GLOBAL SOLUTIONS FOR A SEMILINEAR 2D KLEIN-GORDON EQUATION WITH EXPONENTIAL TYPE NONLINEARITY

S. Ibrahim
Department of Mathematics & Statistics, McMaster University, 1280 main Street West, Hamilton, Ontario, L8S 4L8 Canada.
slim.ibrahim@mcmaster.ca

M. Majdoub
Faculté des Sciences de Tunis, Département de Mathématiques
Campus universitaire 1060, Tunis, Tunisia. *mohamed.majdoub@fst.rnu.tn*

N. Masmoudi
Department of Mathematics, Courant Institute of Mathematical Sciences New York University, 251 Mercer
St. New York, NY 10012, U.S.A. *masmoudi@cims.nyu.edu*

(Submitted by: ...)

Abstract. We prove the existence and uniqueness of global solutions for a Cauchy problem associated to a semilinear Klein-Gordon equation in two space dimension. Our result is based on an interpolation estimate with a sharp constant obtained by a standard variational method.

Accepted for publication: ...
AMS Subject Classifications: ..., ..., ...

1. Introduction

In this paper, we study the well-posedness of the Cauchy problem associated to the semilinear wave equation

$$(\mathbf{E}_\alpha) \qquad (\partial_t^2-\Delta)u+ue^{\alpha u^2}=0,$$

where $u=u(t,x)$ is a real-valued function of $(t,x)\in\mathbb{R}\times\mathbb{R}^2$, $\partial_t=\frac{\partial}{\partial t}$ and Δ is the Laplace operator acting on space variables. We denote by $\Box=\partial_t^2-\Delta$ the wave operator. The initial data (f,g)

$$u(0,x)=f(x) \text{ and } \partial_t u(0,x)=g(x) \tag{1.1}$$

are in the energy space $H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$ and α is a nonnegative real number. H^1 is the usual Sobolev space endowed with the norm

$$\|f\|_{H^1}^2=\|f\|_{L^2}^2+\|\nabla f\|_{L^2}^2.$$

Before going any further, we shall first recall **a** few historic facts about this problem. First, we recall that in space dimensions $d\geq 3$, the defocusing semilinear wave equation with power p reads

$$\Box u+|u|^{p-1}u=0, \tag{1.2}$$

where p is a real number with $p>1$. This problem has been widely investigated and there is a large literature dealing with the local and global solvability of (1.2) in the scale of the Sobolev spaces H^s [11, 12, 17, 20, 22, 23, 26, 29, 36, 37, 38, 42] the uniqueness in suitable subspaces of finite energy solutions, ([27] and the references therein) and the asymptotics of the solutions as t goes to infinity (scattering theory) [3, 5, 9, 10, 16, 13, 14, 31, 33, 35]. For a detailed bibliography, see [44].

Second, it is well-known that the Cauchy problem (1.2) is locally well-posed in the usual Sobolev space $H^s(\mathbb{R}^d)$ if $s>\frac{d}{2}$, or when $\frac12\leq s<\frac{d}{2}$ and $p\leq 1+\frac{4}{d-2s}$. The difference between the conditions for the solvability among $s<\frac{d}{2}$, $s=\frac{d}{2}$, and $s>\frac{d}{2}$ basically comes from the Sobolev embedding $H^s\hookrightarrow L^p$ if $2\leq p\leq\frac{2d}{d-2s}$, $H^s\hookrightarrow L^p$ if $2\leq p<\infty$ and $H^s\hookrightarrow L^p$ if $2\leq p\leq\infty$, respectively (see [29, 30] and the references therein).

Finally, for the global solvability in the energy space $H^1\times L^2$, there are mainly three cases.

The first case is when $p<p_c$ where $p_c=\frac{d+2}{d-2}$, this is the subcritical case. In this case, Ginibre and Velo [12], showed that the problem (1.2) with initial data in $H^1\times L^2$ has a unique global solution in the space $\mathcal{C}(\mathbb{R},H^1(\mathbb{R}^d))\cap\mathcal{C}^1(\mathbb{R},L^2(\mathbb{R}^d))$.

If the exponent p is critical, (which means $p=p_c$), this problem was first

solved by Struwe [42] in the radial case, then by Grillakis [17] in the general case, and Shatah-Struwe [36], [37] in other dimensions. See also [19] for the case of variable metric. Notice that the proof is based on the so-called Strichartz inequalities for the solutions of the linear operator (See [15, 35, 39, 41, 40, 43]).
Finally in the case $p > p_c$, the well-posedness in the energy space is an open problem except for some partial results (for example see [4, 7, 25]). See also [6] for Shrödinger equations.
In dimension $d = 2$, any polynomial nonlinearity is “subcritiacl” with respect to the H^1 norm, so an exponential nonlinearity seems to be a natural critical nonlinearity.
Now we return to the equation $(\mathbf{E}_\alpha)$. Multiplying $(\mathbf{E}_\alpha)$ by $2\partial_t u$ and integrating on $\mathbb{R}^2$, we formally obtain the following conservation law

$$E(u,t) := \|\partial_t u(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2 + \int_{\mathbb{R}^2} \frac{e^{\alpha u^2}-1}{\alpha} dx = E(u,t=0). \quad (1.3)$$

A *priori,* one can estimate the nonlinear part of the energy using the following Moser-Trudinger type inequalities(see [2]).

Proposition 1. *Let $\alpha \in (0,4\pi)$. A constant c_α exists such that*

$$\int_{\mathbb{R}^2} \left(\exp\left(\alpha u(x)^2\right) - 1\right) dx \leq c_\alpha \|u\|_{L^2}^2 \quad (1.4)$$

for all u in $H^1(\mathbb{R}^2)$ such that $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. Moreover, if $\alpha \geq 4\pi$, then (1.4) is false.

Remark 1. We point out that $\alpha = 4\pi$ becomes admissible in (1.4) if we require $\|u\|_{H^1(\mathbb{R}^2)} \leq 1$ rather than $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. Precisely, we have

$$\sup_{\|u\|_{H^1(\mathbb{R}^2)}\leq 1} \int_{\mathbb{R}^2} \left(\exp\left(4\pi u(x)^2\right) - 1\right)\, dx < +\infty$$

and this is false for $\alpha > 4\pi$. See [34] for more details.

Remark 2. Notice that if u is a solution of $(\mathbf{E}_\alpha)$ then for any $\beta > 0$, $\frac{u}{\beta}$ is a solution of $(\mathbf{E}_{\alpha\beta^2})$ and obviously one may choose β such that $\alpha\beta^2 > 4\pi$. Hence, it makes sense to take the initial data in a ball of the energy space.

Remark 3. There are two points of view to deal with the Cauchy problem associated to the equation $(\mathbf{E}_\alpha)$. The first one is to fix the initial data in the unit ball of the energy space and distinguish the cases $\alpha < 4\pi$, $\alpha = 4\pi$ and $\alpha > 4\pi$. The second one is to fix $\alpha = 4\pi$ and discuss the size of the initial data in the energy space. Actually these two points of view are equivalent. We will choose the second one. In all what follows, we suppose $\alpha = 4\pi$.

Definition 1. The Cauchy problem ($\mathbf{E_{4\pi}}$)-(1.1) is said to be *subcritical* if

$$E_0 := \|g\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 + \int_{\mathbb{R}^2} \frac{\exp\left(4\pi f^2\right) - 1}{4\pi} dx < 1.$$

It is *critical* if $E_0 = 1$ and *supercritical* if $E_0 > 1$.

To establish an energy estimate, one has to consider the nonlinearity as a source term in ($\mathbf{E_{4\pi}}$), so we need to estimate it in the $L^1_t(L^2_x)$ norm. To do so, we use (1.4) combined with the so-called Strichartz estimate (See [15]).

Proposition 2 (Strichartz estimate).

$$\|v\|_{L^4_T(\mathcal{C}^{1/4}(\mathbb{R}^2))} \leq C\left[\|\partial_t v(0)\|_{L^2(\mathbb{R}^2)} + \|v(0)\|_{H^1(\mathbb{R}^2)} + \|\Box v + v\|_{L^1_T(L^2(\mathbb{R}^2))}\right] \tag{1.5}$$

But the problem with taking the L^2_x norm is to double 4π and therefore, we loose any control of that term using only (1.4). The following estimate is an L^∞ logarithmic inequality which enables us to establish the link between $\|e^{4\pi u^2} - 1\|_{L^1_T(L^2(\mathbb{R}^2))}$ and dispersion properties of solutions of the linear Klein-Gordon equation.

Proposition 3. *For any real $\lambda > \frac{2}{\pi}$, for any real $\mu > 0$, a constant $C_{\lambda,\mu}$ exists such that, for any function $u \in \mathcal{C}^{1/4}(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, one has*

$$\|u\|_{L^\infty}^2 \leq \lambda \|u\|_\mu^2 \, \log\left(C_{\lambda,\mu} + \frac{\|u\|_{\mathcal{C}^{1/4}}}{\|u\|_\mu}\right) \tag{1.6}$$

where $\|u\|_\mu^2 := \|\nabla u\|_{L^2}^2 + \mu^2 \|u\|_{L^2}^2$.

Recall that $\mathcal{C}^{1/4}(\mathbb{R}^2)$ denotes the space of 1/4-Hölder continuous functions endowed with the norm

$$\|u\|_{\mathcal{C}^{1/4}(\mathbb{R}^2)} := \|u\|_{L^\infty(\mathbb{R}^2)} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{1/4}}.$$

A sketch of the proof of this Proposition is discussed in the Appendix. We refer to [18] for more details. We just point out that the condition $\lambda > \frac{2}{\pi}$ in (1.6) is optimal. Our first result is the following local (in time) existence Theorem.

Theorem 1. *Assume that* $\|\nabla f\|_{L^2(\mathbb{R}^2)} < 1$. *Then, there exists a time* $T > 0$ *and a unique solution* u *to the problem* $(\mathbf{E_{4\pi}})$*-(1.1) in the space*

$$\mathcal{C}_T(H^1(\mathbb{R}^2)) \cap \mathcal{C}^1_T(L^2(\mathbb{R}^2)).$$

Moreover, $u \in L^4_T(\mathcal{C}^{1/4}(\mathbb{R}^2))$ *and satisfies, for all* $0 \leq t < T$, $E(u,t) = E(u,0)$.

Here and below $\mathcal{C}_T(X)$ denotes $\mathcal{C}([0,T);X)$. The proof of this Theorem is based on the combination of the three propositions given above. We derive the local wellposedness using a classical fixed point argument.

Remark 4. The assumption $E_0 \leq 1$ in particular implies that $\|\nabla f\|_{L^2(\mathbb{R}^2)} < 1$ and consequently we have the short time existence of solutions in both subcritical and critical case. So it makes sense to deal with global existence in these cases.

Remark 5. In a recent work, A. Attallah [1] proved a local existence result of solutions to $(\mathbf{E}_\alpha)$ for $\alpha < 4\pi$, under more restrictive assumptions on the initial data by assuming that $f = 0$ and g is radially symmetric with compact support. This result is based on $L^\infty_t(L^2_x)$ estimate of the nonlinear term. Such an estimate seems to be unreasonable in the general case.

As an immediate consequence of Theorem 1 we have the following global existence result.

Theorem 2 **(Subcritical case).**
Assume that $E_0 < 1$, *then the problem* $(\mathbf{E_{4\pi}})$*-(1.1) has a unique global solution* u *in the class*

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)) \cap \mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^2)).$$

Moreover, $u \in L^4_{loc}(\mathbb{R},\ \mathcal{C}^{1/4}(\mathbb{R}^2))$ *and satisfies the energy identity.*

The reasons of Definition 1 is the following. If u denotes the solution given by Theorem 1, where $T^* < \infty$ is the largest time of existence, then the conservation of the total energy gives us, in the subcritical setting, a uniform bound of $\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^2)}$ away from 1 and therefore the solution can be continued in time. Contrary to the critical case, where we loose this uniform control and therefore the total mass of energy can be concentrated in the $\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^2)}$ part. By establishing some local (in space-time) identities, we show that such concentration cannot hold in the critical case and therefore we have the following theorem.

Theorem 3 (**Critical case**).
Assume that $E_0 = 1$, then the problem $(\mathbf{E_{4\pi}})$*-(1.1) has a unique global solution u in the class*

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)) \cap \mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^2)).$$

Moreover, $u \in L^4_{loc}(\mathbb{R},\ \mathcal{C}^{1/4}(\mathbb{R}^2))$ and satisfies the energy identity.

Remark 6. It would be desirable to show that the global solution u is in $L^4(\mathbb{R},\ \mathcal{C}^{1/4}(\mathbb{R}^2))$ globally in time, at least when we replace the nonlinear term by $u\left(e^{4\pi u^2} - 1 - 4\pi u^2\right)$. This question as well as some scattering results will be dealt with in a forthcoming paper.

When the initial data are more regular, we can easily prove that the solution remains regular. More precisely, we have the following theorem.

Theorem 4. *Assume that $(f, g) \in H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$ with $s > 1$ and $\|\nabla f\|_{L^2(\mathbb{R}^2)} < 1$. Then, the solution u given in Theorem 1 is in the space $\mathcal{C}_T(H^s(\mathbb{R}^2)) \cap \mathcal{C}^1_T(H^{s-1}(\mathbb{R}^2))$.*

To the best of the authors's knowledge, Theorem 2 and Theorem 3 are the only results for global solutions of such 2D problems with exponential growth nonlinearities. In [29], Nakumura and Ozawa proved, under an assumption of smallness of the initial data, the existence of global solutions.

The structure of the paper is as follows. In the next section, we present some notations which will be used in the sequel. Section 3 is devoted to the proof of Theorem 1 about local existence. The fourth section deals with global existence. Proofs of Theorems 2-3 are presented there. In the appendix, we give a sketch of the proof of Proposition 3 and some variants of the L^∞ logarithmic inequality.

2. Notations

For $Q \subset \mathbb{R} \times \mathbb{R}^2$ and $S < T$, we define

$$Q_S^T := \{z = (t, x) \in Q \quad \text{such that} \quad S \le t \le T\}, \tag{2.1}$$

the truncated part of Q between the times S and T.
For $z_0 = (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^2$, we define

$$K(z_0) := \{z = (t, x) \in \mathbb{R} \times \mathbb{R}^2 \quad \text{such that } |x - x_0| \le t_0 - t\}$$

the backward cone of vertex z_0, and for fixed t

$$D(t, z_0) := \{x \in \mathbb{R}^2 \quad \text{such that } |x - x_0| \le t_0 - t\}$$

its section at the time t. The mantle of the cone $K(z_0)$, denoted by $M(z_0)$, is defined by

$$M(z_0) := \{z = (t, x) \in \mathbb{R} \times \mathbb{R}^2 \quad \text{such that } |x - x_0| = t_0 - t\}.$$

For $x \in \mathbb{R}^2$ and $r \in \mathbb{R}\backslash\{0\}$, we denote by $B(x, r)$ the ball of $\mathbb{R}^2$ centered at x and of radius $|r|$. If $x = 0$, we use the notation $B(r)$ instead of $B(x, r)$. In particular, we remark that $D(t, z_0) = B(x_0, t_0 - t)$.
For any function $u = u(t, x)$, we define the energy density, the local energy, the flux density and the flux of u by

$$e(u, \partial_t u)(t, x) := (\partial_t u)^2 + |\nabla_x u|^2 + \frac{e^{4\pi u^2} - 1}{4\pi},$$

$$E(u, D(t, z_0)) := \int_{D(t,z_0)} e(u, \partial_t u)(t, x)\, dx$$

$$d_{z_0}(u) := \frac{1}{\sqrt{2}}\Big[\Big|\partial_t u \frac{x}{|x|} + \nabla_x u\Big|^2 + \frac{e^{4\pi u^2} - 1}{4\pi}\Big],$$

$$\text{Flux}(u, M_S^T(z_0)) := \int_{M_S^T(z_0)} d_{z_0}(u)\, d\sigma,$$

respectively. If no ambiguity can occur, we denote $e(u) = e(u, \partial_t u)$. Let v_0 denote the solution of the free Klein-Gordon equation with the same initial data (f, g), namely

$$\Box v_0 + v_0 = 0, \quad v_0(0, x) = f(x) \quad \text{and} \quad \partial_t v_0(0, x) = g(x). \tag{2.2}$$

Recall that for any real number $\mu > 0$ and any function w in $H^1(\mathbb{R}^2)$

$$\|w\|_\mu^2 := \|\nabla w\|_{L^2}^2 + \mu^2 \|w\|_{L^2}^2.$$

In all what follows, we denote by E_0 the initial energy, namely

$$E_0 := \int_{\mathbb{R}^2} \Big\{|\nabla f(x)|^2 + |g(x)|^2 + \frac{e^{4\pi f(x)^2} - 1}{4\pi}\Big\}\, dx.$$

Finally, we mention that, C will be used to denote a constant which may vary from line to line.

3. LOCAL EXISTENCE

In this section we shall prove the local wellposedness for the equation ($\mathbf{E_{4\pi}}$) (Theorem 1). To show the local (small time) existence of solutions of ($\mathbf{E_{4\pi}}$), we use a standard fixed point argument.
We introduce, for any nonnegative time T, the following complete metric space

$$\mathcal{E}_T = \mathcal{C}([0,T], H^1(\mathbb{R}^2)) \cap \mathcal{C}^1([0,T], L^2(\mathbb{R}^2)) \cap L^4([0,T], \mathcal{C}^{1/4}(\mathbb{R}^2))$$

endowed with the norm

$$\|u\|_T := \sup_{0\leq t\leq T} \Big[\|u(t,.)\|_{H^1} + \|\partial_t u(t,.)\|_{L^2}\Big] + \|u\|_{L^4([0,T],\mathcal{C}^{1/4})}.$$

Proof of Theorem 1.
Let us start by proving the existence. For a positive time T and a positive real number δ, we denote by $\mathcal{E}_T(\delta)$ the ball in $\mathcal{E}_T$ of radius δ and centered at the origin. On the ball $\mathcal{E}_T(\delta)$, we define the map Φ by

$$v \longmapsto \Phi(v) := \tilde{v}, \tag{3.1}$$

where

$$\Box\tilde{v} + \tilde{v} = -(v+v_0)\big(e^{4\pi(v+v_0)^2} - 1\big), \quad \tilde{v}(0,x) = \partial_t\tilde{v}(0,x) = 0, \tag{3.2}$$

and v_0 is defined by (2.2).
Now the problem is to show that, if δ and T are small enough, the map Φ is well defined from $\mathcal{E}_T(\delta)$ into itself and it is a contraction.
In order to show that the map Φ is well defined, we need to estimate the term $\|(v+v_0)\big(e^{4\pi(v+v_0)^2} - 1\big)\|_{L^1_T(L^2_x)}$. Indeed, by the Energy estimate, we know that

$$||\tilde{v}||_T \leq C\|(v+v_0)\big(e^{4\pi(v+v_0)^2} - 1\big)\|_{L^1_T(L^2_x)}.$$

By the Hölder inequality and the Sobolev embedding, we have

$$\begin{aligned}\int_{\mathbb{R}^2} (v+v_0)^2 \left(e^{4\pi(v+v_0)^2} - 1\right)^2 dx &\leq \|v+v_0\|^2_{L^{2+2/\varepsilon}} \;\|\big(e^{4\pi(v+v_0)^2} - 1\big)^2\|_{L^{1+\varepsilon}} \\ &\leq \; C\|v+v_0\|^2_{H^1}\,\big(e^{4\pi\|v+v_0\|^2_{L^\infty}}\big)\|e^{4\pi(v+v_0)^2} - 1\|_{L^{1+\varepsilon}}\end{aligned}$$

where ε is a nonnegative real number which will be chosen later.
The first term on the right hand side of the last inequality can be easily estimated

$$\begin{aligned}\|v+v_0\|^2_{H^1} &\leq \Big(\delta + \|v_0\|_{H^1}\Big)^2 \\ &\leq 2\Big(\delta^2 + \|f\|^2_{H^1} + \|g\|^2_{L^2}\Big) := A.\end{aligned}$$

On the other hand, using Proposition 3, we can write

$$e^{4\pi\|v+v_0\|^2_{L^\infty}} \quad \leq \quad \exp\left(4\pi\lambda\|v+v_0\|^2_\mu \log\left(C_{\lambda,\mu} + \frac{\|v+v_0\|_{\mathcal{C}^{1/4}}}{\|v+v_0\|_\mu}\right)\right)$$

for any $\lambda > \frac{2}{\pi}$, $\mu > 0$, and $C_{\lambda,\mu}$ given by Proposition 3. In addition, it is clear that

$$\int_{\mathbb{R}^2} \left(e^{4\pi(v+v_0)^2} - 1\right)^{1+\varepsilon} dx \quad \leq \quad \int_{\mathbb{R}^2} \left(e^{4\pi(1+\varepsilon)(v+v_0)^2} - 1\right) dx.$$

Now, since $\|\nabla f\|_{L^2} < 1$, we choose a real number $\mu > 0$ such that $\|f\|^2_\mu < 1$. By continuity in time of v_0, there exists a nonnegative time T_0 such that for all t in $[0, T_0]$ we have

$$\|v_0(t,.)\|^2_\mu \leq 1 - \eta \tag{3.3}$$

where $\eta := \frac{1}{2}\left(1 - \|f\|^2_\mu\right)$. Using the fact that $||v+v_0||_\mu \leq \delta + \sqrt{1-\eta}$, it follows that

$$e^{4\pi\|v+v_0\|^2_{L^\infty}} \leq \left(C_{\lambda,\mu} + \frac{\|v+v_0\|_{\mathcal{C}^{1/4}}}{\delta+\sqrt{1-\eta}}\right)^{4\pi\lambda(\delta+\sqrt{1-\eta})^2}$$

and, in view of Proposition 1

$$\begin{aligned}\int_{\mathbb{R}^2} \left(e^{4\pi(v+v_0)^2} - 1\right)^{1+\varepsilon} dx \quad &\leq \quad \int_{\mathbb{R}^2} \left(\exp\left(4\pi(1+\varepsilon)(\delta+\sqrt{1-\eta})^2 \frac{(v+v_0)^2}{(\delta+\sqrt{1-\eta})^2}\right) - 1\right) dx \\ &\leq \quad C(\varepsilon,\delta,\eta)\|v+v_0\|^2_{L^2} \\ &\leq \quad C(\varepsilon,\delta,\eta)A\end{aligned}$$

provided that $4\pi(1+\varepsilon)(\delta+\sqrt{1-\eta})^2 < 4\pi$ which is possible since

$$(1+\varepsilon)(\delta+\sqrt{1-\eta})^2 \longrightarrow 1-\eta < 1 \quad \text{as} \quad \varepsilon,\ \delta \to 0.$$

Therefore, for any $0 < T \leq T_0$, we obtain

$$\int_0^T \|(v+v_0)\ (e^{4\pi(v+v_0)^2} - 1)\|_{L^2} dt \quad \leq \quad C\int_0^T \left(C_{\lambda,\mu} + \frac{\|v+v_0\|_{\mathcal{C}^{1/4}}}{\delta+\sqrt{1-\eta}}\right)^{\beta/2} dt$$

where $\beta := 4\pi\lambda(\delta+\sqrt{1-\eta})^2$. Since $4\pi\lambda(\delta+\sqrt{1-\eta})^2 \longrightarrow 8(1-\eta) < 8$ as δ and λ go to $2/\pi$, we can choose δ and λ such that $\beta < 8$. With this choice we get

$$\int_0^T \|(v+v_0)\ (e^{4\pi(v+v_0)^2} - 1)\|_{L^2} dt \leq CT^{1-\beta/8}\left\|C_{\lambda,\mu} + \frac{\|v+v_0\|_{\mathcal{C}^{1/4}}}{\delta+\sqrt{1-\eta}}\right\|^{\beta/2}_{L^4_T}$$

$$
\begin{aligned}
&\le CT^{1-\beta/8}\left(C_{\lambda,\mu}T^{1/4}+\frac{\|v+v_0\|_{L^4_T(\mathcal{C}^{1/4})}}{\delta+\sqrt{1-\eta}}\right)^{\beta/2}\\
&\le CT^{1-\beta/8}\left(C_{\lambda,\mu}T^{1/4}+\frac{\delta+\|f\|_{H^1}+\|g\|_{L^2}}{\delta+\sqrt{1-\eta}}\right)^{\beta/2}.
\end{aligned}
$$

From this inequality it follows immediately that, if T is small enough then Φ goes from $\mathcal{E}_T(\delta)$ to $\mathcal{E}_T(\delta)$.
To prove that Φ is a contraction (at least for T small), we consider two elements v_1 and v_2 in $\mathcal{E}_T(\delta)$ and define

$$v=v_1-v_2,\quad \tilde{v}=\tilde{v_1}-\tilde{v_2},\quad \bar{v}=(1-\theta)(v_0+v_1)+\theta(v_0+v_2)\quad\text{with}\quad 0\le\theta\le 1.$$

We can write

$$(v_1+v_0)(e^{4\pi(v_1+v_0)^2}-1)-(v_2+v_0)(e^{4\pi(v_2+v_0)^2}-1)=v\big[(1+8\pi\bar{v}^2)e^{4\pi\bar{v}^2}-1\big]$$

for some choice of $0\le\theta(t,x)\le 1$. By the energy estimate and the Strichartz inequality we have

$$\Big\|\Phi(v_1)-\Phi(v_2)\Big\|_T\le C\Big\|v\big[(1+8\pi\bar{v}^2)e^{4\pi\bar{v}^2}-1\big]\Big\|_{L^1_T(L^2_x)}.$$

Notice that

$$\int_0^T\|v\big[(1+8\pi\bar{v}^2)e^{4\pi\bar{v}^2}-1\big]\|_{L^2}\,dt\le C_\varepsilon\int_0^T\|v\big(e^{4\pi(1+\varepsilon)\bar{v}^2}-1\big)\|_{L^2}\,dt,$$

for any $\varepsilon>0$ and that by convexity

$$
\begin{aligned}
e^{4\pi(1+\varepsilon)[(1-\theta)(v_0+v_1)+\theta(v_0+v_2)]^2} &\le (1-\theta)e^{4\pi(1+\varepsilon)(v_0+v_1)^2}+\theta e^{4\pi(1+\varepsilon)(v_0+v_2)^2}\\
&\le e^{4\pi(1+\varepsilon)(v_0+v_1)^2}+e^{4\pi(1+\varepsilon)(v_0+v_2)^2},
\end{aligned}
$$

and

$$\|\bar{v}\|_\mu\le\delta.$$

So arguing as before and setting $\beta=4\pi(1+\varepsilon)\lambda(\delta+\sqrt{1-\eta})^2$, we obtain

$$\|\Phi(v_1)-\Phi(v_2)\|_T\le CT^{1-\beta/8}\left(C_{\lambda,\mu}T^{1/4}+\frac{\delta+\|f\|_{H^1}+\|g\|_{L^2}}{\delta+\sqrt{1-\eta}}\right)^{\beta/2}\|v_1-v_2\|_T.$$

If the parameters $\varepsilon>0$, $\lambda>2/\pi$, $\mu>0$ and $\delta>0$ are suitably chosen, then $\beta<8$ and therefore for T small enough, Φ is a contraction map. Notice that this also proves uniqueness in the existence class, namely in $v_0+\mathcal{E}_T(\delta)$.
To prove the conservation of the energy, we multiply the equation by $\partial_t u$ and then integrate by parts. All the computations are justified since the exponential term is in $L^1_T(L^2_x)$.

Now we shall prove the uniqueness in the energy space. Since the uniqueness holds in $v_0+\mathcal{E}_T(\delta)$, it suffices to show that, if $u=v_0+v$ solves $(\mathbf{E_{4\pi}})$-(1.1) with $v\in\mathcal{C}_T(H^1(\mathbb{R}^2))\cap\mathcal{C}_T^1(L^2(\mathbb{R}^2))$, then necessary $v\in\mathcal{E}_T(\delta)$ (at least for T small).
Let $\varepsilon>0$ and $a>1$ be such that $\frac{a}{a-1}\varepsilon$ and $a-1$ are small enough. There exists $0<\mu\leq 1$ and $T_1>0$ such that (3.3) holds and

$$\sup_{0\leq t\leq T_1}\left(\|v(t,.)\|_{H^1}+\|\partial_t v(t,.)\|_{L^2}\right)\leq\varepsilon.$$

Using Strichartz inequality and the fact that v satisfies

$$\Box v+v=-(v+v_0)\big(e^{4\pi(v+v_0)^2}-1\big),\quad v(0,x)=\partial_t v(0,x)=0,\tag{3.4}$$

we just need to estimate the right hand side of (3.4) in $L^1_T(L^2_x)$. Arguing as in the proof of the existence part, and using the following observations

$$(v+v_0)^2\leq av_0^2+\frac{a}{a-1}v^2,$$

$$e^{x+y}-1=(e^x-1)(e^y-1)+(e^x-1)+(e^y-1),$$

we have to estimate the following three terms: For all $t\in]0,T_1[$

$$\begin{aligned}
\mathcal{I}_1(t) &= \int_{\mathbb{R}^2}\left(e^{4\pi(1+\varepsilon)\frac{a}{a-1}v^2(t,x)}-1\right)^2dx,\\
\mathcal{I}_2(t) &= \int_{\mathbb{R}^2}\left(e^{4\pi(1+\varepsilon)av_0^2(t,x)}-1\right)^2dx,\\
\mathcal{I}_3(t) &= \int_{\mathbb{R}^2}\left(e^{4\pi(1+\varepsilon)av_0^2(t,x)}-1\right)^2\left(e^{4\pi(1+\varepsilon)\frac{a}{a-1}v^2(t,x)}-1\right)^2dx.
\end{aligned}$$

For the first term, we use Remark 1 to obtain

$$\mathcal{I}_1(t)\leq C.$$

For the second term, we write

$$\begin{aligned}
\mathcal{I}_2(t) &\leq e^{4\pi(1+\varepsilon)a\|v_0(t,\cdot)\|^2_{L^\infty}}\int_{\mathbb{R}^2}\left(e^{4\pi(1+\varepsilon)av_0^2(t,x)}-1\right)dx\\
&\leq C\left(1+\|v_0(t,\cdot)\|_{\mathcal{C}^{1/4}}\right)^\beta
\end{aligned}$$

where $\beta=4\pi(1+\varepsilon)a(1-\eta)\lambda$.
Finally, for the third term, we have

$$\begin{aligned}
\mathcal{I}_3(t) &\leq e^{4\pi(1+\varepsilon)a\|v_0(t)\|^2_{L^\infty}}\left(\int\left(e^{4\pi(1+\varepsilon)a^2v_0^2(t,x)}-1\right)dx\right)^{1/a}\left(\int\left(e^{8\pi(1+\varepsilon)\frac{a^2}{(a-1)^2}v^2}-1\right)dx\right)^{1-1/a}\\
&\leq C\left(1+\|v_0(t)\|_{\mathcal{C}^{1/4}}\right)^\beta.
\end{aligned}$$

This finishes the proof of the uniqueness in the energy space. ■

4. Global Existence

We start this section by the following remark about the existence time. This remark will be very important to extend the solution.

Remark 7.
In Theorem 1, the time of existence T depends on f and g. However, in the case $\|\nabla f\|^2_{L^2(\mathbb{R}^2)} + \|f\|^2_{L^2(\mathbb{R}^2)} + \|g\|^2_{L^2(\mathbb{R}^2)} < 1$, this time of existence depends only on $\eta = 1 - \|\nabla f\|^2_{L^2(\mathbb{R}^2)} - \|f\|^2_{L^2(\mathbb{R}^2)} - \|g\|^2_{L^2(\mathbb{R}^2)}$. Indeed, in this case one can take $\mu = 1$ and $T_0 = +\infty$. Then, one can see easily that the choice of $\delta, \varepsilon, \beta$ and T depends only on μ, η and T_0.

4.1. **Subcritical Case.**

Recall that by subcitical we mean that $E_0 < 1$, where

$$E_0 = \int_{\mathbb{R}^2} \left\{ |\nabla f(x)|^2 + |g(x)|^2 + \frac{e^{4\pi f(x)^2} - 1}{4\pi} \right\} dx.$$

Let $(\mathcal{P})$ denote the following Cauchy problem

$$(\mathcal{P}) \qquad \left\{ \begin{array}{rcl} \Box u + u e^{4\pi u^2} & = & 0 \\ u(0,x) & = & f(x) \in H^1(\mathbb{R}^2) \\ \partial_t u(0,x) & = & g(x) \in L^2(\mathbb{R}^2). \end{array} \right.$$

Since the assumption $E_0 < 1$ implies $\|\nabla f\|_{L^2} < 1$, it follows that the problem $(\mathcal{P})$ has an unique maximal solution u in the space $\mathcal{E}_{T^*}$ where $0 < T^* \leq +\infty$ is the lifespan of u. We want to show that $T^* = +\infty$ which means that our solution is global in time.
Assume that $T^* < \infty$, then by the energy identity (1.3), we deduce

$$\sup_{t \in (0,T^*)} \|\nabla u(t,.)\|_{L^2(\mathbb{R}^2)} \leq E_0 < 1.$$

Let $0 < s < T^*$ and consider the following Cauchy problem

$$\left\{ \begin{array}{rcl} \Box v + v e^{4\pi v^2} & = & 0 \\ v(s,x) & = & u(s,x) \\ \partial_t v(s,x) & = & \partial_t u(s,x). \end{array} \right.$$

By a fixed point argument, as in the proof of Theorem 1, we can see that there exists a nonnegative τ and an unique solution v to our problem on the interval $[s,\ s+\tau]$. Notice that τ does not depend on s (see Remark 7).

Choosing s close to T^* such that $T^* - s < \tau$ we can prolong the solution u after the time T^* which is a contradiction. ∎

4.2. **Critical Case.**
We consider now the critical case, namely $E_0 = 1$, and we want to prove a global existence result as in the previous section. The situation here is more delicate and the arguments used in the subcritical case do not apply here. Let us briefly explain what the major difficulty is. Since $E_0 = 1$ and by the conservation of the total energy (1.3), it is possible (at least formally) that a concentration phenomena holds, namely

$$\limsup_{t \to T^*} \; \|\nabla u(t,.)\|_{L^2} = 1$$

where u is the maximal solution and $T^* < +\infty$ is the lifespan of u. In this case, we can not apply the previous argument to prolong the solution. The actual proof is based on proving that the concentration phenomenon does not happen.

Proof of Theorem 3.
Let u be the maximal solution of $(\mathcal{P})$ defined on $[0,\ T^*)$. Assume that T^* is finite and let us prove a contradiction. It is easy to see that

$$\sup_{0 \le t < T^*} \|\nabla u(t,.)\|_{L^2} = 1.$$

Otherwise, there exists a real number $0 < \eta < 1$ such that

$$\sup_{0 \le t < T^*} \|\nabla u(t,.)\|_{L^2} \le 1 - \eta$$

and hence we can prolong our solution arguing exactly in the same way as in the subcritical case. The rest of the proof is divided into several steps.

First Step:
We have the following proposition .

Proposition 4. *The maximal solution u satisfies:*

$$\limsup_{t \to T^*} \; \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = 1, \tag{4.1}$$

$$u(t) \quad \overset{t \to T^*}{\longrightarrow} 0 \quad in \quad L^2(\mathbb{R}^2). \tag{4.2}$$

Proof of Proposition 4:
By the energy identity (1.3), we get

$$\forall \;\; 0 \le t < T^*, \quad \int_{\mathbb{R}^2} \left\{ |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 + \frac{e^{4\pi u(t,x)^2} - 1}{4\pi} \right\} dx = 1.$$

So, $\limsup\limits_{t\to T^*} \;\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \leq 1$. Suppose that

$$\limsup_{t\to T^*} \;\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = L < 1.$$

Then, a time t_0 exists such that $0 < t_0 < T^*$ and

$$t_0 < t < T^* \Longrightarrow \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \leq \frac{L+1}{2}.$$

On the other hand, by continuity,

$$\sup_{0\leq t\leq t_0} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = \|\nabla u(t_1)\|_{L^2(\mathbb{R}^2)}, \quad 0 \leq t_1 \leq t_0.$$

Since $\|\nabla u(t_1)\|_{L^2(\mathbb{R}^2)} < 1$, we obtain $\sup\limits_{0\leq t<T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} < 1$ which is a contradiction. This concludes the proof of (4.1).
To prove (4.2), recall that $u \in \mathcal{C}_{T^*}(H^1(\mathbb{R}^2)) \cap \mathcal{C}^1_{T^*}(L^2(\mathbb{R}^2))$. So, we can write

$$\begin{aligned} \|u(t) - u(s)\|_{L^2} &= \left\| \int_s^t \partial_t u(\tau)\, d\tau \right\|_{L^2} \\ &\leq |t-s|. \end{aligned}$$

It follows that $(u(t))$ admits a limit point $\bar{u}$ in $L^2(\mathbb{R}^2)$ as $t \longrightarrow T^*$, and therefore It remains to show that $\bar{u} = 0$. By (1.3) we have

$$\|\nabla u(t)\|^2_{L^2} - 1 = -\|\partial_t u(t)\|^2_{L^2} - \int_{\mathbb{R}^2} \frac{e^{4\pi u^2} - 1}{4\pi}\, dx.$$

Take the lim sup in both sides of this equality as $t \longrightarrow T^*$, we get

$$0 = -\liminf_{t\longrightarrow T^*} \|\partial_t u(t)\|^2_{L^2} - \liminf_{t\longrightarrow T^*} \int_{\mathbb{R}^2} \frac{e^{4\pi \bar{u}^2} - 1}{4\pi}\, dx.$$

Fatou Lemma implies

$$\liminf_{t\longrightarrow T^*} (e^{4\pi u^2} - 1) = 0$$

and hence $\bar{u} = 0$. This ends the proof of Proposition 4. ■

Second Step:
Now, the idea is to show that, in fact, the concentration phenomena holds in the section of some backward cone of vertex $z^* := (T^*, x^*)$. Since the equation ($\mathbf{E_{4\pi}}$) is invariant under time translation, we can assume without loss of generality that $T^* = 0$ and the initial time is $t = -1$. This assumption will be made in all the sequel. We have the following.

Proposition 5. *There exists a point x^* in $\mathbb{R}^2$ such that for all $r > 0$, we have*

$$\limsup_{t \longrightarrow 0^-} \int_{|x-x^*|\leq r} |\nabla u(t)|^2 \; dx = 1. \tag{4.3}$$

Proof of Proposition 5.
By contradiction assume that for any $x \in \mathbb{R}^2$, there exists two positive real numbers r_x and η_x such that

$$\limsup_{t \longrightarrow 0^-} \int_{|y-x|\leq r_x} e(u)(t,y) \; dy \leq 1 - \eta_x. \tag{4.4}$$

In fact, one can choose these two numbers to be uniform with respect to x because otherwise, taking $r_n = \eta_n = 1/n$ with $n \in \mathbb{N}^*$, a sequence (x_n) would exist such that

$$\limsup_{t \longrightarrow 0^-} \int_{|y-x_n|\leq 1/n} e(u)(t,y) \; dy \geq 1 - 1/n.$$

Now since the measure $e(u)(t,y)dy$ is tight, one can extract a subsequence $(x_{\varphi(n)})$ converging to a point x^* in $\mathbb{R}^2$. So, for any $r > 0$ and $n \in \mathbb{N}^*$

$$\limsup_{t \longrightarrow 0^-} \int_{|y-x^*|\leq r} e(u)(t,y)dy \geq 1 - 2/n$$

and therefore

$$\limsup_{t \longrightarrow 0^-} \int_{|y-x^*|\leq r} e(u)(t,y)dy = 1$$

which contradicts our assumption.
Now the idea is to show that in such a situation the solution u can be continued after the blow-up time $T^* = 0$. Let $x \in \mathbb{R}^2$. Define the cut-off function φ_x by $0 \leq \varphi \leq 1$, $\varphi_x \equiv 1$ in $B(x, r/2)$ and $\varphi_x \equiv 0$ in $B(x,r)^c$. Obviously, from (4.4) and Proposition 4, we have

$$\limsup_{t \longrightarrow 0^-} \int_{|y-x|\leq r} e(\varphi_x u, \varphi_x \partial_t u)(t) \; dy \leq 1 - \eta.$$

Now choose a time $t_1 > T^* - r/8$ such that

$$\int_{|y-x|\leq r} e(\varphi_x u, \varphi_x \partial_t u)(t_1) \; dy \leq 1 - \eta/2.$$

From the local theory (Theorem 1), one can solve $(\mathbf{E_{4\pi}})$ with the initial data $(\varphi_x u(t_1, \cdot), \varphi_x \partial u(t_1, \cdot))$ globally in time. By finite speed of propagation, we deduce that u can be continued in the backward light cone of vertex

$(x, t_1 + r/2)$. Hence u is continued at least till the time $T^* + r/8$ which is a contradiction.

A consequence of the Proposition 5 is

Corollary 1. *With the notations of Proposition 5 we have the following*

$$\lim_{t \longrightarrow 0^-} \int_{|x-x^*| \leq -t} |\nabla u(t)|^2 \; dx = 1. \tag{4.5}$$

$$\forall \; t < 0, \qquad \int_{|x-x^*| \leq -t} e(u(t)) \, dx = 1. \tag{4.6}$$

Proof of Corollary 1.
Assume that $x^* = 0$. The proof of (4.5) is straightforward. Indeed, suppose that (4.5) is false. Then, there exists a sequence of negative real number (t_n) tending to zero when n goes to infinity such that

$$\forall \; n \in \mathbb{N}, \quad \int_{|x| \leq -t_n} |\nabla u(t_n)|^2 \; dx \leq 1 - \eta \quad \text{for some } 0 < \eta < 1.$$

Then, arguing as in the proof of the previous proposition, we can prolong our solution which yields a contradiction.
To prove (4.6), fix $\varepsilon > 0$. By (4.5), there exists a time $t_\varepsilon < 0$ such that $\int_{|x| \leq -t} |\nabla u(t)|^2 \; dx \geq 1 - \varepsilon$ for $t_\varepsilon \leq t < 0$. By using the finite speed of propagation, we deduce that

$$\forall \; t < 0, \quad \int_{|x| \leq -t} e(u)(t) \; dx \geq 1 - \varepsilon.$$

Letting ε go to zero, we obtain the desired result. ■

End of the Proof of Theorem 3.
Let $S < T < 0$, K_S^T be the truncated backward cone of vertex $(0,0)$ and M_S^T be its mantle. Multiplying the equation $(E_{4\pi})$ by $\partial_t u$ and u yields

$$\partial_t e(u) - \operatorname{div}_x(2\partial_t u \nabla u) = 0 \tag{4.7}$$

$$\partial_t(\partial_t u \; u) - \operatorname{div}_x(u \nabla u) + |\nabla u|^2 - |\partial_t u|^2 + u^2 \, e^{4\pi u^2} = 0 \tag{4.8}$$

Integrating the conservation laws (4.7) and (4.8) over the backward truncated cone K_S^T, we obtain the following two identities:

$$\int_{B(T)} e(u(T)) \, dx - \int_{B(S)} e(u(S)) \, dx = \frac{-1}{\sqrt{2}} \int_{M_S^T} \left\{ \left| \partial_t u \frac{x}{|x|} + \nabla u \right|^2 + \frac{e^{4\pi u^2} - 1}{4\pi} \right\} d\sigma. \tag{4.9}$$

$$\int_{B(T)} \partial_t u(T)\, u(T)\, dx \quad - \quad \int_{B(S)} \partial_t u(S)\, u(S)\, dx + \frac{1}{\sqrt{2}} \int_{M_S^T} \Big(\partial_t u + \nabla u . \frac{x}{|x|}\Big) u\, d\sigma$$
$$+ \quad \int_{K_S^T} \Big\{|\nabla u|^2 - |\partial_t u|^2 + u^2\, e^{4\pi u^2}\Big\}\, dx\, dt = 0. \qquad (4.10)$$

Notice that these identities are obtained by using an approximation argument as in [37]. Using (4.6), we deduce first that

$$\int_{M_S^T} \Big\{\Big|\partial_t u \frac{x}{|x|} + \nabla u\Big|^2 + \frac{e^{4\pi u^2} - 1}{4\pi}\Big\}\, d\sigma = 0. \qquad (4.11)$$

Since $u(t) \longrightarrow 0$ strongly in L^2 and $\|\nabla u(t)\|_{L^2}^2 \longrightarrow 1$ as t goes to zero, it follows from the energy identity (1.3) that

$$\partial_t u(t) \longrightarrow 0 \quad \text{strongly in} \quad L^2(\mathbb{R}^2).$$

Letting T go to zero in (4.10) and using (4.11), we get

$$-\int_{B(S)} \partial_t u(S)\, u(S)\, dx + \int_{K_S^0} \Big\{|\nabla u|^2 - |\partial_t u|^2 + u^2\, e^{4\pi u^2}\Big\}\, dx\, dt = 0. \quad (4.12)$$

Multiplying (4.12) by $\frac{-1}{S}$ we deduce

$$\int_{B(S)} \partial_t u(S)\, \frac{u(S)}{S}\, dx \leq \frac{1}{S} \int_{K_S^0} |\nabla u|^2\, dx\, dt - \frac{1}{S} \int_{K_S^0} |\partial_t u|^2\, dx\, dt. \quad (4.13)$$

It is clear that

$$\frac{1}{S} \int_{K_S^0} |\nabla u|^2\, dx\, dt \longrightarrow -1 \quad \text{as} \quad S \longrightarrow 0^-$$

and

$$\frac{1}{S} \int_{K_S^0} |\partial_t u|^2\, dx\, dt \longrightarrow 0 \quad \text{as} \quad S \longrightarrow 0^-.$$

Moreover,

$$\frac{u(S)}{S} = \frac{1}{S} \int_0^S \partial_t u(\tau)\, d\tau$$

so, $(\frac{u(S)}{S})$ is bounded in $L^2(\mathbb{R}^2)$ and hence

$$\int_{B(S)} \partial_t u(S)\, \frac{u(S)}{S}\, dx \longrightarrow 0 \quad \text{as} \quad S \longrightarrow 0^-.$$

Taking the limit $S \longrightarrow 0^-$ in (4.13), we get $0 \leq -1$ which yields a contradiction. The proof of Theorem 3 is then completely achieved. ∎

We end this section by giving a rapidly proof of Theorem 4.

Proof of Theorem 4.
We treat the case $s=2$ (the general case is similar except some technical complications). Let $E_c(v,t)=\|\partial_t v(t,.)\|_{L^2}+\|v(t,.)\|_{H^1}$ where $v=\partial u$ and ∂ denotes any partial space derivative.
By the energy estimate, we get

$$\begin{aligned}\partial_t E_c(v,t) &\leq \|vu^2(t,\cdot)\|_{L^2}+\|v(e^{4\pi\,u^2}-1)(t,\cdot)\|_{L^2}+\|vu^2(e^{4\pi\,u^2}-1)(t,\cdot)\|_{L^2}\\ &\leq CE_c(v,t)(C+e^{2\pi\|u(t,\cdot)\|^2_{L^\infty}}).\end{aligned}$$

Using the fact that

$$e^{2\pi\|u(t,\cdot)\|^2_{L^\infty}}\leq C\left(1+\|u(t,\cdot)\|_{\mathcal{C}^{1/4}}\right)^\beta,\quad \beta<4,$$

and the Gronwal's Lemma, we conclude the proof. ∎

Remark 8.
Thanks to the injection $H^s(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$, $s>1$, and the following simple fact

$$e^{4\pi a w^2}-1=(e^{4\pi w^2}-1)(e^{4\pi(a-1)w^2}+\cdots+e^{4\pi w^2}+1),\quad a\in\mathbb{N}^*,$$

the Cauchy problem $(\mathbf{E_{4\pi}})$-(1.1) is locally well-posed in $H^s(\mathbb{R}^2)\times H^{s-1}(\mathbb{R}^2)$ without any assumption on $\|\nabla f\|_{L^2(\mathbb{R}^2)}$.

5. APPENDIX

In this Appendix, we give a sketch of the proof of Proposition 3. We refer to [18] for more details and related inequalities.
We start by showing the following inequality in the unit ball of $\mathbb{R}^2$.

Lemma 1. *For any real number $\lambda>\frac{2}{\pi}$, a constant C_λ exists such that, for any function $u\in H^1_0(B(1))\cup\dot{\mathcal{C}}^{1/4}(B(1))$, we have*

$$\|u\|^2_{L^\infty}\leq\lambda\|\nabla u\|^2_{L^2}\log(C_\lambda+N(u)),\tag{5.1}$$

where $N(u):=\frac{\|u\|_{\dot{\mathcal{C}}^{1/4}}}{\|\nabla u\|_{L^2}}$.

Proof. Without loss of generality, we can normalize $\|u\|_{L^\infty}$ to be equal to 1. Let $H^1_{0,rad}(B(1))$ be the set of all decreasing and radially symmetric functions in $H^1_0(B(1))$. For any parameter $D\geq1$, we denote by K_D the closed convex subset of $H^1_{0,rad}(B(1))$ defined by

$$K_D=\left\{u\in H^1_{0,rad}(B(1));\;\; u(r)\geq 1-Dr^{1/4},\quad r\in[0,1]\right\}.$$

Consider the following problem of minimizing

$$I[u] := \|\nabla u\|^2_{L^2(B(1))},$$

among all the functions belonging to the set K_D. This is a variational problem with obstacle. It is well known (see for example, Kinderlehrer-Stampacchia [24] and L. E. Evans [8]) that it has a unique minimizer u^* which is in the Sobolev space $W^{2,\infty}(B(1))$. Hence the following radially symmetric set

$$\mathcal{O} := \{x \in B_1 : u^*(x) > 1 - D|x|^{1/4}\}.$$

is open and u^* is harmonic in $\mathcal{O}$. On the other hand, note that all harmonic functions on $\mathbb{R}^2$ can only coincide in a unique tangent point with the function $r \to 1 - Dr^{1/4}$. Note that because of the boundary condition at $r = 1$, u^* cannot start to be harmonic near $r = 0$. Therefore there exists, a unique $a \in]0,1[$ such that

$$u^*(r) = 1 - Dr^{1/4} \text{ if } r \in [0,a] \tag{5.2}$$

$$u^*(r) = (1 - Da^{1/4})\frac{\log r}{\log a} \text{ if } r \in [a,1], \tag{5.3}$$

satisfying also the tangent condition

$$a^{1/4} = \frac{1 - Da^{1/4}}{\log(a^{1/4})}. \tag{5.4}$$

In particular, note that $\|u^*\|_{L^\infty} = 1$, $\|u^*\|_{\dot{\mathcal{C}}^{1/4}} = D$, and

$$\|\nabla u^*\|^2_{L^2} = \frac{\pi}{4}D^2\, a^{1/2} - 2\pi\Big(\frac{1 - Da^{1/4}}{\log(a)}\Big)^2 \log(a). \tag{5.5}$$

Substituting D from (5.4) into (5.5) and denoting by $x := a^{1/4} \in]0,1[$, we get the following

$$\|\nabla u^*\|^2_{L^2} = \frac{\pi}{2}\frac{1/2 - \log(x)}{(1 - \log(x))^2}$$

and

$$\|u^*\|_{\dot{\mathcal{C}}^{1/4}} = \frac{1}{x(1 - \log(x))}.$$

Therefore

$$\|\nabla u^*\|^2_{L^2} \log(C_\lambda + N(u^*)) := H(x),$$

where

$$H(x) = 2\pi\alpha\frac{1/2 - \log(x)}{(1 - \log(x))^2} \log\Big(C_\lambda + \frac{1}{x\sqrt{2\pi\alpha(1/2 - \log(x))}}\Big).$$

Taking $C_\lambda = e$ in $H(x)$, we see that $H(x)$ goes to $2\pi\alpha$ as x goes to 0. Hence, for any $\lambda > \frac{1}{2\pi\alpha}$, there exists $x_\lambda > 0$ such that $\lambda H(x) \geq 1$, for any $0 < x < x_\lambda$ and $C_\lambda \geq e$. Now, if $x \in [x_\lambda, 1]$, choosing the constant $C_\lambda > e$ big enough such that

$$\frac{1/2}{(1-\log(x_\lambda))^2}\log(C_\lambda) \geq 1.$$

we see that $\lambda H(x) \geq 1$. Hence, by this choice of C_λ, we see that $\lambda H(x) \geq 1$ for all $0 < x \leq 1$. This achieves the proof of the lemma. ■

Remark 9. Note that the limiting case $\lambda = \frac{2}{\pi}$ is not allowed in (5.1) (see [18]).

Proof of Proposition 3.
¿From the above result we derive the Proposition 3. Indeed, let u be a function in $H^1(\mathbb{R}^2) \cap \mathcal{C}^{1/4}(\mathbb{R}^2)$, $\lambda > \frac{2}{\pi}$ and $0 < \mu \leq 1$. Fix a radially symmetric function φ in $\mathcal{C}_0^\infty(B_4)$ satisfying $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ for r near 0, $|\partial_r\varphi| \leq 1$ and $|\Delta\varphi| \leq 1$. Define φ_μ by $\varphi_\mu(x) = \varphi(\frac{\mu}{2}|x|)$.
Without loss of generality, we can assume that $\|u\|_{L^\infty} = |u(0)|$. Note that in particular one has

$$\|\varphi_\mu u\|_{\dot{\mathcal{C}}^{1/4}} \leq \|u\|_{\mathcal{C}^{1/4}}$$

$$\|\nabla(\varphi_\mu u)\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \frac{\mu^2}{4}\|u\|_{L^2}^2 + 2\int_{\mathbb{R}^2} \varphi_\mu u \nabla\varphi_\mu \nabla u dx.$$

Integrating by parts,

$$2\int_{\mathbb{R}^2} \varphi_\mu u \nabla\varphi_\mu \nabla u dx = -\frac{1}{2}\int_{R^2} \Delta\varphi_\mu^2 u^2 dx = -\frac{\mu^2}{8}\int_{R^2} \Delta\varphi^2(\frac{\mu}{2}x)\ u^2 dx.$$

Hence,

$$\|\nabla(\varphi_\mu u)\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \mu^2\|u\|_{L^2}^2.$$

Applying the result of Lemma 1 and using the fact that for any constant $C > 0$, the function $x \to x^2\log(C_\lambda + \frac{C}{x})$ is increasing, the proof of Proposition 3 is achieved. ■

Acknowledgments:
S. I. would like to thank the Fields Institute & McMaster University for their support. He was supported by the NSERC grant 238452-01.
M .M. is supported by the *Laboratoire d'EDP* of the Faculty of Sciences of Tunis.
N.M. is partially supported by an NSF grant and by an Alfred P. Sloan Fellowship.

References

[1] **A. Atallah Baraket**: *Local existence and estimations for a semilinear wave equation in two dimension space*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat **8**, 1, 1-21, 2004.

[2] **S. Adachi and K. Tanaka**: *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents*, Proc. Amer. Math. Society, V. 128, N. 7, pp. 2051-2057, 1999.

[3] **H. Bahouri and P. Gérard** : *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math. 121, pp. 131-175, 1999.

[4] **P. Brenner and P. Kumlin**: *On wave equations with supercritical nonlinearities*, Arch. Math. **74**, pp. 129-146, 2000.

[5] **H. Bahouri and J. Shatah** : *Global estimate for the critical semilinear wave equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, Vol. 15, $n6$, pp. 783-789, 1998.

[6] **N. Burq, P. Gérard and N. Tvzetkov**: *An instability property of the nonlinear Schrödinger equation on $\mathcal{S}^d$*, Math. Res. Lett. 9, no. 2-3, pp. 323-335, 2002.

[7] **M. Christ, J. Colliander and T. Tao**: *Ill-posedness for nonlinear Schrödinger and wave equations*, Preprint.

[8] **L. C. Evans**: *Partial differential equations*, Graduate studies in Mathematics AMS 1998.

[9] **P. Gérard**: *Oscillations and concentration effects in semilinear dispersive wave equations*, J. Funct. Anal. 133, pp. 50-68, 1996.

[10] **J. Ginibre**: *Scattering theory in the energy space for a class of nonlinear wave equation*, Advanced Studies in Pure Mathematics **23**, 83-103, 1994.

[11] **J. Ginibre, A. Soffer and G. Velo**: *The global Cauchy problem for the critical nonlinear wave equation*, J. Funct. Anal. **110**, 96-130, 1992.

[12] **J. Ginibre and G. Velo**: *The global Cauchy problem for nonlinear Klein-Gordon equation*, Math.Z , **189**, pp. 487-505, 1985.

[13] **J. Ginibre and G. Velo**: *Conformal invariance and time decay for nonlinear wave equations*, II, Ann. Henri Poincaré (Physique Théorique) **47**, 263-276,1987.

[14] **J. Ginibre and G. Velo**: *Scattering theory in the energie space for a class of nonlinear wave equations*, Comm. Math. Phys.J. Functional Analysis , **123**, pp. 535-573, 1989.

[15] **J. Ginibre and G. Velo**: *Generalized Strichartz inequalities for the wave equations*, J. Functional Analysis , **133**, pp. 50-68, 1995.

[16] **M. Grillakis**: *Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity*, Annal. of math. , **132**, pp. 485-509, 1990.

[17] **M. Grillakis**: *Regularity for the wave equation with a critical nonlinearity*, Comm. Pure Appl. Math , **XLVI**, pp. 749-774, 1992.

[18] **S. Ibrahim, M. Majdoub and N. Masmoudi**: *Double logarithmic inequality with a sharp constant*, submitted.

[19] **S. Ibrahim et M. Majdoub**: *Solutions globales de l'équation des ondes semi-linéaire critique à coefficients variables*, Bull. Soc. math. France **131** (1), pp. 1-21, 2003.

[20] **L. V. Kapitanski**: *The Cauchy problem for a semilinear wave equation*, Zap. Nauchn. Sem. Leningrad Otdel. Math. Inst. Steklov(LOMI), **181-182**, pp. 24-64 and 38-85, 1990.

[21] **L. V. Kapitanski**: *Some generalizations of the Strichartz-Brenner inequality*, Leningrad Math. Journ., **1**, # 10, pp. 693-726, 1990.

[22] **L. V. Kapitanski**: *The Cauchy problem for a semilinear wave equation II*, J. of Soviet Math., **62**, Série I, pp. 2746-2777, 1992.

[23] **L. V. Kapitanski**: *Global and unique weak solutions of nonlinear wave equations*, Math. Res. Lett., **1**, pp. 211-223, 1994.

[24] **D. Kinderlehrer and G. Stampacchia**: *An introduction to variational inequalities and their applications*, Academic Press, 1980.

[25] **G. Lebeau**: *Nonlinear optics and supercritical wave equation*, Bull. Soc. R. Sci. Lige 70, No.4-6, 267-306, 2001.

[26] **H. Lindblad and D. Sogge**: *On existence and scattering with minimal regularity for semilinear wave equation*, J. Funct. Anal. **130**, 357-426, 1995.

[27] **N. Masmoudi and F. Planchon**: *Uniquenes for the quintic semilinear wave equation*, Preprint.

[28] **J. Moser**: *Asharp form of an inequality of N. Trudinger*, Ind. Univ. Math. J. **20**, 1077-1092, 1971.

[29] **M. Nakamura and T. Ozawa**: *Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth*, Math. Z. **231**, 479-487, 1999.

[30] **M. Nakamura and T. Ozawa**: *The Cauchy problem for nonlinear wave equations in the Sobolev space of critical order*, Discrete and Continuous Dynamical Systems, V. 5, N. 1, 215-231, 1999.

[31] **H. Pecher**: *Nonlinear small data scattering for the wave and Klein-Gordon equation*, Math. Z. **185**, 261-270, 1984.

[32] **H. Pecher**: *Local solutions of semilinear wave equations in H^{s+1}*, Mathematical Methods in the Applied Sciences **19**, 145-170, 1996.

[33] **H. Pecher**: *L^p - Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen*, I, Math. Z. **150** (19, 159-183, 1976.

[34] **B. Ruf**: *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$*, Preprint.

[35] **I. E. Segal**: *Space-time decay for solutions of the wave equations*, Adv. Math., **22**, pp. 304-311, 1976.

[36] **J. Shatah and M. Struwe**: *Regularity results for nonlinear wave equations*, Ann.of Math., **2**, n° 138 pp. 503-518, 1993.

[37] **J. Shatah and M. Struwe**: *Well-Posedness in the energy space for semilinear wave equation with critical growth*, IMRN, **7**, pp. 303-309, 1994.

[38] **W. Strauss**: *On weak solutions of semilinear hyperbolic equations*, Anais Acad. Brasil Cienc. **42**, 645-651, 1970.

[39] **R. Strichartz**: *A priori estimates for the wave equation and some applications*, J. Funct. Analysis, **5**, pp. 218-235, 1970.

[40] **R. Strichartz**: *A priori estimates for the wave equation and some applications*, J. Funct. Anal. **5**, 218-235, 1970.

[41] **R. Strichartz**: *Restrictions of Fourier transform to quadratic surfaces and decay of solutions of wave equations*, Duke Math. Journ., **44**, pp. 705-714, 1977.

[42] **M Struwe**: *Semilinear wave equations*, Bull. Amer. Math. Soc., **N.S**, n° 26, pp. 53-85, 1992.

[43] **D. Tataru**: *Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation*, Prépublication.

[44] **C. Zuily**: *Solutions en grand temps d'équations d'ondes non linéaire* , Séminaire Bourbaki, **779**, 1993-1994.