BROWNIAN-LAPLACE MOTION AND ITS USE IN FINANCIAL MODELLING.

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Key Words: Laplace motion; generalized normal Laplace (GNL) distribution; Lévy process; option value; Black-Scholes

ABSTRACT.

Brownian-Laplace motion is a Lévy process which has both continuous (Brownian) and discontinuous (Laplace motion) components. The increments of the process follow a *generalized normal Laplace* (GNL) distribution which exhibits positive kurtosis and can be either symmetrical or exhibit skewness. The degree of kurtosis in the increments increases as the time between observations decreases. This and other properties render Brownian-Laplace motion a good candidate model for the motion of logarithmic stock prices. An option pricing formula for European call options is derived and it is used to calculate numerically the value of such an option both using nominal parameter values (to explore its dependence upon them) and those obtained as estimates from real stock price data.

1. INTRODUCTION.

The Black-Scholes theory of option pricing was originally based on the assumption that asset prices follow geometric Brownian motion (GBM). For such a process the logarithmic returns ($\log(P_{t+1}/P_t)$ on the price P_t are independent identically distributed (iid) normal random variables. However it has been recognized for some time now that the logarithmic returns do not behave quite like this, particularly over short intervals. Empirical distributions of the logarithmic returns in high-frequency data usually exhibit excess kurtosis with more probability mass near the origin and in the tails and less in the flanks than would occur for normally distributed data. Furthermore the degree of excess kurtosis is known to increase as the sampling interval decreases (see *e.g.* Rydberg, 2000). In addition skewness can sometimes be present. To accommodate for these facts new models for price movement based on Lévy motion have been developed (see *e.g.* Schoutens, 2003). For any infinitely divisible distribution a Lévy process can be constructed whose increments follow the given distribution. Thus in modelling financial data one needs to find an infinitely divisible distribution which fits well to observed logarithmic returns. A number of such distributions have been suggested including the gamma, inverse Gaussian, Laplace (or variance gamma), Meixner and generalized hyperbolic distributions (see Schoutens, 2003 for details and references).

In this paper a new infinitely divisible distribution – the *generalized normal Laplace* (or GNL) distribution – which exhibits the properties seen in observed logarithmic returns, is introduced. This distribution arises as the convolution of independent normal and generalized Laplace (Kotz *et al.*, 2001, p. 180) components[1]. A Lévy process based on the generalized Laplace (variance-gamma) distribution alone has no Brownian component, only linear deterministic and pure jump components *i.e.* its Lévy-Khintchine triplet is of the form $(\gamma, 0, \nu(dx))$ (see Schoutens, 2003, p. 58). The new distribution of this paper in effect adds a Brownian component to this motion, leading to what will be called *Brownian-Laplace motion*[2].

In the following section the generalized normal Laplace (GNL) distribution is defined and some properties given. Brownian-Laplace motion is then defined as a Lévy process whose increments follow the GNL distribution. In Sec. 3 a pricing formula is developed for European call options on a stock whose logarithmic price follows Brownian-Laplace motion. In Sec. 4 some numerical examples are given.

[1]The generalized asymmetric Laplace distribution is better known as the variance-gamma distribution in the finance literature. It is also known as the Bessel K-function distribution (see Kotz *et al.*, 2001, for a discussion of the terminology and history of this distribution).

[2]An alternative name, which invokes two of the greatest names in the history of mathematics, would be *Gaussian-Laplace motion*

2. THE GENERALIZED NORMAL-LAPLACE (GNL) DISTRIBUTION.

The *generalized normal Laplace* (GNL) distribution is defined as that of a random variable Y with characteristic function

$$\phi(s) = \left[\frac{\alpha\beta\exp(\mu is - \sigma^2 s^2/2)}{(\alpha - is)(\beta + is)}\right]^\rho \tag{1}$$

where α, β, ρ and σ are positive parameters and $-\infty < \mu < \infty$. We shall write

$$Y \sim \text{GNL}(\mu, \sigma^2, \alpha, \beta, \rho)$$

to indicate that the random variable Y follows such a distribution.
Since the characteristic function (**??**) can be written

$$\exp(\rho\mu is - \rho\sigma^2 s^2/2)\left[\frac{\alpha}{\alpha - is}\right]^\rho \left[\frac{\beta}{\beta + is}\right]^\rho$$

it follows that Y can be represented as

$$Y \stackrel{d}{=} \rho\mu + \sigma\sqrt{\rho}Z + \frac{1}{\alpha}G_1 - \frac{1}{\beta}G_2 \tag{2}$$

where Z, G_1 and G_2 are independent with $Z \sim$ N(0,1) and G_1, G_2 gamma random variables with scale parameter 1 and shape parameter ρ, *i.e.* with probability density function (pdf)

$$g(x) = \frac{1}{\Gamma(\rho)} x^{\rho-1} e^{-x}.$$

This representation provides a straightforward way to generate pseudo-random deviates following a GNL distribution. Note that from (**??**) it is easily established that the GNL is infinitely divisible. In fact the n-fold convolution of a GNL random variable also follows a GNL distribution.

The mean and variance of the GNL$(\mu, \sigma^2, \alpha, \beta, \rho)$ distribution are

$$\text{E}(Y) = \rho\left(\mu + \frac{1}{\alpha} - \frac{1}{\beta}\right); \qquad \text{var}(Y) = \rho\left(\sigma^2 + \frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)$$

while the higher order cumulants are (for $r > 2$)

$$\kappa_r = \rho(r-1)!\left(\frac{1}{\alpha^r} + (-1)^r \frac{1}{\beta^r}\right).$$

The parameters μ and σ^2 influence the central location and spread of the distribution, while α and β affect the symmetry. If $\alpha > \beta$ the distribution is skewed to the left, and vice versa. The parameter ρ affects the lengths of the tails. The nature of the tails can be determined from the order of the poles of its characteristic (or moment generating) function (see *e.g.* Doetsch, 1970, p. 231*ff*). Precisely $f(y) \sim c_1 y^{\rho-1} e^{-\alpha y}$ $(y \to \infty)$ and $f(y) \sim c_2(-y)^{\rho-1} e^{\beta y}$ $(y \to -\infty)$, (where c_1 and c_2 are constants). Thus for $\rho < 1$, both tails are fatter than exponential; for $\rho = 1$ they are exactly exponential and for $\rho > 1$ they are less fat than exponential. This exactly mimics the tail behaviour of the generalized Laplace distribution. Thus in the tails the generalized Laplace component of the GNL distribution dominates over the normal component.

The parameter ρ affects all moments. However the coefficients of skewness $(\gamma_1 = \kappa_3/\kappa_2^{3/2})$ and of excess kurtosis $(\gamma_2 = \kappa_4/\kappa_2^2)$ both decrease with increasing ρ (and converge to zero as $\rho \to \infty$) with the shape of the distribution becoming more normal with increasing ρ, (exemplifying the central limit effect since the sum of n iid GNL$(\mu, \sigma^2, \alpha, \beta, \rho)$ random variables has a GNL$(\mu, \sigma^2, \alpha, \beta, n\rho)$ distribution).

When $\alpha = \beta$ the distribution is symmetric. In the limiting case $\alpha = \beta = \infty$ the GNL reduces to a normal distribution.

The family of GNL distributions is closed under linear transformation *i.e.* if $Y \sim$ GNL$(\mu, \sigma^2, \alpha, \beta, \rho)$ then, for constants a and b, $a + bY \sim$ GNL$(b\mu + a/\rho, b^2\sigma^2, \alpha/b, \beta/b, \rho)$.

A closed-form for the density has not been obtained except in the special case $\rho = 1$. In this case the GNL distribution becomes what has been called an (ordinary) *normal-Laplace* (NL) distribution since it can be represented as the convolution of independent normal and Laplace variates (Reed & Jorgensen, 2004). The NL probability density function (pdf) is of the form

$$f(y) = \frac{\alpha\beta}{\alpha+\beta}\, \phi\left(\frac{y-\mu}{\sigma}\right) \left[R\left(\alpha\sigma - (y-\mu)/\sigma\right) + R\left(\beta\sigma + (y-\mu)/\sigma\right)\right], \tag{3}$$

where R is *Mills' ratio* (of the complementary cumulative distribution function (cdf) to the pdf of a standard normal variate):

$$R(z) = \frac{\Phi^c(z)}{\phi(z)} = \frac{1 - \Phi(z)}{\phi(z)}.$$

Fig. 1 shows some histograms of samples of size one million from various GNL distributions, generated using (**??**). In all six cases the distributions have expected value zero. The three distributions in the top row all have the same variance (0.00165) while the three distributions in the lower row have a variance twice as large (0.00331). The distributions in the left and centre panels are symmetric ($\alpha = \beta = 17.5, 12.5$ in left and centre panels respectively) while those the right-hand panels are skewed ($\alpha = 20.00 > \beta = 15.75$). Plotted on top of the histograms are normal densities with mean 0 and variance 0.00165 (top row) and 0.00331 (bottom row). Although the left-hand panels exhibit some kurtosis ($\gamma_2 = 4.68$ and 2.84 in top and bottom rows) it is barely discernible by eye; the centre panels have greater kurtosis ($\gamma_2 = 17.99$ and 8.99 in top and bottom rows) which is clearly visible to the eye. The right-hand panels are left-skewed with $\gamma_1 = -0.390$ and -.276 in top and bottom rows.

2.1. ESTIMATION.

The lack of a closed-form for the GNL density means a similar lack for the likelihood function. This presents difficulties for estimation by maximum likelihood (ML). However it may be possible to obtain ML estimates of the parameters of the distribution using the EM-algorithm and the representation (**??**), but to date this has not been accomplished. An alternative method of estimation is the method of moments. While method-of-moments estimates are consistent and asymptotically normal, they are not generally efficient (not achieving the Cramér-Rao bound) even asymptotically. A further problem is the difficulty in restricting the parameter space (*e.g.* for the GNL the parameters α, β, ρ and σ^2 must be positive), since the moment equations may not lead to solutions in the restricted space.

In the case of a symmetric GNL distribution ($\alpha = \beta$) method-of-moments estimators of the four model parameters can be found analytically. They are

$$\hat{\alpha} = \hat{\beta} = \sqrt{20\frac{k_4}{k_6}}; \quad \hat{\rho} = \frac{100}{3}\frac{k_4^3}{k_6^2}; \quad \hat{\mu} = \frac{k_1}{\hat{\rho}} \quad \text{and } \hat{\sigma}^2 = \frac{k_2}{\hat{\rho}} - \frac{2}{\hat{\alpha}^2}$$

where k_i $(i = 1, 2, 4, 6)$ is the ith. sample cumulant obtained either from the sample moments about zero, using well-established formulae (see *e.g.* Kendall & Stuart, 1969, p. 70) or from a Taylor series expansion of the sample cumulant generating function $\log(\frac{1}{n}\sum_{i=1}^{n} e^{sy_i})$.

For the five parameter (asymmetric) GNL distribution numerical methods must be used in part to solve the moment equations, which can be reduced to a pair of nonlinear equations in two variables *e.g.*

$$12k_3(\alpha^{-5} - \beta^{-5}) = k_5(\alpha^{-3} - \beta^{-3}); \qquad 3k_3(\alpha^{-4} - \beta^{-4}) = k_4(\alpha^{-3} - \beta^{-3})$$

with solutions for the other parameters being obtained analytically from these.

3 .A LÉVY PROCESS BASED ON THE GNL DISTRIBUTION – BROWNIAN-LAPLACE MOTION.

Consider now a Lévy process $\{X_t\}_{t\geq 0}$, say for which the increments $X_{t+\tau} - X_\tau$ have characteristic function $(\phi(s))^t$ where ϕ is the characteristic function (**??**) of the GNL$(\mu, \sigma^2, \alpha, \beta, \rho)$ distribution (such a construction is always possible for an infinitely divisible distribution - see *e.g.* Schoutens, 2003). It is not difficult to show that the Lévy-Khintchine triplet for this process is $(\rho\mu, \rho\sigma^2, \Lambda)$ where Λ is the Lévy measure of asymmetric Laplace motion (see Kotz *et al.*, 2001, p.196). Laplace motion has an infinite number of jumps in any finite time interval (a pure jump process). The extension considered here adds a continuous Brownian component to Laplace motion leading to the name *Brownian-Laplace motion.*

The increments $X_{t+\tau} - X_\tau$ of this process will follow a GNL$(\mu, \sigma^2, \alpha, \beta, \rho t)$ distribution and will have fatter tails than the normal – indeed fatter than exponential for $\rho t < 1$. However as t increases the excess kurtosis of the distribution drops, and approaches zero as $t \to \infty$. Exactly this sort of behaviour has been observed in various studies on high-frequency financial data (*e.g.* Rydberg, 2000) - very little excess kurtosis in the distribution of logarithmic returns over long intervals but increasingly fat tails as the reporting interval is shortened. Thus Brownian-Laplace motion seems to provide a good model for the movement of logarithmic prices.

3.1 OPTION PRICING FOR ASSETS WITH LOGARITHMIC PRICES FOLLOWING BROWNIAN-LAPLACE MOTION.

We consider an asset whose price S_t is given by

$$S_t = S_0 \exp(X_t)$$

where $\{X_t\}_{t\geq 0}$ is a Brownian-Laplace motion with $X_0 = 0$ and parameters $\mu, \sigma^2, \alpha, \beta, \rho$. We wish to determine the risk-neutral valuation of a European call option on the asset with strike price K at time T and risk-free interest rate r.

It can be shown using the Esscher equivalent martingale measure (see *e.g.* Schoutens, 2003, p. 77) that the option value can be expressed in a form similar to that of the Black-Scholes formula. Precisely

$$OV = S_0 \int_\gamma^\infty d_{GNL}^{*T}(x;\theta+1)dx - e^{-rT} K \int_\gamma^\infty d_{GNL}^{*T}(x;\theta)dx \tag{4}$$

where $\gamma = \log(K/S_0)$ and

$$d_{GNL}^{*T}(x;\theta) = \frac{e^{\theta x} d_{GNL}^{*T}(x)}{\int_{-\infty}^{\infty} e^{\theta y} d_{GNL}^{*T}(y)dy} \tag{5}$$

is the pdf of X_T under the risk-neutral measure. Here d_{GNL}^{*T} is the pdf of the T-fold convolution of the generalized normal-Laplace, GNL$(\mu, \sigma^2, \alpha, \beta, \rho)$, distribution and θ is the unique solution to the following equation involving the moment generating function (mgf) $M(s) = \phi(-is)$

$$\log M(\theta+1) - \log M(\theta) = r. \tag{6}$$

The T-fold convolution of GNL$(\mu, \sigma^2, \alpha, \beta, \rho)$ is GNL$(\mu, \sigma^2, \alpha, \beta, \rho T)$ and so its moment generating function is (from (**??**))

$$M(s) = \left[\frac{\alpha\beta \exp(\mu s + \sigma^2 s^2/2)}{(\alpha - s)(\beta + s)}\right]^{\rho T}.$$

This provides the denominator of the expression (**??**) for the risk-neutral pdf.

Now let

$$I_\theta = \int_\gamma^\infty d_{GNL}^{*T}(x;\theta)dx = \frac{1}{M(\theta)} \int_\gamma^\infty e^{\theta x} d_{GNL}^{*T}(x)dx \tag{7}$$

so that

$$OV = S_0 I_{\theta+1} - e^{-rT} K I_\theta.$$

Thus to evaluate the option value we need only evaluate the integral in (??). This can be done using the representation (??) of a GNL random variable as the sum of normal and positive and negative gamma components. Precisely the integral can be written as

$$\int_0^\infty g(u;\alpha) \int_0^\infty g(v;\beta) \int_\gamma^\infty e^{\theta x} \frac{1}{\sigma\sqrt{\rho T}} \phi\left(\frac{x-u+v-\mu\rho T}{\sigma\sqrt{\rho T}}\right) dxdvdu \tag{8}$$

where

$$g(x;a) = \frac{a^{\rho T}}{\Gamma(\rho T)} x^{\rho T - 1} e^{-ax}$$

is the pdf of a gamma random variable with scale parameter a and shape parameter ρT; and ϕ is the pdf of a standard normal deviate. After completing the square in x and evaluating the x integral in terms of Φ^c, the complementary cdf of a standard normal, (??) can be expressed

$$I_\theta = \int_0^\infty g(u;\alpha-\theta) \int_0^\infty g(v;\beta+\theta) \Phi^c\left(\frac{\gamma-u+v-\mu\rho T-\theta\sigma^2\rho T}{\sigma\sqrt{\rho T}}\right) dvdu. \tag{9}$$

For given parameter values the double integral (??) can be evaluated numerically quite quickly and thence the option value computed.

4. NUMERICAL RESULTS.

In this section comparisons are made between the value of a European call option under the assumption that price movements follow geometric Brownian-Laplace motion and the corresponding Black-Scholes option values (assuming geometric Brownian motion). We begin with nominal parameter values and then consider parameter values obtained from fitting to real financial data.

In all examples the strike price is set at $K = 1$ and the risk-free interest rate at $r = 0.05$ per annum. Fig.2 shows the difference between the Black-Scholes (BS) option value and the Brownian-Laplace (BL) option value assuming the (daily) increments for the logarithmic returns process are the six GNL distributions in Fig.1. Recall that for the three top panels the mean and variance are all 0 and 0.00165 respectively; while for the lower panels they are

0 and 0.00331. The BS option values were computed using these values and thus are the same for all three panels in the top row; and likewise for the three panels in the bottom row

In the top-left hand panel the parameters of the GNL distributed increments are $\mu = 0, \sigma^2 = 0.01, \alpha = 17.5, \beta = 17.5, \rho = 0.1$, which results in a symmetric distribution with coefficient of excess kurtosis $\gamma_2 = 4.68$. In the top middle panel α and β are reduced to 12.5 and σ^2 adjusted to 0.00373 (in order to maintain the variance at 0.00165) with other parameters remaining the same. This results in kurtosis with $\gamma_2 = 17.99$. In the top right-hand panel some skewness was introduced by setting $\alpha = 20.00, \beta = 15.75$ and adjusting μ to 0.0135 (in order to maintain the mean at zero). This results in coefficient of skewness $\gamma_1 = -0.390$ and of excess kurtosis $\gamma_2 = 4.94$.

In each panel the the three curves correspond to the difference between the BS and the BL option values with exercise date $T = 10, 30$ and 60 days in advance (in all panels highest peak corresponds to $T = 10$ and the lowest peak to $T = 60$). As can be seen the differences follow W-shaped curves similar to those obtained by Eberlein (2001), under the assumption of generalized hyperbolic Lévy motion. The biggest difference, which occurs "at the money" ($S = 1$) in the middle panel with $T = 10$, is about three tenths of a cent, but nonetheless amounts to 5.9% of the BS value. The corresponding maximum difference in the left-hand panel amounts to 1.5% of the BS value. Schoutens (2003, p.34) presents computed coefficients of kurtosis for various stock indices which range from 1.63 to 40.36 (this latter figure is for data spanning the crash of October, 1987 which resulted in a log return of -0.229). This would suggest that the parameter values considered in the top left-hand and center panels are quite plausible.

The effect of skewness in the distribution of returns can be seen in the top right-hand panel of Fig.1. The BS option value exceeds the BL one for smaller values of S with the situation reversed for larger values. The magnitude of the greatest positive difference (BS overvaluing the option) occurs "out of the money" ($S < K = 1$) and amounts to 3.2% of the BS value. The greatest negative difference occurs "in the money" ($S > K = 1$) and while of similar absolute magnitude amounts to only 0.34% of the (much higher) BS value. The value of the coefficient of skewness $\gamma_1 = -0.390$ used in this example is well within the range of those reported by Schoutens (2003, p.34) which range from -0.21 to -1.66, so the results suggest that ignoring skewness may cause considerable error in the theoretical valuation of options, as indeed is well known.

The bottom panels of Fig.1 show the effect of changing the parameter ρ to 0.2 (from 0.1 in the top panels). This has the effect of increasing the variance of the increments (by a factor of two) but at the same time reducing γ_1 (skewness) by a factor of $\sqrt{2}$ and reducing γ_2 (kurtosis) by a factor of 2. Overall it can be seen that the effect is to dampen the magnitude of the differences between BS and BL option values, but at the same time widen somewhat the range of values of S over which differences occur. The maximum difference (centre panel, $T = 10$, $S = 1$) amounts to 0.23 of a cent or 3.3% BS option value.

We turn now to an example using real data. Fig. 3 shows plots for the price of AT &T stock from Jan. 2, 2004 to Jan. 19, 2005. The Q-Q plot (of quantiles of the logarithmic returns against quantiles of a standard normal) in the lower right panel suggests some excess kurtosis. Also there is a suggestion of skewness to the left. However this seems to be caused largely by the one large negative return. The coefficients of skewness and excess kurtosis for the logarithmic returns are respectively -.667 and 6.07, but if the one large negative return is removed they become .217 and 1.66 (note change of sign of skewness). It could be that the data follow an asymmetric distribution or alternatively that they follow a symmetric somewhat fat-tailed distribution, with one large negative value occurring by chance. In view of this both a symmetric GNL distribution (with 4 parameters) and an asymmetric GNL distribution (with 5 parameters) appear to be plausible models and have in consequence been fitted to the data. The method-of-moments estimates are in the two cases respectively: (i) $\hat{\mu} = -0.000117, \hat{\sigma}^2 = 0.0000731, \hat{\alpha} = \hat{\beta} = 57.35, \hat{\rho} = 0.3938$; and $\hat{\mu} = 0.00698, \hat{\sigma}^2 = 0.000934, \hat{\alpha} = 55.53, \hat{\beta} = 39.50, \hat{\rho} = 0.1412$.

Fig. 4 shows Q-Q plots of logarithmic returns against (a) the 5-parameter GNL distribution; (b) the 4-parameter symmetric GNL distribution and (c) the 4-parameter generalized Laplace (GL) distribution (Kotz *et al.*, 2001). It can be seen that while there is little to choose between the fit of (a) and (b) in the tails, (b) does not fit so well in the upper flank; also both GNL models provide a considerably better fit than the generalized Laplace distribution (c), and the normal distribution (Fig. 3).

The Kolmogorov-Smirnov goodness of fit statistic has values 0.0333 and 0.0736 for the fit of the 5- and 4-parameter GNL distribution, respectively, and a value of 0.0662 for the generalized Laplace distribution.

Fig. 5 shows various differences in calculated option values as a function of current stock price with exercise date $T = 10, 30$ and 60 days ahead. The left-hand panel is the difference between the BS and BL using the fitted symmetric GNL distribution; the centre panel is similar but using the fitted asymmetric GNL distribution; and the right hand panel is the difference between the BL option value using the fitted symmetric and asymmetric GNL distributions. As one would expect, in absolute terms the differences are slightly larger for the fitted asymmetric GNL. However in percentage terms this is not the case. In comparison with the BL option value using the symmetric GNL the BS formula overvalues the option by the largest amount exactly "at the money" (at $S = 1$) with $T = 10$. This over-valuation amounts to 2.3% of the BS value. The biggest undervaluation with $T = 10$ occurs at $S = .925$ and amounts to 13.7% of the BS value. In comparison with the BL option value using the asymmetric GNL the corresponding percentages for the maximum overvaluation and undervaluation are 3.4% (overvaluation at $S = .975$) and 0.52% (undervaluation at $S = 1.075$). Note that the time until exercise affects the magnitude of the difference much less when the asymmetric model is fitted. This is true also for the difference between the BL option values using the fitted asymmetric and symmetric GNL distributions (right hand panel in Fig. 4).

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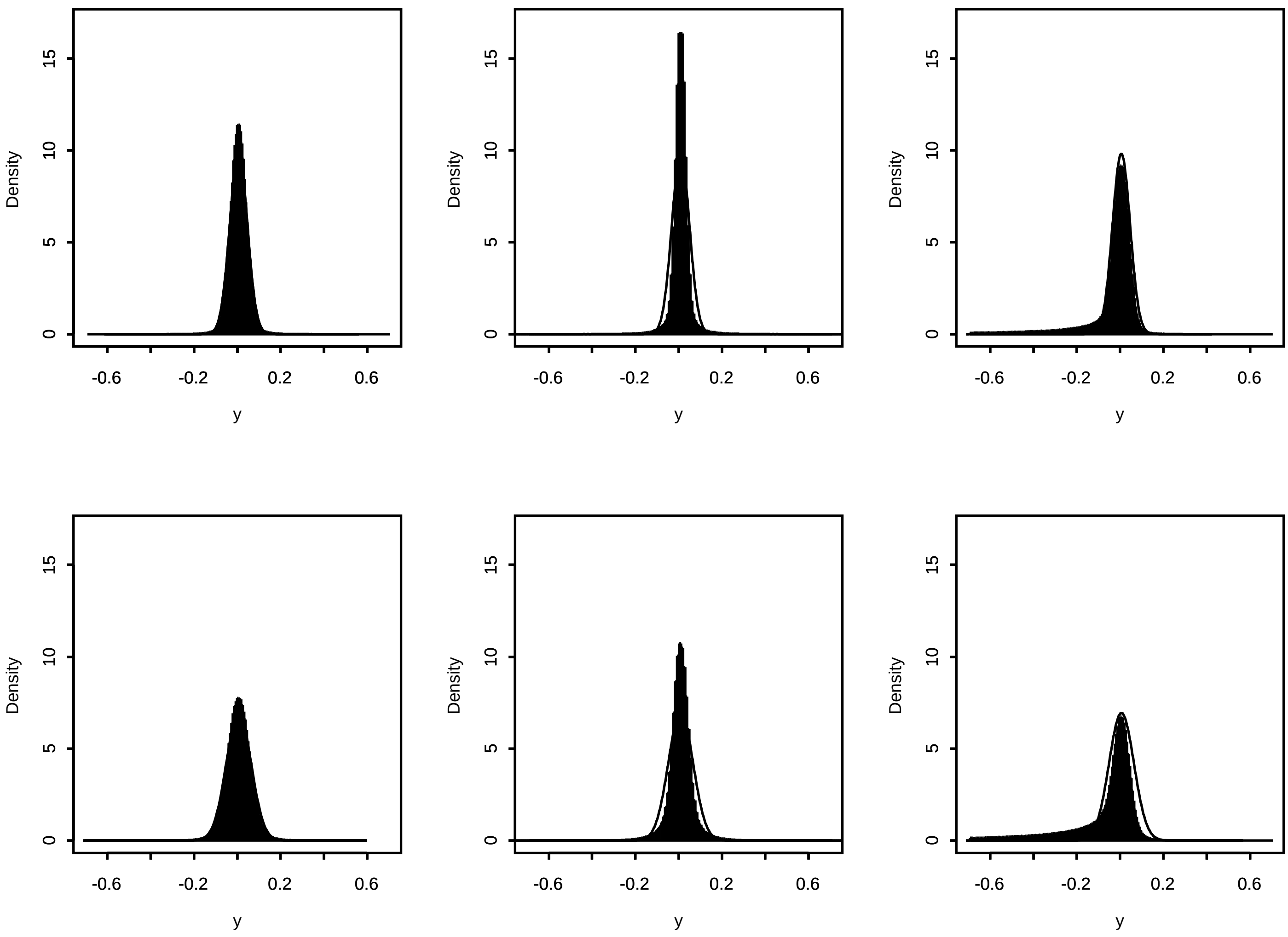

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15
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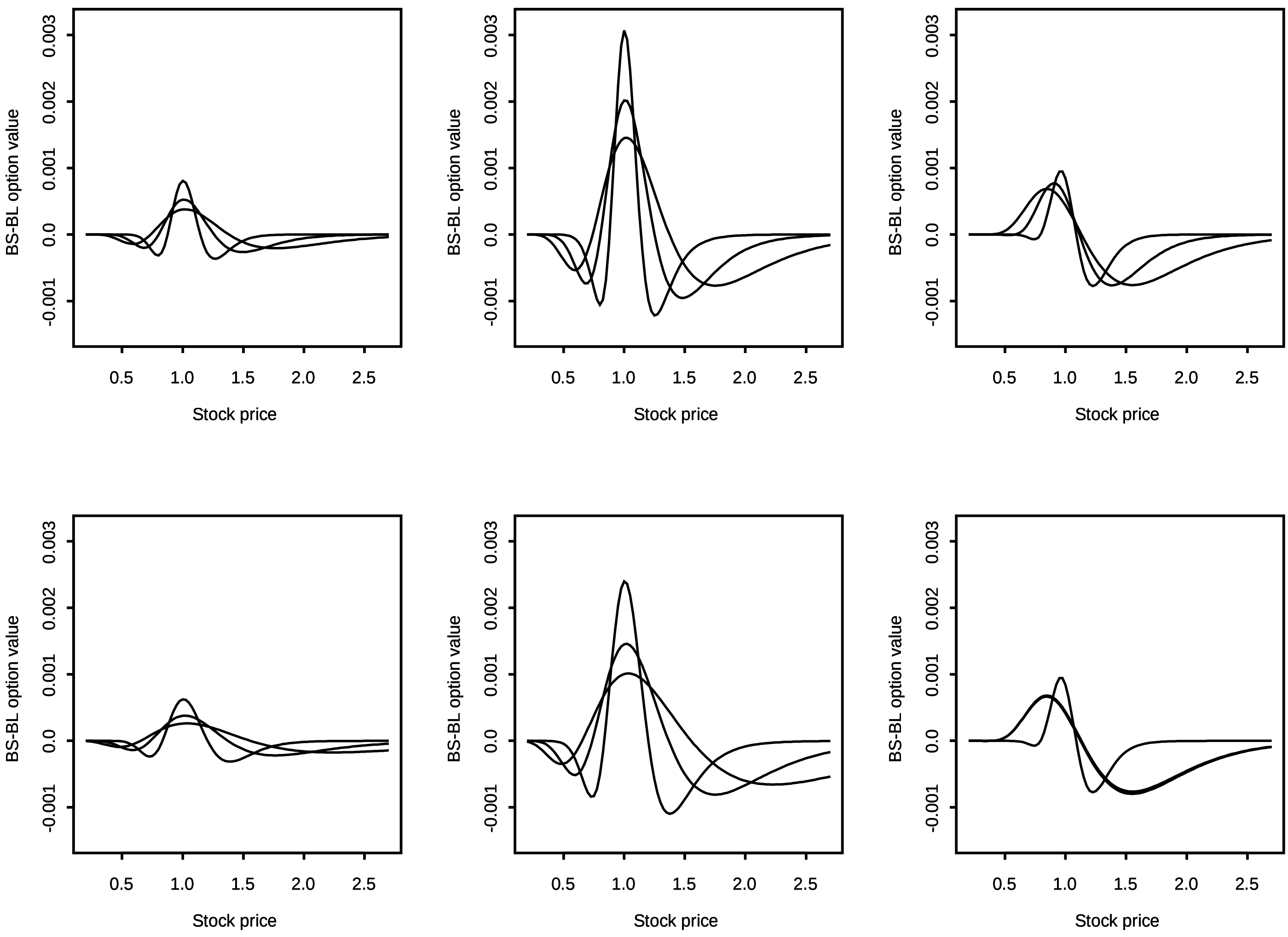
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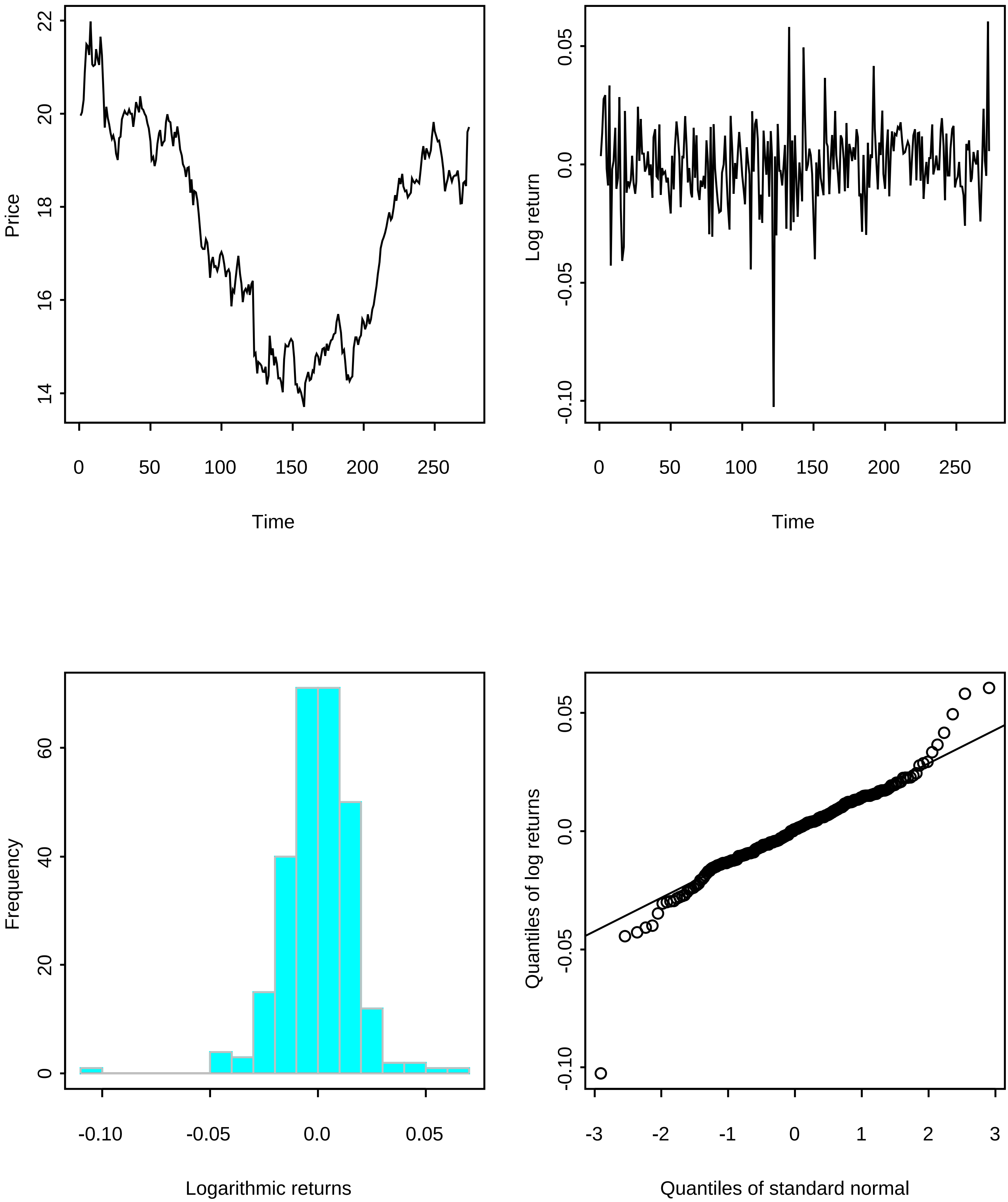
2.5
Stock price



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50
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150
200
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Time
Log return
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-0.10
0
50
100
150
200
250
Time
Frequency
60
40
20
0
-0.10
-0.05
0.0
0.05
Logarithmic returns
Quantiles of log returns
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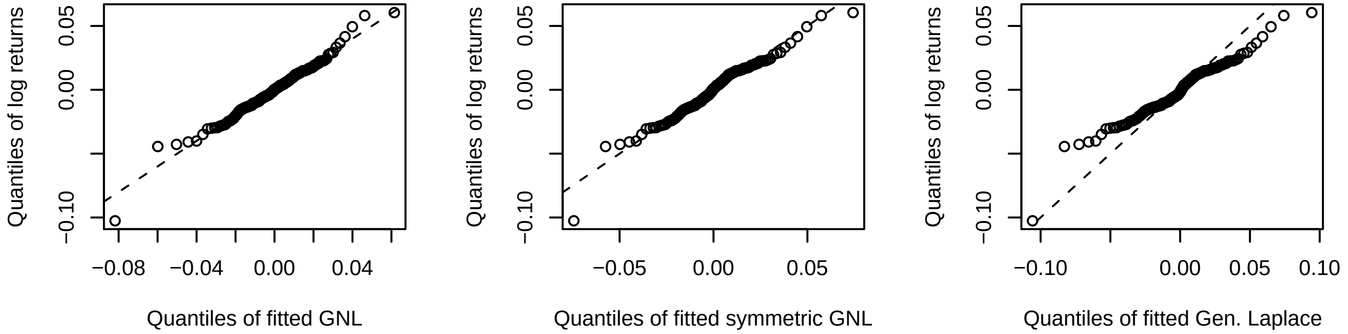
Quantiles of standard normal



Quantiles of log returns
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−0.04
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Quantiles of fitted GNL
Quantiles of log returns
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0.00
−0.10
−0.05
0.00
0.05
Quantiles of fitted symmetric GNL
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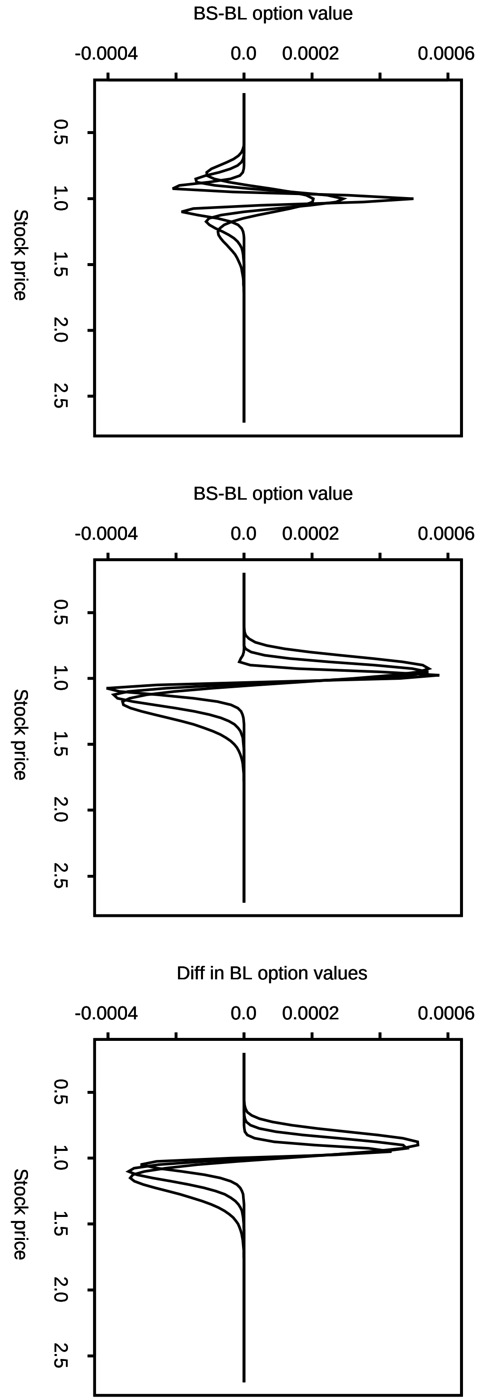
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-0.0004
0.0
0.0002
0.0006
Stock price
0.5
1.0
1.5
2.0
2.5
Diff in BL option values
-0.0004
0.0
0.0002
0.0006
Stock price
0.5
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