points at $0, 1$ and a 2-cycle $\{x_0, y_0\}$.

Lemma 2 *For $f \in \mathcal{C}_\alpha$, and the notation established above.*

(i) $x_n \asymp \left(\frac{1}{n}\right)^{1/\alpha}$;

(ii) $m(J_k) \asymp (\frac{1}{k})^{1+1/\alpha}$;

(iii) *for* $x \in I_k$, $\frac{df}{dx} \asymp k$;

(iv) $m(I_k) \asymp (\frac{1}{k})^{2+1/\alpha}$;

(v) *for* $x \in I_k$, $\operatorname{dist}(x, a) \asymp \left(\frac{1}{k}\right)^{1+\frac{1}{\alpha}}$.

When $\phi(1) = 0$, similar estimates hold for the $\cdot'$ intervals with α replaced by α'.

Proof: For parts (i)–(iv) see [11, Lemma 1]; for part (v) see appendix. □

Remark 2 *When $f \in \mathcal{C}_\alpha$ and $\phi(1) > 0$ the fixed point at 1 is hyperbolic. The corresponding decay rates for $m(I'_k), m(J'_k)$ are exponential, and (iii) does not hold. Some of the estimates and statements below can be modified in this latter case*[3].

[3]In particular, $\{1\}$ is no longer an exceptional point; see the definition in Section 2.

Assumption 1 *For the remainder of the paper we assume that* $\phi(1) = 0$ *so that* f *also has an IFP at 1 with tangency of order* $(1-t)^{1+\alpha'}$ *where* $\alpha' \le \alpha$. *Consequently, the estimates in parts (i), (ii), (iv) and (v) of Lemma 2 reveal decay of the sets* I_k, J_k *which is no faster than for* I'_k, J'_k.

Remark 3 *Below we use a Young-tower built under the first-return time function to the set* $\Delta_0 := \cup_{l=1}^{\infty} I_l \cup I'_l$ *to prove upper bounds on the distribution of first* (σ, δ)*-hyperbolic times* $h_{\sigma,\delta}$ *for certain* (σ, δ). *Indeed, for* $f \in \mathcal{C}_\alpha$ $(\alpha < 1)$, *and any* $\alpha'' \in (\alpha, 1)$, $m\{h_{\sigma,\delta} > n\} = O(n^{-1/\alpha''})$. *The reason for this distribution is that points in* J_l *(whose Lebesgue measure* $\approx l^{-1-1/\alpha}$*) require approximately* l *iterates to achieve enough expansion to be a hyperbolic time. Thus,* $m\{h_{\sigma,\delta} > n\} = \sum_{l>n} m\{h_{\sigma,\delta} = l\} \lesssim l^{\gamma - 1/\alpha}$ *for any* $\gamma > 0$. *Similar bounds are used in [6] to prove decay of correlation results by building a Young tower* Δ' *whose tail set decays in the same way as the distribution of* $h_{\sigma,\delta}$*; the resulting*[4] *decay of correlations are* $O(n^{1-1/\alpha''})$*—close to the typical rate for maps with indifferent fixed points with tangencies of* $O(x^{1+\alpha})$. *Interestingly, direct calculations in [11] where a first-return tower is built over* Δ_0 *give decay of correlations for Hölder observables with rate* $O(n^{-1/\alpha})$ *for maps in* $\mathcal{C}_\alpha$. *The same computations give a CLT for the entire range* $0 < \alpha < 1$. *These improved asymptotics are due to the fact that orbits of* f *experience very rapid expansion when they pass near* a*, giving partial compensation for return from the neighbourhoods of* $\{0, 1\}$*, something that is not accounted for by the hyperbolic times analysis.*

2 Hyperbolic times and sets of exceptional points

Non-uniformity of expansion is expressed relative to a certain (finite) set $\mathcal{S} \subseteq [0, 1]$ of exceptional points. The notation here is precisely as in [4, 3, 2]. Each f is locally C^2 on $[0, 1] \setminus \mathcal{S}$, and must satisfy a non-degeneracy of the following type: there exist constants $B > 1, \beta > 0$ such that such that for *every* $x \in [0, 1] \setminus \mathcal{S}$ we have

$$\frac{1}{B}\mathrm{dist}(x, \mathcal{S})^{\beta} \le \left|\frac{df}{dx}\right| \le B\,\mathrm{dist}(x, \mathcal{S})^{-\beta} \tag{4}$$

and if $y, z \in [0, 1] \setminus \mathcal{S}$ and $|y - z| \le \frac{\mathrm{dist}(z,\mathcal{S})}{2}$ then

$$\left|\log |\tfrac{df}{dx}|\big|_{x=y} - \log |\tfrac{df}{dx}|\big|_{x=z}\right| \le \frac{B}{\mathrm{dist}(z, \mathcal{S})^{\beta}}|y - z|. \tag{5}$$

$\mathrm{dist}(\cdot, \mathcal{S})$ is used to denote the usual Euclidean distance to the set $\mathcal{S}$ (since f is one-dimensional there is no need to impose a separate Lipschitz condition on $(df/dx)^{-1}$).

Lemma 3 *Under the conditions of Assumption 1, for* $f \in \mathcal{C}_\alpha$ *set* $\beta = 1$. *Then there exists a* $B > 1$ *so that* $\mathcal{S} = \{0, a, 1\}$ *satisfies conditions (4) and (5). Hence* $\mathcal{S}$ *is a non-degenerate set of exceptional points for* f.

In case $\phi(1) > 0$ *then only* $x = 0$ *and* $x = a$ *are required to be exceptional points.*

Proof: See appendix. □

[4] And when $0 < \alpha < \frac{1}{2}$, a central limit theorem holds [6, Theorem 2].

Hyperbolic times

Let $\mathcal{S}$ be a non-degenerate exceptional set, $\beta > 0$ as in (4) and (5) and fix $0 < b < \min\{1/2, 1/(4\beta)\}$. Let constants $0 < \sigma < 1$ and $\delta > 0$ be given and define a truncated distance function

$$\mathrm{dist}_\delta(x,\mathcal{S}) := \begin{cases} \mathrm{dist}(x,\mathcal{S}) & \text{if } \mathrm{dist}(x,\mathcal{S}) \le \delta, \\ 1 & \text{if } \mathrm{dist}(x,\mathcal{S}) > \delta. \end{cases} \tag{6}$$

As in [2, 3, 6], n is called a (σ,δ)-hyperbolic time for $x \in [0,1] \setminus \mathcal{S}$ if for all $1 \le l \le n$

$$\prod_{j=n-l}^{n-1} \left| \left(\tfrac{df}{dx} \circ f^j \right)(x))^{-1} \right| \le \sigma^l \qquad \text{and} \qquad \mathrm{dist}_\delta(f^{n-l}(x),\mathcal{S}) \ge \sigma^{bl}. \tag{7}$$

Although orbits escape subexponentially from $\mathcal{S}$ and the rate of growth of derivatives along orbits is not uniform, the essential properties of uniformly expanding maps are captured at hyperbolic times. Since the invariance (and ergodicity [10]) of Lebesgue measure is already at our disposal, establishing the long-run positive density (in time) of hyperbolic times is relatively straightforward.

Lemma 4 *Let $f \in \mathcal{C}_\alpha$. Put $K := -\left(\int_0^a \log(\phi(f(x)))\,dx + \int_a^1 \log(1-\phi(f(x)))\,dx \right)$ where ϕ is the cut function for f. Then for every $\epsilon > 0$ and for every $\sigma \in (e^{-K}, 1)$, there exists a $\delta > 0$ such that for almost every $x \in [0,1]$*

$$\begin{aligned} \lim_{N\to\infty} \tfrac{1}{N} \textstyle\sum_{j=0}^{N-1} \log(\tfrac{df}{dx}(f^j(x))^{-1} \quad & < \log\sigma < 0 \\ \lim_{N\to\infty} \tfrac{1}{N} \textstyle\sum_{j=0}^{N-1} -\log \mathrm{dist}_\delta(f^j(x),\mathcal{S}) \quad & \le \epsilon. \end{aligned} \tag{8}$$

Proof: Apply (Birkhoff's) Ergodic Theorem (for the ergodic system (f, m)) to the function $-\log \frac{df}{dx}$ to obtain the first estimate. For the second estimate, choose $\delta > 0$ such that $\int_0^1 -\log \mathrm{dist}_\delta(x,\mathcal{S})\,dx < \epsilon$ and apply the Ergodic Theorem again. □

This lemma says that for a given choice of b, there is a good choice of (σ,δ) for which there are (many) hyperbolic times for almost every $x \in [0,1] \setminus \mathcal{S}$. The two conditions established in Lemma 4 imply that f is a *non-uniformly expanding map* in the sense of Alves, Bonatti and Viana [4]. It now follows that:

Positive density of hyperbolic times [4, Lemma 5.4] *For every $b > 0$, for every $\sigma \in (e^{-K}, 1)$ there exist $\theta > 0$ and $\delta > 0$ (depending only on f, σ and b) so that for almost every $x \in [0,1]$, for all sufficiently large N there exist (σ,δ)-hyperbolic times $1 \le n_1 < \ldots < n_l \le N$ for x, with $l \ge \theta N$.*

Now that we have established existence of hyperbolic times, we define $h_{\sigma,\delta}(x)$ to be the first $(\sigma,\delta)-$hyperbolic time for x. (If there are no $(\sigma,\delta)-$hyperbolic times for x, set $h_{\sigma,\delta}(x) = \infty$.)

The Young tower partition gives simple lower bounds on $h_{\sigma,\delta}(x)$. Although crude, these lower bounds are still sufficient for our first main result, Theorem 1 below.

Lemma 5 *Let $f \in \mathcal{C}_\alpha$ and fix $\sigma < 1$. Let b and $\delta > 0$ be as above. Let k_1 be minimal such that $\sigma \max_{[0,x_{k_1}]\cup[y_{k_1},1]} \frac{df}{dx} < 1$. Then for $x \in J_k \cup J'_k$ ($k \geq k_1$), $h_{\sigma,\delta}(x) > k - k_1$.*

Proof: Since $\frac{df}{dx}$ is increasing on $(0, a)$ and decreasing on $(a, 1)$ and $\lim_{k\to\infty} \frac{df}{dx}|_{x_k} = 1 = \lim_{k\to\infty} \frac{df}{dx}|_{y_k}$, k_1 is well-defined. Now let $k \geq k_1$, $x \in J_k \cup J'_k$ and fix n with $1 \leq n \leq k - k_1$. For each j with $0 \leq j < n$ we have $f^j(x) \in J_{k-j} \cup J'_{k-j} \subset [0, x_{k_1}) \cup (y_{k_1}, 1]$, so that $\prod_{j=0}^{n-1} |(\frac{df}{dx} \circ f^j(x))^{-1}| > \sigma^n$. Comparing with (7), n cannot be a (σ, δ)-hyperbolic time for x. Since $1 \leq n \leq k - k_1$ was arbitrary, $h_{\sigma,\delta}(x) > k - k_1$. □

Theorem 1 (Lower bounds on hyperbolic times) *Let $f \in \mathcal{C}_\alpha$, with $\sigma < 1$. There is a constant c (depending on σ) such that*

$$m\{h_{\sigma,\delta} \geq n\} \geq c\, n^{-1/\alpha}$$

and $h_{\sigma,\delta}(x)$ fails to be integrable with respect to Lebesgue measure when $\alpha \geq 1$.

Proof: Choose k_1 as in Lemma 5, so $h_{\sigma,\delta}|_{J_{k_1+n}\cup J'_{k_1+n}} > n$. Hence

$$m\{h_{\sigma,\delta} > n\} \geq \sum_{k=k_1+n}^{\infty} m(J_k \cup J'_k) \asymp n^{-1/\alpha}$$

by Lemma 2 (ii). If $\alpha \geq 1$ then

$$\int_0^1 h_{\sigma,\delta}\, dm = \sum_{n=1}^{\infty} m\{h_{\sigma,\delta} \geq n\} = \infty. \qquad \square$$

3 Young towers, large deviations and integrability of the first hyperbolic time when $\alpha < 1$

Suitable choices of (σ, δ) make $h_{\sigma,\delta}$ integrable when $\alpha < 1$. To prove this we distinguish

$$\Delta_0 = [x_0, y_0] = \cup_{j=1}^{\infty} I_j \cup I'_j \qquad (\text{mod } m)$$

as a "good" set, where expansion is very rapid and hyperbolic times are easy to control. The derivative growth condition in (7) is satisfied for $n = 1$ on Δ_0, but for points close to a, the condition on dist_δ fails for $n = 1$. Controlling $h_{\sigma,\delta}$ involves trading expansion with proximity to $\mathcal{S} = \{0, a, 1\}$, and it turns out that getting enough expansion is the difficult part. The idea is to control derivative growth upon successive returns to Δ_0. Long excursions near $\{0, 1\}$ lead to "expansivity deficits" relative to σ^{-n}, with "expansion recovery" by passage through $J_0 \cup J'_0$. This is made quantitatively precise using a large deviations result of Chazottes and Gouëzel [12].

Choice of σ

Choose k_0 such that $\sum_{k\geq k_0} m(J_k \cup J'_k) < m(J_0 \cup J'_0)$. Now choose $\sigma = \sigma(k_0) < 1$ such that

$$\sigma^2 \min_{[x_1,y_1]} \frac{df}{dx} \geq 1 \quad \text{and} \quad \sigma \min_{[x_{k_0},y_{k_0}]} \frac{df}{dx} \geq 1. \tag{9}$$

Define

$$N(x) = \begin{cases} -1 & \text{if } x \in J_0 \cup J'_0, \\ 1 & \text{if } x \in [0, x_{k_0}) \cup (y_{k_0}, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$\int_0^1 N\, dm = \sum_{k=k_0}^{\infty} m(J_k \cup J'_k) - m(J_0 \cup J'_0) < 0. \tag{10}$$

Lemma 6 *Let $f \in \mathcal{C}_\alpha$, $k_0 \geq 1$ and put $H = H(x) := \min\{n \geq 0 \ :\ \sum_{k=0}^{n} N \circ f^k(x) < 0\}$. If $0 < H < \infty$ then $f^H(x) \in J_0 \cup J'_0$ and for all $1 \leq l \leq H$*

$$\prod_{j=H-l}^{H-1} \left|\tfrac{df}{dx} \circ f^j(x)\right|^{-1} = \left|\tfrac{d(f^l)}{dx} \circ f^{H-l}(x)\right|^{-1} \leq \sigma^l.$$

Proof: First, for any $0 \leq n < H$, $\sum_{k=0}^{n} N \circ f^k(x) \geq 0$. Thus

$$N(f^H(x)) = \sum_{k=0}^{H} N \circ f^k(x) - \sum_{k=0}^{H-1} N \circ f^k(x) < 0$$

and hence $N(f^H(x)) = -1$ (since -1 is the only negative value of N). Thus, $f^H(x) \in J_0 \cup J'_0$ and $\sum_{k=0}^{H-1} N \circ f^k(x) = 0$. Therefore

$$\sum_{j=H-l}^{H-1} N \circ f^j = \sum_{k=0}^{H-1} N \circ f^k - \sum_{k=0}^{H-l-1} N \circ f^k \leq 0$$

for each $l \leq H$. To complete the proof, apply (9) to notice that for $x \in J_0 \cup J'_0$, we have $\sigma \frac{df}{dx} > \sigma^{-1}$; if $x \in [0,1] \setminus [x_{k_0}, y_{k_0}]$ then $\frac{df}{dx} \geq 1$ so $\sigma \frac{df}{dx} \geq \sigma$; all other x belong to $[x_{k_0}, y_{k_0}]$ so that $\sigma \frac{df}{dx} \geq 1$. In particular, for each type of x, $\sigma \frac{df}{dx} \geq \sigma^{N(x)}$. Hence

$$\begin{aligned} \prod_{j=H-l}^{H-1} \left|\tfrac{df}{dx} \circ f^j(x)\right|^{-1} &= \sigma^l \prod_{j=H-l}^{H-1} \left|\sigma \tfrac{df}{dx} \circ f^j(x)\right|^{-1} \\ &\leq \sigma^l \prod_{j=H-l}^{H-1} \sigma^{-N\circ f^j(x)} = \sigma^l\, \sigma^{-\sum_{j=H-l}^{H-1} N\circ f^j(x)} \leq \sigma^l. \qquad \square \end{aligned}$$

Choice of δ

Lemma 7 *Let $f \in \mathcal{C}_\alpha$, $\sigma \in (0,1)$ satisfy (9) and let b, k_0 be fixed. Then there is $\delta > 0$ such that whenever $f^n(x) \in J_0 \cup J_0'$ and $1 \le l \le n$,*

$$\mathrm{dist}_\delta(f^{n-l}(x), \{0, a, 1\}) \ge \sigma^{bl}.$$

Proof: Choose k_b such that

$$x \in I_k \cup I_k' \quad (k \ge k_b) \quad \Rightarrow |x - a| \ge \sigma^{b\,k}$$

(note that this is always possible, since there is a constant c such that $|x - a| \ge ck^{-1-1/\alpha}$ for all $x \in I_k^{(\prime)}$). Choose δ small enough that $[a - \delta, a + \delta] \subset \cup_{k \ge k_b}(I_k \cup I_k')$ and $(0, \delta] \cup [1 - \delta, 1) \subset \cup_{k \ge k_b}(J_k \cup J_k')$. Let x be such that $f^n(x) \in J_0 \cup J_0'$. We first show that $\mathrm{dist}_\delta(x, \{0, a, 1\}) \ge \sigma^{bn}$. Either (i) $x \in I_k^{(\prime)}$ for $k \ge k_b$; (ii) $x \in J_k^{(\prime)}$ for $k \ge k_b - 1$; (iii) otherwise. In case (i), for each $j < k$, $f^j(x) \in J_{k-j}^{(\prime)}$. Since $f^n(x) \in J_0^{(\prime)}$ it follows that $n \ge k$. By the choice of k_b and δ,

$$\mathrm{dist}_\delta(x, \{0, a, 1\}) \ge |x - a| \ge \sigma^{b\,k} \ge \sigma^{b\,n}.$$

In case (ii), let $y \in I_{k+1}^{(\prime)}$ be such that $f(y) = x$. Then, since $|\frac{df}{dx}| \ge \sigma^{-1} \ge \sigma^{-b}$ on Δ_0,

$$\mathrm{dist}_\delta(x, \{0, a, 1\}) = \mathrm{dist}(f(y), \{0, 1\}) \ge \sigma^{-b}\mathrm{dist}_\delta(y, a) \ge \sigma^{-b}\sigma^{b(k+1)} \ge \sigma^{bn}.$$

In the final case, $\mathrm{dist}(f^{n-l}(x), \{0, a, 1\}) = 1 > \sigma^{b\,n}$. For $l < n$, $f^l(f^{n-l}(x)) \in J_0 \cup J_0'$ so that the lemma follows from the $l = n$ case using $x' = f^{n-l}(x)$ and $n' = l$. $\square$

Theorem 2 *Let $f \in \mathcal{C}_\alpha, b, k_0$ be fixed, let σ satisfy (9) and choose δ as in Lemma 7. Let H be as defined in Lemma 6. Then $\max\{1, H(x)\}$ is a (σ, δ)–hyperbolic time for x and if $h_{\sigma,\delta}$ is the first (σ, δ)–hyperbolic time then*

$$\int_0^1 h_{\sigma,\delta}(x)\, dx \le \sum_{n=0}^{\infty} m\{x \ : \ H(x) \ge n\}.$$

Proof: If $H = 0$ then $x \in J_0 \cup J_0'$ and 1 is a hyperbolic time. Otherwise, comparing (7) with Lemmas 6 and 7 shows that $H(x)$ is a hyperbolic time. For the integral,

$$\int_0^1 h_{\sigma,\delta}(x)\, dx \le \int_0^1 (1 + H(x))\, dx = \sum_{n=0}^{\infty} m\{x \ : \ H(x) \ge n\}.$$

$\square$

Large deviations on a Young tower

Let $\Delta_0 = [x_0, y_0]$, and partition (modulo sets of measure 0) according to $\Delta_{0,n} = I_n \cup I_n'$ for $n > 0$. For $x \in \Delta_{0,n}$,

$$R(x) := \min\{k > 0 \ : \ f^k(x) \in \Delta_0\} = n + 1$$

so we put

$$\Delta := \cup_{\ell\le n=0}^{\infty}(\Delta_{0,n}\times\{\ell\})\subset\Delta_0\times\mathbb{Z}_+$$

and equip Δ with the measure m_Δ obtained by direct upwards translation of $m|_{\Delta_0}$. The tower map $F:\Delta\circlearrowleft$ is defined in the usual manner: $F(x,\ell)=(x,\ell+1)$ for $\ell<R(x)-1=n$ and $F(x,R(x)-1)=(f^{R(x)}(x),0)$. It is easy to check that

$$m\{R\ge n\}=\sum_{k>n}m(I_k\cup I_k')\asymp n^{-1-1/\alpha} \tag{11}$$

so that $\sum_{k\ge n}m\{R\ge k\}\asymp n^{-1/\alpha}$. Standard arguments[5] show that branches of the map f^R have uniformly bounded distortion on Δ_0, and since f^R is uniformly expanding on Δ_0, the tower map F satisfies the usual regularity conditions [25, 21, 11] for maps on a Young tower[6]. Since Lebesgue measure m is invariant for f, $m|_{\Delta_0}\circ(f^R)^{-1}=m|_{\Delta_0}$ and hence m_Δ is actually F-invariant on Δ. As is usual, $(f,[0,1],m)$ arises as a factor of (F,Δ,m_Δ) via the semi-conjugacy $\Phi(x,\ell)=f^\ell(x)$. Then $\Phi\circ F=f\circ\Phi$, and $m=\Phi_*m_\Delta=m_\Delta\circ\Phi^{-1}$.

Next, lift H to the tower: put $\hat N(x,\ell):=N\circ\Phi(x,\ell)=N(f^\ell(x))$ and

$$\hat H := H\circ\Phi=\min\left\{n\ge 0\ :\ \sum_{k=0}^{n}\hat N\circ F^k<0\right\}.$$

Let

$$\bar N=\int_\Delta \hat N\,dm_\Delta=\int_\Delta N\circ\Phi\,dm_\Delta=\int_0^1 Nd(\Phi_*m_\Delta)=\int_0^1 N\,dm.$$

Note that $\bar N<0$ by (10). Put $\psi:=\hat N-\bar N$. Then ψ has zero mean, and belongs to every Hölder class $C_\beta(\Delta)$ with $\mathrm{Lip}(\psi)=2$ (since it is piecewise constant with respect to the tower partition (see [25, 12, 11]) and $|\psi(x)-\psi(x')|=|\hat N(x)-\hat N(x')|\le 2$).

Lemma 8 (Large Deviations Estimate) *Let $0<\alpha<1$. For every $\epsilon>0$ there is a constant c (not independent of ϵ) such that*

$$m_\Delta\left\{y\in\Delta\ :\ |\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1}\psi\circ F^k(y)|\ge\epsilon\right\}\le c\,n^{-1/\alpha}.$$

Proof: Define $K(x_0,\dots,x_{n-1}):=\sum_{j=0}^{n-1}\psi(x_j)$ and choose $t:=\epsilon\,n$. Since $m\{y\in\Delta_0\ :\ R(y)>n\}\asymp n^{-1-1/\alpha}$, the tail of the Young tower is weak–L^q in the sense of Chazottes and Gouëzel [12, p866] with $q:=1+1/\alpha>2$. Applying [12, Theorem 6.1], there is a constant $C<\infty$ such that

$$m_\Delta\left\{y\in\Delta\ :\ |\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1}\psi\circ F^k(y)|\ge\epsilon\right\}\le C\,(\epsilon\,n)^{-2(q-1)}(n\,\mathrm{Lip}(\psi)^2)^{q-1}. \qquad\square$$

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[5]See Lemma 3 in [11] for example.

[6]If JF denotes the Jacobian $\frac{dm_\Delta\circ F}{dm_\Delta}$ then $\log|JF|\in C_\beta(\Delta)$, where $C_\beta(\Delta)$ is the class of β–Hölder functions, defined with respect to the usual [25] separation time s.

Remark 4 *Using a result of Melbourne and Nicol [21, Theorem 3.1], a weaker version of Lemma 8 can be obtained where the exponent $-1/\alpha$ in the conclusion is replaced by $-1/\alpha''$ for any $\alpha'' > \alpha$ (and $\epsilon < \bar{N}$). This estimate is sufficient to prove the integrability part of our main result (Theorem 3), albiet with a weaker tail estimate for the first hyperbolic time. We are grateful to an anonymous referee who pointed out the stronger estimates that are obtained from Chazottes and Gouëzel's result.*

The large deviations estimate is enough to prove our main theorem:

Theorem 3 *Let $f \in \mathcal{C}_\alpha$. Then the first hyperbolic time for f is integrable if and only if $0 < \alpha < 1$. Specifically, let $\alpha < 1$ and $b < 1$ be fixed. There is a choice of (σ, δ) where $0 < \sigma < 1$ and $\delta > 0$ such that $m\{h_{\sigma,\delta} > n\} \asymp n^{-1/\alpha}$.*

Proof: Theorem 1 deals with the case $\alpha \geq 1$ and provides the lower tail estimate for $h_{\sigma,\delta}$. For $\alpha < 1$,

$$\int h_{\sigma,\delta}\, dm = \sum_{n=0}^{\infty} m\{h_{\sigma,\delta} > n\} < \infty$$

because of the tail distribution of $h_{\sigma,\delta}$, which is established as follows. Let k_0 be large enough that $\bar{N}$ is negative and choose σ to satisfy (9) and δ as in Lemma 7. Then $h_{\sigma,\delta} \leq H + 1$ (compare with Theorem 2). Then

$$m\{h_{\sigma,\delta} > n\} \leq m\{H \geq n\} = \Phi_* m_\Delta\{H \geq n\} = m_\Delta\{H \circ \Phi \geq n\} = m_\Delta\{\hat{H} \geq n\}. \quad (12)$$

For $y \in \{\hat{H} \geq n\}$, $\sum_{k=0}^{n-1} \hat{N} \circ F^k(y) \geq 0$ so that

$$\frac{1}{n}\sum_{k=0}^{n-1} \psi \circ F^k(y) = \frac{1}{n}\sum_{k=0}^{n-1}(\hat{N} \circ F^k(y) - \bar{N}) \geq -\bar{N}.$$

Thus,

$$\{\hat{H} \geq n\} \subseteq \left\{y \in \Delta \;:\; \left|\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1}\psi \circ F^k(y)\right| \geq -\bar{N}\right\}. \quad (13)$$

By Lemma 8, there is a constant $c < \infty$ such that the set on the RHS of (13) has m_Δ–measure bounded by $c\, n^{-\frac{1}{\alpha}}$. □

4 Conclusions

In this paper we have undertaken a detailed study of the asymptotics of *hyperbolic times* for a parameterized family $f_\alpha \in \mathcal{C}_\alpha$, $0 < \alpha < \infty$ of non-uniformly expanding, Lebesgue-measure-preserving maps of the interval. These one-dimensional maps arise naturally as the expanding factors of a class of two-dimensional *generalized baker's transformations* on the unit square.

A central result in the literature, Alves, Bonatti and Viana [4], proves that if a non-uniformly hyperbolic map has the property that almost every x has a (uniform) positive frequency of hyperbolic times, then it admits an absolutely continuous invariant measure.

According to Lemma 4, each of our maps f_α has this property; of course the resulting invariant measure is already known to be Lebesgue.

Therefore our main result concerns the statistics of *first hyperbolic times* h_α for our maps f_α and in particular how the quantities $m\{x : h_\alpha(x) > n\}$ depend on n and α. We show that this is entirely determined by the strength of the (most indifferent) fixed point for the map f_α. In particular,

- h_α is integrable if and only if $0 < \alpha < 1$, corresponding to relatively fast polynomial escape from the indifferent fixed points at $x = 0, 1$ within the range of our family of maps $\mathcal{C}_\alpha$, $0 < \alpha < \infty$ (Theorems 1 and 3).
- For $0 < \alpha < 1$ we establish $m\{h_\alpha > n\} = O(n^{-\frac{1}{\alpha}})$ which implies, by using results from [5, 6], correlation decay for Hölder observables[7] at rate $O(n^{-\frac{1}{\alpha}+1})$.
- Finally, a CLT holds in case $0 < \alpha < 1/2$, again by applying the results of [5, 6].

The conclusions which are obtained in [5, 6] depend on the construction of a suitable Markov or Young tower via first hyperbolic times and their statistics and the analysis of the return times on the resulting tower.

On the other hand, [11] details a specific construction of a Young tower for the maps f_α and proves an improved correlation decay rate of $O(n^{-\frac{1}{\alpha}})$ for all $0 < \alpha < \infty$. It is shown that this correlation decay rate is sharp for Hölder data. The estimates in [11] also imply a CLT for $0 < \alpha < 1$. See Remark 3 for further discussion.

We interpret this as follows. The analysis via hyperbolic times provides a relatively general approach to analysis of nonuniformly hyperbolic systems that leads to a particular Young tower construction related to the hyperbolic times. It is not so surprising that this approach does not always yield optimal results such as sharp estimates on decay of correlation rates or the CLT. In the case of our maps f_α, for example, a dedicated tower construction in [11] can produce optimal results in the form of sharp estimates on correlation decay by bypassing the intermediate construction of hyperbolic times.

5 Appendix

Proof of Lemma 1: (i) The first equation is immediate from Equation (1). For the second, again use Equation (1) and write

$$x_r = 1 - \int_x^1 (1 - \phi(t))\, dt = x + \int_x^1 \phi(t)\, dt = x + a - \int_0^x \phi(t) dt = a + \int_0^x 1 - \phi(t)\, dt$$

and so

$$x_r - a = \int_0^x 1 - \phi(t)\, dt = x - x_l.$$

(ii) Differentiating the expressions in (i) via Lebesgue's theorem gives:

$$\frac{dx_l}{dx} = \phi \text{ and } \frac{dx_r}{dx} = 1 - \phi.$$

[7] See, for example Theorem 3 of [5].

(iii) Apply (i) and (ii):

$$m(T^{-1}[0,x]) = \int_0^{x_l} \mathbf{1}dt + \int_a^{x_r} \mathbf{1}dt = \int_0^x \frac{dx_l}{dx}dx + \int_0^x \frac{dx_r}{dx}\,dx = \int_0^x \mathbf{1}dx = m[0,x]. \qquad \square$$

Proof of Lemma 2 (v): For this part fix k and $x \in I_k$. Since $x_n' \to a$ as $n \to \infty$,

$$\mathrm{dist}(x,a) = x - a = x - x_{k+1} + \sum_{j=k+1}^{\infty}(x_j - x_{j+1}) = a - x_{k+1} + \sum_{j=k+1}^{\infty} m(I_j).$$

In fact this argument shows that $\sum_{j=k+1}^{\infty} m(I_j) \le \mathrm{dist}(x,a) \le \sum_{j=k}^{\infty} m(I_j)$ and both sides are $\asymp \sum_{j=k}^{\infty} j^{-2-1/\alpha} \asymp k^{-1-1/\alpha}$. $\square$

Proof of Lemma 3: First, note that $df/dx = |df/dx| \ge 1$, and $\mathrm{dist}(z,\mathcal{S}) < \max\{a, 1-a\}/2$ for every z, so for the lower bound in (4) holds whenever $B > B_\beta := ((\max\{a,1-a\}/2)^{-\beta}$. We will establish the right hand side inequality after proving (5) For that, it suffices to work on $(0,a)$, since the other interval is similar. By (2), $\frac{df}{dx} = 1/\phi \circ f$, and hence

$$\frac{d}{dx}\log\frac{df}{dx} = -\left(\frac{d\phi/dx}{\phi^2}\right)\circ f.$$

If $t \in J_k$ $(k > 0)$ then $f(t) \in J_{k-1}$ so $f(t) \asymp k^{-1/\alpha}$ and (since ϕ is decreasing), $1 \ge \phi(f(t)) \ge \phi(x_0)$. Moreover, $\frac{d\phi}{dx}|_{x=f(t)} = -\alpha\, c_0(f(t))^{\alpha-1} + o((f(t))^{\alpha-1}) \asymp -k^{1/\alpha-1}$. Hence $\left|\frac{d}{dx}\log\frac{df}{dx}\right| \asymp k^{1/\alpha-1}$. If $t \in J_0$ then $f(t) \in [x_0, y_0]$ and $\phi(f(t)) \asymp 1$ (since ϕ is decreasing and C^1). If $t \in I_k'$ then $f(t) \in J_{k-1}'$ so $1 - f(t) \asymp k^{-1/\alpha'}$, $\phi(f(t)) \asymp k^{-1}$ and $\frac{d\phi}{dx}|_{x=f(t)} \asymp k^{1/\alpha'-1}$. Hence

$$\left|\frac{d}{dx}\log\frac{df}{dx}\right|(t) \asymp \begin{cases} 1 & t \in J_0, \\ k^{1/\alpha-1} & t \in J_k (k>0), \\ k^{1/\alpha'+1} & t \in I_k', \end{cases} \asymp \begin{cases} 1 & t \in J_0, \\ \mathrm{dist}(t,\mathcal{S})^{-(1-\alpha)} & t \in J_k(k>0), \\ \mathrm{dist}(t,\mathcal{S})^{-1} & t \in I_k'. \end{cases}$$

(since $\mathrm{dist}(t,\mathcal{S})|_{J_k} \asymp \mathrm{dist}(t,0)|_{J_k} \asymp k^{-1/\alpha}$ and $\mathrm{dist}(t,\mathcal{S})|_{I_k'} \asymp \mathrm{dist}(t,a)|_{I_k'} \asymp k^{-1/\alpha'-1}$ by Lemma 2). For any $\beta \ge 1$ there is a constant B_0 such that

$$\left|\frac{d}{dx}\log\frac{df}{dx}\right|(t) \le B_0\,\mathrm{dist}(t,\mathcal{S})^{-\beta}. \tag{14}$$

To complete the proof of (5) let $|y - z| < \frac{\mathrm{dist}(z,\mathcal{S})}{2}$. Then the mean value theorem gives a t between y, z such that

$$\log\left|\frac{df/dx|_{x=y}}{df/dx|_{x=z}}\right| = \left|\frac{d}{dx}\log\frac{df}{dx}\right|(t) \times |y-z| \le \frac{B_0}{\mathrm{dist}(t,\mathcal{S})^\beta}\,|y-z|$$

(using 14)). But since $|t - z| \le |z - y| \le \frac{\mathrm{dist}(z,\mathcal{S})}{2}$, $\mathrm{dist}(t,\mathcal{S}) \ge \mathrm{dist}(z,\mathcal{S})/2$ so choosing $B = B_0\,2^\beta$ completes the regularity estimate. A similar argument estimating $\frac{df}{dx} = 1/\phi \circ f$ on the interval $[x_0, a)$ gives

$$|df/dx| = O\left(\mathrm{dist}(x,\mathcal{S})^{-\alpha'/(1+\alpha')}\right),$$

giving the upper bound in (4) for any $\beta \ge 1$. $\square$

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