ON THE LOCAL SOLVABILITY FOR A QUASILINEAR CUBIC WAVE EQUATION

J. E. AZZOUZ, AND S. IBRAHIM

Abstract. This article is concerned with local solvability of the Cauchy problem for a quasilinear cubic wave equation in dimension $d = 3$. Here, we improve the index of regularity of the initial data compared to the one given by classical energy methods.

1. Introduction

This paper is devoted to the construction of local (in time) solutions of the Cauchy problem for a d-dimensional quasilinear wave equation of the type

$$\partial_t^2 u - \Delta u - G(\partial u) \cdot \nabla^2 u = 0, \tag{1.1}$$

where we set $\nabla u = (\partial_1 u, \partial_2 u, ..., \partial_d u)$, $\partial u = (\nabla u, \partial_t u)$ and

$$G \cdot \nabla^2 u = \sum_{1 \le j,k \le d} G^{jk} \partial_j \partial_k u.$$

Quasilinear wave equations appear frequently in general relativity such as Einstein equations or relativistic elasticity, hydrodynamics, minimal surfaces etc. We consider the particular case where the $d \times d$ symmetric matrix G satisfies the following elliptic equation

$$-\Delta G^{jk} = Q_{jk}(\partial u, \partial u) \tag{1.2}$$

where the $(Q_{jk})_{j,k}$ are quadratic forms on $\mathbb{R}^{1+d}$. This is known as the quasilinear cubic wave equation (see [3]). We assume that the initial data

$$(u, \partial_t u)_{|t=0} = (u_0, u_1), \tag{1.3}$$

is in the standard Sobolev space $H^s \times H^{s-1}$.

Recall that using the energy method, one can prove the local well-posedness for the system (1.1)-(1.3) when $s > \frac{d}{2} + \frac{1}{2}$. The crucial fact is to estimate the first derivatives of the metric G in $L^1_T(L^\infty)$. In fact, assuming that $\partial u \in L^\infty_T(H^{s-1})$ with $\frac{d}{2} + \frac{1}{2} < s < \frac{d}{2} + 1$, then the classical law for product shows that $\Delta^{-1}(\partial u)^2 \in H^{2s-\frac{d}{2}}$, and thanks to the Sobolev embedding we get $\partial G \in L^1_T(L^\infty)$. More precisely, we have the following result.

Theorem 1.1. *Let $d \ge 3$, $s > \frac{d}{2} + \frac{1}{2}$ and $(u_0, u_1) \in H^s \times H^{s-1}$.*

Assume that $\|(\nabla u_0, u_1)\|_{\dot{H}^{\frac{d}{2}-1}}$ is small enough. Then, there exists a positive time T and a unique solution u of the system (1.1)-(1.3) satisfying

$$u \in \mathcal{C}([0,T]; H^{\frac{d}{2}+\frac{1}{2}}) \cap \mathcal{C}^1([0,T]; H^{\frac{d}{2}-\frac{1}{2}}).$$

Moreover, a constant C exists (depending only on the initial data) such that $T \ge C\|(\nabla u_0, u_1)\|^{-2}_{\dot{H}^{\frac{d}{2}-1}}$.

1991 *Mathematics Subject Classification.* 35L05.

Here, $\dot{H}^s$ denotes the homogeneous Sobolev space endowed with the semi-norm

$$\|u\|_s^2 := \int_{\mathbb{R}^d} |\ \xi\ |^{2s} |\ \mathcal{F}u(\xi)\ |^2\ d\xi.$$

To improve upon the above existence result, one can use the smoothing properties of equation (1.1). Notice that (1.1) is invariant with respect to the dimensionless scaling $u(t,x) \to u(\lambda t, \lambda x)$. This scaling preserves the Sobolev space of exponent $s_c = \frac{d}{2}$, which is then (heuristically) a lower bound for the range of permissible s. Hence, the above theorem seems to require an extra $\frac{1}{2}$ derivative. The goal of this paper is to try to go as close as possible to the scaling invariant regularity.
Some results in this direction were obtained, in particular, for the equations of the form

$$\partial_t^2 u - \Delta u - g(u)\cdot\nabla^2 u = F(u)Q(\nabla u, \nabla u), \tag{1.4}$$

where

$$g\cdot\nabla^2 u = \sum_{1\le j,k\le d} g^{jk}\partial_j\partial_k u.$$

Q is a quadratic form on $\mathbb{R}^d$, $F \in \mathcal{D}(\mathbb{R})$ and g is a given smooth function, vanishing at 0 and with values in K such that $Id + K$ is a convex subset of positive symmetric matrices.

Recall that in the case of equation (1.4), the energy method allows us to prove the local well-posedness for initial data in $H^s \times H^{s-1}$ with $s > \frac{d+1}{2} + \frac{1}{2}$. We point out that all improvement results are based on Strichartz-type estimates for the wave operator with variable coefficients (as well as on bilinear estimates). When the coefficients are rough, these estimates present a loss of derivative compared to those obtained for the flat wave operator. The first result in this direction was by H. Bahouri-J. Y-Chemin [1] giving the well-posedness for $s > \frac{d+1}{2} + \frac{1}{4}$. Independently, D. Tataru obtained in [14] the same result. Shortly afterward, other improvements were obtained in [2] and in [15]. Later, D. Tataru provided in [16] and [17] a precise relationship between the smoothness of the metric and the corresponding loss in the Strichartz estimates. He pushed down the loss to $\frac{1}{6}^+$. Moreover, in [12], H. Smith-D. Tataru showed that the $\frac{1}{6}$ loss (in Strichartz estimates) is sharp in $d = 3$. In the case when the metric g itself solves an equation of the type (1.4), an important improvement (on the local well-posedness) over the $\frac{1}{6}$ result was proved by S. Klainerman-I. Rodnianski (see [9]). Recently, in regards to equations of the form (1.4), S. Klainerman-I. Rodnianski proved local existence for $s > 2$ for the Einstein vacuum equation in $d = 3$ (see [10]). Moreover, in [13], H. Smith-D. Tataru proved local existence for general equations of the form (1.4) for $s > \frac{7}{4}$ if $d = 2$, and $s > \frac{d+1}{2}$ if $d = 3, 4, 5$.
In the case of equation (1.1), H. Bahouri-J. Y-Chemin proved in [3] the following Theorem.

Theorem 1.2. *Let $d \ge 4$ and denote by $s_d = \frac{d}{2} + \frac{1}{6}$. Assume that $(u_0, u_1) \in H^s \times H^{s-1}(\mathbb{R}^d)$ with $s > s_d$ and $\|(\nabla u_0, u_1)\|_{\frac{d}{2}-1}$ is small enough. Then, there exist a positive time T and a unique solution u of (1.1)-(1.3) such that, for any small positive real number α we have*

$$T^{\frac{1}{6}+\alpha} \ge C_\alpha \|(\nabla u_0, u_1)\|^{-1}_{\frac{d}{2}-\frac{5}{6}+\alpha},$$

$$\partial u \in \mathcal{C}([0,T]; H^{s-1}) \cap L^2_T(\dot{\mathcal{B}}^{\frac{d}{4}-\frac{1}{2}}_{4,2}),\ \text{if } d \ge 5,$$

and

$$\partial u \in \mathcal{C}([0,T]; H^{s-1}) \cap L^2_T(\dot{\mathcal{B}}^{\frac{1}{6}}_{6,2}),\ \text{and } \partial G \in L^1_T(L^\infty)\ \text{if } d = 4.$$

$\dot{\mathcal{B}}^{\sigma}_{p,q}$ *denotes the homogeneous Besov space (see Definition 2.1).*

Note that the proof of Theorem 1.2 strongly depends on the space dimension; if $d \geq 5$ then, by proving the Strichartz inequalities for solutions of the "linearized equation", the authors succeed in exhibiting a Banach space $\mathcal{B}$ containing the solution u and having the property that, if $a \in L^2_T(\mathcal{B})$ then $\partial\Delta^{-1}(a^2) \in L^1_T(L^\infty)$. In particular, this is crucial to get an energy estimate. However, if $d = 4$ the use of Strichartz estimates is not sufficient. To overcome this difficulty, they followed an idea of S. Klainerman and D. Tataru, [11]. They proved microlocal bilinear estimates in the variable coefficients case. Our goal is to show that, using an $L^q(L^r)$ version of the Strichartz inequalities, we can extend the Bahouri-Chemin result to the case $d = 3$, obtaining a better index than that given by the energy method. Before stating the result, we introduce the following notation. For all $q \geq 2$, we define the loss of derivative ρ by

$$\rho(q) = \frac{1}{2} - \frac{2}{3q}. \tag{1.5}$$

We also set

$$s_d(q) = \frac{d}{2} + \rho(q) \tag{1.6}$$

and for all real number $r < d$ satisfying

$$\frac{2}{q} = (d-1)(\frac{1}{2} - \frac{1}{r}) < 1, \tag{1.7}$$

we define

$$\sigma_r = \frac{d}{r} - \frac{1}{2}. \tag{1.8}$$

Our main result is the following.

Theorem 1.3. *Let $s > s_3(6) = \frac{3}{2} + \frac{7}{18}$. There exists $q > 6$, r and σ_r given by (1.7)-(1.8) such that: if the initial data $(u_0, u_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ and $\|(\nabla u_0, u_1)\|_{\frac{3}{2}-1}$ is small enough, then a non trivial time T and a unique solution u of (1.1)-(1.3) exist and they satisfy*

$$\partial u \in \mathcal{C}([0,T]; H^{s-1}(\mathbb{R}^3)) \cap L^q_T(\dot{\mathcal{B}}^{\sigma_r}_{r,2}(\mathbb{R}^3)).$$

Remark 1.4. *In higher dimensions $d \geq 5$, following the same proof given here, we can show the local well-posedness for initial data $(u_0, u_1) \in H^s \times H^{s-1}(\mathbb{R}^d)$ with $s > s_d(2) = \frac{d}{2} + \frac{1}{6}$ and $\|(\nabla u_0, u_1)\|_{\frac{d}{2}-1}$ is small enough. This turns out to be the result of* [3]. *Meanwhile, if $d = 4$ then we obtain a minimal loss of derivative $\rho = \frac{1}{4}$ (which corresponds to the choice $(q, r) = (\frac{8}{3}, 4)$. This is of course not better than the Bahouri-Chemin result given by Theorem 1.2. To get a better result, they proved and used bilinear estimates in* [3].

Remark 1.5. *From the proof of Theorem 1.3 we can derive a lower bound of the time T; writing $s_\alpha := s_3(6) + \alpha = \frac{3}{2} + \rho(q_\alpha) + \frac{\alpha}{2}$ (with a small positive real number α), then a constant C_α exists such that*

$$T^{\frac{1}{18}+\frac{\alpha}{4}} \geq C_\alpha \|\gamma\|^{-1}_{s_\alpha - 1}.$$

To prove Theorem 1.3, we follow the method used in [3] based on a construction of an inductive scheme. The crucial fact is the use of an $L^q(L^r)$ version of the microlocal Strichartz estimates for the linearized equation. (Note that by microlocal estimates we mean estimates satisfied on time intervals which depend on the size of the spatial frequency).

This paper is organized as follows. In section 2, first we give a brief review of the Littlewood -Paley theory and we introduce some notation. Next, we explain the main idea of the result and point out the difficulty we observe to control $\|\partial G\|_{L^1_T(L^\infty)}$ even if u is the solution of the free wave equation. Finally, we state the microlocal Strichartz inequalities we will use. Section 3 is devoted to study some of the properties of the operator $\nabla\Delta^{-1}(a\cdot b)$. Then using paradifferential calculus, we localize the equation at frequencies fixed in a ring and we derive good estimates of the remainder terms. In section 4 we prove Theorem 1.3. First, we establish an *a priori* energy estimate for the solutions of (1.1). Then using Tataru counting method, we deduce the local Strichartz estimates. These estimates and the smallness of the interval $[0,T]$ can be used to close the energy estimate. In section 5, we outline the proof of Theorem 2.7.

2. Notations and preliminary results

2.1. **Some basic facts in Littlewood-Paley theory.** In the following, we give a brief review of the Littlewood-Paley theory. We refer the reader to [4] for a thorough treatment. Denote by $\mathcal{C}_0$ the ring defined by

$$\mathcal{C}_0 = \{\xi \in \mathbb{R}^d \text{ such that } \frac{3}{4} <| \xi |< \frac{8}{3}\},$$

and choose two non-negative radially symmetric functions $\chi \in \mathcal{D}(B(0,4/3))$ and $\varphi \in \mathcal{D}(\mathcal{C}_0)$ such that for all $\xi \in \mathbb{R}^d$

$$\varphi(2^{-k}\xi)\varphi(2^{-k'}\xi) = 0 \quad \text{when} \quad |k-k'| \geq 2$$

$$\chi(\xi)\varphi(2^k\xi) = 0,$$

and

$$\chi(\xi) + \sum_{k\in\mathbb{N}} \varphi(2^k\xi) = 1.$$

Let $\tilde{\mathcal{C}} = B(0,2/3) + \mathcal{C}_0$, then $\tilde{\mathcal{C}}$ is a ring satisfying

$$2^k\tilde{\mathcal{C}} \cap 2^{k'}\tilde{\mathcal{C}} = \emptyset \text{ when } \mid k-k' \mid > 5.$$

Denote by

$$h = \mathcal{F}^{-1}\varphi \text{ and } \tilde{h} = \mathcal{F}^{-1}\chi,$$

and define the operator Δ_k by, for all $u \in \mathcal{S}'(\mathbb{R}^d)$,

$$\Delta_k u = \varphi(2^{-k}D)u = 2^{dk}\int_{\mathbb{R}^d} h(2^k y)u(x-y)dy$$

$$S_k u = \sum_{j\leq k-1} \Delta_j u = \chi(2^{-k}D)u = 2^{dk}\int_{\mathbb{R}^d} \tilde{h}(2^k y)u(x-y)dy.$$

2.2. **Notations.** The Littlewood-Paley decomposition can be used to define the Besov spaces.

Definition 2.1. Let σ be a real number, and (p,q) in $[1,\infty[^2$. Let us state

$$\|u\|_{\dot{\mathcal{B}}^{\sigma}_{p,q}(I\!R^d)} := \Big(\sum_{k\in\mathbb{Z}} 2^{kq\sigma}\|\Delta_k u\|^q_{L^p}\Big)^{\frac{1}{q}}.$$

If $\sigma < \frac{d}{p}$ then the closure in $\mathcal{S}'$ of the compactly supported and smooth functions with respect to this norm is a Banach space. Note that $\dot{\mathcal{B}}^{\sigma}_{2,2}$ is the homogeneous Sobolev space $\dot{H}^{\sigma}$. The above definition can be extended to the case $p = q = \infty$ where $\dot{\mathcal{B}}^{\sigma}_{\infty,\infty}$ is nothing but the homogeneous Hölder space $\dot{\mathcal{C}}^{\sigma}$ with the semi-norm

$$\|u\|_{\dot{\mathcal{C}}^{\sigma}} = \|u\|_{\dot{\mathcal{B}}^{s}_{\infty,\infty}} := \sup_k 2^{k\sigma}\|\Delta_k u\|_{L^\infty}.$$

In all what follows, C denotes a universal constant which may change from line to line. We also make the convention that $(c_k(t))_k$ denotes a sequence which satisfies

$$\sum_{k\in\mathbb{Z}} c_k(t)^2 \leq 1.$$

Typically, we take $c_k(t) = \frac{2^{ks}\|\Delta_k u(t,.)\|_{L^2}}{\|u(t,.)\|_s}$. In the sequel, we set

$$\gamma := \partial u_{|t=0} = (\nabla u_0, u_1).$$

For any real number $0 < \alpha < \frac{2}{9}$, there exists $q_\alpha > 6$ such that $\rho(q_\alpha) = \frac{7}{18} + \frac{\alpha}{2}$. We define

$$s_\alpha := s_3(6) + \alpha = \frac{3}{2} + \rho(q_\alpha) + \frac{\alpha}{2},$$

$$\Gamma^{\alpha}_T(\gamma) := T^{\frac{1}{3q_\alpha}+\frac{\alpha}{2}}\|\gamma\|_{H^{s_\alpha-1}} = T^{\frac{1}{18}+\frac{\alpha}{4}}\|\gamma\|_{H^{s_\alpha-1}}$$

and

$$N^{\alpha}_T(\gamma) := T^{1-\frac{2}{q_\alpha}}\Gamma^{\alpha}_T(\gamma).$$

If $\mathcal{B}$ is a Banach space then then we set $\|u\|_{L^q_T(\mathcal{B})} = \|u\|_{L^q([0,T],\mathcal{B})}$. In the special case $q = \infty$ and $\mathcal{B} = \dot{H}^s$, we simply denote

$$\|u\|_{T,s} := \|u\|_{L^\infty([0,T],\dot{H}^s)}.$$

Definition 2.2. Let $\sigma \in \mathbb{R}$. Denote by $\tilde{L}^q_T(\dot{\mathcal{B}}^{\sigma}_{r,p}(\mathbb{R}^d))$ the set of distributions defined on $]0,T[\times\mathbb{R}^d$ such that

$$\|u\tilde{\|}_{L^q_T(\dot{\mathcal{B}}^{\sigma}_{r,p})} = \|(2^{k\sigma}\|\Delta_k u\|_{L^q_T(L^r)})_{k\in\mathbb{Z}}\|_{l^p}$$

is finite.

Remark 2.3. *The spaces $\tilde{L}^q_T(\dot{\mathcal{B}}^{\sigma}_{r,p}(\mathbb{R}^d))$ are adapted to the method we use. First, we localize in frequency by applying the projector Δ_k on the equation and then we take the time norm before summing with respect to k.*
In particular, in the case $p = q = 2$ and $r = \infty$, we simply denote by $\|u\tilde{\|}_{T,\sigma} := \|u\|_{\tilde{L}^\infty_T(\dot{\mathcal{B}}^{\sigma}_{2,2})}$. Note that we have

$$\|u\|_{T,\sigma} \leq \|\tilde{u}\|_{T,\sigma}$$

and

$$\|u\|_{L^q_T(\dot{\mathcal{B}}^\sigma_{r,p})} \leq \|\tilde{u}\|_{L^q_T(\dot{\mathcal{B}}^\sigma_{r,p})}.$$

Fix a cut-off function $\theta \in \mathcal{D}(]-1,1[)$ whose value is 1 near 0. For any sufficiently smooth function v, we denote by $G_{v,T}$ the truncated metric given by $G_{v,T}(t,x) = \theta(\frac{t}{T})G(\partial v)(t,x)$.

2.3. **Main idea of the result.** Here we want to explain the choice of the parameters ρ, σ and q in any space dimension. The basic fact in the proof of Theorem 1.3 is the energy estimate. This requires the control of

$$\int_0^T \|\partial G(\partial u)(t,.)\|_{L^\infty} dt. \tag{2.9}$$

First, we recall the following law of product in $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^d)$.

Proposition 2.4. *Let $r \geq 2$ and $\frac{d}{2r} < \sigma < \frac{d}{r}$, then for all $a \in \dot{\mathcal{B}}^\sigma_{r,2}(\mathbb{R}^d)$, we have $a^2 \in \dot{\mathcal{B}}^{2\sigma-\frac{d}{r}}_{r,1}(I\!R^d)$.*

In the particular case where $\sigma = \frac{d}{r} - \frac{1}{2}$ and $r < d$, the above proposition implies that if $\partial u \in \dot{\mathcal{B}}^{\frac{d}{r}-\frac{1}{2}}_{r,2}(I\!R^d)$, then $\nabla\Delta^{-1}(\partial u)^2 \in L^\infty$.

Usually, the space $\dot{\mathcal{B}}^{\frac{d}{r}-\frac{1}{2}}_{r,2}$ is determined using Strichartz inequalities. In the constant coefficients case, they are given by the following proposition (see [6]).

Proposition 2.5. *Let $\mathcal{C}_1$ be an ring in $\mathbb{R}^d$ and $u(t,x)$ be a function such that, for a positive real number λ, the function $\mathcal{F}_x u(t)$ is supported in the ring $\lambda\mathcal{C}_1$.*
Then, for any two positive real numbers q and r satisfying (1.7) we have the following estimate

$$\|\partial^{1+j}u\|_{L^q_T(L^r)} \leq \lambda^{\mu+j}\big(\|\partial u_{|t=0}\|_{L^2} + C\|\Box u\|_{L^1_T(L^2)}\big), \tag{2.10}$$

with $\mu = d(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$ and $\Box = \partial_t^2 - \Delta$.

Let us first explain the idea how one can have a control of $\|\partial G(\partial u)\|_{L^1_T(L^\infty)}$ in the simple case where u is the solution of the free wave equation. We want to estimate

$$\int_0^T \|\partial\Delta^{-1}(\partial u \cdot \partial u)(t,.)\|_{L^\infty} dt.$$

We have to estimate an expression of the type

$$\int_0^T \|\Delta^{-1}(\partial^2 u \cdot \partial u)(t,.)\|_{L^\infty} dt.$$

Recall the Bony's decomposition (see [5]).

$$a \cdot b = T_a(b) + T_b(a) + R(a,b),$$

where

$$T_a(b) = \sum_j S_{j-1}(a)\Delta_j b$$

and the remainder term is

$$R(a,b) = \sum_{\substack{j\in\mathbb{Z} \\ -1\le l\le 1}} \Delta_j a \Delta_{j-l} b.$$

Using Hölder inequality and Bernstein's Lemma, we have

$$\|\Delta^{-1}\sum_k S_{k-1}(\partial^2 u)\Delta_k \partial u\|_{L^1_T(L^\infty)} \le CT^{1-\frac{2}{q}}\sum_k 2^{k(\frac{d}{r}-2)}\|S_{k-1}(\partial^2 u)\|_{L^q_T(L^\infty)}\|\Delta_k\partial u\|_{L^q_T(L^r)}.$$

On the other hand, applying Bernstein's Lemma and estimate (2.10) to the first factor in the above sum, we have

$$\begin{aligned}\|S_{k-1}(\partial^2 u)\|_{L^q_T(L^\infty)} &\le C\sum_{k'\le k-2} 2^{k'(\frac{d}{r}+1)}\|\Delta_{k'}\partial u\|_{L^q_T(L^r)} \\ &\le C\sum_{k'\le k-2} 2^{k'(\frac{d}{r}+1)}2^{k'(\frac{d}{2}-\frac{d}{r}-\frac{1}{q})}\|\Delta_{k'}\gamma\|_{L^2}.\end{aligned}$$

Setting $\rho_0(q) = \frac{1}{2}-\frac{1}{q}$ and applying Young's inequality we obtain

$$\|S_{k-1}(\partial^2 u)\|_{L^q_T(L^\infty)} \le C2^{\frac{3k}{2}}\|\gamma\|_{\frac{d}{2}-1+\rho_0(q)}.$$

Therefore $\|\Delta^{-1}T_{\partial^2 u}\partial u\|_{L^1_T(L^\infty)} \le CT^{2\rho_0(q)}\|\gamma\|^2_{\frac{d}{2}-1+\rho_0(q)}$.
The symmetric term can be treated exactly along the same lines. For the remainder term we have, for all $r\ge 2$

$$\|\Delta_p\Delta^{-1}\sum_{\substack{-1\le j\le 1 \\ k\ge p-N_0}} \Delta_k(\partial^2 u)\Delta_{k-j}\partial u\|_{L^1_T(L^\infty)} \le$$

$$CT^{2\rho_0(q)}\sum_{k-p\ge -N_0} 2^{2p(\frac{d}{r}-1)}\|\Delta_k\partial^2 u\|_{L^q_T(L^r)}\|\Delta_{k-j}\partial u\|_{L^q_T(L^r)}.$$

Thanks to Strichartz inequalities (2.10) we can rewrite the above inequality as,

$$\|\Delta_p\Delta^{-1}\sum_{\substack{-1\le j\le 1 \\ k\ge p-N_0}} \Delta_k(\partial^2 u)\Delta_{k-j}\partial u\|_{L^1_T(L^\infty)} \le CT^{2\rho_0}\sum_{k-p\ge -N_0} 2^{2(p-k)(\frac{d}{r}-1)}2^{2k(\frac{d}{2}+\rho_0(q)-1)}\|\Delta_k\gamma\|^2_{L^2}.$$

Applying Young's inequality (since moreover $r<d$), we obtain

$$\|\Delta^{-1}R(\partial^2 u,\partial u)\|_{L^1_T(L^\infty)} \le CT^{2\rho_0(q)}\|\gamma\|^2_{\frac{d}{2}-1+\rho_0(q)}.$$

Therefore,

$$\|\partial G(\partial u)(t,.)\|_{L^1_T(L^\infty)} \le CT^{2\rho_0(q)}\|\gamma\|^2_{\frac{d}{2}-1+\rho_0(q)}.$$

Remark 2.6. *Observe that in the above setting, a loss of derivative $\rho_0 = 0$ corresponds to the choice $q = 2$. If $d = 3$, the pair $(q, r) = (2, \infty)$ is not admissible and therefore it seems hard to reduce the regularity index to that given by scaling arguments using only Strichartz estimates. In our work, we prove an $L^q(L^r)$ version of local Strichartz estimates. The loss of derivative $\rho(q)$ that we obtain is $\rho(q) = \rho_0(q) + 1/3q$, where $1/3q$ is the loss due to the summation of the microlocal Strichartz estimates.*

2.4. **Strichartz inequalities.** Let $\mathcal{G} = (G_\Lambda)_{\Lambda \geq \Lambda_o > 0}$ be a family of smooth, matrix-valued functions defined on $I_\Lambda \times \mathbb{R}^d$ where I_Λ is a time interval containing 0. Denote by

$$\|\mathcal{G}\|_0 := \sup_{\Lambda \geq \Lambda_0} \|\partial G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} + |I_\Lambda| \|\nabla^2 G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} \tag{2.11}$$

and

$$\|\mathcal{G}\|_l := \sup_{\Lambda \geq \Lambda_0} |I_\Lambda| \Lambda^l \|\nabla^{l+2} G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} \quad \text{for } l \geq 1, \tag{2.12}$$

and assume that $\|G_\Lambda\|_{L^\infty}$ is small enough. Let P_Λ be the operator

$$P_\Lambda v := \partial_t^2 v - \Delta v - \sum_{k,l} G_\Lambda^{k,l} \partial_k \partial_l v. \tag{2.13}$$

The Strichartz estimates that we will use are the following

Theorem 2.7. *Let ε_0 be a positive real number and $\mathcal{C}$ be a fixed ring in $\mathbb{R}^d$. Fix $(q, r) \in [2, \infty[^2$ such that $\frac{2}{q} = (d-1)(\frac{1}{2} - \frac{1}{r})$ $q \neq 2$ if $d = 3$, and consider a family $\mathcal{G}$ as above and such that for any l, $\|\mathcal{G}\|_l$ is finite and $\|\mathcal{G}\|_0$ is small enough i.e $\|\mathcal{G}\|_0 \leq \delta$. Then, for any positive real number $\varepsilon \leq \varepsilon_0$, a constant C exists such that if v_Λ is the solution of*

$$(E_\Lambda) \quad \begin{cases} P_\Lambda v_\Lambda = & f \\ \partial v_{\Lambda|\tau=0} = & \gamma, \end{cases}$$

on an interval I_Λ satisfying

$$|I_\Lambda| \leq \Lambda^{2-\varepsilon},$$

and where $f \in L^1(I_\Lambda, L^2)$ and $\gamma \in L^2$ are two functions for which the Fourier transform is included in $\mathcal{C}$ then v_Λ satisfies the following estimate

$$\|\partial v_\Lambda\|_{L^q(I_\Lambda,\, L^r)} \leq C(\|\gamma\|_{L^2} + \|f\|_{L^1(I_\Lambda,\, L^2)}). \tag{2.14}$$

This estimate is established by Bahouri-Chemin in [1]. The proof is based on a dispersive estimate satisfied by an approximate solution to (1.1). We shall outline the proof of Theorem 2.7 in Section 5.

3. Paradifferential calculus

In all what follows, we take $d = 3$. Along this work, we shall deal with quantities of the form $\Delta^{-1}(a.b)$. In the sequel, we summarize some of their properties.

Lemma 3.1. *Assume $\sigma > \frac{3}{2}$, then a constant C exists such that*

$$\|\Delta^{-1}(a \cdot b)\|_{\dot{H}^{\sigma+\frac{1}{2}}} \leq C\big(\|a\|_{\dot{H}^{\sigma-1}} \|b\|_{\dot{\mathcal{C}}^{-\frac{1}{2}}} + \|b\|_{\dot{H}^{\sigma-1}} \|a\|_{\dot{\mathcal{C}}^{-\frac{1}{2}}}\big). \tag{3.15}$$

Moreover, if $\sigma > \frac{3}{2} - \frac{3}{r}$ with $r \geq 1$ then,

$$\|\Delta^{-1}(a \cdot b)\|_{\dot{H}^{\sigma+\frac{1}{2}}} \leq C\big(\|a\|_{\dot{H}^{\sigma-1}} \|b\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}} + \|b\|_{\dot{H}^{\sigma-1}} \|a\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}\big). \tag{3.16}$$

A constant C exists such that

$$\|\Delta^{-1}(a\cdot b)\|_{\dot{\mathcal{B}}^{\frac{3}{2}}_{2,1}}\le C\|a\|_{\dot{H}^{\frac{3}{2}-1}}\|b\|_{\dot{H}^{\frac{3}{2}-1}}. \tag{3.17}$$

Moreover, if $1\le r<3$, then a constant C exists such that

$$\|\nabla\Delta^{-1}(a\cdot b)\|_{\dot{\mathcal{B}}^{\frac{3}{r}}_{r,1}}\le C\|a\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}\|b\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}. \tag{3.18}$$

Proof. The proof of this lemma is an easy application of the paradifferential calculus. We refer the reader to [4] for the proof of (3.15) and (3.17). For the sake of completeness we shall prove (3.18) and (3.16).
We apply Bony's decomposition

$$a\cdot b=T_a(b)+T_b(a)+R(a,b).$$

We begin by proving the following

$$\|a\cdot b\|_{\dot{\mathcal{B}}^{\frac{3}{r}-1}_{r,1}}\le C\|a\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}\|b\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}},$$

which clearly proves (3.18). Using Bernstein's lemma and the fact that $R(a,b)$ has a Fourier transform supported in a ball, an integer $N_0\in\mathbb{N}$ exists such that for all $k\in\mathbb{Z}$,

$$\begin{aligned}\|\Delta_k R(a.b)\|_{L^r} &\le \sum_{\substack{j\ge k-N_0\\ -1\le l\le 1}}\|\Delta_j a\|_{L^\infty}\|\Delta_{j-l}b\|_{L^r}\\ &\le \sum_{\substack{j\ge k-N_0\\ -1\le l\le 1}}2^{j\frac{3}{r}}\|\Delta_j a\|_{L^r}\|\Delta_{j-l}b\|_{L^r}.\end{aligned}$$

Hence,

$$2^{k(\frac{3}{r}-1)}\|\Delta_k R(a.b)\|_{L^r}\le\sum_{j\ge k-N_0}2^{(k-j)(\frac{3}{r}-1)}2^{j(\frac{3}{r}-\frac{1}{2})}\|\Delta_j a\|_{L^r}2^{j(\frac{3}{r}-\frac{1}{2})}\|\Delta_j b\|_{L^r}.$$

Using Young's inequality for sequences and the fact that $r<3$, we obtain

$$\sum_{k\in\mathbb{Z}}2^{k(\frac{3}{r}-1)}\|\Delta_k(R(a.b))\|_{L^r}\le C\|a\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}\|b\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}.$$

To conclude the proof of (3.18), it suffices to estimate the term $\|\Delta_k T_a(b)\|_{L^r}$ and do the same for the symmetric term $T_b(a)$.
Note that the Fourier transform of the function $S_{j-1}(a)\Delta_j b$ is included in a ring of the type $2^j\tilde{\mathcal{C}}$. So

$$\sum_{j\in\mathbb{Z}}\Delta_k(S_{j-1}(a)\Delta_j b)=\sum_{|k-j|\le 5}\Delta_k(S_{j-1}a\Delta_j b).$$

Moreover, applying Bernstein's Lemma and Young's inequality, there exists a sequence (d_j) satisfying $\sum d_j^2=1$ and such that

$$\sum_{l\le j-2}\|\Delta_l a\|_{L^\infty}\le 2^{\frac{j}{2}}d_j\|a\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}.$$

Therefore,

$$2^{k(\frac{3}{r}-1)}\|\Delta_k T_a(b)\|_{L^r} \leq \|a\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}} \sum_{|k-j|\leq 5} 2^{(k-j)(\frac{3}{r}-1)}\big(d_j 2^{j(\frac{3}{r}-\frac{1}{2})}\|\Delta_j b\|_{L^r}\big).$$

Taking the sum in $l^1(\mathbb{Z})$, we deduce (3.18).
To prove (3.16), we choose $\beta > 1$ such that $\frac{1}{\beta} = \frac{1}{2} + \frac{1}{r}$. Applying Bernstein's lemma and Hölder inequality we obtain

$$\begin{aligned} 2^{k(\sigma-\frac{3}{2})}\|\Delta_k R(a,b)\|_{L^2} &\leq \sum_{\substack{j\geq k-N_0\\ -1\leq l\leq 1}} 2^{k(\sigma-\frac{3}{2})}2^{3\frac{k}{r}}\|\Delta_j a \Delta_{j-l} b\|_{L^\beta} \\ &\leq \sum_{j\geq k-N_0} 2^{(k-j)(\sigma-\frac{3}{2}+\frac{3}{r})}2^{j(\frac{3}{r}-\frac{1}{2})}\|\Delta_j a\|_{L^r}2^{j(\sigma-1)}\|\Delta_{j-l} b\|_{L^2}. \end{aligned}$$

The fact that $\sigma > \frac{3}{2} - \frac{3}{r}$ completes the proof. □

To establish an H^s energy estimate for the solutions of (1.1) and for non integer values of s, we also use the paradifferential calculus. The problem is then to study the commutator between a multiplication and the pseudo-differential operator Δ_k.

3.1. **Paralinearization of the equation.**

Lemma 3.2. *Let $s > \frac{3}{2} - \frac{3}{r}$. A constant C exists such that, if u, v and F are three functions satisfying:*
∂u and ∂v are in $L^\infty_T(\dot{H}^{s-1}) \cap L^q_T(\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2})$, $G_{v,T} \in L^1_T(L^\infty)$, $F \in L^1_T(\dot{H}^{s-1})$ and

$$\partial_t^2 u - \Delta u - G_{v,T}\cdot\nabla^2 u = F,$$

then, $u_k := \Delta_k u$ is the solution of

$$\partial_t^2 u_k - \Delta u_k - S_{k-1}(G_{v,T})\cdot\nabla^2 u_k = F_k + R_k(\nabla u, \partial v),$$

where $F_k = \Delta_k F$ and the remainder term $R_k(\nabla u, \partial v)$ satisfies the following estimate

$$\begin{aligned} \|R_k(\nabla u,\partial v)(t,\cdot)\|_{L^2} &\leq Cc_k(t)2^{-k(s-1)}\|\nabla G_{v,T}(t,\cdot)\|_{L^\infty}\|\nabla u(t,.)\|_{s-1} \\ &+ Cc_k(t)2^{-k(s-1)}\|\partial v(t,\cdot)\|_{s-1}\|\partial v(t,.)\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}\|\partial u(t,.)\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}. \end{aligned}$$

Proof. Theorem 2.1 in [3]. We split the product $G_{v,T}\nabla^2 u$ into the two following terms.

$$\begin{aligned} G_{v,T}\nabla^2 u &= \sum_j S_{j-1}(G_{v,T})\cdot\nabla^2 u_j + \sum_j S_{j+2}(\nabla^2 u)\Delta_j G_{v,T} \\ &= R_1 + R_2. \end{aligned}$$

As previously done, the first term

$$R_1 := \sum_{j\in\mathbb{Z}} S_{j-1}(G_{v,T})\nabla^2 u_j$$

is easy to estimate since the Fourier transform of $S_{j-1}(G_{v,T})\nabla^2 u_j$ is supported in the ring $2^j\tilde{\mathcal{C}}$. Hence, we have

$$\begin{aligned}\Delta_k R_1 &= S_{k-1}(G_{v,T})\cdot\nabla^2 u_k+\sum_j\big(S_{j-1}(G_{v,T})-S_{k-1}(G_{v,T})\big)\cdot\Delta_k(\nabla^2 u_j)\\ &+\sum_{|k-j|\le 5}\big[\Delta_k,S_{j-1}(G_{v,T})\big]\nabla^2 u_j.\end{aligned}$$

Using the following estimate on the commutator (for more details see [4] or Lemma 8.2 in [9]),

$$\|[\Delta_k,a]b\|_{L^2}\le C2^{-k}\|\nabla a\|_{L^\infty}\|b\|_{L^2},$$

we get

$$\begin{aligned}\sum_{|k-j|\le 5}\|\big[\Delta_k,S_{j-1}(G_{v,T})\big]\nabla^2 u_j\|_{L^2} &\le C\sum_{|k-j|\le 5}2^{-k}\|\nabla S_{j-1}(G_{v,T})\|_{L^\infty}\|\nabla^2 u_j\|_{L^2}\\ &\le C\|\nabla G_{v,T}\|_{L^\infty}\|\nabla u\|_{s-1}2^{-k(s-1)}\sum_{|k-j|\le 5}2^{(k-j)(s-1)}c_j(t)\\ &\le Cc_k(t)\|\nabla G_{v,T}\|_{L^\infty}\|\nabla u\|_{s-1}2^{-k(s-1)}.\end{aligned}$$

Hence,

$$\sum_{|k-j|\le 5}\|\big[\Delta_k,S_{j-1}(G_{v,T})\big]\nabla^2 u_j\|_{L^2}\le Cc_k(t)2^{-k(s-1)}\|\nabla G_{v,T}\|_{L^\infty}\|\nabla u\|_{s-1}.$$

Similarly, applying Cauchy-Schwartz's inequality and using Bernstein's lemma we have

$$\|\big(S_{j-1}(G_{v,T})-S_{k-1}(G_{v,T})\big)\cdot\nabla^2 u_j\|_{L^2}\le\sum_{l\in[j-2,k-2]}2^{-l}\|\nabla G_{v,T}\|_{L^\infty}2^j\|\nabla u_j\|_{L^2}.$$

Therefore,

$$\|\sum_{|k-j|\le 5}\big(S_{j-1}(G_{v,T})-S_{k-1}(G_{v,T})\big)\cdot\nabla^2 u_j\|_{L^2}\le\|\nabla G_{v,T}\|_{L^\infty}\|\nabla u\|_{s-1}\sum_{\substack{|k-j|\le 5\\ l\in[j-2,k-2]}}2^{j-l}2^{-j(s-1)}c_j.$$

Note that since the number of l, $l\in[j-2,k-2]$ such that $\mid k-j\mid\le 5$ is finite, then

$$\|\sum_{|k-j|\le 5}\big(S_{j-1}(G_{v,T})-S_{k-1}(G_{v,T})\big)\cdot\nabla^2 u_j\|_{L^2}\le C\|\nabla G_{v,T}\|_{L^\infty}\|\nabla u\|_{s-1}\sum_{|k-j|\le 5}2^{-j(s-1)}c_j.$$

Using Young's inequality, we get

$$\|\sum_{j,|k-j|\le 5}\big(S_{j-1}(G_{v,T})-S_{k-1}(G_{v,T})\big)\cdot\nabla^2 u_j\|_{L^2}\le C\|\nabla G_{v,T}\|_{L^\infty}\|\nabla u\|_{s-1}2^{-k(s-1)}c_k(t).$$

Now we estimate the term R_2. The Fourier transform of $S_{j+2}(\nabla^2 u)\Delta_j G_{v,T}$ is included in a ball of the form $B(0,C2^j)$ then

$$\Delta_k R_2=\sum_{j\ge k-N_1}\Delta_k(S_{j+2}(\nabla^2 u)\Delta_j G_{v,T}).$$

Moreover, the following estimate

$$\|S_{j+1}(\nabla^2 u)\|_{L^\infty} \leq 2^{j\frac{3}{2}}\|\nabla u\|_{\dot{\mathcal{C}}^{-\frac{1}{2}}},$$

together with the fact that the space $\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}$ is continuously embedded in $\dot{\mathcal{C}}^{-\frac{1}{2}}$ give

$$\|S_{j+1}(\nabla^2 u)\|_{L^\infty} \leq 2^{j\frac{3}{2}}\|\nabla u\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}.$$

The above estimate and Lemma 3.1 show that

$$\|\Delta_j(G_{v,T})(t)\|_{L^2} \leq Cc_j(t)2^{-j(s+\frac{1}{2})}\|\partial v(t)\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}\|\partial v(t)\|_{s-1}.$$

Using Young's inequality for sequences, the proof of Lemma 3.2 is complete. □

In the proof of Theorem 2.7, we need to localize equation (1.1) in such a way that the frequencies of the metric are much smaller than those of the solution. In fact, the pseudo-differential operator defined above does not have any symbolic calculus and therefore they do not allow the construction of a parametrix for the operator (2.13) in the spirit of Hadamard's method. In the following corollary, we prove a precise paralinearization.

Corollary 3.3. *Let $s > \frac{3}{2}-\frac{3}{r}$. A constant C exists such that, if u, v and F are three functions satisfying:*
∂u and ∂v are in $L^\infty_T(\dot{H}^{s-1}) \cap L^q_T(\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2})$, $G_{v,T} \in L^1_T(L^\infty)$, $F \in L^1_T(\dot{H}^{s-1})$ and such that

$$\partial_t^2 u - \Delta u - G_{v,T}\cdot\nabla^2 u = F,$$

then for any $\delta \in [0,1]$, we have

$$\partial_t^2 u_k - \Delta u_k - S_k^\delta(G_{v,T})\cdot\nabla^2 u_k = F_k + R_k^\delta(\nabla u, \partial v),$$

where

$$S_k^\delta b = S_{k\delta-(1-\delta)\ln_2 T - N_0} b$$

and

$$\begin{aligned}\|R_k^\delta(\nabla u,\partial v)(t,\cdot)\|_{L^1_T(L^2)} &\leq Cc_k 2^{-k(s-1)}(1+(2^kT)^{1-\delta})\Big[\|\nabla G_{v,T}\|_{L^1_T(L^\infty)}\|\nabla u\|_{T,s-1}\\ &+ T^{1-\frac{2}{q}}\|\partial v\|_{T,s-1}\|\partial v\|_{L^q_T(\dot{\mathcal{B}}^{\sigma_r}_{r,2})}\|\partial u\|_{L^q_T(\dot{\mathcal{B}}^{\sigma_r}_{r,2})}\Big].\end{aligned}$$

Proof. Using Lemma 3.2 we can write

$$R_k^\delta(\nabla u,\partial v) = R_k(\nabla u,\partial v) + (S_k^\delta - S_{k-1})(G_{v,T})\cdot\nabla^2 u_k.$$

Hence it suffices to handle $(S_k^\delta - S_{k-1})G_{v,T}\cdot\nabla^2 u_k$.
Note that

$$\|(S_k^\delta - S_{k-1})G_{v,T}\cdot\nabla^2 u_k\|_{L^1_T(L^2)} \leq \|(S_k^\delta - S_{k-1})G_{v,T}\|_{L^1_T(L^\infty)}\|\nabla^2 u_k\|_{L^\infty_T(L^2)}.$$

On the other hand, thanks to Bernstein's lemma we have

$$\begin{aligned}\|(S_k^\delta - S_{k-1})G_{v,T}\|_{L^1_T(L^\infty)} &\leq C \sum_{p\geq k\delta-(1-\delta)\ln_2 T - N_0} 2^{-p}\|\Delta_p(\nabla G_{v,T})\|_{L^1_T(L^\infty)} \\ &\leq C\|\nabla G_{v,T})\|_{L^1_T(L^\infty)} \sum_{p\geq k\delta-(1-\delta)\ln_2 T - N_0} 2^{-p} \\ &\leq C2^{-k\delta+(1-\delta)ln_2 T}\|\nabla G_{v,T}\|_{L^1_T(L^\infty)}.\end{aligned}$$

Noticing that $2^{-k\delta+(1-\delta)ln_2 T} = 2^{-k}(2^kT)^{1-\delta}$, we obtain the desired estimate on the reminder term. □

4. Proof of The main result

Recall that

$$s_\alpha := s_3(6) + \alpha = \frac{3}{2} + \rho(q_\alpha) + \frac{\alpha}{2},$$

$$\Gamma_T^\alpha(\gamma) := T^{\frac{1}{18}+\frac{\alpha}{4}}\|\gamma\|_{H^{s_\alpha-1}} \text{ and } N_T^\alpha(\gamma) := T^{1-\frac{2}{q}}\Gamma_T^\alpha(\gamma). \tag{4.19}$$

To solve (1.1) with initial data $(u_0, u_1) \in H^{s_\alpha} \times H^{s_\alpha - 1}$ with a small $\alpha > 0$, we define the following iterative scheme. First, let $u^{(0)}$ be the solution of the free wave equation

$$\begin{cases} \partial_t^2 u^{(0)} - \Delta u^{(0)} = 0 \\ (u^{(0)}, \partial_t u^{(0)})_{|t=0} = (S_0 u_0, S_0 u_1), \end{cases}$$

and inductively for $n = 0, 1, 2, ...$ define $u^{(n+1)}$ by

$$\begin{cases} \partial_t^2 u^{(n+1)} - \Delta u^{(n+1)} - G_{u^{(n)},T} \cdot \nabla^2 u^{(n+1)} = 0 \\ (u^{(n+1)}, \partial_t u^{(n+1)})_{|t=0} = (S_{n+1}u_0, S_{n+1}u_1). \end{cases}$$

For simplicity, we shall define $G_{n,T} := G_{u^{(n)},T}$. Then, all we need is to show that if T is small enough, the sequence $(u^{(n)})$ is bounded and is a Cauchy sequence in the space $\mathcal{C}([0,T];\dot{H}^{s-1})$. To do so, we introduce the following assertions which we prove by induction.

$$(\mathcal{P}_n) \qquad \begin{cases} \|\partial u^{(n)}\|_{L^q_T(\dot{\mathcal{B}}^{\sigma_r}_{r,2})} \leq C_0 \Gamma_T^\alpha(\gamma) \\ \|\partial u^{(n)}\|_{T,s-1} \leq e^3\|\gamma\|_{s-1} \text{ for any } s \in [\frac{3}{2} - \frac{3}{r} + \alpha, \frac{3}{2} + \rho(q_\alpha) + \alpha]. \end{cases}$$

To prove Theorem 1.3 we show that if $\|\gamma\|_{\frac{d}{2}-1} + N_T^\alpha(\gamma).\Gamma_T^\alpha(\gamma)$ is small enough, then $(\mathcal{P}_1)$ is satisfied and $(\mathcal{P}_n)$ implies $(\mathcal{P}_{n+1})$. First, we point out that under the inductive hypothesis, we have the following *a priori* control of the metric.

Lemma 4.1. *Assume that* $(\mathcal{P}_n)$ *holds, then we have*

$$\|G_{n,T}\|_{L^\infty} \leq C\|\gamma\|^2_{\frac{3}{2}-1} \tag{4.20}$$

and

$$\|\partial G_{n,T}\|_{L^1_T(L^\infty)} \leq C\|\gamma\|^2_{\frac{3}{2}-1} + C_0\big(T^{\frac{7}{18}+\alpha}\|\gamma\|_{s_\alpha-1}\big)^2. \tag{4.21}$$

Proof. This result is an immediate consequence of Lemma 3.1. In fact, (3.17) and (3.18) together with $(\mathcal{P}_n)$ imply (4.20) and (4.21) in the case where ∂ is a space derivative. However, the proof of (4.21) with $\partial = \partial_t$ is quite different. In fact, noticing that

$$\partial_t G_{n,T} = \frac{1}{T}(\partial_t\theta)(\frac{\cdot}{T})G(\partial u^{(n)}) + \theta(\frac{\cdot}{T})\partial_t G(\partial u^{(n)}),$$

and using the equation satisfied by $u^{(n)}$, the term $\partial_t G(\partial_t u^{(n)})$ could be developed as a sum of terms of the type $\Delta^{-1}(\Delta u^{(n-1)}\cdot\partial u^{(n-1)})$ and $\Delta^{-1}(G_{n-1}\nabla^2 u^{(n-1)}\cdot\partial u^{(n-1)})$. Obviously, $\Delta^{-1}(\Delta u^{(n-1)}\cdot\partial u^{(n-1)})$ can be estimated as in (3.18). On the other hand, using the following law of product

$$\|a\cdot b\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}} \leq C\|a\|_{\dot{\mathcal{B}}^{\frac{3}{2}}_{2,1}}\|b\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}, \tag{4.22}$$

we deduce that $G_{n-1}\partial u^{(n)} \in \dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}$, and again applying (3.18), we get (4.21). The proof of Lemma 4.1 is then complete. $\square$

4.1. **Energy estimate.**

The energy estimate satisfied by $u^{(n+1)}$ is the following.

Proposition 4.2. *Assume that $(\mathcal{P}_n)$ is satisfied then, for all real number $s \in]\frac{3}{2}-\frac{3}{r}, \frac{3}{2}+\rho(q_\alpha)+\alpha]$, a constant C exists such that for all $t\in[0,T]$, we have*

$$\|\partial u^{(n+1)}\|_{T,s-1} \leq e^2\|\gamma\|_{s-1}\big(1 + CC_0N^\alpha_T(\gamma)\|\partial u^{(n+1)}\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2})}\big). \tag{4.23}$$

Proof. Recall that according to Lemma 3.2, the sequence $u^{(n+1)}_k := \Delta_k u^{(n+1)}$ satisfies the equation

$$\begin{aligned}\partial_t^2 u^{(n+1)}_k - \Delta u^{(n+1)}_k - S_{k-1}(G_{u^{(n)},T})\cdot\nabla^2 u^{(n+1)}_k = \\ R_k(\nabla u^{(n+1)}, \partial u^{(n)}),\end{aligned} \tag{4.24}$$

with the following estimate

$$\begin{aligned}\|R_k(\nabla u^{(n+1)},\partial u^{(n)})(t,\cdot)\|_{L^2} &\leq Cc_k(t)2^{-k(s-1)}\|\nabla G_{u^{(n)},T}\|_{L^\infty}\|\nabla u^{(n+1)}(t,.)\|_{s-1}\\ &+Cc_k(t)2^{-k(s-1)}\|\partial u^{(n)}(t,\cdot)\|_{s-1}\|\partial u^{(n)}(t,.)\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}\|\partial u^{(n+1)}(t,.)\|_{\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2}}.\end{aligned}$$

Multiplying (4.24) by $\partial_t u^{(n+1)}_k$ and integrating on $\mathbb{R}^3$, we obtain

$$\frac{1}{2}\frac{d}{dt}\big[\|\partial u^{(n+1)}_k\|^2_{L^2}(t) + < S_{k-1}(G_{n,T})\cdot\nabla u^{(n+1)}_k, \nabla u^{(n+1)}_k >_{L^2}\big](t) =$$

$$\frac{1}{2} < S_{k-1}(\partial_{t'}G_{n,T})\cdot\nabla u^{(n+1)}_k, \nabla u^{(n+1)}_k >_{L^2}(t) + < R_k, \partial_t u^{(n+1)}_k >_{L^2}(t) -$$

$$\sum_{1\leq j,l\leq d} < S_{k-1}(\partial_j G^{jl}_{n,T})\cdot\partial_l u^{(n+1)}_k, \partial_t u^{(n+1)}_k >_{L^2}(t).$$

The above estimate on $R_k(\nabla u^{(n+1)},\partial u^{(n)})$ yields,

$$\frac{1}{2}\frac{d}{dt}\big[\|\partial u_k^{(n+1)}\|_{L^2}^2 + < S_{k-1}(G_{n,T})\cdot\nabla u_k^{(n+1)}, \nabla u_k^{(n+1)} >_{L^2}(t)\big] \leq$$

$$C\|\partial G_{n,T}(t,.)\|_{L^\infty}\|\partial u_k^{(n+1)}(t,.)\|_{L^2}^2 +$$

$$C2^{-k(s-1)}c_k(t)\|\partial u_k^{(n+1)}(t,.)\|_{L^2}\|\nabla(G_{n,T})(t,.)\|_{L^\infty}\|\nabla u^{(n+1)}(t,.)\|_{s-1}+$$

$$C2^{-k(s-1)}c_k(t)\|\partial u^{(n)}(t,.)\|_{s-1}\|\partial u^{(n)}(t,.)\|_{\dot{\mathcal{B}}_{r,2}^{\frac{3}{r}-\frac{1}{2}}}\|\partial u^{(n+1)}(t,.)\|_{\dot{\mathcal{B}}_{r,2}^{\frac{3}{r}-\frac{1}{2}}}\|\partial u_k^{(n+1)}(t,.)\|_{L^2}.$$

Multiplying by $2^{2k(s-1)}$, summing and using $(\mathcal{P}_n)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\big[\|\partial u^{(n+1)}\|_{s-1}^2 + h_n(t)\big](t) \leq C\|\partial(G_{n,T})(t,.)\|_{L^\infty}\|\partial u^{(n+1)}(t,\cdot)\|_{s-1}^2+$$

$$C\|\gamma\|_{s-1}\|\partial u^{(n)}(t,.)\|_{\dot{\mathcal{C}}^{-\frac{1}{2}}}\|\partial u^{(n+1)}(t,.)\|_{\dot{\mathcal{C}}^{-\frac{1}{2}}}\|\partial u^{(n+1)}(t,\cdot)\|_{s-1},$$

where we set

$$h_n(t) = \sum_{k\in\mathbb{Z}} 2^{2k(s-1)} < S_{k-1}(G_{n,T})\cdot\nabla u_k^{(n+1)}, \nabla u_k^{(n+1)} >_{L^2}(t).$$

Now, choosing $\|\gamma\|_{\frac{d}{2}-1}$ small enough such that for a constant $0 < c < 1$, the following holds

$$\|\partial u^{(n+1)}(t,\cdot)\|_{s-1}^2 + h_n(t) \leq c^{-1}\|\partial u^{(n+1)}(t,\cdot)\|_{s-1}^2.$$

Therefore, using Gronwall's lemma and the embedding $\dot{\mathcal{B}}_{r,2}^{\sigma_r}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{C}}^{-\frac{1}{2}}(\mathbb{R}^3)$ we deduce that

$$\begin{aligned} \|\partial u^{(n+1)}(t,\cdot)\|_{s-1} &\leq \exp\big(C\int_0^t \|(\partial G_{n,T})(t',.)\|_{L^\infty}\big)\|\gamma\|_{s-1} \\ &\cdot \big[1 + CC_0 N_T^\alpha(\gamma)\|\partial u^{(n+1)}(t,.)\|_{L_T^{q_\alpha}(\dot{\mathcal{B}}_{r,2}^{\frac{3}{r}-\frac{1}{2}})}\big]. \end{aligned} \tag{4.25}$$

The choice $C\|\gamma\|_{\frac{3}{2}-1}^2 + CC_0N_T^\alpha(\gamma)\|\gamma\|_{s_\alpha-1} \leq 2$ completes the proof. □

The following result enables us to obtain an *a priori* control of the remainder term for the precise paralinearization.

Lemma 4.3. *A constant C exists such that under the hypothesis $(\mathcal{P}_n)$ we have for any δ in the interval* $[0,1]$

$$\begin{cases} \partial_t^2 u_k^{n+1} - \Delta u_k^{n+1} - S_k^\delta(G_{n,T})\nabla^2 u_k^{n+1} = R_k^\delta(n) \\ \partial u_k^{n+1}{}_{|t=0} = \gamma_k^{n+1} \end{cases}$$

with $S_k^\delta b = S_{k\delta-(1-\delta)\ln_2 T - N_0} b$ *and*

$$\|R_k^\delta(n)\|_{L_T^1(L^2)} \leq Cc_k 2^{-k(1-\frac{1}{q\alpha})}(2^kT)^{-\frac{1}{18}-\frac{\alpha}{4}}\Gamma_T^\alpha(\gamma)(1+(2^kT)^{1-\delta})(1+CC_0N_T(\gamma)\|\partial u^{n+1}\|_{L_T^{q_\alpha}(\dot{\mathcal{B}}_{r,2}^{\sigma_r})}).$$

Proof. Applying Corollary (3.3) with $\partial u=\partial u^{(n+1)}$, $\partial v=\partial v^{(n)}$ and $s=s_\alpha$, we have

$$\begin{aligned}\|R_k^\delta(n)\|_{L^1_T(L^2)} &\le Cc_k 2^{-k(s_\alpha-1)}(1+(2^kT)^{1-\delta})(\|\nabla G_n\|_{L^1_T(L^\infty)}\|\nabla u^{(n+1)}\|_{T,s_\alpha-1}\\ &+ T^{1-\frac{2}{q_\alpha}}\|\partial u^{(n)}\|_{T,s_\alpha-1}\|\partial u^{(n)}\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\sigma_r}_{r,2})}\|\partial u^{(n+1)}\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\sigma_r}_{r,2})}).\end{aligned}$$

Using $(\mathcal{P}_n)$, (4.20), (4.21) together with the energy estimate (4.23) we obtain

$$\|R_k^\delta(n)\|_{L^1_T(L^2)}\le Cc_k2^{-k(1-\frac{1}{q_\alpha})}(2^kT)^{-\frac{1}{3q_\alpha}-\frac{\alpha}{2}}\Gamma^\alpha_T(\gamma)(1+(2^kT)^{1-\delta})\big(1+CC_0N^\alpha_T(\gamma)\|\partial u^{(n+1)}\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\frac{3}{r}-\frac12}_{r,2})}\big).$$

Thanks to (4.19), the proof is complete. □

Now, we are going to estimate $\|\partial u^{(n+1)}\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\frac{3}{r}-\frac12}_{r,2})}$. We split this study into the two cases of low and high frequencies. The following result deals with the low frequencies.

Corollary 4.4. *Assume that* $(2^kT)^{(\frac{2}{3q_\alpha}-\frac{\alpha}{2})}\le C$ *then, there exists a constant* C *such that under the hypothesis* $(\mathcal{P}_n)$*, we have*

$$\|\partial S_k u^{(n+1)}\|_{\tilde{L}^{q_\alpha}_T(\dot{\mathcal{B}}^{\sigma_r}_{r,2})}\le CT^{\frac{2}{3q_\alpha}-\frac{\alpha}{2}}\Gamma^\alpha_T(\gamma)\big(1+CC_0N^\alpha_T(\gamma)\|\partial u^{(n+1)}\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\frac{3}{r}-\frac12}_{r,2})}\big).$$

Proof. Using Bernstein's inequality, we have

$$2^{2k\sigma_r}\|\partial u_k^{(n+1)}(t,\cdot)\|^2_{L^r}\le C2^{2k(\frac{2}{3q_\alpha}-\frac{\alpha}{2})}\|\partial u^{(n+1)}(t,\cdot)\|^2_{T,s_\alpha-1}.$$

Moreover, thanks to the energy estimate (4.23), we have

$$2^{2k\sigma_r}\|\partial u_k^{(n+1)}\|^2_{L^{q_\alpha}_T(L^r)}\le CT^{\frac{4}{q_\alpha}-\alpha}(2^kT)^{2(\frac{2}{3q_\alpha}-\frac{\alpha}{2})}\Gamma^\alpha_T(\gamma)^2\big(1+CC_0N^\alpha_T(\gamma)\|\partial u^{(n+1)}\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\frac{d}{r}-\frac12}_{r,2})}\big)^2.$$

Choosing α small enough, summing and noticing that $\frac{2}{3q_\alpha}-\frac{\alpha}{2}=\frac19-\alpha$ the proof of the corollary is complete. □

4.2. **Strichartz estimates and the end of the proof of Theorem 1.3.** From the microlocal result (2.14) given in Theorem 2.7, we deduce the following local statement.

Lemma 4.5. *Let* ε *be a positive real number and* G *be a metric such that for a sufficiently small constant* c_0*, we have*

$$\|\partial G\|_{L^1_T(L^\infty)}\le c_0.$$

Fix $q>2$ *and* r *such that* $\frac1q=\frac12-\frac1r$. *A constant* C_ε *exists such that if we set* $\bar{G}_k:=S_k^{\frac23}G$ *and assume that the Fourier transform of* γ_k, $f_k(t,\cdot)$ *and* $u_k(t,\cdot)$ *are supported in the ring* $2^k\mathcal{C}$, *then the solution* u_k *of*

$$(E_k)\quad\begin{cases}\partial_t^2u_k-\Delta u_k-\bar{G}_k\nabla^2u_k=f_k \text{ on }]0,T[\times\mathbb{R}^3\\ \partial u_{k|t=0}=\gamma_k\end{cases}$$

satisfies

$$(4.26)\;\|\partial u_k\|_{L^q_T(L^r)}\le C_\varepsilon2^{k[3(\frac12-\frac1r)-\frac1q]}(2^kT)^{\frac{1}{3q}+\varepsilon}(\|\partial u_k\|_{L^\infty_T(L^2)}+(2^kT)^{-\frac13}\|f_k\|_{L^1_T(L^2)}).$$

Proof. Fix k big enough (this corresponds to the high frequencies case). Suppose that we can construct a finite partition of the interval $[0,T]$;

$$[0,T] = \cup_{l=0}^{l=N(k)} I_{k,l}$$

where $I_{k,l} = [t_{k,l}, t_{k,l+1}]$ and assume that, for every $l = 0, 1, ..., N(k)$ (except probably for $l = N(k)$), the following property holds

$$\frac{|I_{k,l}|}{T(2^kT)^{-\frac{1}{3}-\varepsilon}} + \frac{\|f_k\|_{L^1(I_{k,l},L^2)}}{(2^kT)^{-\frac{1}{3}}\|f_k\|_{L^1_T(L^2)}} + \frac{|I_{k,l}|}{T}(2^kT)^{\frac{2}{3}}\|\nabla\bar{G}_k\|_{L^1(I_{k,l},L^\infty)} = \delta. \tag{4.27}$$

Recall that δ is small enough and it is given by Theorem 2.7. Then we have the following consequences:

- A constant C_δ exists such that the number $N(k)$ of the sub-intervals $I_{k,l}$ is estimated by

$$N(k) \le C_\delta (2^kT)^{\frac{1}{3}+\varepsilon}. \tag{4.28}$$

In fact, denote by $\sigma^{(j)}(k)$ the set of all the l's such that the j^{th} term in (4.27) is the biggest, and decompose $N(k) = N_1(k) + N_2(k) + N_3(k)$, where $N_j(k)$ counts all the l's in $\sigma^{(j)}(k)$. For every $l \in \sigma^{(j)}(k)$, the j^{th} term in (4.27) has to be greater than or equal to $\frac{\delta}{3}$. Therefore we have

$$\frac{|I_{k,l}|}{T(2^kT)^{-\frac{1}{3}-\varepsilon}} \ge \frac{\delta}{3} \quad \text{for all} \quad l \in \sigma^{(1)}(k), \tag{4.29}$$

$$\frac{\|f_k\|_{L^1(I_{k,l},L^2)}}{(2^kT)^{-\frac{1}{3}}\|f_k\|_{L^1_T(L^2)}} \ge \frac{\delta}{3} \quad \text{for all} \quad l \in \sigma^{(2)}(k), \tag{4.30}$$

and

$$\frac{|I_{k,l}|}{T}(2^kT)^{\frac{2}{3}}\|\nabla\bar{G}_k\|_{L^1(I_{k,l},L^\infty)} \ge \frac{\delta}{3} \quad \text{for all} \quad l \in \sigma^{(3)}(k). \tag{4.31}$$

Now after l summation in (4.29) and (4.30), we obtain

$$\begin{aligned} N_1(k) &\le \frac{3(2^kT)^{\frac{1}{3}+\varepsilon}}{T\delta}\Sigma_{l\in\sigma^{(1)}(k)}|I_{k,l}| \\ &\le \frac{3(2^kT)^{\frac{1}{3}+\varepsilon}}{\delta}, \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} N_2(k) &\le \frac{3(2^kT)^{\frac{1}{3}}}{\|f_k\|_{L^1_T(L^2)}\delta}\Sigma_{l\in\sigma^{(2)}(k)}\|f_k\|_{L^1(I_{k,l},L^2)} \\ &\le \frac{3(2^kT)^{\frac{1}{3}}}{\delta} \end{aligned} \tag{4.33}$$

respectively. On the other hand, from (4.31), we deduce that

$$\Big(\frac{3}{2\delta}(2^kT)^{\frac13}\frac{|I_{k,l}|}{T}+(2^kT)^{\frac13}\|\nabla\bar{G_k}\|_{L^1(I_{k,l},L^\infty)}\Big)^2\geq 1. \tag{4.34}$$

Taking the square root of the above inequality and summing over the set $\sigma^{(3)}(k)$ we obtain

$$N_3(k)\leq\frac{3}{2\delta}(2^kT)^{\frac13}+(2^kT)^{\frac13}\|\nabla\bar{G_k}\|_{L^1_T(L^\infty)}. \tag{4.35}$$

From (4.32), (4.33), (4.35) together with the hypothesis on the metric G, we deduce the desired estimate (4.28) on $N(k)$.

- On each sub-interval $I_{k,l}$, the solution u_k satisfies the following microlocal estimate

$$\|\partial u_k\|_{L^q(I_{k,l},L^r)}\leq 2^{k[3(\frac12-\frac1r)-\frac1q]}(\|\partial u_k(t_{k,l})\|_{L^2}+\|f_k\|_{L^1(I_{k,l},L^2)}). \tag{4.36}$$

In fact, rescaling $u_k(t,x)=v_k(2^kt,2^kx)$, it is clear that v_k satisfies

$$\partial_t^2v_k-\Delta v_k-H_k\nabla^2v_k=g_k$$

where $H_k(t,x)=\bar{G_k}(2^{-k}t,2^{-k}x)$ and $g_k(t,x)=2^{2k}f_k(2^kt,2^kx)$. Let us verify that the hypothesis of Theorem 2.7 are satisfied by v_k on the microlocal interval $J_{k,l}:=2^kI_{k,l}$.
First note that choosing $\Lambda=(2^kT)^{\frac13}$, we have

$$|J_{k,l}|\leq(2^kT)^{\frac23-\varepsilon}\leq\Lambda^{2-3\varepsilon}.$$

Second, it is clear that

$$\begin{aligned}\|\partial H_k\|_{L^1(J_{k,l},\,L^\infty)}&=2^{-k}\int_{2^kt_{k,l}}^{2^kt_{k,l+1}}\|\partial\bar{G_k}(2^{-k}t,.)\|_{L^\infty}\\&=\|\partial\bar{G_k}\|_{L^1(I_{k,l},\,L^\infty)}\\&\leq\|\partial G\|_{L^1([0,T],L^\infty)}.\end{aligned}\tag{4.37}$$

In the last inequality we used the fact that $\bar{G_k}:=S_k^{\frac23}G$ and the boundedness of $S_k^{\frac23}$ in L^∞. The smallness of $\|\partial G\|_{L^1([0,T],L^\infty)}$ implies then the smallness of the left hand side of (4.37). Similarly we have

$$\begin{aligned}\|\nabla^2H_k\|_{L^1(J_{k,l},\,L^\infty)}&=2^{-2k}\int_{2^kt_{k,l}}^{2^kt_{k,l+1}}\|\nabla^2\bar{G_k}(2^{-k}t,.)\|_{L^\infty}\\&=2^{-k}\|\nabla^2\bar{G_k}\|_{L^1(I_{k,l},L^\infty)}.\end{aligned}$$

Applying Bernstein's lemma we obtain

$$\|\nabla^2\bar{G_k}\|_{L^\infty}\leq C\frac{(2^kT)^{\frac23}}{T}\|\nabla\bar{G_k}\|_{L^\infty}.$$

Integrating with respect to time we deduce that

$$\|\nabla^2\bar{G_k}\|_{L^1(I_{k,l},\,L^\infty)}\leq C\frac{(2^kT)^{\frac23}}{T}\|\nabla\bar{G_k},\|_{L^1(I_{k,l},\,L^\infty)}.$$

Therefore,

$$
\begin{aligned}
|J_{k,l}| \|\nabla^2 H_k\|_{L^1(J_{k,l},L^\infty)} &= |I_{k,l}| \|\nabla^2 \bar{G}_k\|_{L^1(I_{k,l},L^\infty)} \\
&\leq C|I_{k,l}| \frac{(2^k T)^{\frac{2}{3}}}{T} \|\nabla \bar{G}_k\|_{L^1(I_{k,l},\, L^\infty)} \\
&\leq C\delta. \qquad (4.38)
\end{aligned}
$$

For the last estimate, we have used (4.27). This shows the smallness of $\|(H_k)\|_0$. Applying Theorem 2.7 and using the fact that

$$\|\partial v_k\|_{L^q(J_{k,l},\, L^r)} = 2^{k(\frac{3}{r}+\frac{1}{q}-1)} \|\partial u_k\|_{L^q(I_{k,l},\, L^r)}$$

and

$$\|g_k\|_{L^1(J_{k,l},\, L^2)} = 2^{k(\frac{3}{2}-1)} \|f_k\|_{L^1(I_{k,l},\, L^2)},$$

we obtain

$$\|\partial u_k\|_{L^q(I_{k,l},\, L^r)} \leq 2^{k[3(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}]} \big(\|\partial u_k(t_{k,l})\|_{L^2} + \|f_k\|_{L^1(I_{k,l},\, L^2)}\big),$$

as desired.

- Estimate (4.26) is deduced from (4.36) by summation. Precisely,

$$
\begin{aligned}
\|\partial u_k\|^q_{L^q_T(L^r)} &= \sum_{l=1}^{N(k)} \|\partial u_k\|^q_{L^q(I_{k,l},\, L^r)} \\
&\leq N(k) 2^{qk[3(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}]} \big(\|\partial u_k\|_{L^\infty_T(L^2)} + (2^k T)^{-\frac{1}{3}} \|f_k\|_{L^1_T(L^2)}\big)^q.
\end{aligned}
$$

Using the estimate (4.28) on the number of the sub-intervals we obtain

$$\|\partial u_k\|_{L^q_T(L^r)} \leq C 2^{k[3(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}]} (2^k T)^{\frac{1}{3q}+\varepsilon} \big(\|\partial u_k\|_{L^\infty_T(L^2)} + (2^k T)^{-\frac{1}{3}} \|f_k\|_{L^1_T(L^2)}\big).$$

Now to achieve the proof of Lemma 4.5, it remains to show that such a finite decomposition exists. This is done by induction.
Assume that there exists an increasing sequence $(t_j)_{0\leq j\leq p}$ of points of $[0,T]$ such that $t_p < T$ and, for any $0 \leq j \leq p-1$

$$
\begin{aligned}
\frac{t_{j+1}-t_j}{T(2^k T)^{-\frac{1}{3}-\varepsilon}} &+ \frac{(2^k T)^{\frac{1}{3}}}{\|f_k\|_{L^1_T(L^2)}} \int_{t_j}^{t_{j+1}} \|f_k(t,\cdot)\|_{L^2} dt \\
&+ \frac{t_{j+1}-t_j}{T} (2^k T)^{\frac{2}{3}} \int_{t_j}^{t_{j+1}} \|\nabla \bar{G}_k(t,\cdot)\|_{L^\infty} dt = \delta.
\end{aligned}
$$

As the function

$$
\begin{aligned}
F_p(t) &= \frac{t-t_p}{T(2^k T)^{-\frac{1}{3}-\varepsilon}} + \frac{(2^k T)^{\frac{1}{3}}}{\|f_k\|_{L^1_T(L^2)}} \int_{t_p}^{t} \|f_k(\tau)\|_{L^2} d\tau \\
&+ \frac{t-t_p}{T} (2^k T)^{\frac{2}{3}} \int_{t_p}^{t} \|\nabla \bar{G}_k(\tau)\|_{L^\infty} d\tau
\end{aligned}
$$

is increasing on the interval $[t_p,T]$ then, either the interval $[t_p,T]$ satisfies the condition (4.27) (but with an inequality $< \delta$ instead), then t_{p+1} does not exist. Note that this does not affect the order of the number $N(k)$. Or, a unique t_{p+1} exists in the interval $]t_p,T[$ such that $F_p(t_{p+1}) = \delta$. This is a finite procedure because of the compactness of $[0,T]$. □

As a consequence of Theorem 2.7, we have the following corollary

Corollary 4.6. *If $T^{\frac{7}{18}+\alpha}\|\gamma\|_{s_{\alpha-1}}$ is small and the constant C_0 is large enough then, assertion $(\mathcal{P}_n)$ implies assertion $(\mathcal{P}_{n+1})$.*

Proof. For $2^kT\geq C$, we use the Strichartz estimates (4.26). We have

$$2^{k(\frac{3}{r}-\frac{1}{2})}\|\partial u_k^{(n+1)}\|_{L^{q_\alpha}_T(L^r)}\leq C_\varepsilon 2^{k(1-\frac{1}{q_\alpha})}(2^kT)^{\frac{1}{3q_\alpha}+\varepsilon}\big[(\|\partial u_k^{(n+1)}\|_{L^\infty_T(L^2)}+(2^kT)^{-\frac{1}{3}}\|R^\delta_k(n)\|_{L^1_T(L^2)}\big].$$

Observe that taking $\delta=\frac{2}{3}$ in Lemma 4.3, we have the following estimate on the remainder term

$$\|R_k^{\frac{2}{3}}(n)\|_{L^1_T(L^2)}\quad\leq\quad C2^{-k(1-\frac{1}{q_\alpha})}(2^kT)^{-\frac{1}{3q_\alpha}-\frac{\alpha}{2}}\Gamma^\alpha_T(\gamma)(1+(2^kT)^{\frac{1}{3}})(1+CC_0N_T(\gamma)\|\partial u^{n+1}\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\sigma_r}_{r,2})}).$$

Now, combining the energy estimate (4.23) and the inductive hypothesis $(\mathcal{P}_n)$ to the above estimate we obtain

$$2^{k(\frac{3}{r}-\frac{1}{2})}\|\partial u_k^{(n+1)}\|_{L^{q_\alpha}_T(L^r)}\leq C_\varepsilon(2^kT)^{\varepsilon-\frac{\alpha}{2}}\Gamma^\alpha_T(\gamma)(1+CC_0N^\alpha_T(\gamma)\|\partial u^{(n+1)}(t,.)\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}}_{r,2})}).$$

On the other hand if $2^kT\leq C$, then Corollary 4.4 claims that

$$\sum_{k\in\mathbb{Z}:2^kT\leq C}2^{k(\frac{3}{r}-\frac{1}{2})}\|\partial u_k^{(n+1)}\|_{L^{q_\alpha}_T(L^r)}\leq C_\varepsilon\Gamma^\alpha_T(\gamma)(1+CC_0N^\alpha_T(\gamma)\|\partial u^{n+1}(t,.)\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}})}).$$

Finally, observe that $T^{2(\frac{7}{18}+\alpha)}\|\gamma\|^2_{s_\alpha-1}=\Gamma^\alpha_T(\gamma)N^\alpha_T(\gamma)$ and if $T^{\frac{7}{18}+\alpha}\|\gamma\|_{s_\alpha-1}$ is small enough then

$$\|\partial u^{(n+1)}\|_{L^{q_\alpha}_T(\dot{\mathcal{B}}^{\frac{3}{r}-\frac{1}{2}})}\leq C_0\Gamma^\alpha_T(\gamma)$$

and

$$\|\partial u^{n+1}\|_{T,s-1}\leq e^2\|\gamma\|_{s-1}(1+C\Gamma^\alpha_T(\gamma)N^\alpha_T(\gamma)).$$

This completes the proof of $(\mathcal{P}_{n+1})$. ☐

5. Sketch of the proof of Theorem 2.7

Let's recall the following fundamental result due to H. Bahouri-J-Y. Chemin (see [3] and [1]).

Theorem 5.1. *Let P_Λ be the operator given by (2.13) and denote by $(v_\Lambda)_{\Lambda\geq\Lambda_0}$ the family of solutions of*

$$\begin{aligned}P_\Lambda v_\Lambda &= 0\\ (v_\Lambda,\partial v_\Lambda)_{|t=0} &= (\gamma^0,\gamma^1).\end{aligned}$$

For any integer N, there exist two functions $\mathcal{I}^\pm_\Lambda(\gamma)$ defined on $I_\Lambda\times\mathbb{R}^3$ with

$$|I_\Lambda|\leq\Lambda^{2-\varepsilon},$$

and satisfying

$$\|\partial(v_\Lambda - \mathcal{I}^+_\Lambda(\gamma) - \mathcal{I}^-_\Lambda(\gamma))\|_{L^\infty_{I_\Lambda}(L^2)} \leq C\Lambda^{-N}\|\gamma\|_{L^2} \tag{5.39}$$

and

$$\|\mathcal{I}^\pm_\Lambda(\gamma)(\tau,\cdot)\|_{L^\infty} \leq \frac{C}{\tau}\|\gamma\|_{L^1}. \tag{5.40}$$

Remark 5.2. *The above result stays true if v_Λ solves the wave equation with "conservative Laplacian" i.e*

$$\tilde{P}_\Lambda v_\Lambda := \partial_t^2 v_\Lambda - \partial_j(\tilde{G}^{jk}_\Lambda \partial_k v_\Lambda) = 0. \tag{5.41}$$

where, we set $\tilde{G}^{jk}_\Lambda = G^{jk}_\Lambda + \delta_{jk}$. Therefore, in the sequel we assume that v_Λ solves (5.41).

Proof. Note that since the Fourier transform of v_Λ is included in $\mathcal{C}$, then Bernstein lemma together with (5.39) and (5.40) show the dispersive estimate

$$\|v_\Lambda(\tau,\cdot)\|_{L^\infty} \leq \frac{C}{\tau}\|\gamma\|_{L^1}.$$

Interpolating the above inequality with the energy estimate we obtain,

$$\|v_\Lambda(\tau)\|_{L^r} \leq \frac{C}{\tau^{\gamma(r)}}\|\gamma\|_{L^{\bar{r}}} \tag{5.42}$$

where $q, r \in]2,\infty[$ such that $\frac{1}{q} = \frac{1}{2} - \frac{1}{r}$ and

$$\gamma(r) = 2\Big(\frac{1}{2} - \frac{1}{r}\Big) \text{ and } \frac{1}{r} + \frac{1}{\bar{r}} = 1.$$

The proof of Theorem 2.7 can be achieved using a variation of the so called TT^* method (described in [6]), for non autonomous equations. In the sequel, we follow the idea of Klainerman [8] and Klainerman-Rodnianski [9].
Let P denotes the projection onto functions whose Fourier transform is supported in $\mathcal{C}$. Let $\mathcal{H} := \dot{H}^1 \times L^2$, $X = L^q_T(L^r)$, $X' = L^{\bar{q}}_T(L^{\bar{r}})$. For two real valued vector functions $\underline{u} := (u_0, u_1)$ and $\underline{v} := (v_0, v_1)$ in $\mathcal{H}$ we define

$$< \underline{u}, \underline{v} > := \int_{\mathbb{R}^3} u_1 v_1 + \tilde{G}^{jk}_\Lambda(t=0)\partial_j u_0 \partial_k v_0,$$

where we set $\tilde{G}^{jk}_\Lambda = G^{jk}_\Lambda + \delta_{jk}$.
For a space-time function $\Psi(t,x)$, we denote by $\Psi[0] := (\Psi(0), \partial_t\Psi(0))$. Given $\underline{u} \in \mathcal{H}$, t and s two real numbers, denote by

$$\Phi(t,s,\underline{u}) = (\phi, \partial_t\phi),$$

where the function ϕ (uniquely) solves (5.41) with $(\phi(s,s,\underline{u}), \partial_t\phi(s,s,\underline{u})) = \underline{u}$.
First we prove (2.14) for $\partial_t v_\Lambda$. Set $\phi = v_\Lambda$, and define the operator A by

$$A\underline{u} = -P\partial_t\Phi(t,0,\underline{u}).$$

The goal is to show that $A : \mathcal{H} \longrightarrow X$ is bounded operator with an operator norm $\|A\|_{\mathcal{H}\to X} = M$. It is clear that (2.14) can be derived from (5.42) with a large constant depending on Λ.

Using this as a *bootstrap* assumption we have to establish a uniform bound with respect to Λ. To do so, it is sufficient to exhibit the expression of AA^*, prove that

$$AA^* : X' \longrightarrow X$$

is bounded and establish the relation between the norm operations

$$\|AA^*\|_{X' \to X} = M^2.$$

By definition of A^* we have

$$< A^* f, \underline{u} > := (f, A\underline{u})_{L^2} = -\int_0^T \int_{\mathbb{R}^3} \partial_t \phi P f.$$

Let Ψ solve $\tilde{P}_\Lambda \Psi = Pf$ with $(\Psi, \partial_t \Psi)_{t=T} = 0$. Integrating by parts (in time), we obtain

$$< A^* f, \underline{u} > = < \underline{u}, \Psi[0] + R(f) >,$$

with $R(f) : X' \longrightarrow \mathcal{H}$ given by

$$< \underline{u}, R(f) > = -\int_0^T \int_{\mathbb{R}^3} \psi \tilde{P}_\Lambda \partial_t \phi dx dt.$$

Therefor,

$$AA^* f = A\psi[0] + AR(f).$$

Using the definition of A and Duhamel's formula, we can write

$$A\Psi[0] = P \int_0^T \partial_t \Phi(t, s, (0, Pf(s))) ds,$$

with $F(s) = (0, Pf(s))$. Applying the dispersive inequality (5.42), we obtain

$$\|P\partial_t \Phi(t, s, (0, Pf(s)))\|_{L^r} \leq \frac{C}{|t-s|^{\gamma(r)}} \|Pf(s)\|_{L^{\bar{r}}}.$$

The Hardy-Littlewood-Sobolev inequality implies that

$$\|A\Psi[0]\|_{L^q(L^r)} = \|P\partial_t \Phi(t, s, F(s)))\|_{L^q(L^r)} \leq C\|f\|_{L^{\bar{q}}(L^{\bar{r}})} \tag{5.43}$$

as desired. Note that C is Λ independent constant.
Now we estimate the term $AR(f)$. According to the *bootstrap* assumption, we have

$$\|AR(f)\|_{L^q(L^r)} \leq M\|R(f)\|_{\mathcal{H}}.$$

On th other hand, using the definition of $< \underline{u}, R(f) >$, we have

$$\begin{aligned} \|R(f)\|_{\mathcal{H}} : &= \sup_{\|\underline{u}\|_{\mathcal{H}} \leq 1} < \underline{u}, R(f) > \\ &= \sup_{\|\underline{u}\|_{\mathcal{H}} \leq 1} -\int_0^T \int_{\mathbb{R}^3} \psi \tilde{P}_\Lambda \partial_t \phi dx dt. \end{aligned}$$

Now observe that $\tilde{P}_\Lambda \partial_t \phi = \partial_t \tilde{P}_\Lambda \phi + \partial_j \big(\partial_t \tilde{G}_\Lambda^{jk} \partial_k \phi\big)$, and since ϕ solves (5.41) then

$$\tilde{P}_\Lambda \partial_t \phi = \partial_j \big(\partial_t \tilde{G}_\Lambda^{jk} \partial_k \phi\big).$$

Therefore, after (a space) integration by part

$$\begin{aligned}\|R(f)\|_{\mathcal{H}} &= \sup_{\|\underline{u}\|_{\mathcal{H}}\le 1}\int_0^T\int_{\mathbb{R}^3}\partial_j\psi\partial_t\tilde{G}_\Lambda^{jk}\partial_k\phi\\ &\le \|\partial_t\tilde{G}_\Lambda^{jk}\|_{L^1(L^\infty)}\|\partial\psi\|_{L^\infty(L^2)}\|\partial\phi\|_{L^\infty(L^2)}.\end{aligned}$$

Thanks to the energy estimate applied to ϕ, the $L^1(L^\infty)$ bound on the metric $\tilde{G}_\Lambda$ and the fact that $\|\underline{u}\|_{\mathcal{H}}\le 1$, we deduce that

$$\|AR(f)\|_{L^q(L^r)} \le \frac{M}{4}\|\partial\psi\|_{L^\infty(L^2)}. \tag{5.44}$$

The following Lemma enables us to estimate $\|\partial\psi\|_{L^\infty(L^2)}$.

Lemma 5.3. *Let ψ be the solution to $\tilde{P}_\Lambda\psi = Pf$ with $\psi(T)=\partial_t\psi(T)=0$. Then,*

$$\|\partial\psi\|_{L^\infty(L^2)} \le 2M\|f\|_{L^{\bar{q}}(L^{\bar{r}})}.$$

Note that the above Lemma together with (5.43) and (5.44) imply the following bound

$$M^2 = \|AA^*\|_{X'\to X} \le C + \frac{M^2}{2}$$

and therefore, $M^2 \le 2C^2$ as desired.
To prove Lemma 5.3, we consider a time $t\in[0,T)$ and define ϕ to be the solution to $\tilde{P}_\Lambda\phi = 0$ with initial data $\phi(t)=u_0$, $\partial_t\phi(t)=u_1$, and $\|\underline{u}\|_{\mathcal{H}}\le 1$.. Recall that ψ solves $\tilde{P}_\Lambda\psi = Pf$ with zero initial data at time $t=T$. Multiplying $\tilde{P}_\Lambda\phi$ by $\partial_t\psi$ and $\tilde{P}_\Lambda\psi$ by $\partial_t\phi$ and we integrate in $[t,T]\times\mathbb{R}^3$ to get the identity

$$\int_{R^3}\left(\partial_t\psi\partial_t\phi + \tilde{G}_\Lambda^{jk}\partial_j\psi\partial_k\phi\right)(t)dx = -\int_0^T\int_{R^3}\left(\partial_t\phi Pf + \partial_t\tilde{G}_\Lambda^{jk}\partial_j\psi\partial_k\phi\right)dxdt$$

Hence,

$$\|\partial\psi\|_{L^\infty(L^2)} \le \|P\partial_t\phi\|_{L^q(L^r)}\|f\|_{L^{\bar{q}}(L^{\bar{r}})} + C\|\partial\tilde{G}_\Lambda^{jk}\|_{L^1(L^\infty)}\|\partial\psi\|_{L^\infty(L^2)}\|\partial\phi\|_{L^\infty(L^2)}.$$

From the *bootstrap* assumption, we know that $\|P\partial_t\phi\|_{L^q([t,T],L^r)} \le M\|\underline{u}\|_{\mathcal{H}} \le M$. Moreover, using the energy estimate $\|\partial\phi\|_{L^\infty(L^2)} \le 2\|u\|_{\mathcal{H}} \le 2$, and therefore,

$$\|\partial\psi\|_{L^\infty(L^2)} \le M\|f\|_{L^{\bar{q}}(L^{\bar{r}})} + C\|\partial\tilde{G}_\Lambda^{jk}\|_{L^1(L^\infty)}\|\partial\psi\|_{L^\infty(L^2)}.$$

Since $\|\partial\tilde{G}_\Lambda^{jk}\|_{L^1(L^\infty)}$ is small enough, then

$$\|\partial\psi\|_{L^\infty(L^2)} \le 2M\|f\|_{L^{\bar{q}}(L^{\bar{r}})}$$

as desired.
Now we use the above result to prove (2.14) for a space derivative $\partial_j\phi$. Let f be a function in $L^{\bar{q}}(L^{\bar{r}})$. As before, we estimate

$$\mathcal{I} := \int_0^T\int_{R^3} P\partial_l\phi f dxdt$$

by introducing the function ψ solution to $\tilde{P}_\Lambda \psi = Pf$ with data $\psi(T) = \partial_t \psi(T) = 0$. Hence integrating by parts,

$$\mathcal{I} = \int_0^T \int_{R^3} \psi \tilde{P}_\Lambda \partial_l \phi dxdt + \int_{R^3} \partial_l \phi(0) \partial_t \psi(0) + \partial_l \psi(0) \partial_t \phi(0) dx.$$

Commuting $\tilde{P}_\Lambda$ and ∂_l as before we obtain

$$\mid \int_0^T \int_{R^3} \psi \tilde{P}_\Lambda \partial_l \phi dxdt \mid \leq \|\partial G_\Lambda^{jk}\|_{L^1(L^\infty)} \|\partial \phi\|_{L^\infty(L^2)} \|\partial \psi\|_{L^\infty(L^2)}.$$

Also,

$$\int_{R^3} \partial_l \phi(0) \partial_t \psi(0) + \partial_l \psi(0) \partial_t \phi(0) dx \leq \|\partial \phi(0)\|_{L^2} \|\partial \psi\|_{L^\infty(L^2)}.$$

Applying the energy estimate we obtain

$$\|\partial \phi\|_{L^\infty(L^2)} \leq 2 \|\partial \phi(0)\|_{L^2}.$$

Moreover, Lemma 5.3 implies

$$\|\partial \psi\|_{L^\infty(L^2)} \leq 2M \|f\|_{L^{\bar{q}}(L^{\bar{r}})}$$

with the bound M obtained in the previous step. In particular M does not depend on Λ. Therefore, thanks to the bound $\|\partial \tilde{G}_\Lambda^{jk}\|_{L^1(L^\infty)}$, we deduce

$$\mathcal{I} \leq CM \|\partial \phi(0)\|_{L^2} \|f\|_{L^{\bar{q}}(L^{\bar{r}})}$$

which proves that

$$\|P \partial_l \phi\|_{L^q(L^r)} \leq CM \|\partial \phi(0)\|_{L^2}$$

as desired. The case of inhomogeneous equation can be deduced from the above result by a standard technique. We refer to [8] for more details. □

6. Acknowledgments

The authors thank the anonymous referee for his deep and helpful comments.
Many thanks are due to Professors Jean-Yves Chemin, Hajer Bahouri and Nader Masmoudi for fruitful discussions.
This paper was written while the second author was visiting the Mathematics section at the Abdus Salam International Center for Theoretical Physics, Trieste, Italy, and the Department of Mathematics and Statistics at McMaster University, Hamilton, Canada. He is grateful to all colleagues and the staff for their generous hospitality.

References

[1] **H. Bahouri and J.-Y. Chemin**: *Équations d'ondes quasilinéaires et inégalités de Strichartz*, American Journal of Mathematics, **121**, pp. 1337-1377, 1999.

[2] **H. Bahouri and J.-Y. Chemin**: *Équations d'ondes quasilinéaires et effets dispersif*, Inte. Math. Research Notes, **21**, pp. 1141-1178, 1999.

[3] **H. Bahouri and J.-Y. Chemin**: *Microlocal analysis, bilinear estimates and cubic quasilinear wave equation*, Astérisque no. 284, Société Mathématiques de France, 2003.

[4] **J.-Y. Chemin**: *Fluides parfaits incompressibles*, Astérisque no. 230, Société Mathématiques de France, 1995.

[5] **J. M. Bony**: *Calcul symbolique et propagation dews singularités pour les équations aux dérivées partielles non-linéaires*, Ann. École normale supérieur, **14**, pp. 209-246, 1981.

[6] **J. Ginibre and G. Velo**: *Generalized Strichartz inequalities for the wave equations*, J. Functional Analysis , **133**, pp. 50-68, 1995.

[7] **S. Klainerman**: *The null condition and global existence to non linear wave equations*, Comm. Pures and appl. Math. **38**, 631-641 1985.

[8] **S. Klainerman**: *A commuting vectorfields approach to Strichartz-type inequalities and applications to quasi-linear wave equations*, Internat. Math. Res. Notices, 221 274 2001.

[9] **S. Klainerman and I. Rodnianski**: *Improved local well posedness for quasilinear wave equations in dimensions three*, Duke Math. J. **117**, no. 1, pp 1-124, 2003.

[10] **S. Klainerman and I. Rodnianski**: *Rough solutions of the Einstein-vacuum equations*, Ann. of Math. **161**, pp 1143-1193, 2005.

[11] **S. Klainerman and D. Tataru**: *On the optimal local regularity for the Yang Mills equation in* $\mathbb{R}^{4+1}$, J. of the Amer. Math. Society **12**, pp. 93-116, 1999.

[12] **H. F. Smith and D. Tataru**: *Sharp counterexamples for Strichartz estimates for low regularity metrics*, Math. Res. Lett. **9**, no. 2-3, 199–204, 2002

[13] **H. F. Smith and D. Tataru**: *Sharp local well-posedness results for the nonlinear wave equations*, Ann. of Math. **162**, pp 291-366, 2005.

[14] **D. Tataru**: *Local and global results for wave maps*, Comm. Partial Differential Equations, **23**, no. 9-10, 1781–1793, 1998.

[15] **D. Tataru**: *Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation*, Amer. J. Math. **122**, no. 2, 349–376, 2000.

[16] **D. Tataru**: *Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation*, **II**, Amer. J. Math.**123**, 385–423, 2001.

[17] **D. Tataru**: *Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation*, **III**, J. Amer. Math. Soc. **15**, 419–442, 2002.

Institut Préparatoire aux Études d'Ingenieurs de Tunis , Tunis, Tunisie
E-mail address: `Jameleddine.Azzouz@issatso.rnu.tn`

and

Department of Mathematics & Statistics, Arizona State University, Arizona, USA
E-mail address: `ibrahim@math.asu.edu`