STREAMLINES CONCENTRATION AND APPLICATION TO THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

ERIC FOXALL, SLIM IBRAHIM AND TSUYOSHI YONEDA

Department of Mathematics and Statistics, University of Victoria
PO Box 3060 STN CSC, Victoria, BC, Canada, V8W 3R4

Department of Mathematics and Statistics, University of Victoria
PO Box 3060 STN CSC, Victoria, BC, Canada, V8W 3R4

Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan

Keywords: Navier-Stokes equations, Streamline, Stokes Theorem

Mathematics Subject Classification: 35Q30, 76D05, 76M99

Abstract. For a smooth domain D containing the origin, we consider a vector field $u \in C^1(D \setminus \{0\}, \mathbb{R}^3)$ with $\operatorname{div} u \equiv 0$ and exclude certain types of possible isolated singularities at the origin, based on the geometry of streamlines that go near that possible singular point.

1. Introduction

In this paper we consider divergence-free smooth vector fields $u \in C^1(D \setminus \{0\}, \mathbb{R}^3)$ defined on a domain D of $\mathbb{R}^3$ containing the origin which may have a singular point at the origin. We give a definition based on streamline concentration towards the eventual singularity, and we show that if there is sufficient streamline concentration, then the vector field cannot be an L^2 function[1]. Therefore, this result rules out a certain geometric situation (streamline concentration) at a possible singular time for incompressible fluid equations such as the 3D Navier-Stokes equations. Before going any further, let us briefly recall a few results about the 3D Navier-Stokes equations on $\mathbb{R}^3$. The equations ruling the flow of an incompressible viscous fluid on $\mathbb{R}^3$ are

$$\begin{cases} \partial_t v - \triangle v + \operatorname{div}(v \otimes v) + \nabla p = 0, \\ \operatorname{div}(v) = 0, \quad v|_{t=0} = v_0 \end{cases} \tag{1.1}$$

in which

v is a vector-valued function representing the velocity of the fluid, and p is the pressure. The initial value problem of the above equation is endowed with the condition that $v(0, \cdot) = v_0 \in L^2(\mathbb{R}^3)$.

A finite energy *weak solution* to the Navier-Stokes equations (1.1) over a time interval $(0, T)$ is a pair (v, p) satisfying

(1) equation (1.1) in the distributional sense,
(2) $(v, p) \in L^\infty([0, T], L^2) \cap L^2([0, T], \dot{H}^1) \times L^{\frac{5}{3}}_{loc}((0, T) \times \mathbb{R}^3)$

S. I. is partially supported by NSERC# 371637-2009 grant and a start up fund from University of Victoria.

T. Y. is partially supported by PIMS Post-doc fellowship at the University of Victoria, and partially supported by NSERC# 371637-2009.

[1] we define this situation precisely in the next section

(3) the energy inequality, for $0 < t < T$

$$\|v(t,\cdot)\|_{L^2}^2 + 2\int_0^t \|\nabla v(t',\cdot)\|_{L^2}^2\, dt' \le \|v(0,\cdot)\|_{L^2}^2. \tag{1.2}$$

For a divergence free initial data $v_0 \in (L^2(\mathbb{R}^3))^3$, the existence of global in time and finite energy *weak solutions* to the Navier-Stokes equations is due to the pioneer works of J. Leray [13] in the case $D = \mathbb{R}^3$ and E. Hopf [10] in the case of the torus. Moreover, neither the uniqueness nor the global regularity are known. These questions are the outstanding problems of regularity for solutions to the Navier-Stokes equations. Recall that the space-time singular set $S(u)$ of u is defined as follows.

Definition 1.1. *A point $(x_0, t_0) \notin S(u)$ if there exists a parabolic cylinder $Q_{(x_0,t_0)}(r) := \{|x - x_0| < r\} \times (t_0 - r^2, t_0)$ about (x_0, t_0) such that the solution $u \in L^\infty(Q_{(x_0,t_0)}(r))$.*

Modern regularity theory for solutions to equation (1.1) began with the works of Prodi [14], Serrin [16], Ladyzhenskaya [12] implying that if

$$u \in L_t^p(L_x^q)(Q_{(x_0,t_0)}(r)), \quad \text{for} \quad \frac{3}{q} + \frac{2}{p} < 1,$$

then $\partial_x^k u \in \mathcal{C}^\alpha((Q_{(x_0,t_0)}(r/2)))$ for some $0 < \alpha < 1$ and therefore u is regular. Later on, M. Struwe [17] extended this to the case (of scaling invariant pair) i.e. $\frac{3}{q} + \frac{2}{p} = 1$, and recently this was extended to the limit case $u \in L_t^\infty(L_x^3)$ by L. Escauriaza, G. Seregin, and V. Sverak (see their famous work [8]). After the appearance of the Prodi-Serrin-Ladyzhenskaya criterion, many different regularity cirteria and Liouville type theorem of solutions to (1.1) were established (see [1], [2], [6] and [11]).

We would like to mention a regularity criterion in [18] by A. Vasseur (see also [4]). He gave a regularity criterion for solutions u to (1.1) in terms of the integral condition $\mathrm{div}(\frac{u}{|u|}) \in L^p(0,\infty; L^q(\mathbb{R}^3))$ with $\frac{2}{p} + \frac{3}{q} \leqslant \frac{1}{2}$ imposed on the scalar quantity $F = \mathrm{div}(\frac{u}{|u|})$. Note that the case $(p, q) = (6, \infty)$ is included.

Concerning the analysis of the singular set $S(u)$, we recall the following facts: First, by definition, the set $S(u)$ is closed, and thanks to the result of C. Foias and R. Temam [9], the $\frac{1}{2}$-dimensional Hausdorff measure of the set of singular times $\tau(u) := \mathrm{proj}_t S(u)$[2] is zero. Next, V. Scheffer [15] and then L. Caffarelli, R. Kohn and L. Nirenberg [3] showed the best result concerning *partial regularity* of *suitable weak* solutions[3] of the Navier-Stokes equations stating that the parabolic one-dimensional Hausdorff measure of $S(u)$ is zero. Finally, a consequence of the latter result is a bound on the spatial singular set for each time slice $S_T := S(u) \cap \{t = T\}$ which has at most one-dimensional Hausdorff measure.

In this paper, we focus on the vector field at a possible singular time $T \in \tau(u)$, and examine the geometry of its streamlines. Recall that in [5], C-H. Chan and the third author proposed a possible scenario for an isolated space singularity at a possible blow-up time by using the energy inequality and regularity criterions especially [8] and [18]. They constructed a divergence free velocity field u within a *streamtube* segment with increasing twisting (i.e., increasing swirl).

The construction of such a vector field u demonstrates the way in which *excessive* increase of twisting of streamlines can result in the *blow up* of the quantities $\|u\|_{L^\alpha(\mathbb{R}^3)}$ (for some $2 < \alpha < 3$) and $\|\mathrm{div}(\frac{u}{|u|})\|_{L^6(\mathbb{R}^3)}$ while at the same time preserving the finite energy property $u \in L^2(\mathbb{R}^3)$ of the fluid. Note that the increasing swirl streamtube is not included in the sufficient concentration streamlines case. The device of streamtube has already proposed as the vortex-tube (see[7]).

In this work, we show that if "enough" streamlines of a smooth and divergence free vector

[2] the map $(x, t) \mapsto t$

[3] roughly, these are weak solutions satisfying the local energy inequality instead of the global one (1.2).

field concentrate towards a possible isolated singular point,[4] then the vector field cannot be an L^2 function. The main idea is to costruct an appropriate "streamline flux tube" and apply Stokes' Theorem.

2. A classification of divergence vector fields

Definition 2.1. *(Streamline) Let D be a smooth domain containing the origin and $u : D \setminus \{0\} \to \mathbb{R}^3$ be a smooth vector field. For a starting point $\eta \in D$, we define a streamline $\gamma_\eta(s) : [0, \infty) \to \mathbb{R}^3$ as the curve solving*

$$\partial_s \gamma_\eta(s) = u(\gamma_\eta(s)) \quad \text{for} \quad s > 0 \quad \text{with} \quad \gamma_\eta(0) = \eta. \tag{2.1}$$

One may assume that streamlines are global, because otherwise, they go towards the possible singular point at the origin.

The following definition is the key to classify the divergence-free vector field with a possible isolated singularity at the origin. Let B_α be the open ball with radius α centered at the origin.

Definition 2.2. *For $\alpha > r$ let*

$$A_r^\alpha = \{\eta \in \partial B_\alpha : \gamma_\eta(s) \in B_r \text{ for some } s > 0, \text{ and } \gamma_\eta(s') \in B_\alpha \text{ for } 0 < s' < s\}.$$

The above definition excludes the streamlines entering the ball B_α infinitely many times before entering B_r. If it happens and a streamline enters B_α finitely many times before getting into B_r, then one can re-parametrize the time so that its last entrance occurs at time $s = 0$.

Remark 2.3. For streamlines from A_r^α we have the following properties

- $|A_r^\alpha|$ is monotone decreasing with respect to α and increasing with respect to r. Indeed,
$$|A_r^\alpha| \geq |A_{r'}^\alpha| \quad \text{for} \quad r > r', \quad |A_r^\alpha| \geq |A_r^{\alpha'}| \quad \text{for} \quad \alpha < \alpha'.$$
- Without loss of generality, we can assume that streamlines from A_r^α are globally defined.
- From definition of A_r^α we cannot have stagnation points of the fluid (i.e. $u(\gamma_\eta(s)) = 0$ for some $s > 0$).

Definition 2.4. *(Stream-surface & flux-tube) Let $D \subset \mathbb{R}^3$ be a surface and s be such that $\gamma_\eta(s)$ is defined for each $\eta \in D$.*

- *A stream-surface $S^D(s)$ is defined as $S^D(s) = \bigcup_{\eta \in D} \gamma_\eta(s)$.*
- *A flux-tube $T^D(s)$ is given by $T^D(s) = \bigcup_{0 \leq s' \leq s} S^D(s')$.*
- *The mantle of the flux-tube $T^D(s)$ is $\partial T^D(s)$.*

For $|x| \neq 0$ denote by $\hat{n}(x) = x/|x|$. Smoothness and membership in C^1 are used interchangeably. The main result reads as follows.

Theorem 2.5. *If for some $\alpha > 0$ and for some $C > 0$ independent of r, $|\int_{A_r^\alpha} u \cdot \hat{n} d\sigma| \geq C r^{1/2}$ as $r \to 0$, then $u \notin L^2(\mathbb{R}^3)$.*

The following special case is worth noting. See Figure 1.

Corollary 2.6. *Suppose for some $\alpha > 0$ and for $A \subset \partial B_\alpha$ that $\int_A u \cdot \hat{n} d\sigma \neq 0$ and $A_r^\alpha \supset A$ for $0 < r < \alpha$. Then $u \notin L^2(\mathbb{R}^3)$.*

Proof. It follows from the definition of A_r^α that $u \cdot \hat{n}$ has constant (negative) sign on A_r^α. Let $C = |\int_A u \cdot \hat{n} d\sigma| > 0$, then for $0 < |r| < \min\{1, \alpha\}$, $|\int_{A_r^\alpha} u \cdot \hat{n} d\sigma| \geq |\int_A u \cdot \hat{n} d\sigma| \geq C r^{1/2}$. □

[4]note that such singular set has a zero one-dimensional Hausdorff measure.

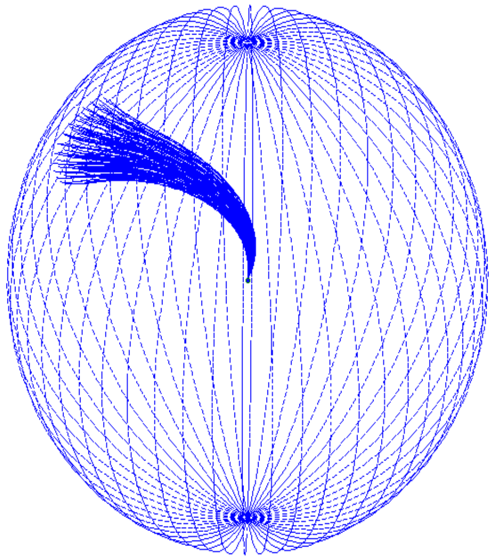

FIGURE 1. The set A of Corollary 2.6, with streamlines going to the origin

The proof of Theorem 2.5 proceeds in a few steps. First of all suppose that

$$\int_{\partial B_r} |u \cdot \hat{n}| d\sigma \geq \left| \int_{A_r^\alpha} u \cdot \hat{n} d\sigma \right|$$

for each r (this is proved in a moment). Then, Jensen's inequality gives

$$\frac{1}{|\partial B_r|} \int_{\partial B_r} |u|^2 d\sigma \geq \left(\frac{1}{|\partial B_r|} \int_{\partial B_r} |u| d\sigma \right)^2 \tag{2.2}$$

or

$$\int_{\partial B_r} |u|^2 d\sigma \geq \left(\frac{1}{|\partial B_r|} \int_{\partial B_r} |u| d\sigma \right)^2 \tag{2.3}$$

and by assumption

$$\left(\frac{1}{|\partial B_r|} \int_{\partial B_r} |u| d\sigma \right)^2 \geq \frac{1}{4\pi r^2} \left| \int_{A_r^\alpha} u \cdot \hat{n} d\sigma \right|^2 \geq \frac{1}{4\pi r^2} Cr = \frac{C}{4\pi r}$$

from which it follows that

$$\|u\|_{L^2} \geq \left(\int_0^\epsilon \int_{\partial B_r} |u|^2 d\sigma dr \right)^{1/2} \geq \left(\int_0^\epsilon \frac{C}{4\pi r} \right)^{1/2} = \infty$$

where $\epsilon > 0$ is such that $| \int_{A_r^\alpha} u \cdot \hat{n} d\sigma| \geq Cr^{1/2}$ for $0 < r \leq \epsilon$.

Now, to prove that $\int_{\partial B_r} |u \cdot \hat{n}| d\sigma \geq \left| \int_{A_r^\alpha} u \cdot \hat{n} d\sigma \right|$ observe first of all that $\int_{A_r^\alpha} u \cdot \hat{n} d\sigma = \int_{\text{reg } A_r^\alpha} u \cdot \hat{n} d\sigma$ where $\text{reg } A_r^\alpha = \{\eta \in A_r^\alpha : (u \cdot \hat{n})(\eta) \neq 0\}$. Since α is fixed, let A_r denote $\text{reg } A_r^\alpha$. From the definition of A_r^α it follows that $(u \cdot \hat{n})(\eta) < 0$ for $\eta \in A_r$.

Lemma 2.7. *Let $D \subset \partial B_\alpha$ have piecewise smooth boundary and $(u \cdot \hat{n})(\eta) < 0$ for $\eta \in D$. Suppose that $S^D(s) \subset B_r$ for some $s > 0$ and that $S^D(s') \subset B_\alpha$ for $0 < s' \leq s$. Then*

$$\int_D u \cdot \hat{n} d\sigma = \int_{D^*} u \cdot \hat{n} d\sigma$$

where $D^* \equiv T^D(s) \cap \partial B_r$. *Also, if* D_1 *and* D_2 *are two such sets with* $D_1 \cap D_2 = \emptyset$, *then* $D_1^* \cap D_2^* = \emptyset$.

Proof. The function $\gamma_\eta : D \times [0, s] \to T^D(s)$ is onto and it follows from the theory of ordinary differential equations and from $u \in C^1$ that $\gamma_\eta \in C^1$. Also, γ_η is injective, which follows from uniqueness of solutions and from the fact that for each $\eta \in D$, $\gamma_\eta(s) \notin D$ for $s > 0$. From these properties it can be shown that $\partial T^D(s) = D \cup S^D(s) \cup T^{\partial D}(s)$. Piecewise smoothness of $\partial T^D(s)$ then follows from the piecewise smoothness of ∂D and smoothness of solutions to the vector field. Let $T = \{x \in T^D(s) : r < |x| < \alpha\}$ and let $V = \{x \in T^{\partial D}(s) : r < |x| < \alpha\}$, and let D^* be as defined above. Note that T has piecewise smooth boundary since it is the intersection of two sets with piecewise smooth boundary. Write $\partial T = D \cup D^* \cup V$. If $x \in V$ then a part of the streamline through x lies in V, therefore $u(x)$ is in the tangent space of V at x. Then, applying the divergence theorem and using $\operatorname{div} u \equiv 0$ gives the stated result. Observe that the implication $D_1 \cap D_2 = \emptyset \Rightarrow D_1^* \cap D_2^* = \emptyset$ follows from the uniqueness of solutions in the same way as above. ☐

Claim 2.8. A_r *is open. Moreover, for each* $\eta \in A_r$ *there is a* $\delta > 0$ *such that* $D \equiv \{\xi \in \partial B_\alpha : |\xi - \eta| < \delta\}$ *satisfies the assumptions of the above lemma.*

Proof. Let $\eta \in A_r$ and s be as in the definition of A_r^α. Then $(u\cdot\hat{n})(\eta) < 0$. By continuity there exists $\delta > 0$ so that $E \equiv \{\xi \in \partial B_\alpha : |\xi - \eta| \le \delta\}$ has $(u\cdot\hat{n})(\lambda) < 0$ for $\xi \in E$. E is compact, and by a property of compact sets, there exists $\alpha > 0$ so that $\operatorname{dist}(\xi, E) < \alpha$ implies $(u\cdot\hat{n})(\xi) < 0$. Let $t = \inf\{s' > 0 : |\gamma_\eta(s') - \eta| > \alpha/2\}$ and let $\beta(s) = \inf\{|\gamma_\eta(s') - \partial B_\alpha| : t \le s' \le s\}$. Observe that $\beta > 0$ since the sets $\{\gamma_\eta(s') : t \le s' \le s\}$ and ∂B_α are compact and disjoint. Let $\beta' > 0$ be such that $|\xi - \gamma_\eta(s)| < \beta'$ implies $\xi \in B_r$. Let $\alpha' = \min\{\alpha/2, \beta, \beta'\}$. By continuous dependence on initial data, there is a $\delta' > 0$, $\delta' \le \delta$ so that $|\xi - \eta| < \delta'$ implies $|\gamma_\xi(s') - \gamma_\eta(s')| < \alpha'$ for $0 \le s' \le s$. For these ξ, $|\gamma_\xi(s') - E| < \alpha$ for $0 \le s' \le t$ and so $(u \cdot \hat{n})(\gamma_\xi(s')) < 0$ for $0 \le s' \le t$, from which it follows that $\gamma_\xi(s') \in B_\alpha$ for $0 < s' \le t$. Then, $|\gamma_\xi(s) - \gamma_\eta(s)| < \beta'$ implies $\gamma_\xi(s) \in B_r$, and $|\gamma_\xi(s') - \gamma_\eta(s')| < \beta$ implies $\gamma_\xi(s') \in B_\alpha$, for $t \le s' \le s$. Therefore δ' gives D that satisfies the claim. ☐

End of the proof of Theorem 2.5. Since A_r is open it is Lebesgue measurable. It follows that for each $\epsilon > 0$, by a theorem for measurable sets there exists K closed, $K \subset A_r$ such that $m(A_r \backslash K) < \epsilon$, where m denotes Lebesgue measure. For each $\eta \in A_r$ let D_η be as in the above claim, then $\{D_\eta\}_{\eta \in K}$ is an open cover of K. Since K is a closed and bounded subset of $\mathbb{R}^3$, it is compact and therefore from the above cover one can take a finite subcover $\{D_{\eta_i}\}_{1 \le i \le k}$. Let $E_1 = D_{\eta_1}$ and for $2 \le i \le k$ let $E_i = D_{\eta_i} \setminus E_{i-1}$; then the E_i are pairwise disjoint and have piecewise smooth boundary, and $\bigcup_{i=1}^k E_i$ covers K. For each i let $E_i^* = T^{E_i}(s) \cap \partial B_r$. Then

$$\int_{\bigcup_{i=1}^k E_i} u \cdot \hat{n} d\sigma = \int_{\bigcup_{i=1}^k E_i^*} u \cdot \hat{n} d\sigma$$

using $\int_{E_i} u \cdot \hat{n} d\sigma = \int_{E_i^*} u \cdot \hat{n} d\sigma$ (from Lemma 2.7) for each i and $E_i \cap E_j = \emptyset$ implies that $E_i^* \cap E_j^* = \emptyset$. Since $\bigcup_{i=1}^k E_i^* \subset \partial B_r$ and $m(A_r \setminus \bigcup_{i=1}^k E_i) \le m(A_r \setminus K) < \epsilon$ it follows that

$$\int_{\partial B_r} |u \cdot \hat{n} d\sigma| \ge \left| \int_{A_r} u \cdot \hat{n} d\sigma \right| - \epsilon \|u\|_{L^\infty(\partial B_\alpha)}$$

Since $u \in C^1(D \setminus \{0\}, \mathbb{R}^3)$ by assumption then $\|u\|_{L^\infty(\partial B_\alpha)} < \infty$. Moreover, since $\epsilon > 0$ is arbitrary we have

$$\int_{\partial B_r} |u \cdot \hat{n} d\sigma| \ge \left| \int_{A_r} u \cdot \hat{n} d\sigma \right| = \left| \int_{A_r^\alpha} u \cdot \hat{n} d\sigma \right|$$

as claimed.

Remark 2.9.

- Note that condition $|\int_{A_r^\alpha} u\cdot\hat{n}d\sigma| \geq Cr^{1/2}$ in the theorem implicitly requires that the Lebesgue measure of the set A_r^α is non zero for some $\alpha > 0$ and any $0 < r < \alpha$. The example of a rotating vector field $u(x) = \frac{(x_2, -x_1, 0)}{|x|^\gamma}$ shows that for any $\alpha > 0$, and for any $r < \alpha$ the set A_r^α is empty. Moreover, this example shows that the vector field u can be in L^2 as well as not in L^2 depending whether or not $\gamma < 4$ or $\gamma > 4$.
- We can easily generalize the main theorem (Theorem 2.5) to L^p spaces $(1 \leq p \leq \infty)$. In fact, we just use Hölder inequality instead of Jensen's inequality which is used in (2.2) and (2.3). More precisely we have the following statement:

  If for some $\alpha > 0$ and for some $C > 0$ independent of r, $|\int_{A_r^\alpha} u\cdot\hat{n}d\sigma| \geq Cr^{2(1-1/p)}$ as $r \to 0$, then $u \notin L^p(\mathbb{R}^3)$.

Acknowledgments. The third author thanks the Pacific Institute for the Mathematical Sciences for support of his presence there during the academic year 2010/2011. This paper developed during a stay of the third author as a PostDoc at the Department of Mathematics and Statistics, University of Victoria.

References

[1] J. T. Beale, T. kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94 (1984) 61-66.

[2] H. Beirao da Veiga. A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$. *Chinese Ann. Math. Ser. B*, 16 (1995) 407-412. A Chinese summary appears in Chinese Ann. Math. Ser. A 16 (1995), 797.

[3] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35 (1982) 771-831.

[4] C. H. Chan, Smoothness criteria for Navier-Stokes equations in terms of regularity along the stream lines. *Methods Appl. Anal.*, 17 (2010) 81-103.

[5] C. H. Chan and T. Yoneda, On possible isolated blow-up phenomena and regularity criterion of the 3D Navier-Stokes equation along the streamlines, submitted.

[6] C.C. Chen,R. M. Strain,T. P. Tsai, H. T. Yau, Lower bounds on the blow-up rate of the axisymmetric Navier-Stokes equations. II. *Comm. Partial Differential Equations*, 34 (2009) 203-232.

[7] A. J. Chorin and J. E. Marsden, A mathematical introduction to fluid mechanics, Thied edition. *Springer-Verlag*, New York, 1993.

[8] L. Escauriaza, G. Seregin, and V. Sverak. $L_{3,\infty}$-solutions of the Navier-Stokes equations and backward uniqueness. *Russian Math. Surveys.*, 58 (2003) 211-250.

[9] C. Foias and R. Temam. Some analytic and geometric properties of the solutions of the evolution Navier-Stokes equations. *J. Math. Pures Appl.(9)*, 58 (1979) 339-368.

[10] E. Hopf. Uber die Anfangswertaufgabe fur die hydrodynamischen Grundgleichungen. *Math. Nachr.*, 4 (1951) 213-231.

[11] G. Koch, N. Nadirashvili, G. A. Seregin, A. V. Sverak, Liouville theorems for the Navier-Stokes equations and applications. *Acta Math.*, 203 (2009) 83-105.

[12] O.A. Ladyzhenskaya. Uniqueness and smoothness of generalized solutions of Navier-Stokes equations. *Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.*, 5 (1967) 169-185.

[13] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta. Math.*, 63 (1934) 183-248.

[14] G. prodi. Un teorema di unicita per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl. (4)*, 48 (1959) 173-182.

[15] V. Scheffer. Hausdorff measure and the Navier-Stokes equations. *Comm. Math. Phys.*, 55 (1977) 97-112.

[16] J. Serrin. The initial value problem for the Navier-Stokes equations. In *Nonlinear Problems Proc. Sympos., Madison, Wis.*, pages 69-98. Univ. of Wisconsin Press, Madison, Wis., 1963.

[17] M. Struwe. On partial regularity results for the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41 (1988) 437-458.

[18] A. Vasseur. Regularity criterion for 3D Navier-Stokes equations in terms of the direction of the velocity. *Appl. Math.*, 54 (2009) 47-52.

E-mail address: E. Foxall: e.t.foxall@gmail.com
E-mail address: S. Ibrahim: ibrahim@math.uvic.ca
URL: http://www.math.uvic.ca/~ibrahim/
E-mail address: T. Yoneda: yoneda@math.sci.hokudai.ac.jp