A NOTE ON FIRE FREQUENCY CONCEPTS AND DEFINITIONS.

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Abstract

The concepts of hazard of burning, fire interval and fire cycle are considered. It is claimed that the current notion of fire cycle is poorly defined (since the time required to burn a specified area is a random variable). It is shown that the *expected* time to burn an area equal to the study area normally exceeds the fire interval (the average time between fires at any location). In view of this it is recommended that the notion of fire cycle in its current form be abandoned.

Keywords: hazard of burning, fire cycle, fire interval, local hazard, area-wide hazard.

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1 Introduction.

The basic theoretical concepts of fire history were laid out by Johnson and Van Wagner (1985) and subsequently reiterated and refined by Johnson and Gutsell (1994). In the former paper the authors emphasize (p.218) that the fire history models used "can be interpreted on a per element basis or in terms of the proportion of the universe". Thus the *average fire interval* (the average of return times between fires at a point) is identified with the *fire cycle*, defined as the time required to burn an area equal in area to that of the universe.

The reasoning behind this duality of interpretations is based on the identification of the per annum *probability* of a fire at a point (element) with the *proportion* of the area burned in a given year. Such an identification is not strictly true. Fires are random processes and the actual proportion burned in any year will be a random variable. However the *expected* value of this random variable (*i.e.* the expected proportion burned) will be equal to the per annum probability. One might hope for a similar relationship, involving expectations, between return time and fire cycle *i.e.* that the *expected* time between fires at any point is the same as the *expected* time to burn an area equal to the area of the universe. Unfortunately, as is shown in the Appendix, this is not true. In fact the latter expected time exceeds the former one, except in one unrealistic special case[1]. Thus the identification of fire interval

[1]This is when all fires are of the same size, and the study area is an integer multiple of this size; in this case the two expectations are equal.

and fire cycle is not valid.

In order to avoid confusion in future fire history studies, and especially in simulation studies in which areas “burned” are generated on a computer, the notions of fire cycle, fire interval *etc.* need to be clarified. This is purpose of this paper. The main recommendation is that the notion of fire cycle, as the time required to burn an area equal to the area of the study area, be abandoned. Rather it is better to think “element-wise”, *i.e.* in terms of the hazard of burning at any point and its reciprocal the expected fire interval at the point. The notion of fire cycle as originally defined results from a deterministic way of thinking and as its stands is inadequate and can lead to confusion.

2 Definitions of basic concepts.

The notion of the *hazard of burning* was introduced by Johnson and Gutsell (1994). We shall distinguish between a *local* hazard of burning and an *area-wide* hazard of burning.

The *local hazard of burning at a point* $\mathbf{x}$ in the study area, at time t, can be defined as

$$\lambda(t;\mathbf{x}) = \lim_{dt\to 0}\left\{\mathrm{P}(\text{fire at location }\mathbf{x}\text{ in }[t,t+dt])/dt\right\}. \quad (1)$$

Clearly this has units $(\text{time})^{-1}$ *e.g. per annum.* In contrast to the local hazard of burning, the *area-wide hazard of burning* can be defined as

$$\Lambda(t) = \lim_{dt\to 0}\left\{\mathrm{P}(\text{fire ignited somewhere in study area in }[t,t+dt])/dt\right\}. \quad (2)$$

which again has units $(\text{time})^{-1}$.

How do these two concepts relate? Clearly $\Lambda(t) \geq \lambda(t; \mathbf{x})$ for all points $\mathbf{x}$ in the study area. Also one can write

$$\lambda(t; \mathbf{x}) = \Lambda(t) \int_A h(\mathbf{x}, \mathbf{y}; t) f(\mathbf{y}; t) dy \tag{3}$$

where $h(\mathbf{x}, \mathbf{y}; t)$ is the conditional probability of a fire ignited at point $\mathbf{y}$ spreading to $\mathbf{x}$ at time t; and $f(\mathbf{y}; t)$ is the probability density function of where an ignition occurs over the study area A, given that one occurs at time t. Letting $p(t, \mathbf{x})$ denote the integral (so that $p(t, \mathbf{x})$ is the conditional probability of a fire occurring at $\mathbf{x}$, given that a fire starts somewhere in the study area at time t) leads to

$$\lambda(t; \mathbf{x}) = \Lambda(t) p(t, \mathbf{x}) \tag{4}$$

Note that the area-wide hazard of burning will in general depend on the size of the study area (as area increases so will the area-wide hazard). Because of this, it is not very useful in characterizing aspects of the fire ecology.

The *fire interval at location* $\mathbf{x}$ is defined as the expected time between fires at that location. In general with a time-varying hazard of burning this will depend on the time t of the most recent fire and can be shown (see Appendix) to be

$$FI_t(\mathbf{x}) = \int_0^\infty \exp\left[-\int_0^z \lambda(t + s; \mathbf{x}) ds \right] dz. \tag{5}$$

Note that this depends on the hazard of burning for all times beyond t. Without further assumptions it is of little practical use. The usual simplifying assumptions are those of spatial and temporal homogeneity, the latter at least over suitably long epochs. Spatial homogeneity is a realistic assumption if the study area can be partitioned into bio-geographically homogeneous sub-areas.

2.1 Temporal homogeneity.

Suppose that the above hazard rates are constant over some epoch, so that the local hazard of burning at location $\mathbf{x}$ is a constant $\lambda(\mathbf{x})$ (and the area-wide hazard is a constant Λ). In this case the formula (**??**) for the fire interval at $\mathbf{x}$ reduces to (using (**??**))

$$FI(\mathbf{x}) = \frac{1}{\lambda(\mathbf{x})} = \frac{1}{\Lambda p(\mathbf{x})}. \tag{6}$$

The fire interval has units of time.

Note that one could also define an *area-wide fire interval.* In the time-homogeneous case this would simply be the reciprocal of the area-wide hazard of burning *i.e.* $1/\Lambda$. However, like the *area-wide hazard of burning* it will depend on the size of the study area, and so is of limited usefulness.

2.2 Spatial homogeneity.

If in addition to temporal homogeneity, there is spatial homogeneity, then the local hazard of burning will not depend on location $\mathbf{x}$ (*i.e.* $\lambda(\mathbf{x}) \equiv \lambda$ for all $\mathbf{x}$) nor will the local fire interval

$$FI = \frac{1}{\lambda} = \frac{1}{\Lambda p} \tag{7}$$

where p is the conditional probability of a fire occurring at any specific point given that a fire occurs somewhere in the study area.

2.3 The fire cycle.

Johnson and Van Wagner (1985) equate the local fire interval FI (assuming spatial and temporal homogeneity) with the *fire cycle* FC, which they define as the time required to burn an area equal in area to the study area. This definition emerges from a deterministic way of thinking (in which fixed *proportions* of the study area are burned every year). Clearly in the real world the time required to burn a fixed area will not be fixed, but rather be a random variable. One could modify the Johnson-Van Wagner definition of fire cycle to be the *expected* time required to burn an area equal in area to the study area.

However with this definition the fire cycle is no longer necessarily equal to the local fire interval FI. Indeed it is shown in the Appendix that if fires occur (anywhere in the study area) in a *Poisson process*[2] at rate Λ, and the average area burned per fire is μ, then the expected time EFC, say, to burn an area A equal to the size of the study area satisfies

$$EFC \geq \frac{A}{\Lambda\mu} \tag{8}$$

Note that μ/A is the expected fraction of the study area burned in any

[2] *i.e.* independently of one another with the probability of a fire in $(t, t + dt)$ being Λdt for all times t. Note that one can also work in discrete time and have fires occurring in a given year with a fixed probability π, say. Similar results pertain in this case – see Appendix.

fire and can be thought of (under the assumptions of homogeneity) as the probability p that the fire burns any particular location, given that a fire is ignited somewhere in the study area. Thus using (**??**) the above inequality can be expressed as

$$EFC \geq \frac{1}{\Lambda p} = FI \tag{9}$$

In most cases the inequality is strict. Indeed there appears to be only one case in which it holds as an equality – that is when every fire is the same size (area burned $= \mu$ with probability one) and the total study area is an integer multiple of μ (*i.e.* $A = k\mu$ for some $k = 1, 2, \ldots$).

In the Appendix some other specific examples are considered. One is when the size of fires is exponentially distributed, with mean μ. In this case the expected time to burn an area A is

$$EFC = FI + \frac{1}{\Lambda}.$$

If fires are infrequent then $1/\Lambda$ will be large and the expected fire cycle considerably larger than the fire interval. A similar result pertains in the case when the size of fires follows a gamma distribution. Explicit formulas for the EFC are obtained using the gamma distribution with shape parameter $\kappa = 2$ and 3.

Also results are obtained for the case when fires are all of the same size. In this case, provided the area of the study area is an integer multiple of the size of a fire, $EFC = FI$. This is essentially the (deterministic) case contemplated by Johnson and Van Wagner (1985) when they developed the notion of fire cycle and claimed its identity with the fire interval.

3 Estimation.

Maximum likelihood estimation of the local hazard of burning, and its reciprocal the fire interval, for stand-replacing fires using time-since fire map data has been described by Reed *et al.* (1998). The question of determining change points between epochs of temporal homogeneity has been discussed by Reed (2000). Methods for the maximum likelihood estimation of the local hazard of burning and the fire interval for other fires using fire scar data have been presented by Reed & Johnson (2004).

4 Conclusions.

In view of the difference between the local fire interval and the (expected) fire cycle, it is recommended that to avoid confusion, the original definition of the fire cycle (as the time required to burn an area equal in area to the study area) be no longer used. Firstly it is not well-defined – by this definition the fire cycle is a random variable – and secondly, even if the expected value of this random time is used, it does not coincide with the local fire interval (or the reciprocal of the local hazard of burning). It is recommended either that the notion of the fire cycle no longer be used; or if it is that it be defined as identical to the local fire interval *i.e.* that the fire cycle be defined as *the expected time between fires at any given location in the study area.*

The heretofore accepted duality of fire history concepts proposed by Johnson and Van Wagner (1985) ("per element" notion or "proportion of the universe" notion) occurs only in a (theoretical but imaginary) deterministic world. Persisting with this duality can cause confusion and error, especially in simulation studies. In view of this it is recommended that concepts based on the proportion of the study area (universe) be no longer used and instead only the "element based" notions of local hazard of burning and local fire interval be used to describe fire history.

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Appendix.

Expected time between fires at a particular location.

Given that a fire has just occurred at time t at location $\mathbf{x}$, let $FI_t(\mathbf{x})$ denote the expected time until the next fire at this location. This can be written (using a well-known result for the expectation of a non-negative r.v.) as

$$FI_t(\mathbf{x}) = \int_0^\infty S(z|t)\ dz$$

where $S(z|t)$ is the conditional probability of no fire at $\mathbf{x}$ in the time interval $(t, t+z]$ given a fire at $\mathbf{x}$ at time t. But this 'survivor function' relates to the local hazard of burning as

$$S(z|t) = \exp\left[-\int_0^z \lambda(t+s;\mathbf{x})\ ds\right]$$

leading to the result (**??**).

Relationship between EFC and FI under assumptions of homogeneity.

Assume temporal and spatial homogeneity, and suppose that the area-wide hazard of burning is Λ. Assuming independence of fires this implies that the number of fires $N(t)$ occurring by time t is a Poisson process with

$$\mathrm{P}(N(t) = n) = \frac{e^{-\Lambda t}(\Lambda t)^n}{n!}, \qquad n = 0, 1, \ldots$$

Suppose that the areas burned in fires are independently, identically distributed (iid) random variables (rvs) with mean μ. Then the total area burnt by time t is a random variable

$$S_t = X_1 + X_2 + \ldots + X_{N(t)}$$

where $X_1, X_2, \ldots$ are iid rvs.

Now let $T^\star(A)$ be the time when the total area burned first reaches A, the area of the study area (*i.e.* $T^\star(A) = \min(t : S_t \geq A)$). When the dependence on A of this time is not important we shall simply write $T^\star$, so that $T^\star$ is the fire cycle as defined by Johnson and Van Wagner (1985). But note, this is a random variable, so the definition is not precise. If the expected value of this time is considered, using conditional expectation one can write

$$\mathrm{E}(T^\star) = \mathrm{E}(\mathrm{E}(T^\star|N(T^\star)) = \frac{1}{\Lambda}\mathrm{E}(N(T^\star)) \tag{10}$$

since the expected time between fires is $1/\Lambda$. Now $X_1, X_2, \ldots$ forms a *renewal process* and $N(T^\star)$ is a *stopping time* for such a process. It follows by Wald's theorem (see *e.g.* Grimmett and Strirzaker, 1992) that $\mathrm{E}(X_1 + X_2 + \ldots + X_{N(T^\star)}) = \mathrm{E}(N(T^\star))\mathrm{E}(X_i)$ so that

$$\mathrm{E}(N(T^\star)) = \frac{\mathrm{E}(X_1 + X_2 + \ldots + X_{N(T^\star)})}{\mu} \tag{11}$$

The numerator of the rhs is greater or equal to A. Thus it follows, using (**??**), that

$$\mathrm{E}(T^\star) \geq \frac{A}{\Lambda\mu} \tag{12}$$

Now $\mathrm{E}(T^\star)$ is the expected value of the fire cycle (EFC); and μ/A is the expected proportion of the study area burned in any fire. Under the assumption of spatial homogeneity it is the conditional probability p that a fire occurs at any particular location in study area given that a fire is ignited somewhere in the study area. Thus the right-hand side of (**??**) is equal to $\frac{1}{\Lambda p} = \frac{1}{\lambda}$ where λ is the local hazard of burning. Thus (**??**) states that the expected fire cycle is greater or equal to the local fire interval; or $EFC \geq FI$.

To evaluate the expected fire cycle in specific cases we examine the cumulative distribution function (cdf) of total area S_t, burned by time t. It is

$$\begin{aligned} F_S(s) \quad = \mathrm{P}(S_t \leq s) \quad &= \sum_{n=0}^{\infty} \mathrm{P}(S_t \leq s | N(t) = n) \frac{e^{-\Lambda t}(\Lambda t)^n}{n!} \\ &= \sum_{n=0}^{\infty} F_n(s) \frac{e^{-\Lambda}(\Lambda t)^n}{n!} \end{aligned} \tag{13}$$

where F_n is the cdf of the n-fold convolution of X_i *i.e.* it is the cdf of $X_1 + X_2 + \ldots + X_n$.

Now if $T^\star(A)$ is the time required to burn an area A;

$$\begin{aligned} \mathrm{P}(T^\star(A) \geq t) &= \mathrm{P}(S_t \leq A) \\ &= \sum_{n=0}^{\infty} F_n(A) \frac{e^{-\Lambda t}(\Lambda t)^n}{n!} \end{aligned} \tag{14}$$

The expected value of a continuous non-negative random variable Y, say can be computed using $\mathrm{E}(Y) = \int_0^\infty \mathrm{P}(Y \geq y) dy$. Thus

$$\begin{aligned} \mathrm{E}(T^\star(A)) &= \int_0^\infty \mathrm{P}(T^\star \geq t)\ dt \\ &= \sum_{n=0}^{\infty} \frac{F_n(A)}{n!} \int_0^\infty e^{-\Lambda t}(\Lambda t)^n \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{F_n(A)}{n!} \frac{\Gamma(n+1)}{\Lambda} \\
&= \frac{1}{\Lambda} \sum_{n=0}^{\infty} F_n(A) \qquad (15)
\end{aligned}
$$

where $\Gamma()$ is the usual gamma function and $F_0(A)$ is the *Heaviside step function* which assumes value zero for $A \leq 0$ and value 1 for $A > 0$.

In general closed-form expressions for $F_n(A)$ are not available. However in some cases one can evaluate (**??**) using *Laplace transforms*.

The Laplace transform of a probability density function (pdf), $f(x)$, say, of a random variable X with nonnegative support is

$$\tilde{f}(s) = \int_0^{\infty} e^{-sx} f(x) \; dx = \mathrm{E}\left(e^{-sX}\right).$$

Also the Laplace transform of the cdf $F(x)$ of such a random variable is $\tilde{f}(s)/s$ and that of the Heaviside function is $1/s$. Furthermore the pdf of the n-fold convolution of the r.v. X is

$$\tilde{f}_n(s) = \left[\tilde{f}(s)\right]^n.$$

Using these results one can obtain the Laplace transform $\tilde{\tau}(s)$ of $\tau(A) = \mathrm{E}(T^{\star}(A))$ in (**??**). It is

$$
\begin{aligned}
\tilde{\tau}(s) &= \frac{1}{\Lambda}\left[\frac{1}{s} + \sum_{n=1}^{\infty} \frac{\left[\tilde{f}(s)\right]^n}{s}\right] \\
&= \frac{1}{\Lambda s} \frac{1}{1-\tilde{f}(s)} \qquad (16)
\end{aligned}
$$

We now consider some special cases:

(a) Fire size exponentially distributed.

Suppose the size of fires is exponentially distributed with mean μ *i.e.* with pdf $f(x) = (1/\mu)e^{-x/\mu}$ for $x > 0$. The Laplace transform of f is $\tilde{f}(s) = 1/(1+\mu s)$ and in consequence $\tilde{\tau}(s) = (1+\Lambda\mu s)/(\mu s^2)$. This can be inverted to yield $\tau(A) = \mathrm{E}(T^\star(A)) = A/(\Lambda\mu) + 1/\Lambda$. As discussed in the text $A/(\Lambda\mu)$ is the fire interval (FI), so the expected fires cycle (EFC) satisfies

$$EFC = FI + \frac{1}{\Lambda}$$

If fires are infrequent (small Λ) the difference between the EFC and FI can be large.

(b) Fire size following a gamma distribution.

Suppose the size of fires has pdf

$$f(x) = \left(\frac{\kappa}{\mu}\right)^{\kappa} \frac{1}{\Gamma(\kappa)}\, x^{\kappa-1}\, e^{-\frac{\kappa}{\mu}x} \qquad \kappa > 1.$$

Like the exponential distribution above this has mean μ and a long tail to the right. However unlike the exponential distribution its mode is not at zero, but rather at $\frac{\kappa-1}{\kappa}\mu$. The Laplace transform of f is $\tilde{f}(s) = 1/(1+\mu s/\kappa)^{\kappa}$. For specific integer values of κ one can, with a little work, invert the Laplace transform $\tilde{\tau}(s)$ to yield the EFC. For example with $\kappa = 2$

$$EFC = FI + \frac{1}{4\Lambda}\left[3 + e^{-4A/\mu}\right].$$

With $\kappa = 3$

$$EFC = FI + \frac{1}{3\Lambda}\left[2 + e^{-\frac{9A}{2\mu}}\left(\cos(\frac{3\sqrt{3}A}{2\mu}) + \frac{1}{\sqrt{3}}\sin(\frac{3\sqrt{3}A}{2\mu})\right)\right].$$

Again, as the case of exponentially distributed fire size ($\kappa = 1$), in both of these cases if fires are infrequent (small Λ) the difference between the EFC and FI can be large.

(c) Fire of constant size. If fires are of constant size μ then $\tilde{f}(s) = e^{-\mu s}$ and $\tilde{\tau}(s) = 1/(\Lambda s(1 - e^{-\mu s}))$. This is the Laplace transform of a step function with steps of height $1/\Lambda$ at $0, \mu, 2\mu, \ldots$; or in other words of the function $\{A/\Lambda\mu\}$ where $\{z\}$ is the *ceiling* function *i.e.* $\{z\}$ = smallest integer $\geq z$. Thus

$$EFC = \frac{1}{\Lambda}\left\{\frac{A}{\mu}\right\}$$

This is equal to the fire interval FI if A is an integer multiple of μ, but otherwise exceeds FI.

Not that the above results can be replicated using a discrete-time formulation of the problem (to reflect the fact that fires occur only during a fire season). In this case assume a probability Θ of a fire being ignited anywhere in the study area in a given season. It is not difficult to show that all of the above results hold with Λ replaced by Θ. Again the difference between expected fire cycle and the fire return interval will be large if fires are infrequent (small Θ).