HEREDITARY PROPERTIES OF SPECTRAL ISOMETRIES

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ABSTRACT. We prove that a surjective spectral isometry between von Neumann algebras of type I is a Jordan isomorphism. This is based on a study of some hereditary properties of spectral isometries.

The objective of this note is to study the behaviour of surjective spectral isometries with respect to restrictions to corners, quotients, or direct products. Combining these results with theorems on the structure of spectral isometries we shall obtain that every surjective spectral isometry has to be a Jordan isomorphism for some new classes of C^*-algebras.

Throughout, A and B will denote unital semisimple Banach algebras over the field $\mathbb{C}$ of complex numbers, unless specified otherwise. A linear mapping $T: A \to B$ is called a *spectral isometry* if $r(Tx) = r(x)$ for every element $x \in A$, where $r(\cdot)$ stands for the spectral radius. We recall a few basic facts about spectral isometries:

(i) *Every surjective spectral isometry is bounded.*
This follows from [**Aup1**, Theorem 5.5.2].

(ii) *Every spectral isometry is injective.*
This follows from the identity $r(x) = r(Tx) = r(Tx + Ta) = r(x + a)$ valid for each $a \in A$ with $Ta = 0$ and all $x \in A$, which entails that $a = 0$ by Zemánek's characterisation of the radical; see [**Aup1**, Theorem 5.3.1].

(iii) *Every surjective spectral isometry maps the centre $Z(A)$ of A onto the centre $Z(B)$ of B.*
This follows from Pták's characterisation of the centre; see [**MS1**, Proposition 4.3].

(iv) *Let $T: A \to B$ be a unital surjective spectral isometry, that is, $T1 = 1$. Then $T_{|Z(A)}$ is an algebra isomorphism from $Z(A)$ onto $Z(B)$.*
This is a consequence of (ii), (iii) and Nagasawa's theorem; see [**Aup1**, Theorem 4.1.17] or [**MS1**, Corollary 4.4].

In the following we will assume that T is a unital surjective spectral isometry from A onto B. By property (iv) above, every central idempotent e in A is mapped onto a central idempotent Te. We will now study the behaviour of T when restricted to the corner determined by e.

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Lemma 1. *Let $e \in A$ be a central idempotent, and let $f = Te$. Then $(1-f)T(ex)$ is quasi-nilpotent for every $x \in A$.*

Proof. Suppose that $r((1-f)T(ex)) > 0$ for some $x \in A$. By rotating x by a suitable complex number of modulus 1, if necessary, we can assume that $r((1-f)T(ex)) \in \sigma((1-f)T(ex))$. Hence, $r(\varepsilon\,(1-f)T(ex) + 1 - f) > 1$ for all $\varepsilon > 0$. On the other hand, $r(\varepsilon\, ex + 1 - e) = 1$ for all small enough $\varepsilon > 0$. Consequently, for such ε,

$$1 = r(\varepsilon\, ex + 1 - e) = r(\varepsilon\, T(ex) + 1 - Te) \geq r(\varepsilon\,(1-f)T(ex) + 1 - f) > 1,$$

a contradiction. □

By means of this, we obtain a linear subspace $(1-f)T(eA)$ of B consisting of quasi-nilpotent elements. We shall apply this fact to show that the restriction of T to eA is a spectral isometry onto $(Te)B$.

Proposition 2. *Let $e \in A$ be a central idempotent, and let $f = Te$. Then $T_e\colon eA \to fB$, $ex \mapsto fT(ex)$ is a unital spectral isometry from eA onto fB.*

Proof. Since, by Lemma 1,

$$r(ex) = r(T(ex)) = r(fT(ex) + (1-f)T(ex)) = r(fT(ex)) \qquad (x \in A),$$

T_e is a unital spectral isometry. By property (ii) above, it follows that T_e is injective. As T is surjective, every element $y \in fB$ can be uniquely written as $y = fT(ex) + fT((1-e)x)$ with $x \in A$. In order to establish the surjectivity of T_e, we need to show that $fT((1-e)x) = 0$.

Suppose that $fT(ex')$ is quasi-nilpotent for some $x' \in A$. Since T_e is a spectral isometry, ex' is quasi-nilpotent. For each $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} r\big(\lambda\, fT(ex') + fT((1-e)x)\big) &\leq r\big(T(\lambda\, ex' + (1-e)x)\big) \\ &= r(\lambda\, ex' + (1-e)x) = r((1-e)x). \end{aligned}$$

As a result, the subharmonic function $\lambda \mapsto r\big(\lambda\, fT(ex') + fT((1-e)x)\big)$ is bounded on $\mathbb{C}$, hence it must be constant. Since $fT((1-e)x)$ is quasi-nilpotent by Lemma 1, we infer that $r\big(fT(ex') + fT((1-e)x)\big) = 0$ for all quasi-nilpotent $fT(ex')$ and each $x \in A$.

Suppose now that $q \in fB$ is quasi-nilpotent. As T is a surjective spectral isometry, there is a unique quasi-nilpotent $p \in A$ such that

$$q = Tp = fT(ep) + fT((1-e)p).$$

Clearly, both ep and $(1-e)p$ are quasi-nilpotent as well. Hence, $fT(ep)$ is quasi-nilpotent. From the above we deduce that

$$\begin{aligned} r(q + fT((1-e)x)) &= r\big(fT(ep) + fT((1-e)p) + fT((1-e)x)\big) \\ &= r\big(fT(ep) + fT((1-e)(p+x))\big) = 0. \end{aligned}$$

By Zemánek's characterisation of the radical [**Aup1**, Theorem 5.3.1] it follows that $fT((1-e)x) \in \operatorname{rad}(fB) = \{0\}$, as claimed. As a result, $y = fT(ex)$ and so T_e is surjective. □

Theorem 3. *Let $T: A \to B$ be a unital surjective spectral isometry between the unital semisimple Banach algebras A and B. For each idempotent $e \in Z(A)$, the image $f = Te$ is a central idempotent in B and the restriction of T to eA is a unital surjective spectral isometry onto fB.*

Proof. The arguments in the proof of Proposition 2 show that $fT((1-e)A) = \{0\} = (1-f)T(eA)$. Therefore, $T(eA) \subseteq fB$ and $T((1-e)A) \subseteq (1-f)B$. In fact, we have equality in both cases by Proposition 2. As a result, T_e is nothing other than $T_{|eA}$, from which the claim follows. $\square$

In the recent past, a lot of attention has been devoted to the question under what additional assumptions (on T or on the algebras A and B) every unital surjective spectral isometry has to be a Jordan isomorphism; see, e.g., [**Aup2**, **Aup3**, **BSmo**, **CH**, **MM1**, **MS2**, **Sem**, **Sou**]. It was surmised in [**MS1**] that this holds whenever A and B are C^*-algebras. From Theorem 3 above we can derive the following consequences, which extend some of the known results.

Corollary 4. *Suppose that A_1 and A_2 are unital semisimple Banach algebras with the property that every unital spectral isometry from A_i, $i = 1, 2$ onto a unital semisimple Banach algebra is a Jordan isomorphism. Then $A_1 \oplus A_2$ has the same property.*

Proof. Put $A = A_1 \oplus A_2$ and let $T: A \to B$ be a unital surjective spectral isometry onto a unital semisimple Banach algebra B. Letting $e = 1 \oplus 0$ we obtain unital surjective spectral isometries $T_e: eA \to fB$ and $T_{1-e}: (1-e)A \to (1-f)B$, where $f = Te$, by Proposition 2. Since, by hypothesis, both T_e and T_{1-e} are Jordan isomorphisms, so is T. $\square$

It is well known that every unital spectral isometry from $M_n(\mathbb{C})$ onto itself is a Jordan isomorphism, see [**Aup2**, Proposition 2]. We can now extend this to arbitrary finite-dimensional algebras.

Corollary 5. *Every unital surjective spectral isometry between finite-dimensional semisimple Banach algebras is a Jordan isomorphism.*

Proof. Let $T: A \to B$ be such a spectral isometry. Since T is injective, we have $\dim A = \dim B = n$. By Wedderburn's theorem, both A and B are finite direct sums of full matrix algebras. As $T_{|Z(A)}$ is an isomorphism from $Z(A)$ onto $Z(B)$, every maximal orthogonal family of minimal idempotents in $Z(A)$ has to be mapped onto a maximal orthogonal family of minimal idempotents in $Z(B)$. Therefore, $A = \bigoplus_{i=1}^k M_{n_i}$ with $\sum_{i=1}^k n_i^2 = n$ and $B = \bigoplus_{j=1}^k M_{\ell_j}$ with $\ell_j = n_i$ for exactly one pair (i, j). (Here, we have already used Theorem 3.) Let e_i be the minimal central idempotent in A such that $e_i A \cong M_{n_i}$, $1 \le i \le k$, and $f_i = Te_i$, so that $f_i B \cong M_{\ell_j}$ with $\ell_j = n_i$. By Theorem 3, $T_{|e_i A}: e_i A \to f_i B$ is a unital surjective spectral isometry for each i wherefore, by Aupetit's theorem mentioned above, it is a Jordan isomorphism. Consequently, T itself must be a Jordan isomorphism. $\square$

The description of spectral isometries on matrix algebras does not extend to algebras of operators on arbitrary Banach spaces, see [**BSmo**]. It does, however, extend to Hilbert space, see [**BSmo**] or [**Sem**]. Using this fact, we obtain the following extension of Corollary 5. Let $B(H)$ stand for the C^*-algebra of all bounded linear operators on a Hilbert space H.

Corollary 6. *Let $A = \prod_{i\in I} B(H_i)$ and $B = \prod_{j\in J} B(K_j)$, where H_i and K_j are Hilbert spaces. Then every unital surjective spectral isometry from A onto B is a Jordan isomorphism.*

Proof. The restriction of T to $Z(A)$ is an isomorphism between the commutative C^*-algebras $Z(A)$ and $Z(B)$; therefore, it is *-preserving. It follows that $|I| = \dim Z(A) = \dim Z(B) = |J|$ and that T sends minimal projections in $Z(A)$ onto minimal projections in $Z(B)$. Let $e_i \in Z(A)$ be a minimal projection. By Theorem 3, $T_{|e_iA}$ is a unital spectral isometry onto f_iB, where $f_i = Te_i$. As $e_iA \cong B(H_i)$ and $f_iB \cong B(K_i)$, [**BSmo**, Theorem 1] entails that $T_{|e_iA}$ is a Jordan isomorphism for each i, from which the assertion follows. ☐

Remark. There are a number of variations on this theme. For instance, the direct product in the statement of Corollary 6 can be replaced by the direct sum (c_0-direct sum) of C^*-algebras. Moreover, the algebra $B(H)$ can be replaced by a unital purely infinite simple C^*-algebra in view of [**MM1**, Theorem B].

Another application of Theorem 3 enables us to reduce the open conjecture in the situation of von Neumann algebras to the finite case.

Corollary 7. *Let A be a von Neumann algebra, and let $A = A_1 \oplus A_2$ be the decomposition of A into its finite part A_1 and its properly infinite part A_2. Suppose that $T: A \to B$ is a unital surjective spectral isometry onto a unital semisimple Banach algebra B. Then T is a Jordan isomorphism if and only if $T_{|A_1}$ is a Jordan isomorphism.*

Proof. Let e be the central projection in A such that $A_1 = eA$, and let $f = Te$. By Theorem 3, $T_{|A_1}: A_1 \to B_1$ and $T_{|A_2}: A_2 \to B_2$ are unital surjective spectral isometries, where $B_1 = fB$ and $B_2 = (1-f)B$. By [**MS2**, Theorem 3.6], $T_{|A_2}$ is a Jordan isomorphism since A_2 is properly infinite. Thus, the statement follows from Corollary 4. ☐

Remark. There is further information available in the case of a finite von Neumann algebra A. Let τ denote the canonical centre-valued trace on A, and let N denote its kernel. Then $A = Z(A) \oplus N$. By [**FH1**, Theorème 3.2], $N = [A, A]$, the linear span of the commutators in A, and by [**PT1**, Theorem 3], $[A, A] = \mathcal{N}^{(2)}(A)$, the linear span of all nilpotent elements of index two. Property (iii) above entails that $TZ(A) = Z(B)$, whereas [**MS2**, Lemma 3.1] yields that, whenever $a \in A$, $a^2 = 0$ if and only if $(Ta)^2 = 0$. It follows that $T\mathcal{N}^{(2)}(A) = \mathcal{N}^{(2)}(B)$ and that $B = Z(B) \oplus \mathcal{N}^{(2)}(B)$. (Since B is semisimple there are no non-zero central nilpotent elements in B.) In order to establish the conjecture for finite von Neumann algebras it therefore suffices to show that every bijective bi-continuous linear mapping between $\mathcal{N}^{(2)}(A)$ and $\mathcal{N}^{(2)}(B)$ is given by the restriction of a Jordan isomorphism. For matrix algebras, this was obtained in [**BPW**].

Our next corollary is another piece of evidence that the conjecture on spectral isometries is expected to be true for von Neumann algebras.

Corollary 8. *Let A be a von Neumann algebra, and let $T: A \to B$ be a unital surjective spectral isometry onto a unital semisimple Banach algebra B. Then*

$$T(za) = (Tz)(Ta) \qquad (z \in Z(A),\, a \in A).$$

Proof. The proof of Theorem 3 shows that $T(ea) = (Te)(Ta)$ for all projections $e \in Z(A)$ and all $a \in A$. The claim therefore follows from the spectral theorem applied in $Z(A)$ and the boundedness of T. $\square$

From this result it follows in particular that T maps every ultraweakly closed ideal in A onto a closed ideal in B. It is therefore natural to ask about the permanence properties of spectral isometries with respect to quotients. We call a linear mapping T a *spectral contraction* if $r(Tx) \le r(x)$ for all x in the domain.

Proposition 9. *Let $T\colon A \to B$ be a spectral isometry from the C^*-algebra A onto the C^*-algebra B. Suppose that $I \subseteq A$ and $J \subseteq B$ are closed ideals in A and B, respectively, such that $TI \subseteq J$. Then the induced mapping $\hat{T}\colon A/I \to B/J$ is a spectral contraction. If $TI = J$ then $\hat{T}$ is a spectral isometry as well.*

Proof. Take $a \in A$. By the spectral radius formula in quotient C^*-algebras, see [**Ped**, Theorem 2] or [**MW**, Corollary on p. 274], we have $r(a+I) = \inf_{x\in I} r(a+x)$. Applying this we find that

$$\begin{aligned} r(\hat{T}(a+I)) &= r(Ta+J) = \inf_{y\in J} r(Ta+y) \\ &\le \inf_{x\in I} r(Ta+Tx) = \inf_{x\in I} r(a+x) \\ &= r(a+I). \end{aligned}$$

If $TI = J$, the inequality sign in the above estimate turns into an equality sign. $\square$

We can rephrase this result in terms of extensions as follows. Suppose that

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

yields A as an extension of the C^*-algebra B by the C^*-algebra I (so that $B \cong A/I$). If $T\colon A \to A$ is a surjective spectral isometry and I is a closed ideal of A such that $TI = I$ then both $T_{|I}\colon I \to I$ and the induced mapping $\hat{T}\colon B \to B$ are spectral isometries. The converse question seems to be more difficult to decide.

Question. Suppose that, with the above notation, both $T_{|I}$ and $\hat{T}$ are surjective spectral isometries. Does T have to be a spectral isometry?

The next result gives a positive answer to this question at least when the ideal is the ideal $K(H)$ of compact operators on a Hilbert space H.

Theorem 10. *Let H_1 and H_2 be infinite dimensional Hilbert spaces. Suppose that $T\colon K(H_1) \to K(H_2)$ is a surjective spectral isometry. Then T extends to a spectral isometry from $B(H_1)$ onto $B(H_2)$. Consequently, T is of the form $x \mapsto \mu axa^{-1}$, $x \in B(H_1)$ or of the form $x \mapsto \mu ax^ta^{-1}$, $x \in B(H_1)$ for some $\mu \in \mathbb{C}$ with $|\mu| = 1$ and some invertible operator $a \in B(H_1, H_2)$.*

For the proof we will need the following result, which is an immediate consequence of [**MS2**, Lemma 3.1].

Lemma 11. *Let $T\colon A \to B$ be a surjective spectral isometry between the semisimple Banach algebras A and B. For each $a \in A$, $a^n = 0$ if and only if $(Ta)^n = 0$.*

Proof of Theorem 10. Ahmed: Can we prove this in this formulation? $\square$

In every von Neumann algebra there is a large class of ideals which is preserved by spectral isometries. Recall that a *Glimm ideal* I in a von Neumann algebra A is of the form $I = AM$, where M is a maximal ideal in the centre of A. (The ideal AM is closed by Cohen's factorisation theorem.) We denote by A_I the *Glimm quotient* $A_I = A/I$. Corollary 8 together with Proposition 9 enable us to deduce the following result.

Proposition 12. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the von Neumann algebras A and B. For each Glimm ideal I in A, $J = TI$ is a Glimm ideal in B. Thus, the induced operator $\hat{T}\colon A_I \to B_J$ is a unital surjective spectral isometry.*

Proof. Let M be a maximal ideal of the centre $Z(A)$ and let $I = AM$ be its corresponding Glimm ideal. Since $T_{|Z(A)}$ is an isomorphism between $Z(A)$ and $Z(B)$, $N = TM$ is a maximal ideal of $Z(B)$. By Corollary 8, $TI = T(AM) \subseteq BN$. Applying the same argument to T^{-1}, we obtain $TI = BN$. Thus, $J = TI$ is a Glimm ideal in B. The second assertion now follows from Proposition 9. $\square$

Every Glimm quotient A_I of a von Neumann algebra A is a primitive C^*-algebra, by [**H**, Theorem 4.7]. Since the Glimm ideals separate the points of A, it thus suffices to show that every unital surjective spectral isometry between primitive C^*-algebras is a Jordan isomorphism in order to establish the conjecture for general von Neumann algebras.

We apply this method to give a positive answer for type I von Neumann algebras.

Theorem 13. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the von Neumann algebras A and B. If A or B is of type I, then T is a Jordan isomorphism.*

Proof. We may assume that A is of type I; the other case is treated by considering T^{-1} instead. By Corollary 7, we may further suppose that A is finite. Therefore $A = \prod_n A_n$, where each A_n is an n-homogeneous von Neumann algebra, $n \in \mathbb{N}$ and hence of the form $A_n = C(X_n) \otimes M_n(\mathbb{C})$ with X_n a hyperstonean space [**T**, Theorem V.1.27]. As $Z(A) = \prod_n C(X_n) \otimes \mathbb{C}$, every maximal ideal M in $Z(A)$ is of the form $M = \prod_n I_n \otimes \mathbb{C}$, where all I_n but one are equal to A_n and there is n_0 such that $I_{n_0} = \{f \in C(X_{n_0}, M_{n_0}(\mathbb{C})) \mid f(x_0) = 0\}$. As a result, every Glimm ideal I of A is of the form $I = \prod_n I_n \otimes M_n(\mathbb{C})$ with the same restriction on the n's. The mapping $(f_n) + I \mapsto f_{n_0}(x_0)$ establishes an isomorphism from A/I onto $M_{n_0}(\mathbb{C})$; therefore the Glimm quotient A_I has dimension n_0^2. By Proposition 12, $J = TI$ is a Glimm ideal in B and the induced mapping $\hat{T}\colon A_I \to B_J$ is a unital surjective spectral isometry. As B/J is primitive and of dimension n_0^2, it is also isomorphic to $M_{n_0}(\mathbb{C})$. Aupetit's result [**Aup2**, Proposition 2] thus entails that $\hat{T}$ is a Jordan isomorphism. Since the Glimm ideals separate the points (in A and B, respectively), we conclude that T is a Jordan isomorphism. $\square$

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