THREE LINEAR PRESERVER PROBLEMS

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Abstract. Linear preserver problems are questions about characterising linear maps on spaces of matrices or spaces of operators (or more generally on rings or algebras) that preserve certain properties. We present an exposition of three such problems on preserving invertibility or commutativity or rank one.

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INTRODUCTION.

What came to be called the "linear preserver problems" are questions on characterising linear maps on spaces of matrices or spaces of operators (or more generally on algebras) that preserve certain properties. There has been a great deal of research in this area, especially on spaces of matrices, with results dating back to 1897 (see Theorem 0 below). We refer the reader to the expository articles [LT1, LT2]. There has been also some research activity for maps on Banach algebras, algebras of operators, abstract rings, ... etc. Possibly the earliest result on this subject is Frobenius' characterization, in 1897, of determinant preserving linear maps which we state presently. The transpose of matrix x is denoted by x^t.

THEOREM 0 (Frobenius [Fr]) *Let ϕ be a determinant preserving map on the space of all (real or complex) $n \times n$ matrices, i.e.,* $\det\phi(a) = \det(a)$ *for every matrix a, then there exists invertible matrices b and c with* $\det(bc) = 1$, *such that either $\phi(a) = b\,x\,c$ for every x or $\phi(a) = b\,x\,c$ for every x.*

Three of the most appealing linear preserver problems, in my view, are *invertibility preservers,* because of its connection with algebra isomorphisms and Jordan isomorphisms, *commutativity preservers,* because of its connection with Lie isomorphisms and *rank one preservers,* because many others preserver problems are reduced to it. In this expository article, we will concentrate on these three problems. The discussion that follow will be far from encyclopedic, and the emphasis will reflect the author's experience.

1. INVERTIBILITY PRESERVING MAPS AND JORDAN ISOMORPHISMS

Let A and B be algebras with identity. A linear map ϕ from A to B, is called *unital* if $\phi(1) = 1$ and is called *invertibility preserving* if $\phi(a)$ is invertible in B for every invertible element $a \in A$. It is called an *anti-homomorphism* if $\phi(ab) = \phi(b)\phi(a)$ for every a and $b \in A$, and a *Jordan homomorphism* if $\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ for all a and $b \in A$, or equivalently $\phi(a^2) = \big(\phi(a)\big)^2$ for every $a \in A$. As usual, a bijective anti-homomorphism (respectively Jordan homomorphism) is called an anti-isomorphism (respectively a Jordan isomorphism). It is quite obvious that every isomorphism and every anti-isomorphism is a Jordan isomorphism. It is also quite obvious that every isomorphism and every anti-isomorphism is unital and preserves invertibility. Indeed [So], every Jordan isomorphism is also unital and preserves invertibility. One may ask whether the converse is true. Is every unital invertibility preserving map a Jordan isomorphism. Such questions were raised by Kaplansky [K] in the context of rings and of Banach algebras. It is easily seen (see Example 1.n below) that the answer to such a sweeping question is negative, and then we may ask about sufficient conditions on the algebras to allow us to make such a conclusion.

The earliest result along these lines is that of Dieudonné [D] for maps on the space M_n of all $n \times n$ matrices. (The article [D] deals with bijective "semi-linear" maps that preserve non-invertibility.) Another related result is in [MP] dealing with not necessarily bijective maps on the space of (real or complex) matrices that preserve invertibility. The results is that such maps are of the form given in Theorem 0.1, with no restriction on $\det(bc)$. In particular, if the map is also unital, then $bc = 1$ and the map is indeed a Jordan isomorphism. Infinite dimensional generalisations of this result appeared in [JS] for spectrum preserving maps and in [So] for invertibility preserving maps. We state these results here. We denote the algebra of all bounded linear operators on a Banach space X by $\mathcal{L}(X)$. The dual of X is denoted by X' and the adjoint of an opertor T is denoted by T^*.

THEOREM 1.1. ([JS], [S]) *Let X and Y be Banach spaces over the complex field and let ϕ be a unital bijective linear map from $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$. Then the following conditions are equivalent.*

(a) *ϕ preserves invertibility.*

(b) *ϕ is a Jordan isomorphism.*

(c) *ϕ is either an isomorphism or an anti-isomorphism.*

(d) *Either*

(i) *$\phi(T) = A^{-1} T A$ for every $T \in \mathcal{L}(X)$, where $A : Y \to X$ is an isomorphism;*

or

(ii) *$\phi(T) = B^{-1} T^* B$ for every $T \in \mathcal{L}(X)$, where $B : Y \to X'$ is an isomorphism.*

In particular such maps are automatically continuous in any of the usual topologies on $\mathcal{L}(X)$.

We should point out that the equivalence of (b) and (c) is true for any additive map from a ring onto a prime ring [H1, pp. 47-51]. and that the equivalence of (c) and (d) include a classical theorem asserting that every automorphism of $\mathcal{L}(X)$ is inner, i.e., of the form $x \mapsto a^{-1} x a$. Furthermore, the case of a nonunital map ϕ can be reduced to the unital case by considering the map ψ defined by $\psi(x) = \phi(1)^{-1} \phi(x)$. We state the conclusion formally. Consequently such a map takes one of the forms $\phi(T) = A T B$ or $\phi(T) = A T^* B$ for invertible operators A and B between the relevant spaces.

The proof of Theorem 1.1 in [S] and the related result in [JS] proceed by first characterising rank one operators in terms of the spectrum. This implies that an invertibility preserving map preserves the property of having rank one. The spectrum of an element a is denoted by $\operatorname{spec}(a)$.

THEOREM 1.2. ([JS], [So]) *For an operator $R \in \mathcal{L}(X)$, the following conditions are equivalent:*

(i) rank $R \leq 1$

(ii) *For every $T \in \mathcal{L}(X)$ and every distinct scalars α and β,*

$$\operatorname{spec}(T + \alpha R) \cap \operatorname{spec}(T + \beta R) \subseteq \operatorname{spec}(T).$$

(iii) For every $T \in \mathcal{L}(X)$, there exists a compact subset K_T of the complex plane, such that

$$\operatorname{spec}(T + \alpha R) \cap \operatorname{spec}(T + \beta R) \subseteq K_T.$$

In a different direction, results of Gleason [G] and Kahane-Zelazko [KZ], refined by Zelazko [Z] show that every unital invertibility preserving linear map from a Banach algebra A into a semi-simple commutative Banach algebra B is multiplicative. (See also [RS]). Additional related results are in [Au], [CHNRR], and [Ru]. Articles [CHNRR] and [Ru] contain similar results on invertibility preserving positive linear maps on C^*-algebras and von-Neumann algebras respectively.

The commutativity assumption in [G] and [KZ] is quite crucial. It would be a major advance if the conclusion holds for noncommutative algebras. More precisely, we pose this question.

Question. Let A be a semi-simple Banach algebra and let ϕ be a unital bijective linear map on A. If ϕ preserves invertibility, must it be a Jordan isomorphism?

Aupetit [Au2] has recently announced a proof when A is a von-Neumann algebra. Perhaps the next step is to prove the result for C^*-algebras.

We close this section by a counterexample. Another example may be found in [Au1; p.28].

EXAMPLE. Let A be the algebra of 4×4 matrices of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, where A, B, C are 2×2 matrices, and let

$$\phi \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} A & B^t \\ 0 & C \end{bmatrix}$$

It is straightforward to verify that ϕ is unital and preserves invertibility, but that it is not a Jordan homomorphism.

Other examples may be constructed by taking A to be a radical algebra with identity adjoined and ϕ a bijective unital linear mapping sending the radical to itself.

2. COMMUTATIVITY PRESERVING MAPS AND LIE ISOMORPHISMS

A linear map φ from an algebra A to an algebra B is said to be *commutativity preserving* if $\varphi(a)$ commutes with $\varphi(b)$ for every pair of commuting elements a, b in A. It is said to be *commutativity preserving in both directions* when the condition $ab = ba$ holds if and only if $\varphi(a)\varphi(b) = \varphi(b)\varphi(a)$. It is called a *Lie homomorphism* if $\varphi([a, b]) = [\varphi(a), \varphi(b)]$, where $[x, y]$ denotes $xy - yx$. The terminology is justifiable by the fact that under this operation $(a, b) \mapsto [a, b]$, the algebra A becomes a Lie algebra. In fact, it is a standard result (see, for eg. [Hu, Chapter V]) that every Lie algebra } may be embedded as a Lie subalgebra of an associate algebra – the universal enveloping algebra of } – equipped with the product $[a, b]$. As usual, a Lie isomorphism is a bijective Lie homomorphism.

We note that if α is an (associative) isomorphism or the negative of an anti-isomorphism from A to B and γ is a linear map from A into the centre of B, such that $\gamma(ab - ba) = 0$ for every a and b in A, then $\alpha + \gamma$ is a Lie isomorphism, provided it is injective. We may ask for sufficient conditions on algebras A and B for the converse to hold.

Evidently every Lie isomorphism φ between algebras A and B preserves commutativity in both directions, as does the map $c\varphi+\tau$ for a non-zero scalar c and a linear map τ from A into the centre of B. Again we may ask for sufficient conditions on algebras A and B for the converse to hold.

Commutativity preserving linear maps on spaces of matrices or that preserve commutativity in both directions. As in several other algebras, the linear maps that preserve zero Lie brackets in both directions differ only slightly from those that preserve all Lie brackets.

Lie isomorphisms between rings and between self-adjoint operator algebras have been considered by several authors, see [Ma1], [Ma2], [Mi], [Br]. Quite frequently, the Lie isomorphisms are closely related to the associative isomorphisms are described above. Commutativity preserving linear maps on spaces of matrices or operators have also received a great deal of attention. They have been considered by [W] and [PW] for the algebra M_n, in [CL] for the space of symmetric matrices, in [CJR] for the space of self adjoint operators on Hilbert space, in [O] for the algebra $B(X)$ of all bounded linear operators on a Banach space X, and in [BM] for von-Neumann algebras. In the majority of cases, it is shown that under some injectivity or surjectivity conditions, commutativity preserving maps (possibly in both directions) takes one of two forms

$$T \mapsto cA^{-1}TA + f(T)I$$

or

$$T \mapsto cA^{-1}T^{\dagger}A + f(T)I$$

where c is a scalar, $T^{\dagger}$ is either the adjoint or the transpose or some other anti-isomorphism (depending on the space considered), and where A is an invertible operator (perhaps a unitary), and f a linear functional. Consequently, the results may be stated as showing that every such a map is a linear combination of a Lie isomorphism and a map with central range. recently algebras of triangular or block-triangular matrices and their infinite dimensional generalisations have received a lot of attention. In the remainder of this section, we will discuss results on commutativity preservers and on Lie isomorphisms for some such algebras. Let $T_n(F)$ denote the algebra of upper triangular n by n matrices over an arbitrary field F. The "transpose" of an $n \times n$ matrix A with respect to the "anti-diagonal", i.e., the diagonal that includes the positions $(j, n-j)$ is denoted by T^{+}. It is easy to see that the mapping $T \mapsto T^{+}$ is an anti-isomorphism. Indeed it a composition of the usual transpose and an inner automorphism induced by the matrix $J := [\delta_{i,n-i}]$, where $\delta_{i,j}$ is the Kronecker delta symbol.

THEOREM 2.1. [MS] *Let F be an arbitrary field and φ a linear map from $T_n(F)$, the algebra of upper triangular matrices, into itself. Assume that $n \geq 3$. The following conditions are equivalent.*

(a) *φ preserves commutativity in both directions.*

(b) *There exists a non-zero scalar $c \in F$, a linear functional f on T_n and an invertible matrix $S \in T_n$ such that φ takes one of the following forms.*

$$\varphi(T) = cS^{-1}TS + f(T)I$$

or

$$\varphi(T) = cS^{-1}T^{+}S + f(T)I$$

(c) *There exists a Lie isomorphism α of $T_n(F)$ a non-zero scalar $c \in F$, and a linear mapping f from $T_n(F)$ into its centre such that $\varphi = c\alpha + f$.*

The above result is false for $n = 2$.

COROLLARY 2.2 *Let $\varphi : T_n(F) \to T_n(F)$ be a linear map. The following are equivalent. Then φ is a Lie automorphism of $T_n(F)$ if and only if φ takes one of the following forms:*

$$\varphi(T) = S^{-1}TS + \mathrm{tr}\,(TD)I,$$

or

$$\varphi(T) = -S^{-1}T^{+}JS + \mathrm{tr}\,(TD)I,$$

where $S \in T_n(F)$ is invertible, tr denotes the trace and D is a diagonal matrix with tr $(D) \neq -1$.

We note that the result above implies that every Lie isomorphism $\varphi = \alpha + \tau$, where α is either an automorphism or the negative of an anti-automorphism and τ maps $T_n(F)$ into its centre and annihilates all commutators $AB - BA$. The above corollary had also been proven in [Do] for matrices over rings in which the only idempotents are 0 and 1.

The above result about Lie isomorphisms have been extended in [MS2] to one of the best known infinite dimensional generalisations of T_n, nest algebras on a Hilbert space. In order to avoid the technical difficulties that usually surround this subject, we will only present an example. Let H be a Hilbert Space with an orthonormal basis $\mathcal{B} = \{e_n : n \in \mathbf{Z}\}$, and let T_∞ be the algebra of all bounded linear operators on H whose matrix with respect to the basis above is an upper triangular (doubly infinite) matrix. The *bilateral shift* is the operator S on H determined by $Se_n = e_{n+1}$. Consisting with previous notations, if an operator C is determined by the matrix $[c_ij]$ relative to the basis $\mathcal{B}$, then C^{+} will denote the operator whose matrix with respect to the same basis is $[c_{-j,-i}]$.

THEOREM 2.3. *Let $\varphi : T_\infty \to T_\infty$ be a linear map. Then φ is a Lie automorphism of T_∞ if and only if φ takes one of the following forms:*

$$\varphi(T) = S^{-n}A^{-1}TAS^{n} + \tau(T)I,$$

or

$$\varphi(T) = -S^{-n}A^{-1}T^{+}AS^{n} + \tau(T)I,$$

where A is an invertible element of T_∞, S is the bilateral shift, n is an integer, and τ is a linear functional on T_∞ that annihilates all commutators.

Also in the finite dimensional spaces, the following result about block triangular algebras was proved in [MS2]. We start with a definition.

For every finite sequence of positive integers $n_1, n_2, \dots n_k$, satisfying $n_1 + n_2 + \dots + n_k = n$, we associate an algebra $\mathcal{T}(n_1, n_2, \dots n_k)$ consisting of all $n \times n$ matrices of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ 0 & A_{22} & \dots & A_{2k} \\ \vdots & & & \\ 0 & 0 & \dots & A_{kk} \end{bmatrix}$$

where A_{ij} is an $n_i \times n_j$ matrix. We call such an algebra a *block upper triangular algebra.*

THEOREM 2.4. *Let $\mathcal{A} = \mathcal{T}(n_1, n_2, \dots n_r)$ and Let $\mathcal{B} = \mathcal{T}(m_1, m_2, \dots m_s)$ be block upper triangular algebras in M_n and M_m respectively, and let φ be a Lie isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then $m = n$, $r = s$ and there exists an invertible matrix $B \in \mathcal{B}$ and a linear functional τ on $\mathcal{A}$ satisfying $\tau(I) \neq -1$ such that either*
(a) $n_i = m_i$ *and* $\varphi(T) = B^{-1}TB + \tau(T)I$, *or*
(b) $n_i = m_{r-i}$ *and* $\varphi(T) = B^{-1}T^{+}B + \tau(T)I$.

The mapping τ is given by $\tau(T) = \mathrm{tr}\,(TD)$, where D is a diagonal matrix such that tr $(D) \neq -1$ and the diagonal entries in every one of the blocks that determine $\mathcal{A}$ are identical.

3. RANK ONE PRESERVING MAPS

A map φ from a space $\mathcal{S}_1$ of matrices into a space $\mathcal{S}_2$ of matrices is said to *preserve matrices of rank one* if $\varphi(T)$ is of rank one whenever T has rank one. It is said *preserve rank one matrices in both directions* when $\varphi(T)$ is of rank one if and only if T has rank one.

Characterizing linear maps on spaces of matrices or operators that preserve rank 0ne operators has been an active area of research for quite a while. Of all linear preserver problems, this is arguably the most basic. Indeed several other questions about preservers may be reduced to, or solved with the help of, rank-one preservers. This has already been noted in §1 above and has been observed long ago in [Ma]. In addition to maps that preserve invertibility or spectrum as discussed in §1, preserving commutativity ([CL], [Ra], [W]) quite often involve rank-one preservers. Classifying isomorphisms of several types of operator algebras is frequently accomplished by exploiting the fact that they preserve rank one operators; see, e.g. [Da; Chapter 17].

Although the forms of rank one preservers are very similar to the forms of other preservers disacussed in the previous sections, we describe them slightly differently. By a *left multiplication* on an algebra A we mean a mapping L_a defined by $L_a(x) = ax$, for every $x \in A$, where a is an element of A. *Right multiplications* R_a are defined analogously.

The linear rank one preservers on the space of all $n \times n$ matrices was characterized by Marcus and Moyls [11]. They show that every such map is a composition of a left multiplication L_A by an invertible matrix A, a right multiplication R_B by an invertible matrix B, and possibly the transpose map. For related results, and a summary of similar results obtained from 1960 until 1989, we refer to [Lo] and the references therein.

In this section, we discuss more recent results about additive (not necessarily linear) maps that preserve rank one especially on triangular matrix algebras. In [OS2], Omladic and Semrl characterized surjective additive maps on the space of finite rank operators on real or complex Banach spaces. In case of finite dimensional spaces, they show that every such a map is a composition of the three types of maps described above and a fourth type induced by an automorphism of the underlying field, which we describe presently.

Assume that $c \mapsto \tilde{c}$ is an automorphism of the underlying field F, and $C = [c_{ij}] \in M_{mn}(F)$. We denote the matrix $[\tilde{c}_{ij}]$ by $\tilde{C}$. Evidently the map $C \mapsto \tilde{C}$ preserves every rank. We say that $C \mapsto \tilde{C}$ is the map *induced* on the space of matrices by the field-automorphism $c \mapsto \tilde{c}$.

We shall make use of the transpose with respect to the anti-diagonal $T \mapsto T^+$ described in §2.

We now define another type of rank one preservers which appears in [BS]

(3.1) Let each of $f_1, f_2, \dots f_n$ be an additive mapping from F to F such that f_1 is bijective, and let $\mathbf{f} = (f_1, f_2, \dots f_n)$. Define a mapping $\hat{\mathbf{f}}$ on a triangular algebra $\mathcal{A} = \mathcal{T}(n_1 \dots n_k)$, with $n_1 = 1$, by

$$\hat{\mathbf{f}}\left(\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ 0 & c_{22} & \dots & c_{2n} \\ \vdots & & & \\ 0 & 0 & \dots & c_{nn} \end{bmatrix}\right) = \begin{bmatrix} f_1(c_{11}) & f_2(c_{11}) + c_{12} & \dots & f_n(c_{11}) + c_{1n} \\ 0 & c_{22} & \dots & c_{2n} \\ \vdots & & & \\ 0 & 0 & \dots & c_{nn} \end{bmatrix}$$

This is a surjective additive mapping on $\mathcal{A}$ and it preserves rank one matrices, but only when $n_1 = 1$.

(3.2) For $\mathbf{f}$ and $f_1, f_2, \dots f_n$ as above, define a mapping $\check{\mathbf{f}}$ on a triangular algebra $\mathcal{A} = \mathcal{T}(n_1 \dots n_k)$, with $n_k = 1$, in a similar fashion except that the "action" is on the last column instead of the first row, more precisely $\check{\mathbf{f}}(C) = (\hat{\mathbf{f}}(C^+))^+$. Again this is an additive mapping on $\mathcal{A}$ preserving rank one matrices, but only when $n_k = 1$.

We now present a result from [BS].

THEOREM 3.3. [BS] *Let $\mathcal{A} = \mathcal{T}(n_1 \dots n_k)$ be a block upper triangular algebra in $M_n(F)$, such that $\mathcal{A} \neq \mathcal{T}_2(F)$. Let $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ be a surjective additive mapping that preserves rank one matrices. Then φ is a composition of some or all of the following maps:*

(i) Left multiplication by an invertible matrix in $\mathcal{A}$.
(ii) Right multiplication by an invertible matrix in $\mathcal{A}$.
(iii) The map $C \mapsto \tilde{C}$, induced by a field automorphism $a \mapsto \tilde{a}$ of F.
(iv) The map $\hat{\mathbf{f}}$ defined in 3.1 above, but only when $n_1 = 1$.
(v) The map $\check{\mathbf{f}}$ defined in 3.2 above, but only when $n_k = 1$.
(vi) The transpose with respect to the antidiagonal $T \mapsto T^+$. This is present only when $\mathcal{A} = \mathcal{A}^+$, i.e., $n_j = n_{k-j+1}$ for every j.

COROLLARY 3.4 *If φ is as in Theorem 3.3, then:*
(a) φ is injective;
(b) φ preserves every rank, i.e., rank $\varphi(T)$ = rank T, for every $T \in \mathcal{A}$.

REMARK When φ is linear, then obviously maps of type (iii), (iv), and (v) in Theorem 3.3 cannot be present. This is catually true for maps on spaces of matrices much more general than triangular algebras, see [BS].

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