DOUBLE LOGARITHMIC INEQUALITY WITH A SHARP CONSTANT

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Abstract. We prove a Log Log inequality with a sharp constant. We also show that the constant in the Log estimate is "almost" sharp. These estimates are applied to prove a Moser-Trudinger type inequality for solutions of a 2D wave equation.

1. Introduction and statement of the results

By the Sobolev embedding theorem, it is well known that the Sobolev space $H^1(\mathbb{R}^2)$ is embedded in all Lebesgue spaces $L^p(\mathbb{R}^2)$ for $2 \leq p < +\infty$ but not in $L^\infty(\mathbb{R}^2)$. Moreover, H^1 functions are in a so-called Orlicz space i.e their exponential powers are integrable functions. Precisely, we have the following Moser-Trudinger inequality (see [1], [10], [11]).

Proposition 1.1. *There exists a universal positive constant C such that, for all $u \in H^1(\mathbb{R}^2)$, we have*

$$\|u\|_{H^1(\mathbb{R}^2)} \leq 1 \Longrightarrow \int_{\mathbb{R}^2} \left(e^{4\pi u(x)^2} - 1\right) dx \leq C. \tag{1.1}$$

In this paper, we show that we can control the L^∞ norm by the H^1 norm and a stronger norm with a logarithmic growth or double logarithmic growth. The inequality is sharp for the double logarithmic growth.

Recall that H^1 is the usual Sobolev space endowed with the norm $\|u\|_{H^1}^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2$. For any real number $\alpha \in]0,1[$, we denote by $\dot{\mathcal{C}}^\alpha$ the sub-space of α- Hölder continuous functions endowed with the semi-norm

$$\|u\|_{\dot{\mathcal{C}}^\alpha} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Also, we denote $\|u\|_{\mathcal{C}^\alpha} := \|u\|_{\dot{\mathcal{C}}^\alpha} + \|u\|_{L^\infty}$ and define $N_\alpha(u)$ to be the ratio $N_\alpha(u) := \frac{\|u\|_{\dot{\mathcal{C}}^\alpha}}{\|\nabla u\|_{L^2}}$. For any bounded domain Ω in $\mathbb{R}^2$, define $H_0^1(\Omega)$ to be the completion in the Sobolev space $H^1(\Omega)$ of smooth and compactly suppported functions.
The main result of this paper is the following.

Theorem 1.2 (Double logarithmic inequality)**.** *Let $\alpha \in]0,1[$ and B_1 be the unit ball in $\mathbb{R}^2$. Any function in $H_0^1(B_1) \cap \dot{\mathcal{C}}^\alpha(B_1)$ is bounded. Moreover, a positive constant C_0 exists such that for any function $u \in H_0^1(B_1) \cap \dot{\mathcal{C}}^\alpha(B_1)$, one has*

$$\|u\|_{L^\infty}^2 \leq \frac{1}{2\pi\alpha} \|\nabla u\|_{L^2}^2 \log\left[e^3 + C_0 N_\alpha(u) \sqrt{\log(2e + N_\alpha(u))}\right] \tag{1.2}$$

and, the constant $\frac{1}{2\pi\alpha}$ in (1.2) is sharp.

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Note that $\log(e) = 1$. Our second result concerns the following logarithmic inequality.

Theorem 1.3 (Logarithmic inequality). *Let α be in $]0,1[$. For any real number $\lambda > \frac{1}{2\pi\alpha}$, a constant C_λ exists such that, for any function $u \in H_0^1(B_1) \cap \dot{\mathcal{C}}^\alpha(B_1)$, we have*

$$\|u\|_{L^\infty}^2 \leq \lambda \|\nabla u\|_{L^2}^2 \log(C_\lambda + N_\alpha(u)). \tag{1.3}$$

Moreover, the above inequality does not hold for $\lambda = \frac{1}{2\pi\alpha}$.

2. A Littlewood-Paley proof

To prove the fundamental theorems, we start by showing that inequality (1.3) can easily be obtained with an unknown absolute constant C instead of $\frac{1}{2\pi\alpha}$. To do so, we give a brief recall of the Littlewood-Paley theory and we refer the reader to [4] for a thorough treatment. Denote by $\mathcal{C}_0$ the annular ring defined by

$$\mathcal{C}_0 = \{\xi \in \mathbb{R}^2 \text{ such that } \frac{3}{4} <| \xi |< \frac{8}{3}\},$$

and choose two non-negative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0,4/3))$ and $\mathcal{D}(\mathcal{C}_0)$ such that for all $\xi \in \mathbb{R}^2$

$$\chi(\xi) + \sum_{k\in\mathbb{N}} \varphi(2^{-k}\xi) = 1.$$

Denote by $h = \mathcal{F}^{-1}\varphi$ and define the frequency projector Δ_k by, for all $u \in \mathcal{S}'(\mathbb{R}^2)$,

$$\Delta_k u = \varphi(2^{-k}D)u = 2^{2k} \int_{\mathbb{R}^2} h(2^k y) u(x-y)dy,$$

and

$$\tilde{\Delta}_0 = \Sigma_{k\leq 0} \Delta_k.$$

Recall that

$$\|\nabla u\|_{L^2} \sim \Big(\sum_{k\in\mathbb{Z}} 2^{2k} \|\Delta_k u\|_{L^2}^2 \Big)^{\frac{1}{2}}$$

and

$$\|u\|_{\dot{\mathcal{C}}^\alpha} \sim \sup_k 2^{k\alpha} \|\Delta_k u\|_{L^\infty}.$$

We have the following result in the whole space.

Proposition 2.1. *Let α be in $]0,1[$. A positive constant C exists such that for any function $u \in \mathcal{C}^\alpha(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, one has*

$$\|u\|_{L^\infty(\mathbb{R}^2)}^2 \leq C\|u\|_{L^2(\mathbb{R}^2)}^2 + C\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 \log(e + \frac{\|u\|_{\dot{\mathcal{C}}^\alpha(\mathbb{R}^2)}}{\|\nabla u\|_{L^2(\mathbb{R}^2)}}). \tag{2.4}$$

Proof. Write

$$u = \tilde{\Delta}_0 u + \sum_{j=1}^{\infty} \Delta_j u = \tilde{\Delta}_0 u + \sum_{j=1}^{N-1} \Delta_j u + \sum_{j=N}^{\infty} \Delta_j u,$$

where N is a non-negative integer which will be chosen later.
Using Bernstein's inequality, we get

$$\begin{aligned}\|u\|_{L^\infty} &\leq C\|\tilde{\Delta}_0 u\|_{L^2} + C\sum_{j=1}^{N-1} 2^j\|\Delta_j u\|_{L^2} + \sum_{j=N}^{\infty} 2^{-j\alpha}(2^{j\alpha}\|\Delta_j u\|_{L^\infty})\\ &\leq C\|u\|_{L^2} + C\sqrt{N}\Big(\sum_{j=1}^{N-1} 2^{2j}\|\Delta_j u\|_{L^2}^2\Big)^{1/2} + \Big(\sum_{j=N}^{\infty} 2^{-j\alpha}\Big)\|u\|_{\dot{\mathcal{C}}^\alpha}\\ &\leq C\|u\|_{L^2} + C\sqrt{N}\,\|\nabla u\|_{L^2} + \frac{2^{-\alpha N}}{1-2^{-\alpha}}\,\|u\|_{\dot{\mathcal{C}}^\alpha}.\end{aligned}$$

So

$$\|u\|_{L^\infty}^2 \leq 2C^2\|u\|_{L^2}^2 + 2C^2N\,\|\nabla u\|_{L^2}^2 + 2\frac{2^{-2\alpha N}}{(1-2^{-\alpha})^2}\,\|u\|_{\dot{\mathcal{C}}^\alpha}^2.$$

Denoting by $]x[$ the integer part of the real number x and choosing

$$N := \mathrm{Max}\Big(1, 1 + \Big]2\log_2\frac{\|u\|_{\dot{\mathcal{C}}^\alpha}^2}{\|\nabla u\|_{L^2}^2}\Big[\Big),$$

the proof of Proposition 2.1 is achieved. ∎

Clearly, if u is supported in B_1 then using the Poincaré inequality, we get

$$\|u\|_{L^\infty}^2 \leq C\|\nabla u\|_{L^2}^2\log(C_0 + N(u)). \tag{2.5}$$

3. Proof of theorem 1.2

To prove (1.2) and the fact that the constant is sharp, it is sufficient to show that

$$2\pi\alpha = \inf_{u\in H_0^1(B_1)\cap\dot{\mathcal{C}}^\alpha(B_1)} \frac{\|\nabla u\|_{L^2}^2\log\Big[e^3 + C_0N_\alpha(u)\sqrt{\log(2e+N_\alpha(u))}\Big]}{\|u\|_{L^\infty}^2}. \tag{3.6}$$

for any C_0 big enough. Let us start by proving the sharpness of the constant. Defining $u_k(x) = f_k(-2\log|x|)$, where for any non-negative integer k

$$\begin{aligned} f_k(t) &= \sqrt{\tfrac{k}{2\pi}}\tfrac{t}{k} \text{ if } t\leq k\\ f_k(t) &= \sqrt{\tfrac{k}{2\pi}} \text{ if not }.\end{aligned}$$

An easy computation shows that

$$\|\nabla u_k\|_{L^2}^2 = 2, \quad \|u_k\|_{\dot{\mathcal{C}}^\alpha} = Ck^{\frac{1}{2}-\alpha}\exp\frac{\alpha k}{2}$$

and therefore, after taking the limit as $k\to\infty$, we deduce that

$$2\pi\alpha \geq \inf_{u\in H_0^1(B_1)\cap\dot{\mathcal{C}}^\alpha(B_1)} \frac{\|\nabla u\|_{L^2}^2\log\Big[e^3 + C_0N_\alpha(u)\sqrt{\log(2e+N_\alpha(u))}\Big]}{\|u\|_{L^\infty}^2}.$$

These functions was introduced in [1] and [9] to show the optimality of the exponent 4π in Trudinger-Moser inequality (see [10]).

To prove (1.2), we start by noticing that for any function u, the norms $\|\nabla u\|_{L^2}$ and $\|u\|_{\dot{\mathcal{C}}^\alpha}$ are non-increasing under symmetric non-increasing rearrangements, while $\|u\|_{L^\infty}$ remains unchanged.

Using the fact that for all $C > 0$

$$t \to f(t) := t^2 \log\left[e^3 + \frac{C}{t}\sqrt{\left[\log(2e + \frac{1}{t})\right]}\right]$$

is increasing, it is sufficient to check the minimizer figured in (3.6) in the class of non-negative, non-increasing and radially symmetric functions.

Without loss of generality, we can normalize $\|u\|_{L^\infty}$ to be equal to 1. Moreover, we will assume that $\|u\|_{\dot{\mathcal{C}}^\alpha} \geq 1$ because in the contrary case, the proof is similar.

Let $H^1_{0,rad}(B_1)$ be the space of all non-increasing and radially symmetric functions in $H^1_0(B_1)$. For any parameter $D \geq 1$, we denote by K_D the closed convex subset of $H^1_{0,rad}(B_1)$ defined by

$$K_D = \{u \in H^1_{0,rad}(B_1) : u(r) \geq 1 - Dr^\alpha, \quad r \in [0,1]\}. \tag{3.7}$$

To get the result, it is sufficient to prove that

$$\begin{aligned} 2\pi\alpha &\leq \inf_{D\geq 1} \inf_{\{u\in K_D\}} \|\nabla u\|^2_{L^2} \log\left[e^3 + \frac{C_0 D}{\|\nabla u\|_{L^2}}\sqrt{\log(2e + \frac{D}{\|\nabla u\|_{L^2}})}\right] \\ &\leq \inf_{D\geq 1} \inf_{\{u\in K_D,\, \|u\|_{L^\infty}=1,\, \|u\|_{\dot{\mathcal{C}}^\alpha}=D\}} \|\nabla u\|^2_{L^2} \log\left[e^3 + \frac{C_0 D}{\|\nabla u\|_{L^2}}\sqrt{\log(2e + \frac{D}{\|\nabla u\|_{L^2}})}\right]. \end{aligned}$$

Consider the following problem of minimizing

$$I[u] := \|\nabla u\|^2_{L^2(B_1)}, \tag{3.8}$$

among all the functions belonging to the set K_D. This is a variational problem with obstacle. It is well known (see for example, Kinderlehrer-Stampacchia [8] and L. C. Evans [5]) that it has a unique minimizer u^* which is variationally characterized by

$$\int_{B_1} \nabla u^* \cdot \nabla v \, dx \geq \|\nabla u^*\|^2_{L^2(B_1)}, \tag{3.9}$$

for any $v \in K_D$. Moreover u^* is in the Sobolev space $W^{2,\infty}(B_1)$. Hence the following radially symmetric set

$$\mathcal{O} := \{x \in B_1 : u^*(x) > 1 - D|x|^\alpha\}$$

is open and u^* is harmonic in $\mathcal{O}$. On the other hand, note that any radially symmetric harmonic functions in $\mathbb{R}^2$ can only coincide in a unique tangent point with the function $r \to 1 - Dr^\alpha$. Note also that because of the boundary condition at $r = 1$, u^* cannot start to be harmonic near $r = 0$. Therefore there exists, a unique $a \in]0,1[$ such that

$$\begin{aligned} u^*(r) &= 1 - Dr^\alpha \text{ if } r \in [0,a] \\ u^*(r) &= (1 - Da^\alpha)\frac{\log r}{\log a} \text{ if } r \in [a,1], \end{aligned} \tag{3.10}$$

satisfying also the tangent condition

$$a^\alpha = \frac{1 - Da^\alpha}{D|\log(a^\alpha)|}. \tag{3.11}$$

Note that if $D \to 1$ then $a \to 1$ and therefore (3.11) still makes sense in the limit case. Also, because of the regularity of u^* at $r = 0$ it is necessary that $a \neq 1$. In particular, note that $\|u^*\|_{L^\infty} = 1$, $\|u^*\|_{\dot{C}^\alpha} = D$, and

$$\|\nabla u^*\|_{L^2}^2 = \pi\alpha D^2 a^{2\alpha} - 2\pi(\frac{1 - Da^\alpha}{\log(a)})^2 \log(a). \tag{3.12}$$

Substituting D from (3.11) into (3.12), we get the following

$$\|\nabla u^*\|_{L^2}^2 = 2\pi\alpha \frac{1/2 - \log(a^\alpha)}{(1 - \log(a^\alpha))^2}.$$

Denoting by $x := a^\alpha \in]0,1[$, then we have

$$\|\nabla u^*\|_{L^2}^2 = 2\pi\alpha \frac{1/2 - \log(x)}{(1 - \log(x))^2} \tag{3.13}$$

and

$$\|u^*\|_{\dot{C}^\alpha} = \frac{1}{x(1 - \log(x))}. \tag{3.14}$$

Setting

$$g(x) := \frac{1}{x\sqrt{2\pi\alpha(1/2 - \log(x))}},$$

and

$$F_C(x) := \frac{\frac{1}{2} - \log(x)}{(1 - \log(x))^2} \log\left[e^3 + Cg(x)\sqrt{\log(2e + g(x))}\right],$$

it is sufficient to show that a constant C_0 exists such that for all $0 < x \leq 1$, the function F_{C_0} satisfies

$$F_{C_0}(x) \geq 1. \tag{3.15}$$

First, observe that for every $0 < x \leq 1$

$$\frac{\frac{1}{2} - \log(x)}{(1 - \log(x))^2} \geq \frac{1}{(2 - \log(x))}.$$

Hence for any $C > 0$, (3.15) holds if $2 - \log x \leq 3$, namely if $x \geq 1/e$.

In the sequel, we suppose that $x \leq 1/e$, hence

$$\begin{aligned} F(x) \geq \frac{1}{(2 - \log(x))}\Big[&- \log(x) + \log(\frac{C_0}{\sqrt{2\pi\alpha}}) - \frac{1}{2}\log(1/2 - \log(x)) + \frac{1}{2}\log(\log(2e + g(x)))\Big] \\ &\geq 1 + \frac{1}{(2 - \log(x))}\Big[\log(\frac{C_0}{e^2\sqrt{2\pi\alpha}}) + \frac{1}{2}\log\Big(\frac{\log(2e + g(x))}{(1/2 - \log(x))}\Big)\Big]. \end{aligned} \tag{3.16}$$

The function $h(x) = \frac{\log(2e+g(x))}{(1/2-\log(x))}$ is bounded away from zero on $(0, 1/e)$. Hence, we can find C_0 big enough such that the second term on the right hand side of (3.16) is non negative. This achieves the proof of Theorem 1.2. ■

4. Proof of theorem 1.3

The proof of Theorem 1.3 is similar to that of Theorem 1.2. Indeed, consider u^* the minimizer of the Dirichlet norm (3.8) among all functions in K_D defined in (3.7). Note that according to (3.13) and (3.14), we have

$$\|\nabla u^*\|_{L^2}^2 \log(C_\lambda + N_\alpha(u^*)) := H(x),$$

where

$$H(x) = 2\pi\alpha \frac{1/2 - \log(x)}{(1-\log(x))^2} \log\Big(C_\lambda + \frac{1}{x\sqrt{2\pi\alpha(1/2-\log(x))}}\Big).$$

Taking $C_\lambda = e$ in $H(x)$, we see that $H(x)$ goes to $2\pi\alpha$ as x goes to 0. Hence, for any $\lambda > \frac{1}{2\pi\alpha}$, there exists $x_\lambda > 0$ such that $\lambda H(x) \geq 1$, for any $0 < x < x_\lambda$ and $C_\lambda \geq e$. Now, if $x \in [x_\lambda, 1]$, choosing the constant $C_\lambda > e$ big enough such that

$$\frac{1/2}{(1-\log(x_\lambda))^2} \log(C_\lambda) \geq 1,$$

we see that $\lambda H(x) \geq 1$. Hence, by this choice of C_λ, we see that $\lambda H(x) \geq 1$ for all $0 < x \leq 1$. This achieves the proof of (1.3).

Now, let us prove that (1.3) does not hold for $\lambda = \frac{1}{2\pi\alpha}$. More precisely, we will prove that a sequence of functions $(u_n)_n$ exists such that $u_n \in H_0^1(B_1) \cap \dot{\mathcal{C}}^\alpha(B_1)$ and for n big enough the following holds

$$\|u_n\|_{L^\infty}^2 > \frac{1}{2\pi\alpha} \|\nabla u_n\|_{L^2}^2 \log(n^{1/4} + n^{1/4} N_\alpha(u_n)). \tag{4.17}$$

Let u_n be the radially symmetric function defined by

$$u_n(r) = 1 - e^n r^\alpha \text{ if } r \in [0, a_n], \text{ and } u_n(r) = (1 - e^n a_n^\alpha)\frac{\log r}{\log a_n} \text{ if } r \in [a_n, 1],$$

where a_n is chosen such that $a_n^\alpha := x_n$ is the unique solution in $(0,1)$ of the equation $x = \frac{1-e^n x}{e^n|\log(x)|}$. Notice indeed, that the function $h(x) = e^n(x + x|\log(x)|)$ is increasing on $(0,1)$. Hence, we see easily that

$$\frac{e^{-n}}{n\log(n)} \leq x_n \leq \frac{e^{-n}}{n}. \tag{4.18}$$

Obviously, this construction is inspired from the minimizer of the variational problem with obstacle described in Section 3 where we have chosen $D_n = e^n$. Hence, according to (3.13) and (3.14), we have

$$\|\nabla u_n\|_{L^2}^2 = 2\pi\alpha \frac{1/2 - \log(x_n)}{(1-\log(x_n))^2}$$

and

$$\|u_n\|_{\dot{\mathcal{C}}^\alpha} = \frac{1}{x_n(1-\log(x_n))}.$$

Now to prove (4.17), it is sufficient to prove that for n big enough we have

$$h_n := \frac{\frac{1}{2}-\log(x_n)}{(1-\log(x_n))^2}\log\Big[n^{1/4} + \frac{n^{1/4}}{x_n\sqrt{2\pi\alpha(1/2-\log(x_n))}}\Big] < 1.$$

Note that using (4.18), we have

$$h_n < \frac{\frac{1}{2}+n+\log(n)+\log\log n}{(1+\log(n)+n)^2}\log\Big[n^{1/4} + \frac{n^{1/4}e^n n\log n}{\sqrt{2\pi\alpha n}}\Big]$$

Hence $h_n < 1 - \frac{1}{4}\frac{\log n}{n} + o(\frac{\log n}{n})$ which is strictly less than 1 if n is sufficiently large. The proof of (4.17) is achieved. ■

5. Case of the whole space

Theorems 1.2 and 1.3 were stated in the ball of radius one. If the function u is supported in a bigger ball $B_R = B(0,R)$ then a simple scaling argument shows that

$$\|u\|^2_{L^\infty(B_R)} \le \frac{1}{2\pi\alpha}\|\nabla u\|^2_{L^2(B_R)}\log\Big[e^3 + C_0R^\alpha N_\alpha(u)\sqrt{\log\big(2e+R^\alpha N_\alpha(u)\big)}\Big].$$

Remark 5.1. Using symmetric non-increasing rearrangement of functions, the results of Theorem 1.2 and Theorem 1.3 remain true for any bounded and regular domain Ω of $\mathbb{R}^2$. Precisely, if $f\in H^1_0(\Omega)\cap\dot{\mathcal{C}}^\alpha(\Omega)$ then, its corresponding symmetric non -increasing function, usually denoted by $f^\star$, is in $f^\star\in H^1_0(B_R)\cap\dot{\mathcal{C}}^\alpha(B_R)$, where $R=\sqrt{\frac{|\Omega|}{2\pi}}$. We refer to [12], [2] for the definition, the properties and applications of rearrangements of functions. Applying Theorem 1.2 and Theorem 1.3 results to $f^\star$ and using the fact that

$$\|f^\star\|_{L^\infty} = \|f\|_{L^\infty}$$
$$\|\nabla f^\star\|_{L^2} \le \|\nabla f\|_{L^2}, \quad \|f^\star\|_{\dot{\mathcal{C}}^\alpha} \le \|f\|_{\dot{\mathcal{C}}^\alpha}$$

we get the result for general domain.

Note that this estimate can not be extended to the whole space since R^α diverges. Instead, we have the following result concerning the whole space.

Corollary 5.2. *Let $\alpha\in]0,1[$. For any $\lambda > \frac{1}{2\pi\alpha}$ and any $0<\mu\le 1$, a constant $C_\lambda>0$ exists such that, for any function $u\in H^1(\mathbb{R}^2)\cap\mathcal{C}^\alpha(\mathbb{R}^2)$*

$$\|u\|^2_{L^\infty} \le \lambda(\|\nabla u\|^2_{L^2}+\mu^2\|u\|^2_{L^2})\log\Big(C_\lambda + \frac{8^\alpha\mu^{-\alpha}\|u\|_{\mathcal{C}^\alpha}}{\sqrt{\|\nabla u\|^2_{L^2}+\mu^2\|u\|^2_{L^2}}}\Big), \tag{5.19}$$

Proof. Let u be a function in $H^1(\mathbb{R}^2)\cap\mathcal{C}^\alpha(\mathbb{R}^2)$, $\lambda>\frac{1}{2\pi\alpha}$ and $0<\mu\le 1$. Fix a radially symmetric function φ in $\mathcal{C}^\infty_0(B_4)$ satisfying $0\le\varphi\le 1$, $\varphi\equiv 1$ for r near 0, $|\partial_r\varphi|\le 1$ and $|\Delta\varphi|\le 1$. Define φ_μ by $\varphi_\mu(x)=\varphi(\frac{\mu}{2}|x|)$.
Without loss of generality, we can assume that $\|u\|_{L^\infty}=|u(0)|$. Note that in particular one has

$$\|\varphi_\mu u\|_{\dot{\mathcal{C}}^\alpha} \le \|u\|_{\mathcal{C}^\alpha}$$

$$\|\nabla(\varphi_\mu u)\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \frac{\mu^2}{4}\|u\|_{L^2}^2 + 2\int_{\mathbb{R}^2} \varphi_\mu u \nabla\varphi_\mu \nabla u dx.$$

Integrating by parts,

$$2\int_{\mathbb{R}^2} \varphi_\mu u \nabla\varphi_\mu \nabla u dx = -\frac{1}{2}\int_{R^2} \Delta\varphi_\mu^2 u^2 dx = -\frac{\mu^2}{8}\int_{R^2} \Delta\varphi^2(\frac{\mu}{2}x)\ u^2 dx.$$

Hence,

$$\|\nabla(\varphi_\mu u)\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \mu^2\|u\|_{L^2}^2.$$

Applying the result of Theorem 1.3 and using the fact that for any constant $C > 0$, the function $x \to x^2 \log(C_\lambda + \frac{C}{x})$ is increasing, the proof of Corollary 5.2 is achieved. ∎

We also have the following result

Corollary 5.3. *Let $\alpha \in]0, 1[$. For any $\lambda > \frac{1}{2\pi\alpha}$, a constant $C_\lambda > 0$ exists such that, for any function $u \in H^1(\mathbb{R}^2) \cap \mathcal{C}^\alpha(\mathbb{R}^2)$*

$$\|u\|_{L^\infty} \leq \|u\|_{L^2} + \|\nabla u\|_{L^2}\sqrt{\lambda \log(e + C_\lambda \frac{\|u\|_{\mathcal{C}^\alpha}}{\|\nabla u\|_{L^2}})}. \tag{5.20}$$

For the proof of Corollary 5.3, we take the Littlewood-Paley decomposition of u, $u = \tilde{\Delta}_0 u + v$ where $v = \sum_{j=1}^\infty \Delta_j u$. Hence $\|v\|_{L^2} \leq C\|\nabla v\|_{L^2}$ and $\|v\|_{\mathcal{C}^\alpha} \leq \|u\|_{\mathcal{C}^\alpha}$. So

$$\|u\|_{L^\infty} \leq \|\tilde{\Delta}_0 u\|_{L^\infty} + \|v\|_{L^\infty}.$$

Then, we apply Corollary 5.2 to v with λ' and μ' such that $\lambda'(1 + C^2\mu'^2) < \lambda$. ∎

Of course, we have similar inequalities for the Log Log inequality (1.2) in $\mathbb{R}^2$ with the sharp constant $\frac{1}{2\pi\alpha}$.

6. Application to the wave equation

Corollary 5.2 is useful in studying 2D-nonlinear wave equations with exponential nonlinearities, and the constant $\frac{1}{2\pi\alpha}$ is crucial for local wellposedness results (see [7] for further discussion). In particular from Corollary 5.2 we can derive a Moser-Trudinger type inequality for the solution of the linear Klein-Gordon. Precisely, let $(f, g) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ such that $\|f\|_{H^1}^2 + \|g\|_{L^2}^2 \leq 1$. Denote by v the solution of the 2D linear Klein-Gordon equation

$$\begin{aligned} \partial_t^2 v - \Delta v + v &= 0 \\ v(0,\cdot) = f \quad , \quad & \partial_t v(0,\cdot) = g. \end{aligned}$$

Since the energy $\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^2)}^2 + \|v(t,\cdot)\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_t v(t,\cdot)\|_{L^2(\mathbb{R}^2)}^2$ is conserved, $v(t,\cdot)$ remains in the unit ball of H^1 uniformly in time. So according to (1.1) we have

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}^2} \left(e^{4\pi v(t,x)^2} - 1\right) dx \leq C$$

which means that $\exp(4\pi v^2(t,\cdot)) - 1 \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^2))$. To solve the 2D linear Klein-Gordon equation with an exponential nonlinearity, we would like that $\exp(4\pi v^2(t,\cdot)) - 1 \in L^1_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$. This is the object of the following result.

Proposition 6.1. *For any $T > 0$, a non-negative constant C_T exists such that*

$$\int_0^T \| \exp(4\pi v^2(t,\cdot)) - 1\|_{L^2(\mathbb{R}^2)}\, dt \le C_T.$$

Proof. For any $\mu > 0$, denote by

$$E_\mu(t) := \|\nabla v(t,\cdot)\|^2_{L^2(\mathbb{R}^2)} + \mu^2 \|v(t,\cdot)\|^2_{L^2(\mathbb{R}^2)}.$$

Recall that since $v \in \mathcal{C}(\mathbb{R},\ H^1) \cap \mathcal{C}^1(\mathbb{R},\ L^2)$, $E_\mu(t)$ is a continuous function of t. The energy conservation satisfied by v shows that

$$\|\partial_t v(t,\cdot)\|^2_{L^2(\mathbb{R}^2)} + E_1(t) = E_1(0) + \|g\|^2_{L^2} \le 1.$$

Now, fix $\mu < 1$ and $T > 0$. There exists a time $\tau = \tau(\mu, T)$ such that

$$\sup_{t\in[0,T]} E_\mu(t) = E_\mu(\tau) < 1.$$

For almost every t we have

$$(6.21)\quad \int_{\mathbb{R}^2} \big(\exp(4\pi v^2(t,x)) - 1\big)^2 dx \le \| \exp(4\pi v^2(t,\cdot)) - 1\|_{L^1} \exp(4\pi \|v(t,\cdot)\|^2_{L^\infty}).$$

Note that, thanks to conservation of the energy and Moser-Trudinger inequality, the first factor in the above inequality is uniformly bounded. On the other hand, choosing $\alpha = \frac{1}{4}$ in (5.19) we obtain, for any $\lambda > \frac{2}{\pi}$

$$\exp(2\pi \|v(t,\cdot)\|^2_{L^\infty}) \le \big(e + \frac{\|v(t,\cdot)\|_{\mathcal{C}^{1/4}}}{E_\mu(\tau)^{1/2}}\big)^{2\pi\lambda E_\mu(\tau)}.$$

Since $E_\mu(\tau) < 1$, one can choose $\lambda > \frac{2}{\pi}$ such that $\beta := 2\pi\lambda E_\mu(\tau) < 4$. Hence, we have

$$\begin{aligned}\int_0^T \exp(2\pi\|v(t,\cdot)\|^2_{L^\infty}) dt &\le C\int_0^T \big(e + \frac{\|v(t,\cdot)\|_{\mathcal{C}^{1/4}}}{E_\mu(\tau)^{1/2}}\big)^\beta dt \\ &\le CT^{1-\frac{\beta}{4}} \int_0^T \big(e + \frac{\|v(t,\cdot)\|_{\mathcal{C}^{1/4}}}{E_\mu(\tau)^{1/2}}\big)^4 dt.\end{aligned}$$

Now, thanks to the so-called Strichartz estimates (see [6]), we have $v \in L^4(\mathbb{R},\ \mathcal{C}^{1/4}(\mathbb{R}^2))$ and therefore Proposition 6.1 is proved. ■

Remark 6.2. Recall that in [3], a similar result was proved in a particular setting, namely, $f = 0$ and g is radially symmetric with compact support.

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