

# Existence of a ground state and blow-up problem for a nonlinear Schrödinger equation with critical growth

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## 1 Introduction

In this paper, we consider the following nonlinear Schrödinger equation.

$$2i \frac{\partial \psi}{\partial t} + \Delta \psi + \mu |\psi|^{p-1} \psi + |\psi|^{\frac{4}{d-2}} \psi = 0, \quad (\text{NLS})$$

where  $\psi = \psi(x, t)$  is a complex-valued function on  $\mathbb{R}^d \times \mathbb{R}$  ( $d \geq 3$ ),  $\Delta$  is the Laplace operator on  $\mathbb{R}^d$ ,  $\mu > 0$  and  $1 + \frac{4}{d} < p < 2^* - 1$  ( $2^* := \frac{2d}{d-2}$ ).

The nonlinearity of (NLS) is attractive and contains the energy-critical term  $|\psi|^{\frac{4}{d-2}} \psi$  in the scaling sense. It is known (see [7, 12]) that:

- (i) for any  $t_0 \in \mathbb{R}$  and datum  $\psi_0 \in H^1(\mathbb{R}^d)$ , there exists a unique solution  $\psi \in C(I_{\max}, H^1(\mathbb{R}^d))$  to (NLS) with  $\psi(t_0) = \psi_0$ , where  $I_{\max}$  denotes the maximal interval on which  $\psi$  exists;
- (ii) the solution  $\psi$  enjoys the following conservation laws:

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2} \quad \text{for any } t \in I_{\max}, \quad (1.1)$$

$$\mathcal{H}(\psi(t)) = \mathcal{H}(\psi_0) \quad \text{for any } t \in I_{\max}, \quad (1.2)$$

where

$$\mathcal{H}(u) := \|\nabla u\|_{L^2}^2 - \mu \frac{2}{p+1} \|u\|_{L^{p+1}}^{p+1} - \frac{d-2}{d} \|u\|_{L^{2^*}}^{2^*}. \quad (1.3)$$

We are interested, like in [3], in the existence of standing wave and blowup problem for (NLS). Here, the standing wave means a solution to (NLS) of the form  $\psi(x, t) = e^{\frac{i}{2}t\omega} Q(x)$ , so that  $Q$  must satisfy the elliptic equation

$$-\Delta u + \omega u - \mu |u|^{p-1} u - |u|^{\frac{4}{d-2}} u = 0. \quad (1.4)$$

When  $\mu = 0$ , it is well-known that the equation (1.4) has no solution for  $\omega > 0$ . Moreover, if the subcritical perturbation is repulsive, i.e.,  $\mu < 0$ , then there is no non-trivial solution; Indeed, using the Pohozaev identity, we can verify that any solution  $Q \in H^1(\mathbb{R}^d)^1$  to (1.4) with  $\mu < 0$  obeys that

$$0 = \omega \|Q\|_{L^2}^2 - \mu \left(1 - \frac{d(p-1)}{2(p+1)}\right) \|Q\|_{L^{p+1}}^{p+1}, \quad (1.5)$$

which, together with  $\omega > 0$  and  $p < 2^* - 1$ , shows  $Q$  is trivial. Thus,  $\mu > 0$  is necessary for the existence of non-trivial solution to (1.4).

<sup>1</sup>The solution  $Q \in H^1(\mathbb{R}^d)$  also belongs to  $L_{loc}^\infty(\mathbb{R}^d)$  (see, e.g., Appendix B in [11]).

Our first aim is to seek a special solution to (1.4) called ground state via a certain variational problem;  $Q$  is said to be the ground state, if it is a solution to (1.4) and

$$\mathcal{S}_\omega(Q) = \min\{\mathcal{S}_\omega(u) : u \text{ is a solution to (1.4)}\}, \quad (1.6)$$

where

$$\mathcal{S}_\omega(u) := \omega \|u\|_{L^2}^2 + \mathcal{H}(u). \quad (1.7)$$

More precisely, we look for the ground state as a minimizer of the variational problem below:

$$m_\omega := \inf \left\{ \mathcal{S}_\omega(u) \mid u \in H^1(\mathbb{R}^d) \setminus \{0\}, \mathcal{K}(u) = 0 \right\}, \quad (1.8)$$

where

$$\begin{aligned} \mathcal{K}(u) &:= \left. \frac{d}{d\lambda} \mathcal{S}_\omega(T_\lambda u) \right|_{\lambda=1} = \left. \frac{d}{d\lambda} \mathcal{H}(T_\lambda u) \right|_{\lambda=1} \\ &= 2 \|\nabla u\|_{L^2}^2 - \mu \frac{d(p-1)}{p+1} \|u\|_{L^{p+1}}^{p+1} - 2 \|u\|_{L^{2^*}}^{2^*}, \end{aligned} \quad (1.9)$$

and  $T_\lambda$  is the  $L^2$ -scaling operator defined by

$$(T_\lambda u)(x) := \lambda^{\frac{d}{2}} u(\lambda x) \quad \text{for } \lambda > 0. \quad (1.10)$$

We can verify the following fact (see [3, 8] for the proof):

**Proposition 1.1.** *Any minimizer of the variational problem for  $m_\omega$  becomes a ground state of (1.4).*

We remark that the existence of the ground state is studied by many authors (see, e.g., [1, 2, 3]); In particular, in [1], they proved the existence of a ground state for a class of elliptic equations including (1.4) through a minimization problem different from ours. An advantage of our variational problem is that we can prove the instability of the ground state found through it. Indeed, we can prove the blowup result (see Theorem 1.2 below).

In order to find the minimizer of the variational problem for  $m_\omega$ , we also consider the auxiliary variational problem:

$$\tilde{m}_\omega = \inf \left\{ \mathcal{I}_\omega(u) \mid u \in H^1(\mathbb{R}^d) \setminus \{0\}, \mathcal{K}(u) \leq 0 \right\}, \quad (1.11)$$

where

$$\begin{aligned} \mathcal{I}_\omega(u) &:= \mathcal{S}_\omega(u) - \frac{2}{d(p-1)} \mathcal{K}(u) \\ &= \omega \|u\|_{L^2}^2 + \frac{d(p-1)-4}{d(p-1)} \|\nabla u\|_{L^2}^2 + \frac{4-(d-2)(p-1)}{d(p-1)} \|u\|_{L^{2^*}}^{2^*}. \end{aligned} \quad (1.12)$$

Note that the subcritical potential term disappears in the functional  $\mathcal{I}_\omega$ . The problem (1.12) is good to treat, since the functional  $\mathcal{I}_\omega$  is positive and the constraint is  $\mathcal{K} \leq 0$  which is stable under the Schwarz symmetrization. Moreover, we have the following;

**Proposition 1.2.** *Assume  $d \geq 3$ . Then, for any  $\omega > 0$  and  $\mu > 0$ , we have;*

(i)  $m_\omega = \tilde{m}_\omega$

(ii) *Any minimizer of the variational problem for  $\tilde{m}_\omega$  is also a minimizer for  $m_\omega$ , and vice versa.*

We state our main theorems:

**Theorem 1.1.** *Assume  $d \geq 4$ . Then, for any  $\omega > 0$  and  $\mu > 0$ , there exists a minimizer of the variational problem for  $m_\omega$ . We also have  $m_\omega = \tilde{m}_\omega > 0$ .*

**Theorem 1.2.** *Assume  $d = 3$ . If  $\mu$  is sufficiently small, then we have;*

- (i) *the variational problem for  $m_\omega$  has no minimizer.*
- (ii) *there is no ground state for the equation (1.4).*

We see from Theorem 1.1 together with Proposition 1.1 that when  $d \geq 4$ , a ground state exists for any  $\omega > 0$  and  $\mu > 0$ .

We give the proofs of Proposition 1.2 and Theorem 1.1 in Section 2. The proof of Theorem 1.1 is based on the idea of Brezis and Nirenberg [5]. Since we consider the variational problem different from theirs, we need a little different argument in the estimate of  $m_\omega$  (see Lemma 2.2). We give the proof of Theorem 1.2 in Section 3.

Next, we move to the blowup problem. Using the variational value  $m_\omega$ , we define the set  $A_{\omega,-}$  by

$$A_{\omega,-} := \left\{ u \in H^1(\mathbb{R}^d) \mid \mathcal{S}_\omega(u) < m_\omega, \mathcal{K}(u) < 0 \right\}. \quad (1.13)$$

We can see that  $A_{\omega,-}$  is invariant under the flow defined by (NLS) (see Lemma 4.1 below). Furthermore, we have the following result:

**Theorem 1.3.** *Assume  $d \geq 4$ ,  $\omega > 0$  and  $\mu > 0$ . Let  $\psi$  be a solution to (NLS) starting from  $A_{\omega,-}$ , and let  $I_{\max}$  be the maximal interval on which  $\psi$  exists. If  $\psi$  is radially symmetric, then  $I_{\max}$  is bounded.*

We will prove Theorem 1.3 in Section 4.

Besides the blowup result, we can also prove the scattering result; Put

$$A_{\omega,+} := \left\{ u \in H^1(\mathbb{R}^d) \mid \mathcal{S}_\omega(u) < m_\omega, \mathcal{K}(u) > 0 \right\}. \quad (1.14)$$

Then, we have

**Theorem 1.4.** *Assume  $d \geq 4$ ,  $\omega > 0$  and  $\mu > 0$ . Let  $\psi$  be a solution to (NLS) starting from  $A_{\omega,+}$ . Then,  $\psi$  exists globally in time and satisfies*

$$\psi(t) \in A_{\omega,+} \quad \text{for any } t \in \mathbb{R}, \quad (1.15)$$

and there exists  $\phi_+, \phi_- \in H^1(\mathbb{R}^d)$  such that

$$\lim_{t \rightarrow +\infty} \left\| \psi(t) - e^{\frac{i}{2}t\Delta} \phi_+ \right\|_{H^1} = \lim_{t \rightarrow -\infty} \left\| \psi(t) - e^{\frac{i}{2}t\Delta} \phi_- \right\|_{H^1} = 0. \quad (1.16)$$

We will give the proof of Theorem 1.4 in a forthcoming paper. In fact, we will prove the scattering result for a more general nonlinearity.

## 2 Proofs of Proposition 1.2 and Theorem 1.1

In this section, we prove Proposition 1.2 and Theorem 1.1. In the proofs, we often use the following easy facts:

For any  $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ , there exists a unique  $\lambda(u) > 0$  such that

$$\mathcal{K}(T_\lambda u) \begin{cases} > 0 & \text{for any } 0 < \lambda < \lambda(u), \\ = 0 & \text{for } \lambda = \lambda(u), \\ < 0 & \text{for any } \lambda > \lambda(u). \end{cases} \quad (2.1)$$

$$\frac{d}{d\lambda} \mathcal{S}_\omega(T_\lambda u) = \frac{1}{\lambda} \mathcal{K}(T_\lambda u) \quad \text{for any } u \in H^1(\mathbb{R}^d) \text{ and } \lambda > 0. \quad (2.2)$$

Moreover, we see from (2.1) and (2.2) that for any  $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ ,

$$\mathcal{S}_\omega(T_\lambda u) < \mathcal{S}_\omega(T_{\lambda(u)} u) \quad \text{for any } \lambda \neq \lambda(u), \quad (2.3)$$

$$\mathcal{S}_\omega(T_\lambda u) \text{ is concave for } \lambda > \lambda(u). \quad (2.4)$$

Let us begin with the proof of Proposition 1.2.

*Proof of Proposition 1.2.* We first show the claim (i):  $m_\omega = \tilde{m}_\omega$ . Let  $\{u_n\}$  be any minimizing sequence for  $\tilde{m}_\omega$ , i.e.,  $\{u_n\}$  is a sequence in  $H^1(\mathbb{R}^d) \setminus \{0\}$  such that

$$\lim_{n \rightarrow \infty} \mathcal{I}_\omega(u_n) = \tilde{m}_\omega, \quad (2.5)$$

$$\mathcal{K}(u_n) \leq 0 \quad \text{for any } n \in \mathbb{N}. \quad (2.6)$$

Using (2.1), for each  $n \in \mathbb{N}$ , we can take  $\lambda_n \in (0, 1]$  such that  $\mathcal{K}(T_{\lambda_n} u_n) = 0$ . Then, we have

$$m_\omega \leq \mathcal{S}_\omega(T_{\lambda_n} u_n) = \mathcal{I}_\omega(T_{\lambda_n} u_n) \leq \mathcal{I}_\omega(u_n) = \tilde{m}_\omega + o_n(1), \quad (2.7)$$

where we have used  $\lambda_n \leq 1$  to derive the second inequality. Hence, we obtain  $m_\omega \leq \tilde{m}_\omega$ . On the other hand, it follows from

$$\tilde{m}_\omega \leq \inf_{\substack{u \in H^1 \setminus \{0\} \\ \mathcal{K}=0}} \mathcal{I}_\omega(u) = \inf_{\substack{u \in H^1 \setminus \{0\} \\ \mathcal{K}=0}} \mathcal{S}_\omega(u) = m_\omega \quad (2.8)$$

that  $\tilde{m}_\omega \leq m_\omega$ . Thus, we have proved that  $m = m_\omega$ .

Next, we shall show the claim (ii). Let  $Q$  be any minimizer for  $\tilde{m}_\omega$ , i.e.,  $Q \in H^1(\mathbb{R}^d) \setminus \{0\}$  with  $\mathcal{K}(Q) \leq 0$  and  $\mathcal{I}_\omega(Q) = \tilde{m}_\omega$ . Since  $\mathcal{I}_\omega = \mathcal{S}_\omega - \frac{2}{d(p-1)} \mathcal{K}$  (see (1.12)) and  $m_\omega = \tilde{m}_\omega$ , it is sufficient to show that  $\mathcal{K}(Q) = 0$ . Suppose the contrary that  $\mathcal{K}(Q) < 0$ , so that there exists  $0 < \lambda_0 < 1$  such that  $\mathcal{K}(T_{\lambda_0} Q) = 0$ . Then, we have

$$\tilde{m}_\omega \leq \mathcal{I}_\omega(T_{\lambda_0} Q) < \mathcal{I}_\omega(Q) = \tilde{m}_\omega, \quad (2.9)$$

which is a contradiction. Hence,  $\mathcal{K}(Q) = 0$  and therefore  $Q$  is also a minimizer for  $m_\omega$ .  $\square$

Now, employing the idea of Brézis and Nirenberg [5], we prove Theorem 1.1. To this end, we introduce another variational value  $\sigma$ :

$$\sigma := \inf \left\{ \|\nabla u\|_{L^2}^2 \mid u \in \dot{H}^1(\mathbb{R}^d), \|u\|_{L^{2^*}} = 1 \right\}. \quad (2.10)$$

Then, we have;

**Proposition 2.1.** *Assume that for  $\omega > 0$  and  $\mu > 0$ , we have*

$$m_\omega < \frac{2}{d} \sigma^{\frac{d}{2}}. \quad (2.11)$$

*Then, there exists a minimizer of the variational problem for  $m_\omega$ .*

*Proof of Proposition 2.1.* Let  $\{u_n\}$  be a minimizing sequence of the variational problem for  $\tilde{m}_\omega$ . Extracting some subsequence, we may assume that

$$\mathcal{I}_\omega(u_n) \leq 1 + \tilde{m}_\omega \quad \text{for any } n \in \mathbb{N}. \quad (2.12)$$

We denote the Schwarz symmetrization of  $u_n$  by  $u_n^*$ . Then, we easily see that

$$\mathcal{K}(u_n^*) \leq 0 \quad \text{for any } n \in \mathbb{N}, \quad (2.13)$$

$$\lim_{n \rightarrow \infty} \mathcal{I}_\omega(u_n^*) = \tilde{m}_\omega, \quad (2.14)$$

$$\sup_{n \in \mathbb{N}} \|u_n^*\|_{H^1} < \infty \quad \text{for any } n \in \mathbb{N}. \quad (2.15)$$

Since  $\{u_n^*\}$  is radially symmetric and bounded in  $H^1(\mathbb{R}^d)$ , there exists a radially symmetric function  $Q \in H^1(\mathbb{R}^d)$  such that

$$\lim_{n \rightarrow \infty} u_n^* = Q \quad \text{weakly in } H^1(\mathbb{R}^d), \quad (2.16)$$

$$\lim_{n \rightarrow \infty} u_n^* = Q \quad \text{strongly in } L^q(\mathbb{R}^d) \text{ for } 2 < q < 2^*, \quad (2.17)$$

$$\lim_{n \rightarrow \infty} u_n^*(x) = Q(x) \quad \text{for almost all } x \in \mathbb{R}^d. \quad (2.18)$$

This function  $Q$  is a candidate for a minimizer of the variational problem (1.8). We shall show that it is actually a minimizer.

Let us begin with the non-triviality of  $Q$ . Suppose the contrary that  $Q$  is trivial. Then, it follows from (2.13) and (2.17) that

$$0 \geq \limsup_{n \rightarrow \infty} \mathcal{K}(u_n^*) = 2 \limsup_{n \rightarrow \infty} \left\{ \|\nabla u_n^*\|_{L^2}^2 - \|u_n^*\|_{L^{2^*}}^{2^*} \right\}, \quad (2.19)$$

so that

$$\limsup_{n \rightarrow \infty} \|\nabla u_n^*\|_{L^2}^2 \leq \liminf_{n \rightarrow \infty} \|u_n^*\|_{L^{2^*}}^{2^*}. \quad (2.20)$$

Moreover, this together with the definition of  $\sigma$  (see (2.10)) gives us that

$$\limsup_{n \rightarrow \infty} \|\nabla u_n^*\|_{L^2}^2 \geq \sigma \liminf_{n \rightarrow \infty} \|u_n^*\|_{L^{2^*}}^{2^*} \geq \sigma \limsup_{n \rightarrow \infty} \|\nabla u_n^*\|_{L^2}^{\frac{2(d-2)}{d}}, \quad (2.21)$$

so that

$$\sigma^{\frac{d}{2}} \leq \limsup_{n \rightarrow \infty} \|\nabla u_n^*\|_{L^2}. \quad (2.22)$$

Hence, if  $Q$  is trivial, then we have that

$$\begin{aligned} \tilde{m}_\omega &= \lim_{n \rightarrow \infty} \mathcal{I}_\omega(u_n^*) \\ &\geq \lim_{n \rightarrow \infty} \left\{ \frac{d(p-1)-4}{d(p-1)} \|\nabla u_n^*\|_{L^2}^2 + \frac{4-(d-2)(p-1)}{d(p-1)} \|u_n^*\|_{L^{2^*}}^{2^*} \right\} \\ &\geq \frac{2}{d} \liminf_{n \rightarrow \infty} \|\nabla u_n^*\|_{L^2}^2 \geq \frac{2}{d} \sigma^{\frac{d}{2}}. \end{aligned} \quad (2.23)$$

However, this contradicts the hypothesis (2.11). Thus,  $Q$  is non-trivial.

We shall show that  $\mathcal{K}(Q) = 0$ . By the Brezis-Lieb Lemma (see [4]), we have

$$\mathcal{I}_\omega(u_n^*) - \mathcal{I}_\omega(u_n^* - Q) - \mathcal{I}(Q) = o_n(1). \quad (2.24)$$

Since  $\lim_{n \rightarrow \infty} \mathcal{I}_\omega(u_n^*) = \tilde{m}_\omega$  and  $\mathcal{I}_\omega$  is positive, we see from (2.24) that

$$\mathcal{I}_\omega(Q) \leq \tilde{m}_\omega. \quad (2.25)$$

First, we suppose that  $\mathcal{K}(Q) < 0$ , which together with the definition of  $\tilde{m}_\omega$  (see (1.11)) and (2.25) implies that  $\mathcal{I}_\omega(Q) = \tilde{m}_\omega$ . Moreover, we can take  $\lambda_0 \in (0, 1)$  such that  $\mathcal{K}(T_{\lambda_0}Q) = 0$ . Then, we have

$$\tilde{m}_\omega \leq \mathcal{I}_\omega(T_{\lambda_0}Q) < \mathcal{I}_\omega(Q) = \tilde{m}_\omega. \quad (2.26)$$

This is a contradiction. Therefore  $\mathcal{K}(Q) \geq 0$ .

Second, suppose the contrary that  $\mathcal{K}(Q) > 0$ . Then, it follows from (2.13) and

$$\mathcal{K}(u_n^*) - \mathcal{K}(u_n^* - Q) - \mathcal{K}(Q) = o_n(1) \quad (2.27)$$

that  $\mathcal{K}(u_n^* - Q) < 0$  for any sufficiently large  $n \in \mathbb{N}$ , so that there exists a unique  $\lambda_n \in (0, 1)$  such that  $\mathcal{K}(T_{\lambda_n}(u_n^* - Q)) = 0$ . Then, we easily verify that

$$\begin{aligned} \tilde{m}_\omega &\leq \mathcal{I}_\omega(T_{\lambda_n}(u_n^* - Q)) \\ &\leq \mathcal{I}_\omega(u_n^* - Q) = \mathcal{I}_\omega(u_n^*) - \mathcal{I}_\omega(Q) + o_n(1) \\ &= \tilde{m}_\omega - \mathcal{I}_\omega(Q) + o_n(1). \end{aligned} \quad (2.28)$$

Hence, we conclude  $\mathcal{I}_\omega(Q) = 0$ . However, this contradicts the fact that  $Q$  is non-trivial. Thus,  $\mathcal{K}(Q) = 0$ .

Since  $Q$  is non-trivial and  $\mathcal{K}(Q) = 0$ , we have

$$m_\omega \leq \mathcal{S}_\omega(Q) = \mathcal{I}_\omega(Q). \quad (2.29)$$

Moreover, it follows from (2.24) and Proposition 1.2 that

$$\mathcal{I}_\omega(Q) \leq \liminf_{n \rightarrow \infty} \mathcal{I}_\omega(u_n^*) \leq \tilde{m}_\omega = m_\omega. \quad (2.30)$$

Combining (2.29) and (2.30), we obtain that  $\mathcal{S}_\omega(Q) = \mathcal{I}_\omega(Q) = m_\omega$ . Thus, we have proved that  $Q$  is a minimizer for the variational problem (1.8).  $\square$

We shall give the proof of Theorem 1.1. In view of Proposition 2.1, it suffices to show the following lemma.

**Lemma 2.2.** *Assume  $d \geq 4$ . Then, for any  $\omega > 0$  and  $\mu > 0$ , we have*

$$m_\omega < \frac{2}{d} \sigma^{\frac{d}{2}}. \quad (2.31)$$

*Proof of Lemma 2.2.* In view of Proposition 1.2, it suffices to show that

$$\tilde{m}_\omega < \frac{2}{d} \sigma^{\frac{d}{2}}. \quad (2.32)$$

Let  $W$  be the Talenti function

$$W(x) := \left( \frac{\sqrt{d(d-2)}}{1+|x|^2} \right)^{\frac{d-2}{2}}. \quad (2.33)$$

Put  $W_\varepsilon(x) := \varepsilon^{-\frac{d-2}{2}} W(\varepsilon^{-1}x)$  for  $\varepsilon > 0$ . It is known that

$$-\Delta W - |W|^{\frac{4}{d-2}} W = 0, \quad (2.34)$$

$$\sigma^{\frac{d}{2}} = \|\nabla W_\varepsilon\|_{L^2}^2 = \|W_\varepsilon\|_{L^{2^*}}^{2^*} \quad \text{for any } \varepsilon > 0. \quad (2.35)$$

We easily see that  $W_\varepsilon$  belongs to  $L^2(\mathbb{R}^d)$  for  $d \geq 5$ . When  $d = 4$ ,  $W_\varepsilon$  no longer belong to  $L^2(\mathbb{R}^4)$  and we need a cut-off function in our proof: Let  $b \in C^\infty(\mathbb{R}^4)$  be a function such that

$$b(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases} \quad (2.36)$$

We define the function  $w_\varepsilon$  by

$$w_\varepsilon := \begin{cases} bW_\varepsilon & \text{when } d = 4, \\ W_\varepsilon & \text{when } d \geq 5. \end{cases} \quad (2.37)$$

Then, for any  $\varepsilon < 1$ , we can verify (see p.35 of [13]) that

$$\|\nabla w_\varepsilon\|_{L^2}^2 = \sigma^{\frac{d}{2}} + O(\varepsilon^{d-2}), \quad (2.38)$$

$$\|w_\varepsilon\|_{L^{2^*}}^{2^*} = \sigma^{\frac{d}{2}} + O(\varepsilon^d), \quad (2.39)$$

$$\|w_\varepsilon\|_{L^2}^2 = \begin{cases} C_1 \varepsilon^2 |\log \varepsilon| + O(\varepsilon^2) & \text{if } d = 4, \\ C_1 \varepsilon^2 + O(\varepsilon^{d-2}) & \text{if } d \geq 5, \end{cases} \quad (2.40)$$

$$\|w_\varepsilon\|_{L^{p+1}}^{p+1} = C_2 \varepsilon^{d - \frac{(d-2)(p+1)}{2}} + O(\varepsilon^{\frac{(d-2)(p+1)}{2}}), \quad (2.41)$$

where  $C_1$  and  $C_2$  are positive constants independent of  $\varepsilon$ . Moreover, it follows from (2.1) that;

For any  $\varepsilon \in (0, 1)$ , there exists a unique  $\lambda_\varepsilon > 0$  such that

$$0 = \mathcal{K}(T_{\lambda_\varepsilon} w_\varepsilon) = 2\lambda_\varepsilon \|\nabla w_\varepsilon\|_{L^2}^2 - \mu \frac{d(p-1)}{p+1} \lambda_\varepsilon^{\frac{d(p-1)}{2}} \|w_\varepsilon\|_{L^{p+1}}^{p+1} - 2\lambda_\varepsilon^{2^*} \|w_\varepsilon\|_{L^{2^*}}^{2^*}. \quad (2.42)$$

Combining (2.38)–(2.41) and (2.42), we obtain that

$$\lambda_\varepsilon^{\frac{4}{d-2}} = 1 - C \lambda_\varepsilon^{\frac{d(p-1)-4}{2}} \varepsilon^{d - \frac{(d-2)(p+1)}{2}} + O(\varepsilon^{d-2}), \quad (2.43)$$

where  $C > 0$  is some constant which is independent of  $\varepsilon$ . Since  $\frac{4}{d-2} > \frac{d(p-1)}{2} - 2$ , we also find from (2.43) that

$$\lim_{\varepsilon \downarrow 0} \lambda_\varepsilon = 1. \quad (2.44)$$

Put  $\lambda_\varepsilon = 1 + \alpha_\varepsilon$ , where  $\lim_{\varepsilon \downarrow 1} \alpha_\varepsilon = 0$ . Then, (2.43) together with the expansion  $(1 + \alpha_\varepsilon)^q = 1 + q\alpha_\varepsilon + o(\alpha_\varepsilon)$  ( $q > 0$ ) gives us that

$$1 + \frac{4}{d-2}\alpha_\varepsilon + o(\alpha_\varepsilon) = 1 - C \left\{ 1 + \frac{d(p-1)-4}{2}\alpha_\varepsilon + o(\alpha_\varepsilon) \right\} \varepsilon^{d - \frac{(d-2)(p+1)}{2}} + O(\varepsilon^{d-2}), \quad (2.45)$$

so that

$$\alpha_\varepsilon = -C_0 \varepsilon^{d - \frac{(d-2)(p+1)}{2}} + O(\varepsilon^{d-2}) \quad \text{for any sufficiently small } \varepsilon > 0, \quad (2.46)$$

where  $C_0 > 0$  is some constant which is independent of  $\varepsilon$ . When  $d \geq 4$ ,  $d - 2 > d - \frac{(d-2)(p+1)}{2}$  and hence we find from (2.46) that

$$\lambda_\varepsilon = 1 - C_0 \varepsilon^{d - \frac{(d-2)(p+1)}{2}} + O(\varepsilon^{d-2}) \quad \text{for any sufficiently small } \varepsilon > 0. \quad (2.47)$$

Using (2.38)–(2.41) and (2.47), we obtain that

$$\begin{aligned} \tilde{m}_\omega &\leq \mathcal{I}_\omega(T_{\lambda_\varepsilon} w_\varepsilon) \\ &= \omega \|w_\varepsilon\|_{L^2}^2 + \frac{d(p-1)-4}{d(p-1)} \lambda_\varepsilon^2 \|\nabla w_\varepsilon\|_{L^2}^2 + \frac{4-(d-2)(p-1)}{d(p-1)} \lambda_\varepsilon^{2^*} \|w_\varepsilon\|_{L^{2^*}}^{2^*} \\ &\leq \omega C_1 \varepsilon^2 + \frac{d(p-1)-4}{d(p-1)} \lambda_\varepsilon^2 \sigma^{\frac{d}{2}} + \frac{4-(d-2)(p-1)}{d(p-1)} \lambda_\varepsilon^{2^*} \sigma^{\frac{d}{2}} + O(\varepsilon^{d-2}) \\ &\leq \omega C_1 \varepsilon^2 + \frac{d(p-1)-4}{d(p-1)} \left\{ 1 - 2C_0 \varepsilon^{d - \frac{(d-2)(p+1)}{2}} \right\} \sigma^{\frac{d}{2}} \\ &\quad + \frac{4-(d-2)(p-1)}{d(p-1)} \left\{ 1 - 2^* C_0 \varepsilon^{d - \frac{(d-2)(p+1)}{2}} \right\} \sigma^{\frac{d}{2}} + o(\varepsilon^{d - \frac{(d-2)(p+1)}{2}}) \\ &= \frac{2}{d} \sigma^{\frac{d}{2}} - \frac{16}{d(d-2)} C_0 \varepsilon^{d - \frac{(d-2)(p+1)}{2}} + \omega C_1 \varepsilon^2 + o(\varepsilon^{d - \frac{(d-2)(p+1)}{2}}) \end{aligned} \quad (2.48)$$

for any sufficiently small  $\varepsilon > 0$ .

Since  $d - \frac{(d-2)(p+1)}{2} < 2$  for  $p > 1$ , we obtain  $\tilde{m}_\omega < \frac{2}{d} \sigma^{\frac{d}{2}}$ .  $\square$

### 3 Non-existence result in 3D

In this section, we will give the proof of Theorem 1.2. Before proving the theorem, we prepare the following lemma.

**Lemma 3.1.** *Assume  $d \geq 3$ ,  $\omega > 0$  and  $\mu > 0$ . Let  $u$  be a minimizer of the variational problem for  $m_\omega$ . Then, we have that*

$$\|u\|_{H^1} \lesssim 1, \quad (3.1)$$

where the implicit constant is independent of  $\mu$ .



*Proof of Lemma 3.1.* Fix a non-trivial function  $\chi \in C_c^\infty(\mathbb{R}^d)$  and put

$$\tilde{\chi} := \left( \frac{\|\nabla\chi\|_{L^2}^2}{\|\chi\|_{L^{2^*}}^{2^*}} \right)^{\frac{1}{2^*-2}} \chi. \quad (3.2)$$

It is easy to see that

$$\|\nabla\tilde{\chi}\|_{L^2}^2 = \left( \frac{\|\nabla\chi\|_{L^2}^2}{\|\chi\|_{L^{2^*}}^{2^*}} \right)^{\frac{2}{2^*-2}} \|\nabla\chi\|_{L^2}^2 = \left( \frac{\|\nabla\chi\|_{L^2}^2}{\|\chi\|_{L^{2^*}}^{2^*}} \right)^{\frac{2^*}{2^*-2}} \|\chi\|_{L^{2^*}}^{2^*} = \|\tilde{\chi}\|_{L^{2^*}}^{2^*}, \quad (3.3)$$

so that  $\mathcal{K}(\tilde{\chi}) < 0$ . Hence, we obtain that

$$\tilde{m}_\omega \leq \mathcal{I}_\omega(\tilde{\chi}). \quad (3.4)$$

Note here that  $\mathcal{I}_\omega(\tilde{\chi})$  is independent of  $\mu$ .

Now, we take any minimizer  $u$  for  $m_\omega$ . Then, Proposition 1.2 shows that  $u$  is also a minimizer for  $\tilde{m}_\omega$ . Hence, we see from (3.4) that

$$\|u\|_{H^1} \lesssim \mathcal{I}_\omega(u) = \tilde{m}_\omega \leq \mathcal{I}_\omega(\tilde{\chi}), \quad (3.5)$$

where the implicit constant is independent of  $\mu$ . Thus, we have completed the proof.  $\square$

Now, we are in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* We first prove the claim (i). Suppose the contrary that there exists a minimizer  $u$  of the variational problem for  $m_\omega$ . It follows from Proposition 1.2 that  $u$  is also a minimizer for  $\tilde{m}_\omega$ . Let  $u^*$  be the Schwarz symmetrization of  $u$ . Then, we easily see that  $u^*$  is a minimizer for  $\tilde{m}_\omega$  as well as  $u$ . Hence, using Proposition 1.2 again, we find that  $u^*$  is a radially symmetric solution to the elliptic equation (1.4). Moreover, Lemma 1 in [2] gives us that  $u^* \in C^2(\mathbb{R}^d)$ .

Put  $\phi(r) := ru^*(r)$  ( $r = |x|$ ). We see that  $\phi$  satisfies the equation

$$\phi'' - \phi + \mu r^{1-p} \phi^p + r^{-4} \phi^5 = 0, \quad r > 0. \quad (3.6)$$

Let  $g \in W^{3,\infty}([0, \infty))$ . Multiplying  $g\phi'$  by the equation (3.6) and integrating the resulting equation, we obtain

$$\begin{aligned} & -\frac{g(0)}{2}(\phi'(0))^2 - \frac{1}{2} \int_0^\infty g'(\phi')^2 + \frac{1}{2} \int_0^\infty g' \phi^2 \\ & + \mu \int_0^\infty \left( \frac{p-1}{p+1} r^{-p} g - \frac{1}{p+1} r^{1-p} g' \right) |\phi|^{p+1} + \int_0^\infty \left( \frac{2}{3} r^{-5} g - \frac{r^{-4}}{6} g' \right) |\phi|^6 = 0. \end{aligned} \quad (3.7)$$

On the other hand, multiplying the equation (3.6) by  $g'\phi/2$  and integrating the resulting equation, we obtain

$$-\frac{1}{2} \int_0^\infty g'(\phi')^2 + \frac{1}{4} \int_0^\infty g''' \phi^2 - \frac{1}{2} \int_0^\infty g' \phi^2 + \frac{\mu}{2} \int_0^\infty r^{1-p} g' |\phi|^{p+1} + \frac{1}{2} \int_0^\infty r^{-4} |\phi|^6 g' = 0. \quad (3.8)$$

Subtracting (3.8) from (3.7), we have the identity

$$\begin{aligned} & -\frac{g(0)}{2}(\phi'(0))^2 + \frac{1}{4} \int_0^\infty (g''' - 4g')\phi^2 \\ & = \mu \int_0^\infty \left\{ \frac{p-1}{p+1}g - \frac{p+3}{2(p+1)}rg' \right\} r^{-p}|\phi|^{p+1} + \frac{2}{3} \int_0^\infty \{g - rg'\} r^{-5}|\phi|^6. \end{aligned} \quad (3.9)$$

We take  $g$  such that

$$\begin{cases} g'''(r) - 4g'(r) = 0, & r > 0, \\ g(0) = 0. \end{cases} \quad (3.10)$$

It is easy to verify that

$$g(r) := \frac{1 - e^{-2r}}{2} \text{ is a solution to (3.10).} \quad (3.11)$$

Besides, we see from an elementary calculus that

$$g(r) - rg'(r) = \frac{1 - e^{-2r}}{2} - re^{-2r} \geq 0 \quad \text{for any } r > 0, \quad (3.12)$$

$$(g(r) - rg'(r))' = 2re^{-2r} > 0 \quad \text{for any } r > 0. \quad (3.13)$$

Taking this function, we see from the formula (3.9) that

$$0 = \mu \int_0^\infty \left\{ \frac{p-1}{p+1}g - \frac{p+3}{2(p+1)}rg' \right\} r^{-p}|\phi|^{p+1} + \frac{2}{3} \int_0^\infty \{g - rg'\} r^{-5}|\phi|^6. \quad (3.14)$$

We shall show that

$$\left| \int_0^\infty \left\{ \frac{p-1}{p+1}g - \frac{p+3}{2(p+1)}rg' \right\} r^{-p}|\phi|^{p+1} \right| \lesssim 1, \quad (3.15)$$

$$\liminf_{\mu \rightarrow 0} \int_0^\infty \{g - rg'\} r^{-5}|\phi|^6 \gtrsim 1, \quad (3.16)$$

where the implicit constants are independent of  $\mu$ . Once we obtain these estimates, taking  $\mu \rightarrow 0$  in (3.9), we derive a contradiction. Thus, the claim (i) holds.

We first prove (3.15). It is easy to verify that the right-hand side of (3.15) is bounded by

$$\begin{aligned} \int_0^\infty (1 - e^{-2r})r|u^*(r)|^{p+1} dr + \int_0^\infty e^{-2r}r^2|u^*(r)|^{p+1} dr & \lesssim \int_0^\infty r^2|u^*(r)|^{p+1} dr \\ & \sim \|u\|_{L^{p+1}(\mathbb{R}^3)}^{p+1} \lesssim \|u\|_{H^1}^{p+1}, \end{aligned} \quad (3.17)$$

where the implicit constants are independent of  $\mu$ . Combining (3.17) with Lemma 3.1, we obtain the estimate (3.15). Next, we prove (3.16). It follows from (3.12), (3.13) and the mean value theorem that

$$\int_0^\infty \{g - rg'\} r^{-5}|\phi|^6 = \int_0^\infty \{g - rg'\} r|u^*|^6 \gtrsim \int_0^\infty r^2|u^*|^6 \sim \|u\|_{L^6}^6, \quad (3.18)$$

where the implicit constants are independent of  $\mu$ . Here, using the Sobolev embedding,  $\mathcal{K}(u) = 0$ , the Hölder inequality and Lemma 3.1, we obtain that

$$\|u\|_{L^6}^2 \lesssim \|\nabla u\|_{L^2}^2 \lesssim \|u\|_{L^6}^{\frac{3(p-1)}{2}} + \|u\|_{L^6}^6 \quad \text{for all } 0 < \mu \leq 1. \quad (3.19)$$

Hence, we find that

$$1 \lesssim \|u\|_{L^6}^6 \quad \text{for all } 0 < \mu \leq 1. \quad (3.20)$$

The estimate (3.18) together with (3.20) gives (3.16).

We shall prove the claim (ii). Let  $Q$  be a ground state of the equation (1.4). We see from (1.6) and Proposition 1.1 that  $Q$  is also a minimizer for  $m_\omega$ . However, this contradicts the claim (i). Thus, (ii) follows.  $\square$

## 4 Blowup result

In this section, we prove Theorem 1.3. The key lemma is the following.

**Lemma 4.1.** *Assume  $d \geq 4$ ,  $\omega > 0$  and  $\mu > 0$ . Let  $\psi$  be a solution to (NLS) starting from  $A_{\omega,-}$ , and let  $I_{\max}$  be the maximal interval where  $\psi$  exists. Then, we have*

$$\psi(t) \in A_{\omega,-} \quad \text{for any } t \in I_{\max}, \quad (4.1)$$

$$\sup_{t \in I_{\max}} \mathcal{K}(\psi(t)) < 0. \quad (4.2)$$

*Proof of Lemma 4.1.* Since the action  $\mathcal{S}_\omega(\psi(t))$  is conserved with respect to  $t$ , we have  $\mathcal{S}_\omega(\psi(t)) < m_\omega$  for any  $t \in I_{\max}$ . Thus, it suffices for (4.1) to show that

$$\mathcal{K}(\psi(t)) < 0 \quad \text{for any } t \in I_{\max}. \quad (4.3)$$

Suppose the contrary that  $\mathcal{K}(\psi(t)) > 0$  for some  $t \in I_{\max}$ . Then, it follows from the continuity of  $\psi(t)$  in  $H^1(\mathbb{R}^d)$  that there exists  $t_0 \in I_{\max}$  such that  $\mathcal{K}(\psi(t_0)) = 0$ . Then, we see from the definition of  $m_\omega$  that  $\mathcal{S}_\omega(\psi(t_0)) \geq m_\omega$ , which is a contradiction. Thus, (4.3) holds.

Next, we shall prove (4.2). Since  $\mathcal{K}(\psi(t)) < 0$  for any  $t \in I_{\max}$ , we see from (2.1);

$$\text{For any } t \in I_{\max}, \text{ there exists } 0 < \lambda(t) < 1 \text{ such that } \mathcal{K}(T_{\lambda(t)}\psi(t)) = 0. \quad (4.4)$$

This together with the definition of  $m_\omega$  shows that

$$\mathcal{S}_\omega(T_{\lambda(t)}\psi(t)) \geq m_\omega. \quad (4.5)$$

Moreover, it follows from the concavity (2.4) together with (2.2), (4.3) and (4.5) that

$$\begin{aligned} \mathcal{S}_\omega(\psi(t)) &> \mathcal{S}_\omega(T_{\lambda(t)}\psi(t)) + (1 - \lambda(t)) \frac{d}{d\lambda} \mathcal{S}_\omega(T_\lambda\psi(t)) \Big|_{\lambda=1} \\ &= \mathcal{S}_\omega(T_{\lambda(t)}\psi(t)) + (1 - \lambda(t)) \mathcal{K}(\psi(t)) \\ &> m_\omega + \mathcal{K}(\psi(t)). \end{aligned} \quad (4.6)$$

Fix  $t_0 \in I_{\max}$ . Then, we have from (4.6) and the conservation law of  $\mathcal{S}_\omega$  that

$$\mathcal{K}(\psi(t)) < -m_\omega + \mathcal{S}_\omega(\psi(t)) = -m_\omega + \mathcal{S}_\omega(\psi(t_0)). \quad (4.7)$$

Hence, (4.2) follows from the hypothesis  $\mathcal{S}_\omega(\psi(t_0)) < m_\omega$ .  $\square$

Let us move to the proof of Theorem 1.3.

*Proof of Theorem 1.3.* The proof is based on the idea of [9, 10]. We introduce the generalized version of virial identity; Let  $\rho$  be a smooth function on  $\mathbb{R}$  such that

$$\rho(x) = \rho(4 - x) \quad \text{for all } x \in \mathbb{R}, \quad (4.8)$$

$$\rho(x) \geq 0 \quad \text{for all } x \in \mathbb{R}, \quad (4.9)$$

$$\int_{\mathbb{R}} \rho(x) dx = 1, \quad (4.10)$$

$$\text{supp } \rho \subset (1, 3), \quad (4.11)$$

$$\rho'(x) \geq 0 \quad \text{for all } x < 2. \quad (4.12)$$

Put

$$w(r) := r - \int_0^r (r - s)\rho(s) ds \quad \text{for } r \geq 0, \quad (4.13)$$

$$W_R(x) := R^2 w\left(\frac{|x|^2}{R^2}\right) \quad \text{for } x \in \mathbb{R}^d \text{ and } R > 0. \quad (4.14)$$

Then, for any solution  $\psi \in C(I_{\max}, H^1(\mathbb{R}^d))$  to (NLS) and  $t_0 \in I_{\max}$ , putting  $\psi_0 := \psi(t_0)$ , we have the identity

$$\begin{aligned} & \int_{\mathbb{R}^d} W_R |\psi(t)|^2 \\ &= \int_{\mathbb{R}^d} W_R |\psi_0|^2 + (t - t_0) \Im \int_{\mathbb{R}^d} \nabla W_R \cdot \nabla \psi_0 \bar{\psi}_0 + \int_{t_0}^t \int_{t_0}^{t'} \mathcal{K}(\psi(t'')) dt'' dt' \\ & \quad - \int_{t_0}^t \int_{t_0}^{t'} \int_{\mathbb{R}^d} \left\{ 2 \int_0^{\frac{|x|^2}{R^2}} \rho(r) dr |\nabla \psi(t'')|^2 + \frac{4|x|^2}{R^2} \rho\left(\frac{|x|^2}{R^2}\right) \left| \frac{x}{|x|} \cdot \nabla \psi(t'') \right|^2 \right\} dt'' dt' \\ & \quad + \int_{t_0}^t \int_{t_0}^{t'} \int_{\mathbb{R}^d} \left\{ d \int_0^{\frac{|x|^2}{R^2}} \rho(r) dr + \frac{2|x|^2}{R^2} \rho\left(\frac{|x|^2}{R^2}\right) \right\} \left\{ \mu \frac{p-1}{p+1} |\psi(t'')|^{p+1} + \frac{2}{d} |\psi(t'')|^{2^*} \right\} dt'' dt' \\ & \quad - \frac{1}{4} \int_{t_0}^t \int_{t_0}^{t'} \int_{\mathbb{R}^d} \Delta^2 W_R |\psi(t'')|^2 dt'' dt' \quad \text{for any } t \in I_{\max} \text{ and } R > 0. \end{aligned} \quad (4.15)$$

We easily verify that

$$|\nabla W_R(x)|^2 \leq 4W_R(x) \quad \text{for any } R > 0 \text{ and } x \in \mathbb{R}^d \quad (4.16)$$

$$\|W_R\|_{L^\infty} \lesssim R^2, \quad (4.17)$$

$$\|\nabla W_R\|_{L^\infty} \lesssim R, \quad (4.18)$$

$$\|\Delta^2 W_R\|_{L^\infty} \lesssim \frac{1}{R^2} \|\rho\|_{W^{2,\infty}(\mathbb{R})}. \quad (4.19)$$

Now, let  $\psi$  be a solution to (NLS) starting from  $A_{\omega,-}$ . We see from Lemma 4.1 that

$$\varepsilon_0 := - \sup_{t \in I_{\max}} \mathcal{K}(\psi(t)) > 0. \quad (4.20)$$

Our aim is to show that the maximal existence interval  $I_{\max}$  is bounded. We divide the proof into three steps.

**Step 1.** We claim that;

There exists  $m_* > 0$  and  $R_* > 0$  such that for any  $R \geq R_*$ ,

$$m_* < \inf \left\{ \int_{|x| \geq R} |v(x)|^2 dx \mid \begin{array}{l} v \in H_{rad}^1(\mathbb{R}^d), \quad \mathcal{K}^R(v) \leq -\frac{\varepsilon_0}{4}, \\ \|v\|_{L^2} \leq \|\psi_0\|_{L^2} \end{array} \right\}, \quad (4.21)$$

where  $H_{rad}^1(\mathbb{R}^d)$  is the set of radially symmetric functions in  $H^1(\mathbb{R}^d)$ , and

$$\begin{aligned} \mathcal{K}^R(v) := & \int_{\mathbb{R}^d} \left\{ 2 \int_0^{\frac{|x|^2}{R^2}} \rho(r) dr + \frac{4|x|^2}{R^2} \rho \left( \frac{|x|^2}{R^2} \right) \right\} |\nabla v|^2 \\ & - \int_{\mathbb{R}^d} \left\{ d \int_0^{\frac{|x|^2}{R^2}} \rho(r) dr + \frac{2|x|^2}{R^2} \rho \left( \frac{|x|^2}{R^2} \right) \right\} \left\{ \mu \frac{p-1}{p+1} |v|^{p+1} + \frac{2}{d} |v|^{2^*} \right\}. \end{aligned} \quad (4.22)$$

Let  $R > 0$  be a sufficiently large constant to be chosen later, and let  $v$  be a function such that

$$v \in H_{rad}^1(\mathbb{R}^d), \quad (4.23)$$

$$\mathcal{K}^R(v) \leq -\frac{\varepsilon_0}{4}, \quad (4.24)$$

$$\|v\|_{L^2} \leq \|\psi_0\|_{L^2}. \quad (4.25)$$

Then, we see from (4.24) that

$$\begin{aligned} & \frac{\varepsilon_0}{4} + \int_{\mathbb{R}^d} \left\{ 2 \int_0^{\frac{|x|^2}{R^2}} \rho(r) dr + \frac{4|x|^2}{R^2} \rho \left( \frac{|x|^2}{R^2} \right) \right\} |\nabla v|^2 \\ & \leq -\mathcal{K}^R(v) + \int_{\mathbb{R}^d} \left\{ 2 \int_0^{\frac{|x|^2}{R^2}} \rho(r) dr + \frac{4|x|^2}{R^2} \rho \left( \frac{|x|^2}{R^2} \right) \right\} |\nabla v|^2 \\ & = \int_{\mathbb{R}^d} \left\{ d \int_0^{\frac{|x|^2}{R^2}} \rho(r) dr + \frac{2|x|^2}{R^2} \rho \left( \frac{|x|^2}{R^2} \right) \right\} \left\{ \mu \frac{p-1}{p+1} |v|^{p+1} + \frac{2}{d} |v|^{2^*} \right\}. \end{aligned} \quad (4.26)$$

To estimate the right-hand side of (4.26), we employ the following inequality (see Lemma 6.5.11 in [6]); Assume that  $d \geq 1$ . Let  $\kappa$  be a non-negative and radially symmetric function

in  $C^1(\mathbb{R}^d)$  with  $|x|^{-(d-1)} \max\{-\frac{x \cdot \nabla \kappa}{|x|}, 0\} \in L^\infty(\mathbb{R}^d)$ . Then, we have

$$\begin{aligned} & \|\kappa^{\frac{1}{2}} u\|_{L^\infty} \\ & \lesssim \|u\|_{L^2}^{\frac{1}{2}} \left\{ \left\| |x|^{-(d-1)} \kappa \frac{x \cdot \nabla u}{|x|} \right\|_{L^2}^{\frac{1}{2}} + \left\| |x|^{-(d-1)} \max\left\{-\frac{x \cdot \nabla \kappa}{|x|}, 0\right\} \right\|_{L^\infty}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} \right\} \end{aligned} \quad (4.27)$$

for any  $u \in H_{rad}^1(\mathbb{R}^d)$ ,

where the implicit constant depends only on  $d$ .

Now, put

$$\kappa_1(x) := 2 \int_0^{\frac{|x|^2}{R^2}} \rho(r) dr + \frac{4|x|^2}{R^2} \rho\left(\frac{|x|^2}{R^2}\right), \quad (4.28)$$

$$\kappa_2(x) := d \int_0^{\frac{|x|^2}{R^2}} \rho(r) dr + \frac{2|x|^2}{R^2} \rho\left(\frac{|x|^2}{R^2}\right), \quad (4.29)$$

so that

$$\mathcal{K}^R(v) = \int_{\mathbb{R}^d} \kappa_1 |\nabla v|^2 - \int_{\mathbb{R}^d} \kappa_2 \left( \mu \frac{p-1}{p+1} |v|^{p+1} + \frac{2}{d} |v|^{2^*} \right). \quad (4.30)$$

Then, we can verify that

$$\text{supp } \kappa_j \subset \{|x| \geq R\} \quad \text{for } j = 1, 2, \quad (4.31)$$

$$\|\kappa_j\|_{L^\infty} \lesssim 1 \quad \text{for } j = 1, 2, \quad (4.32)$$

$$\sup_{x \in \mathbb{R}^d} \max \left\{ -\frac{x \cdot \nabla \sqrt{\kappa_2(x)}}{|x|}, 0 \right\} \lesssim \frac{1}{R}, \quad (4.33)$$

$$1 \lesssim \inf_{|x| \geq R} \frac{\kappa_1(x)}{\kappa_2(x)}. \quad (4.34)$$

The inequality (4.27) together with (4.32), (4.33) and (4.25) yields that

$$\begin{aligned} & \int_{\mathbb{R}^d} \kappa_2 \frac{2}{d} |v|^{2^*} \leq \|\sqrt{\kappa_2}^{\frac{1}{2}} v\|_{L^\infty}^{\frac{4}{d-2}} \int_{\mathbb{R}^d} \kappa_2^{\frac{d-3}{d-2}} |v|^2 \\ & \lesssim \frac{1}{R^{\frac{2(d-1)}{d-2}}} \|v\|_{L^2(|x| \geq R)}^{\frac{2}{d-2}} \left\{ \|\sqrt{\kappa_2} \nabla v\|_{L^2}^{\frac{2}{d-2}} + \frac{1}{R^{\frac{2}{d-2}}} \|v\|_{L^2}^{\frac{2}{d-2}} \right\} \|v\|_{L^2}^2 \\ & \leq \frac{\|\psi_0\|_{L^2}^2}{R^{\frac{2(d-1)}{d-2}}} \left( \|v\|_{L^2(|x| \geq R)} \|\sqrt{\kappa_2} \nabla v\|_{L^2} \right)^{\frac{2}{d-2}} + \frac{\|\psi_0\|_{L^2}^{2^*}}{R^{2^*}}, \end{aligned} \quad (4.35)$$

where the implicit constant depends only on  $d$  and  $\rho$ . Moreover, applying the Young inequality ( $ab \leq \frac{a^r}{r} + \frac{b^{r'}}{r'}$  with  $\frac{1}{r} + \frac{1}{r'} = 1$ ) to the right-hand side of (4.35), we obtain that

$$\int_{\mathbb{R}^d} \kappa_2 \frac{2}{d} |v|^{2^*} \lesssim \|v\|_{L^2(|x| \geq R)}^2 \|\sqrt{\kappa_2} \nabla v\|_{L^2}^2 + \left( \frac{\|\psi_0\|_{L^2}^2}{R^{\frac{2(d-1)}{d-2}}} \right)^{\frac{d-2}{d-3}} + \frac{\|\psi_0\|_{L^2}^{2^*}}{R^{2^*}}. \quad (4.36)$$

Similarly, we have

$$\int_{\mathbb{R}^d} \kappa_2 \frac{2}{d} |v|^{p+1} \lesssim \|v\|_{L^2(|x| \geq R)}^2 \|\sqrt{\kappa_2} \nabla v\|_{L^2}^2 + \left( \frac{\|\psi_0\|_{L^2}^2}{R^{\frac{(d-1)(p-1)}{2}}} \right)^{\frac{4}{5-p}} + \frac{\|\psi_0\|_{L^2}^{2^*}}{R^{\frac{d(p-1)}{2}}}. \quad (4.37)$$

If  $R$  is so large that

$$\left( \frac{\|\psi_0\|_{L^2}^2}{R^{\frac{(d-1)(p-1)}{2}}} \right)^{\frac{4}{5-p}} + \frac{\|\psi_0\|_{L^2}^{2^*}}{R^{\frac{d(p-1)}{2}}} + \left( \frac{\|\psi_0\|_{L^2}^2}{R^{\frac{2(d-1)}{d-2}}} \right)^{\frac{d-2}{d-3}} + \frac{\|\psi_0\|_{L^2}^{2^*}}{R^{2^*}} \ll \varepsilon_0, \quad (4.38)$$

then (4.26) together with (4.36) and (4.38) shows that

$$\int_{\mathbb{R}^d} \left( \kappa_1 - C \|v\|_{L^2(|x| \geq R)}^2 \kappa_2 \right) |\nabla v|^2 \leq -\frac{\varepsilon_0}{8} \quad (4.39)$$

for some constant  $C > 0$  depending only on  $d, \mu, p$  and  $\rho$  (the notation  $A \ll \varepsilon_0$  means that the quantity  $A$  is much smaller than  $\varepsilon_0$ ). Hence, we conclude that

$$\inf_{|x| \geq R} \frac{\kappa_1(x)}{\kappa_2(x)} < C \|v\|_{L^2(|x| \geq R)}^2, \quad (4.40)$$

which together with (4.34) gives us the desired result.

**Step 2.** Let  $m_*$  and  $R_*$  be constants found in (4.21), and let  $T_{\max} := \sup I_{\max}$ . Then, our next claim is that

$$\sup_{t \in [t_0, T_{\max})} \int_{|x| \geq R} |\psi(x, t)|^2 dx \leq m_* \quad (4.41)$$

for any  $R > R_*$  satisfying the following properties:

$$\|\Delta^2 W_R\|_{L^\infty} \|\psi_0\|_{L^2}^2 \leq \varepsilon_0, \quad (4.42)$$

$$\int_{|x| \geq R} |\psi_0(x)|^2 dx < m_*, \quad (4.43)$$

$$\frac{1}{R^2} \left( 1 + \frac{1}{\varepsilon_0} \|\nabla \psi_0\|_{L^\infty}^2 \right) \int_{\mathbb{R}^d} W_R |\psi_0|^2 < m_*. \quad (4.44)$$

Here, we remark that it is possible to take  $R$  satisfying (4.44) (see [9]).

In order to prove (4.41), we introduce

$$T_R := \sup \left\{ T > t_0 \mid \sup_{t_0 \leq t < T} \int_{|x| \geq R} |\psi(x, t)|^2 dx \leq m_* \right\} \quad \text{for } R > 0, \quad (4.45)$$

and prove that  $T_R = T_{\max}$  for any  $R > 0$  satisfying (4.42)–(4.44).

It follows from (4.43) together with the continuity of  $\psi(t)$  in  $L^2(\mathbb{R}^d)$  that  $T_R > t_0$ .

We suppose the contrary that  $T_R < T_{\max}$ . Then, we have

$$\int_{|x| \geq R} |\psi(x, T_R)|^2 dx = m_*. \quad (4.46)$$

Hence, we see from the definition of  $m_*$  (see (4.21)) together with the mass conservation law (1.1) that

$$-\frac{\varepsilon_0}{4} \leq \mathcal{K}^R(\psi(T_R)). \quad (4.47)$$

Combining the generalized virial identity (4.15) with (4.47) and (4.42), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} W_R |\psi(T_R)|^2 &< \int_{\mathbb{R}^d} W_R |\psi_0|^2 + T_R \Im \int_{\mathbb{R}^d} \nabla W_R \cdot \nabla \psi_0 \overline{\psi_0} - \frac{T_R^2}{2} \varepsilon_0 \\
&\quad + \frac{T_R^2}{8} \varepsilon_0 + \frac{T_R^2}{8} \varepsilon_0 \\
&= \int_{\mathbb{R}^d} W_R |\psi_0|^2 + \frac{1}{\varepsilon_0} \left| \int_{\mathbb{R}^d} \nabla W_R \cdot \nabla \psi_0 \overline{\psi_0} \right|^2 \\
&\quad - \frac{\varepsilon_0}{4} \left( T_R - \frac{2}{\varepsilon_0} \Im \int_{\mathbb{R}^d} \nabla W_R \cdot \nabla \psi_0 \overline{\psi_0} \right)^2 \\
&\leq \left( 1 + \frac{1}{\varepsilon_0} \|\nabla W_R\|_{L^\infty}^2 \right) \int_{\mathbb{R}^d} W_R |\psi_0|^2.
\end{aligned} \tag{4.48}$$

Hence, we see from (4.44) that

$$\int_{\mathbb{R}^d} W_R |\psi(T_R)|^2 < R^2 m_*. \tag{4.49}$$

Since  $W_R(x) \geq R^2$  for  $|x| > R$ , (4.49) gives us that

$$\int_{|x| \geq R} |\psi(x, T_R)|^2 dx = \frac{1}{R^2} \int_{|x| \geq R} R^2 |\psi(x, T_R)|^2 dx \leq \frac{1}{R^2} \int_{\mathbb{R}^d} W_R |\psi(T_R)|^2 < m_*, \tag{4.50}$$

which contradicts (4.46). Thus, we have proved  $T_R = T_{\max}$  and therefore (4.41) holds.

Similarly, putting  $T_{\min} := \inf I_{\max}$ , we have

$$\sup_{t \in (T_{\min}, t_0]} \int_{|x| \geq R} |\psi(x, t)|^2 dx \leq m_* \tag{4.51}$$

for any  $R > R_*$  satisfying (4.42)–(4.44).

**Step 3** We complete the proof of Theorem 1.3. Take  $R > R_*$  satisfying (4.42)–(4.44). Then, it follows from the definition of  $m_*$  together with (4.41) and (4.51) that

$$-\frac{\varepsilon_0}{4} \leq \mathcal{K}^R(\psi(t)) \quad \text{for any } t \in I_{\max}. \tag{4.52}$$

Hence, we see from the generalized virial identity (4.15) together with (4.52) and (4.42) that

$$\int_{\mathbb{R}^d} W_R |\psi(t)|^2 \leq \int_{\mathbb{R}^d} W_R |\psi_0|^2 + t \Im \int_{\mathbb{R}^d} \nabla W_R \cdot \nabla \psi_0 \overline{\psi_0} - \frac{\varepsilon_0 t^2}{2} \quad \text{for any } t \in I_{\max}. \tag{4.53}$$

Supposing  $\sup I_{\max} = +\infty$  or  $\inf I_{\max} = -\infty$ , we have from (4.53) that

$$\int_{\mathbb{R}^d} W_R |\psi(t)|^2 < 0 \quad \text{for some } t \in I_{\max}, \tag{4.54}$$

which is a contradiction. Thus,  $I_{\max}$  is bounded.  $\square$



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