# GEOMETRIC-OPTICS FOR NONLINEAR CONCENTRATING WAVES IN FOCUSING AND NON-FOCUSING TWO GEOMETRIES

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ABSTRACT. With the methods used in [1] and [4], we prove that, in absence of focus, nonlinear geometrical optics of the critical wave equation with variable coefficients, is reduced to linear geometrical optics combined with wave operators for the critical wave equation with coefficients fixed on concentrating points. On the odd-dimensional spheres, we prove that passing through a focus is generated by a modified scattering operator.

## 1. INTRODUCTION

In this paper, we consider a sequence $\underline{u} := (u_n)_n$, solution of the equation

$$(\partial_t^2 - \Delta_{\mathcal{M}})u + \mid u \mid^{p_c - 1} u = 0, \ I\!R_t \times \mathcal{M}_x^d, \tag{1.1}$$

where, $\mathcal{M}$ is either the d-dimensional sphere $S^d$ and in that case $\Delta_{\mathcal{M}}$ is assumed to be the Laplace-Beltrami operator on $S^d$, or $\mathcal{M}$ is the whole space $I\!R^d$ with a local perturbation of the Laplace operator, namely $\Delta_{\mathcal{M}} = \mathrm{div}_x(A(x)\nabla_x .)$, and $A$ is a matrix valued function $A$ satisfying:
*There exist two constants* $0 < c_0 \leq 1$ *and* $R_0 > 0$ *such that*

$$(\mathcal{H}) \qquad \begin{cases} (i \quad) & c_0 \mid \xi \mid^2 \leq A(x)\xi.\xi \leq c_0^{-1} \mid \xi \mid^2, \quad \forall x, \ \xi \in I\!R^d \\ (ii \quad) & A(x) \equiv Id, \quad \forall x \in I\!R^d, \ \textit{with} \ \mid x \mid \geq R_0. \end{cases}$$

Under the condition $(\mathcal{H})(i)$, the matrix $A(x)$ is positive definite for a hyperbolic equation, and so its inverse $A^{-1}(x)$ defines a riemannian metric on $I\!R^d$. Notice that in the constant case, namely when $A(x) \equiv Id$, the operator $\partial_t^2 - \Delta_{\mathcal{M}}$ is the usual d'alembertian operator on $I\!R_t \times I\!R_x^d$, defined by $\Box := \partial_t^2 - \Delta_x$.
We assume that $d \geq 3$. The exponent, $p_c = \frac{d+2}{d-2}$, corresponds to the critical Lebesgue space in which the homogenous Sobolev space $\dot{H}^1(\mathcal{M})$ is embedded. We recall that $\dot{H}^1(\mathcal{M}) = \{g \in L^{p_c+1}(\mathcal{M}) \text{ such that } \|\nabla g\|_{L^2(\mathcal{M})} < \infty\}$.

The initial Cauchy data, $(u_n, \partial_t u_n)_{|t=0} = (\varphi_n, \psi_n)$, are supposed to be

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bounded in the energy space $\mathcal{E}(\mathcal{M}) := (\dot{H}^1 \times L^2)(\mathcal{M})$, with the norm

$$\|(g_0, g_1)\|_{\mathcal{E}} := \|(\nabla g_0, g_1)\|_{L^2(\mathcal{M})^2}.$$

Our aim is to describe the sequence $\underline{u}$ by means of a sequence of "simpler functions" in the energy space $\mathcal{E}$. A crucial role is played here by the sequence $\underline{v}$, solution of the linear wave equation

$$(\partial_t^2 - \Delta_{\mathcal{M}})v = 0, \tag{1.2}$$

with the same Cauchy data at time $t = 0$, that is

$$(v_n, \partial_t v_n)_{|t=0} = (u_n, \partial_t u_n)_{|t=0}. \tag{1.3}$$

Let us first state a few facts about solutions of (1.1). In the case $\mathcal{M} = I\!R^d$ and $\Delta_{\mathcal{M}} = \mathrm{div}_x(A(x)\nabla_x .)$, global existence of solutions of equation (1.1), in the energy space, was recently proved by the author and M. Majdoub in [9], [10]. They used a method introduced by Shatah-Struwe, when the coefficients are constant, (see [15] and [16]) . We emphasize the fact that these results use the so-called Strichartz estimates, in a crucial way.

**Proposition 1.1** ([Strichartz estimates.])**.** *Let $T$ be a positive real number and $g$ a function satisfying $(g, \partial_t g) \in \mathcal{C}(I\!R, \mathcal{E})$. Assume that*

$$h := [\partial_t^2 g - \mathit{div}_x(A(x)\nabla_x g)] \in L^1([0,T], L^2(I\!R^d)),$$

*then we have*

$$\|g\|_{L^q([0,T],L^r(I\!R^d))} \le c_{q,T}\Big\{\|\nabla_{t,x} g_{|t=0}\|_{L^2(I\!R^d)^2} + \int_0^T \|h(t)\|_{L^2(I\!R^d)} dt\Big\}, \tag{1.4}$$

*with* $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - 1, \quad q \ge \frac{d+1}{d-1}$ *and* $q > 2$ *if* $d = 3$.

Notice that in the constant case, the constant $c_{q,T}$ does not depend on $T$ and therefore the previous estimates are global; that is
every solution $g$ of $\Box g = 0$ satisfies

$$\|g\|_{L^q(I\!R, L^r(I\!R^d))} \le \tilde{c}_q E_0^{\frac{1}{2}}(g, 0), \tag{1.5}$$

where, for all function $g(t,x)$, $E_0(g,t) := \|(g, \partial_t g)(t,.)\|_{\mathcal{E}}^2$ denotes the energy of $g$ at time $t$. Inequalities (1.4) were obtained by Ginibre-Velo [8] in the constant case, and by Kapitanski [12] when coefficients are smooth. Recent results established by Smith [17] and by Smith-Sogge [18], generalizing these estimates for less regular coefficients and for exterior of strictly convex compact sets.
In the case of the sphere, the non-concentration result proved in [10], along with the method followed by [15] and [16] imply, in fact in an easy way, existence and uniqueness of solutions when the data are in the energy space: we will not write the details here.

On the other hand, in the constant case, Bahouri-Gérard proved in [1] a

structure theorem for solutions of (1.1), up to remainder terms, small both in energy and in Strichartz norms. More precisely, they showed that nonlinear geometrical optics are reduced to linear geometrical optics combined with the same scattering operator as solution passes through a point of concentration. Here, we prove similar results but with a family of scattering operators depending on the concentrating points.
Before stating our results, let us introduce the particular case of a nonlinear concentrating wave.
We denote by $r_0 > 0$ a lower bound for the injectivity radius on $\mathcal{M}$. Fix an even cut-off function $\theta \in \mathcal{C}_0^\infty(B(0, r_0))$ satisfying $\theta(x) \equiv 1$ on $B(0, \frac{r_0}{2})$. For all $x_1,\ x_2 \in \mathcal{M}$, let $\theta_{x_2}(x_1) = \theta(\exp_{x_2}^{-1}(x_1))$. Here, exp is the exponential map, and $B(x, r)$ is the euclidian ball with center $x$ and radius $r$. We also denote by

$$B'(x, r) := \{y \in \mathcal{M} \text{ such that } \mid \exp_x^{-1}(y) \mid_x < r\},$$

the geodesic ball. ( $\mid \ . \ \mid_x$ denoting the norm in $T_x\mathcal{M}$ with respect to the riemannian metric).

**Definition 1.2.** Given an element $[(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$ in the space $\mathcal{E} \times (I\!\!R_+^* \times \mathcal{M} \times I\!\!R)^{I\!\!N}$ such that $\lim_n (h_n, x_n, t_n) = (0, x_\infty, t_\infty)$, a linear concentrating wave $\underline{v} := (v_n)_n$ associated with the data $[(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$ is a solution of (1.2) satisfying,

$$\begin{cases} (v_n, \partial_t v_n)(t_n, .) &= {h_n}^{1-\frac{d}{2}} \theta(\exp_{x_n}^{-1} .)(\varphi, \frac{1}{h_n}\psi)(\frac{\exp_{x_n}^{-1} \cdot}{h_n}) \\ & + \ o(1), \quad \text{in } \mathcal{E}(B'(x_n, r_0)) \\ (v_n, \partial_t v_n)(t_n, .) &= o(1) \quad \text{in } \mathcal{E}(\overline{B}'^c(x_n, r_0)). \end{cases}$$

*Remark* 1.3. (1) In the following, we will call concentrating data, the element $[(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$ appearing in the Definition 1.2.

(2) Two concentrating data are said to be equivalent if, the corresponding linear concentrating waves are equivalent in the energy space. From now on, we will not distinguish between a concentrating data and its associate linear concentrating wave; we will usually write $\underline{v} = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$.

(3) Notice that in the constant case, we have $\exp_{x_n}^{-1}(x) = x - x_n$ and $r_0$ is in fact equal to $+\infty$. In the case of $(I\!\!R^d, \mathrm{div}_x(A(x)\nabla_x .))$, we can still use the previous fact and then get the same definition of a linear concentrating wave as in [1], [4]. In fact, we have to estimate, according to the $L^2$ norm, expressions of the type ${h_n}^{-\frac{d}{2}} \theta(\exp_{x_n}^{-1} .) g(\frac{\exp_{x_n}^{-1} \cdot}{h_n}) - {h_n}^{-\frac{d}{2}} g(\frac{x - x_n}{h_n})$ in the geodesic ball $B'(x_n, r_0)$. For simplicity, let us suppose that $g \in \mathcal{C}_0^\infty(I\!\!R^d)$.

Putting the change of variables $x = \exp_{x_n}(h_n y)$, hence for $n$ sufficiently large, we can write

$$\int_{h_n|y|<r_0} \mid g(y) - g(\frac{\exp_{x_n}(h_n y) - x_n}{h_n}) \mid^2 dy = (i) + (ii),$$

with,

$$(i) \quad = \quad \int_{|y|<R} \mid g(y) - g(\frac{\exp_{x_n}(h_n y) - x_n}{h_n}) \mid^2$$

and

$$(ii) \quad = \quad \int_{R<|y|<\frac{r_0}{h_n}} \mid g(y) - g(\frac{\exp_{x_n}(h_n y) - x_n}{h_n}) \mid^2 .$$

The positive real number $R$ is chosen such that the euclidian ball $B(0, c_\infty R)$ contains the support of $g$. The constant $0 < c_\infty \leq 1$ is given by the local equivalence between the geodesic and the euclidian distances ( see Proposition 2.1 in [10]), that is

$$c_\infty \mid x - x_n \mid \leq \mid \exp_{x_n}^{-1} x \mid_{x_n} \leq c_\infty^{-1} \mid x - x_n \mid . \tag{1.6}$$

Using the dominated-convergence Theorem, it is easy to show that $(i)$ goes to 0 as $n$ tends to infinity. On the other hand,

$$\begin{aligned} (ii) \quad &\leq \quad \int_{R<|y|<\frac{r_0}{h_n}} \mid g\frac{\exp_{x_n}(h_n y) - x_n}{h_n} \mid^2 dy \\ &\leq \quad {h_n}^{-\frac{d}{2}} \int_{Rh_n<|\exp_{x_n}^{-1}(x)|_{x_n}<\frac{r_0}{2}} \mid g(\frac{x - x_n}{h_n}) \mid^2 dx. \end{aligned}$$

The above choice of $R$ and (1.6) give the desired result.

**Definition 1.4.** Let $\underline{v}$ be a linear concentrating wave and $\underline{u}$ be the solution of

$$\begin{cases} (\partial_t^2 - \Delta_{\mathcal{M}})u_n + \mid u_n \mid^{p_c - 1} u_n = 0 \\ (u_n, \partial_t u_n)_{|t=0} = (v_n, \partial_t v_n)_{|t=0}. \end{cases} \tag{1.7}$$

Then, the sequence $\underline{u}$ is called the nonlinear concentrating wave associated to $\underline{v}$.

The profile decomposition of bounded sequences in $\dot{H}^1$, (see [7]) and the propagation results of $h$-oscillations proved in [4], along with the method followed by [1] and [4], reduces the study to the particular case of a nonlinear concentrating wave. Similar results for the case of general data, $(\varphi_n, \psi_n)$, would be easy to prove.

In this paper, our goal is to describe every nonlinear concentrating wave $\underline{u}$ in the high frequency approximation; namely, up to relatively compact sequences $(r_n)_n$ according to the Energy-Strichartz norm

$$|||\ r\ |||_I := \sup_I \|(r, \partial_t r)(t, .)\|_{\mathcal{E}} + \|r\|_{L^{p_c}(I, L^{2p_c})},$$

where $I$ is a time interval of $I\!R$. It is a matter of finding linear concentrating wave $\underline{f}$ such that $\underline{u} = \underline{f} + \underline{r}$ with $\lim_n ||| r_n |||_I = 0$.

In the first part of this paper, we deal with the case $(I\!R^d, \mathrm{div}(A(x)\nabla_x .))$. The idea is the following. We rescale the Cauchy problem associated to (1.2) by introducing the "microscopic" variables $s = \frac{t-t_n}{h_n}$ and $y = \frac{x-x_n}{h_n}$ (which preserve the energy norm). Thus we have

$$\partial_s^2 \tilde{v}_n - \mathrm{div}_y(A(h_n . + x_n)\nabla_y \tilde{v}_n) = 0 \text{ and } (\tilde{v}_n, \partial_s \tilde{v}_n)_{|s=0} = (\varphi, \psi),$$

where $v_n(t,x) = h_n^{\frac{d}{2}-1} \tilde{v}_n(s,y)$. Therefore, one may be tempted to consider the "rescaled Laplace operator" $\mathrm{div}_y(A(h_n y + x_n)\nabla_y .)$ to be "equivalent" to the operator $\mathrm{div}_y(A(x_\infty)\nabla_y .)$, and then use the scattering theory when coefficients are fixed on the point of concentration $x_\infty$. We prove that this heuristic makes sense for time sufficiently close to the concentration time $t_\infty$. However, when we are away from that time, geometrical focus could hold.

**Definition 1.5.** A given point $x_\infty \in \mathcal{M}$ is said to be a focus if there exist a point $x \in \mathcal{M}$ and a time $t \in I\!R$ such that the set

$$\mathcal{F}_{x_\infty}(x,t) := \{\xi \in S_x^*\mathcal{M} : \quad \exp{}_x t\xi = x_\infty\}$$

of directions of geodesics stemming from a point $x$ and reaching $x_\infty$ in a time $t$, has a positive surface measure.

Here, $S^*\mathcal{M}$ is the unit cosphere bundle and the measure $\mid \Omega \mid$ of a set $\Omega$ is the one induced by the metric of $\mathcal{M}$ on $T_x^*\mathcal{M}$. For instance, any two antipodal points $x_1$ and $x_2$ on the sphere $S^d$ satisfy

$$\mid \mathcal{F}_{x_1}(x_2, t = \pi) \mid = \mid S^d \mid .$$

Our first result is the following

**Theorem 1.6.** *Let $\underline{v} = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t})]$ be a linear concentrating wave. We denote by $\underline{u}$ its nonlinear associated concentrating wave. There exist two linear concentrating waves denoted by $[(\varphi_\infty, \psi_\infty)_\pm, \underline{h}, \underline{x}, \underline{t}]$, such that:*
*For all interval $I$ of $I\!R$ containing 0 and not $t_\infty$, satisfying the following non-focusing property*

$$(\mathcal{H}_\mathcal{G})(I, x_\infty) : \qquad \mid \mathcal{F}_{x_\infty}(y, t_\infty - t) \mid = 0 \quad \forall t \in I,\ y \in I\!R^d,$$

*we have,*

**(i)** $\overline{\lim_n} ||| u_n - [(\varphi_{\infty,-}, \psi_{\infty,-}), \underline{h}, \underline{x}, \underline{t}] |||_{K^-_{n,\lambda}} \longrightarrow 0$ *as* $\lambda \to +\infty$

**(ii)** $\overline{\lim_n} ||| u_n - [(\varphi_{\infty,+}, \psi_{\infty,+}), \underline{h}, \underline{x}, \underline{t}] |||_{K^+_{n,\lambda}} \longrightarrow 0$ *as* $\lambda \to +\infty,$

*where, $K$ is a compact subset of $I$ and $K^\pm_{n,\lambda} = \{t \in K :\ \pm(t_\infty - t) \le -\lambda h_n\}$.*

*Remark* 1.7. (1) The linear profiles $(\varphi_\infty, \psi_\infty)_\pm$, given by the above theorem, are well defined according to the asymptotic states of solutions of

$$\partial_t^2 u - \mathrm{div}_y (A(x_\infty)\nabla_y u) + \mid u \mid^{p_c - 1} u = 0.$$

See Proposition 3.1 for the precise definition.

(2) The geometric condition $(\mathcal{H}_\mathcal{G})(I, x_\infty)$, takes into account the fact that in variable coefficients case, the linear solution $\underline{v}$ can concentrate possibly several times between times $t = 0$ and $t = t_\infty$, because of possible focus. The example of the sphere $S^d$ shows that, there are intervals $I$ for which this condition is not satisfied. Nevertheless, in the general case of a riemannian manifold, it is enough to suppose that the time of concentration is rather small so that $(\mathcal{H}_\mathcal{G})(I, x_\infty)$ is satisfied. It is a property of the exponential map.

(3) $(\mathcal{H}_\mathcal{G})(I, x_\infty)$ is globally satisfied, ( i.e $I = I\!\!R - \{t_\infty\}$), when the metric is flat because the geodesics are the right-hand sides stemming from $x_\infty$, which never intersect. We will give a larger class of such metrics (see the example in section 3).

In the second part of this paper, we study the same problem on the sphere $S^d$; geodesics from a point $x_\infty \in S^d$, all meet the antipodal point at time $t = \pi$, and periodically at time $t = \pi + k\pi$, with $k \in \mathbb{Z}$.

**Theorem 1.8.** *Let* $\underline{v} = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$ *be a linear concentrating wave on the sphere* $S^d$ *and* $\underline{u}$ *its associate nonlinear concentrating wave. We assume that* $t_\infty = \lim_n t_n \in ]0, \pi]$*, then we have*

**(i)** $\overline{\lim_n} ||| \; u_n - [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}] \; |||_{]t_\infty - \pi + \Lambda h_n, t_\infty - \Lambda h_n[} \to 0; \;\; \Lambda \to +\infty.$

**(ii)** $\overline{\lim_n} ||| \; u_n - [S(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}] \; |||_{]t_\infty + \Lambda h_n, t_\infty + \pi - \Lambda h_n[} \to 0; \;\; \Lambda \to +\infty.$

*Moreover if d is odd, then beyond the first focus we have*

**(iii)** $\overline{\lim_n} || \mid u_n - [S\mathcal{A}_d S(\varphi, \psi), \underline{h}, -\underline{x}, \underline{t}] \mid ||_{]t_\infty + \pi + \Lambda h_n, t_\infty + 2\pi - \Lambda h_n[} \to 0; \;\; \Lambda \to +\infty,$
*where* $\mathcal{A}_d$ *is an involution isometry of the energy space defined in Lemma 4.2.*

When $t_\infty$ is arbitrarily chosen, we will replace the scattering operator $S$ by one of the appropriate wave operator given in Proposition 3.1. For arbitrary intervals, we prove that each focus crossing is described by the above nonlinear scattering operator, composed with the map $\mathcal{A}_d$. They are iterated as many times as the solution passes through a focus. Precisely, by induction we can easily deduce the next corollary.

**Corollary 1.9.** *Using the same notations of Theorem 1.6, then for all $j \in Z..$, we have*

$$\overline{\lim_n} ||| \ u_n - [\tilde{S}^{(j)} S(\varphi, \psi), \underline{h}, (-1)^j \underline{x}, \underline{t}] \ |||_{]t_\infty + j\pi + \Lambda h_n, t_\infty + (j+1)\pi - \Lambda h_n[} \to 0 \ \textit{as} \ \Lambda \to +\infty.$$

*Here, $\tilde{S} = S \circ \mathcal{A}_d$ and $\tilde{S}^{(j)} = \tilde{S} \circ \tilde{S} \circ ..... \circ \tilde{S}$, $j$ times.*

*Remark* 1.10. When $d$ is an even integer, we believe that the result of Theorem 1.8 is still true (eventually with another scattering operator). Our method only proves the first statement of Theorem 1.8 nevertheless, beyond the first focus we need more careful study to exhibit the new profile of $\pi$-time translated concentrating waves.

We notice that similar results, were obtained by [3], in the context of semi-classical Shrödinger equation with isotropic harmonic potential.
The paper is organized as follows. In section 2, we give some properties of linear concentrating waves on the whole space $I\!R^d$. In section 3, we study nonlinear concentrating waves and then we prove Theorem 1.6. The last section is devoted to the study of nonlinear concentrating waves on the sphere and to the proof of Theorem 1.8.

## 2. Study of a Linear Concentrating Wave on $I\!R^d$

First, we fix a concentrating wave $\underline{v} = [(\varphi, \psi), \underline{h}, \underline{x}, t_\infty]$ and an interval $I$ of $I\!R$ containing 0. For the sake of simplicity, we suppose that $d = 3$, $t_n = t_\infty > 0$, $\lim_n x_n = x_\infty = 0$ and $A(0) = Id$. The other cases can be treated in a slightly similar way. In this section, we denote by $\square_A = \partial_t^2 - \text{ div } (A(.)\nabla)$. Thus we have

$$\square_A v_n = 0, \quad (v_n, \partial_t v_n)_{|t=t_\infty}(x) = \frac{1}{\sqrt{h_n}} (\varphi, \frac{1}{h_n}\psi)(\frac{x - x_n}{h_n}). \tag{2.8}$$

Finally, we denote by $(\varphi_\Lambda, \psi_\Lambda)_{\Lambda > 0}$, a $(\mathcal{C}_0^\infty(I\!R^d))^2$ approximation of $(\varphi, \psi)$ in the energy space $\mathcal{E}$.
We split the proof of Theorem 1.6 in several lemmas.

**Lemma 2.1.** *Let $f(s, y)$ be a regular function defined on $I\!R \times I\!R^d$ such that for all $\Lambda \in I\!R^+$, its restriction on $[-\Lambda, \Lambda] \times I\!R$ is supported in some compact set $K_\Lambda$. Then we have,*

$$\overline{\lim_n} \ \| div_y [(A(h_n . + x_n) - A(x_\infty)) \nabla_y f] \|_{L^1([-\Lambda, \Lambda], L^2(I\!R^3))} = 0.$$

**Proof.** Let $\Lambda > 0$ and $K_\Lambda$ be the support of the restriction of the function $f$ on $[-\Lambda, \Lambda] \times I\!R^3$. We denote by $K_\Lambda^1$ the compact of $I\!R^3$, obtained as the projection of $K_\Lambda$ on $I\!R^3$. Therefore, for all real $s \in [-\Lambda, \Lambda]$, we have supp $f(s, .) \subset K_\Lambda^1$. The proof of the lemma is now easy; it is enough to notice that we have

$$\begin{aligned}\|\mathrm{div}_y[(A(h_n.+x_n) &- A(0))\nabla_y f]\|_{L^1([-\Lambda,\Lambda],L^2(I\!R^3))}\\ &\leq c\Big\{\Lambda h_n \sup_{y\in I\!R^3}|\partial A(y)|\sup_{s\in I\!R}E_0^{\frac12}(f,s)\Big\}+\\ &\quad c\quad\Big\{\Lambda\sup_{y\in K_\Lambda}\mid A(h_n y+x_n)-A(0)\mid\sup_{s\in I\!R}E_0^{\frac12}(\partial f,s)\Big\}. \end{aligned}\tag{2.9}$$

The following result shows that using Lemma 2.1, we can fix the coefficients of the operator $\Box_A$, for times $t$ close to the time of concentration $t_\infty$. Precisely we have

**Lemma 2.2.** *Let $\underline{v}$ be the solution of (2.8). Setting $\underline{v}^\circ_\Lambda := (v^\circ_{n,\Lambda})_n$ the sequence satisfying*

$$\Box v^\circ_{n,\Lambda}=0,\quad (v^\circ_{n,\Lambda},\partial_t v^\circ_{n,\Lambda})_{|t=t_\infty}(x)=\frac{1}{\sqrt{h_n}}(\varphi_\Lambda,\frac{1}{h_n}\psi_\Lambda)(\frac{x-x_n}{h_n}).$$

*Then we have*

$$\overline{\lim_n}\ |||\ v_n-v^\circ_{n,\Lambda}\ |||_{[t_\infty-\Lambda h_n,t_\infty+\Lambda h_n]}\longrightarrow 0\ \text{as}\ \Lambda\longrightarrow+\infty.$$

**Proof.** In the "microscopic" variables $s=\frac{t-t_\infty}{h_n}$ and $y=\frac{x-x_n}{h_n}$, we can write

$$v_n(t,x)=\frac{1}{\sqrt{h_n}}\tilde v_n(\frac{t-t_\infty}{h_n},\frac{x-x_n}{h_n})=\frac{1}{\sqrt{h_n}}\tilde v_n(s,y)$$

$$v^\circ_{n,\Lambda}(t,x)=\frac{1}{\sqrt{h_n}}\tilde v^\circ_\Lambda(\frac{t-t_\infty}{h_n},\frac{x-x_n}{h_n})=\frac{1}{\sqrt{h_n}}\tilde v^\circ_\Lambda(s,y),$$

where the "rescaled functions" $\tilde v_n$ and $\tilde v^\circ_\Lambda$ satisfy

$$\Box_{A(h_n.+x_n)}\tilde v_n=0,\quad (\tilde v_n,\partial_s\tilde v_n)_{|s=0}=(\varphi,\psi),$$

$$\Box\tilde v^\circ_\Lambda=0,\quad (\tilde v^\circ_\Lambda,\partial_s\tilde v^\circ_\Lambda)_{|s=0}=(\varphi_\Lambda,\psi_\Lambda).$$

We set $r_{n,\Lambda}:=v_n-v^\circ_{n,\Lambda}$. The rescaled function $\tilde r_{n,\Lambda}$ is then the solution of

$$\begin{cases}\Box_{A(h_n.+x_n)}\tilde r^\circ_{n,\Lambda} &= \mathrm{div}_y[(A(h_n.+x_n)-A(0))\nabla_y\tilde v^\circ_\Lambda]\\ \\ (\tilde r^\circ_{n,\Lambda},\partial_s\tilde r^\circ_{n,\Lambda})_{|s=0} &= (\varphi-\varphi_\Lambda,\psi-\psi_\Lambda).\end{cases}$$

The condition $(\mathcal{H})$ and the energy estimate imply that

$$\begin{aligned}\sup_{s\in[-\Lambda,\Lambda]}E_0^{\frac12}(\tilde r^\circ_{n,\Lambda},s) &\leq c\Big\{\parallel\nabla(\varphi-\varphi_\Lambda)\parallel_{L^2(I\!R^3)}+\parallel(\psi-\psi_\Lambda)\parallel_{L^2(I\!R^3)}\Big\}\\ &+ c\Big\{\|\mathrm{div}_y[(A(h_n.+x_n)-A(0))\nabla_y\tilde v^\circ_\Lambda]\|_{L^1([-\Lambda,\Lambda],L^2(I\!R^3))}\Big\}.\end{aligned}$$

By finite propagation speed, the solution $\tilde v^\circ_\Lambda$ is supported in a fixed compact set denoted by $K_\Lambda$. Taking the limit in $n$ and using Lemma 2.1, we obtain

$$\overline{\lim_n}\sup_{s\in[-\Lambda,\Lambda]}E_0^{\frac12}(\tilde r^\circ_{n,\Lambda},s)\leq c\Big\{\parallel\nabla(\varphi-\varphi_\Lambda)\parallel_{L^2(I\!R^3)}+\parallel(\psi-\psi_\Lambda)\parallel_{L^2(I\!R^3)}\Big\},$$

and hence $\overline{\lim_n} \sup_{s\in[-\Lambda,\Lambda]} E_0(\tilde{r}^\circ_{n,\Lambda}, s) \longrightarrow 0$ as $\Lambda \to +\infty$.
On the other hand, Lemma 2.1 and Strichartz inequalities (1.4) applied to the function $r^\circ_{n,\Lambda}$ on the interval $[t_\infty - \Lambda h_n, t_\infty + \Lambda h_n]$ give

$$\overline{\lim_n} \|\tilde{r}^\circ_{n,\Lambda}\|_{L^5([-\Lambda,\Lambda],L^{10}(I\!R^3))} \longrightarrow 0 \text{ as } \Lambda \to +\infty,$$

which concludes the proof of our lemma.

The next result shows that linear concentrating waves have the following non-concentration property under the condition of "smallness" of the time of concentration $t_\infty$. The proof of this result uses microlocal defect measures, introduced, independently by P. Gérard [5] and L. Tartar [19].

**Lemma 2.3.** *Assume that the geometric condition $(\mathcal{H}_\mathcal{G})(I, x_\infty)$ is satisfied on the time interval $I$ and in the point $x_\infty$. Then, for all compact $K \subset I$, the linear concentrating wave $\underline{v}$ satisfies*

$$(2.10)\quad \overline{\lim_n} \sup_{t\in K\backslash[t_\infty-\Lambda h_n, t_\infty+\Lambda h_n]} \|v_n(t,.)\|_{L^6(I\!R^3)} \longrightarrow 0 \text{ as } \Lambda \to +\infty$$

**Proof.** For the sake of simplicity, we suppose that $I = ]T_1, t_\infty[\cup]t_\infty, T_2[$ with $T_i \in I\!R$ and $T_1 < t_\infty < T_2$. Let $K$ be a compact subset of $I$. Arguing by contradiction, we suppose that (2.10) does not occur. Thus, there exist a constant $c > 0$, a real subsequence $(\Lambda_j)_j$ tending to $+\infty$ and a subsequence $(t_{n_j})_j$ of $(t_n)_n$, convergent to $\tau \in K$ such that

$$(2.11)\quad | t_{n_j} - t_\infty | > \Lambda_j h_{n_j} \text{ and } \lim_j \|f_{n_j}(t_{n_j}, .)\|_{L^6(I\!R^3)} \longrightarrow c.$$

We split the proof of Lemma 2.3 in two parts. First, we consider the case when $\tau \neq t_\infty$.
The sequence $g_j := f_{n_j}(t_{n_j}, .)$ is weakly convergent to 0 in $\dot{H}^1$. Up to extracting a subsequence, we can suppose that $| \nabla g_j(x) |^2 := | f_{n_j}(t_{n_j}, x) |^2$ is weakly convergent to a positive Radon measure $\alpha$ on $I\!R^3$; hence

$$\alpha = w^* - \lim | \nabla g_j(x) |^2 .$$

The idea is to construct a positive microlocal Radon measure denoted by $\mu$ on $I\!R^3 \times S^2$ satisfying

$$(2.12)\qquad \forall y \in I\!R^3, \quad \mu(\{y\} \times S^2) = 0,$$

and for a positive constant $c$

$$\alpha \leq c\beta.$$

Here $\beta := \Pi_1\mu$ and $c$ is a positive constant. These properties particularly imply, that for any point $y \in R^3$, $\alpha(\{y\}) \leq c.\beta(\{y\}) = \mu(\{y\} \times S^2) = 0$, which gives, via concentration-compactness Lemma of P. L. Lions [13], $\lim_j \|g_j\|_{L^6(I\!R^3)} = 0$, contradicting (2.11).
To compute $\alpha(\{y\})$, we use the microlocal defect measures associated to the sequence of solution of the free wave equation (1.2). ( We refer to [6] for the details in the constant case and to [**?**] and [11] for the variable coefficients

case).
For all $t \in I\!\!R$, we can associate to the sequence $(A^{\frac{1}{2}}\nabla_x f_{n_j}(t,.), \partial_t f_{n_j}(t,.))$ a non-negative Radon measure denoted by $\mu^t$ on $I\!\!R^3 \times S^2$ and defined as:

*For all classic pseudo-differential operator $B$ of order 0, one has*

$$(B\partial_t f_{n_j}(t,.), \partial_t f_{n_j}(t,.)) + (BA\nabla f_{n_j}(t,.), \nabla f_{n_j}(t,.)) \overset{j\to+\infty}{\longrightarrow} \tag{2.13}$$

$$\int_{I\!\!R^3\times S^2} \sigma_0(B) d\mu^t.$$

*with a locally uniform convergence in $t$. We recall that $\sigma_0(B)$ represents the principle symbol of the operator $B$.*

Moreover, one can decompose the measure $\mu^t$ in the following way

$$\mu^t = \frac{1}{2}(\mu^t_+ + \mu^t_-),$$

where the measure $\mu^t_\pm$ satisfies the following transport equation

$$\left\{ \begin{array}{c} \{\partial_t \pm H_{\sqrt{A(x)\xi.\xi}} \pm d(x,\xi)\}\mu^t_\pm = 0 \\ \mu^t_{\pm|t=t_\infty} = \mu^{t_\infty}_\pm, \end{array} \right.$$

with,

$$d(x,\xi) = \sum_{j=1}^{3} \frac{\partial^2}{\partial_{\xi_j}\partial_{x_j}} \sqrt{A(x)\xi.\xi} - \sum_{1\le j,k\le 3} \frac{\frac{\partial}{\partial_{x_k}} a_{kj}(x)\xi_j}{2\sqrt{A(x)\xi.\xi}}.$$

$H_{\sqrt{A(x)\xi.\xi}}$ denotes the hamiltonian flow on $I\!\!R^3 \times S^2$ and the measure $\mu^{t_\infty}_\pm$ is defined as:

*For all classical pseudo-differential operator $B$ of order 0, one has*

$$(B(\psi_{n_j} \pm i\sqrt{AD.D}\varphi_{n_j}), \psi_{n_j} \pm i\sqrt{AD.D}\varphi_{n_j}) \overset{j\to+\infty}{\longrightarrow} \int_{I\!\!R^3\times S^2} \sigma_0(B) d\mu^{t_\infty}_\pm.$$

An easy computation shows that if we set

$$g_\pm(\xi) = (2\pi)^{-3} \int_\infty^{+\infty} |\overset{\wedge}{\psi}(r\xi) \pm i \mid A^{\frac{1}{2}}(x_\infty)\xi \mid \overset{\wedge}{\varphi}(r\xi) \mid^2 \; r^2 dr,$$

then the measure $\mu^{t_\infty}_\pm$ is given by :

$$\mu^{t_\infty}_\pm = g_\pm(\xi)\delta_{x-x_\infty} \otimes d\sigma(\xi). \tag{2.14}$$

Taking $B(x, D_x) = b(x) \in \mathcal{C}^\infty_0(I\!\!R^3)$ in (2.13), then it easily follows that for all real $t \in I\!\!R$, we have

$$\Pi_1\mu^t := \beta^{(t)} = w^* - \lim\{\mid \partial_t f_n(t,.) \mid^2 + \mid A^{\frac{1}{2}}\nabla f_n(t,.) \mid^2\}.$$

On the other hand, for all positive function $\lambda \in \mathcal{C}_0^\infty(I\!R^3)$, we have

$$\int_{I\!R^3} | \nabla f_{n_j}(t_{n_j}, .) |^2 \lambda(x)dx \leq c \int_{I\!R^3} [| \partial_t f_{n_j}(t_{n_j}, .) |^2 + | A^{\frac{1}{2}} \nabla f_{n_j}(t_{n_j}, .) |^2] \lambda(x)dx.$$

Since $\lim_j t_{n_j} = \tau$, then the uniform convergence in (2.13) and the geometric condition $(\mathcal{H}_{\mathcal{G}})(I, x_\infty)$ show that $\alpha \leq c\beta^{(\tau)}$. It remains now to verify (2.12). Denoting by $\Phi_t$ the hamiltonian flow $H_{\sqrt{A(x)\xi.\xi}}$, then we have

$$\begin{aligned} \mu_\pm^\tau(x,\xi) &= \exp(-\int_{t_\infty}^{\tau} d(\Phi_{s-\tau}(x,\xi))ds).\mu_\pm^{t_\infty}(\Phi_{t_\infty-\tau}(x,\xi)) \\ &= f^\tau(x,\xi).\Phi_{-(t_\infty-\tau)}\mu_\pm^{t_\infty}, \end{aligned}$$

with $f^\tau(x,\xi) = \exp(-\int_{t_\infty}^{\tau} d(\Phi_{s-\tau}(x,\xi))ds)$. Hence, for all $y \in I\!R^3$ we have

$$\mu_\pm^\tau(\{y\} \times S^2) \leq C(t_\infty, \tau) \int_{\Phi_{t_\infty-\tau}(\{y\}\times S^2)} d\mu_\pm^{t_\infty}(x,\xi),$$

where the constant $C(t_\infty, \tau) = \sup\{f^\tau(x,\xi)/(x,\xi) \in \operatorname{supp} \Phi_{-(t_\infty-\tau)}\ \mu_\pm^{t_\infty}\}$. Using (2.14), and the fact that $\Pi_1 \circ \Phi_{t_\infty-\tau}(y,\xi) = \exp_y((t_\infty - \tau)\xi)$, we get

$$\begin{aligned} \int_{\Phi_{t_\infty-\tau}(\{y\}\times S^2)} d\mu_\pm^{t_\infty}(x,\xi) &= \int_{\mathcal{F}_{x_\infty}(y,t_\infty-\tau)} g(\xi)d\sigma(\xi) \\ &\leq C(t_\infty,\tau). \mid \mathcal{F}_{x_\infty}(y, t_\infty - \tau) \mid . \end{aligned}$$

But the time $t_\infty - \tau$ is in the interval $t_\infty - I$, so the geometric condition shows that for any point $y \in I\!R^3$, we have $\mu_\pm^\tau(\{y\} \times S^2) = 0$ and then $\mu^\tau(\{y\} \times S^2) = 0$. This clearly leads to a contradiction in the case $\tau \neq t_\infty$. When $\tau = t_\infty$, we set $\varepsilon_j =\mid t_\infty - t_{n_j} \mid$, $\tilde{h}_j = \frac{h_{n_j}}{\varepsilon_j}$ and define the sequence $\tilde{f}_j$ as $\tilde{f}_j(s,y) = \sqrt{\varepsilon_j} f_{n_j}(t_\infty - \varepsilon_j s, -\varepsilon_j y)$.
Note that since $\mid t_\infty - t_{n_j} \mid\geq \Lambda_j h_{n_j}$ and $\lim_j \Lambda_j = +\infty$, then $\lim_j \tilde{h}_{n_j} = 0$.
The sequence $(\tilde{f}_j)_j$ is the solution of

$$\begin{cases} \Box_{A(-\varepsilon_j.)} \tilde{f}_j = 0 \\ (\tilde{f}_j, \partial_s \tilde{f}_j)_{|s=0} = \frac{1}{\sqrt{\tilde{h}_j}}(\varphi, \frac{1}{\tilde{h}_j}\psi)(\frac{x}{\tilde{h}_j}). \end{cases}$$

To conclude the proof, it remains to show that $\lim_j \|\tilde{f}_j(1,.)\|_{L^6(I\!R^3)} = 0$. The idea is to proof that the microlocal energy defect measure $\mu^s$ associated to the sequence

$$(A^{\frac{1}{2}} \nabla_y \tilde{f}_j(s,.), \partial_s \tilde{f}_j(s,.)),$$

propagates along the curves of the hamiltonian flow with constant coefficients $H_{|\xi|}$.
Let $q$ be a symbol in the class $S^1_{1,0}$ given by

$$q(x,\xi) = \sqrt{A(x)\xi.\xi} - i\frac{\Sigma\frac{\partial}{\partial x_k}a_{kj}(x)\xi_j}{2\sqrt{A(x)\xi.\xi}} + \frac{\Sigma\frac{\partial}{\partial x_j}\sqrt{A(x)\xi.\xi}\ \frac{\partial}{\partial \xi_j}\sqrt{A(x)\xi.\xi}}{2\sqrt{A(x)\xi.\xi}}.$$

Denote by $Q$ its associate the pseudo-differential operator defined on $I\!R^3_x \times I\!R^3_\xi \backslash \{0\}$. First, notice that we can rewrite $\square_A$ as follows

$$\square_A = (\partial_t - iQ)(\partial_t + iQ) + R_0(x, D_x),$$

where $R_0(x, D_x)$ is a 0 order pseudo-differential operator. Denote by $Q_j$ the pseudo-differential operator with symbol $q_j(y,\xi) := q(\varepsilon_j y, \xi)$, hence we can write

$$\square_{A(-\varepsilon_j .)} - (\partial_s - iQ_j)(\partial_s + iQ_j) = R_0(\varepsilon_j y, D_y), \tag{2.15}$$

Defining $\tilde{f}^s_{j,\pm} := (\partial_s \pm iQ_j)\tilde{f}_j(s,.)$, then (2.15) shows that $\tilde{f}^s_{j,\pm}$ satisfies

$$(\partial_t \mp iQ)\tilde{f}^s_{j,\pm} = R_0(\varepsilon_j y, D_y)\tilde{f}_j(s,.) = o(1) \text{ as } j \to +\infty.$$

Second, by virtue of (2.15) it easily follows that, for all classic pseudo-differential operator $B$ of order 0, we have

$$\frac{d}{ds}(B\tilde{f}^s_{j,\pm}, \tilde{f}^s_{j,\pm}) = \pm i((BQ_j - Q_j^* B)\tilde{f}^s_{j,\pm}, \tilde{f}^s_{j,\pm}) + o(1) \text{ as } j \to +\infty. \tag{2.16}$$

Finally, we can observe that

$$BQ_j - Q_j^* B = B(Q_j - \mid D \mid) - (Q_j^* - \mid D \mid)B + (B \mid D \mid - \mid D \mid B),$$

and notice that for all multi-index $\alpha$, $\beta \in \mathbb{N}^3$, we have

$$\lim_j \sup_{x \in I\!R^3, \xi \in I\!R^3_*} \mid D^\alpha_x D^\beta_\xi (q_j(x,\xi) - \mid \xi \mid) \mid \to 0 \text{ as } \quad j \to +\infty.$$

Therefore, the symbol $(q_j(x,\xi) - \mid \xi \mid)$ tends to 0 in the class of pseudo-differential symbols $S^1_{1,0}$, which proves, by taking the limit in (2.16), that the measure $\mu^s_\pm$ satisfies $(\partial_s \mp H_{|\xi|})\mu^s_\pm = 0$. Hence our lemma follows.

## 3. Study of a Nonlinear Concentrating Wave on $I\!R^d$

Let $\underline{u}$ be the nonlinear concentrating wave associated to the linear concentrating wave $\underline{v}$. Namely, $\underline{u}$ is the solution of

$$\square_A u_n + u_n^5 = 0, \quad (u_n, \partial_t u_n)_{|t=0} = (v_n, \partial_t v_n)_{|t=0}. \tag{3.17}$$

Recall that for the d'alembertian operator is studied a scattering operator $S$ as well as wave operators $\Omega_\pm$ defined as follows (see [1])

**Proposition 3.1.** **(i)** *To every solution of*

$$\Box_{A(x_\infty)} v^\infty = 0, \quad (v^\infty, \partial_s v^\infty)_{|s=0} = (\varphi, \psi),$$

*corresponds a unique function $u_\pm^\infty$, such that*

$$(u_\pm^\infty, \partial_t u_\pm^\infty) \in \mathcal{C}(I\!R, \mathcal{E}), \quad u_\pm^\infty \in L^5(I\!R, L^{10}(I\!R^3))$$

*and satisfying*

$$\Box_{A(x_\infty)} u_\pm^\infty + \mid u_\pm^\infty \mid^4 u_\pm^\infty = 0, \quad \lim_{s \to \pm\infty} E_0(u_\pm^\infty - v^\infty, s) = 0.$$

**(ii)** *The wave operators*

$$\Omega_\pm^\infty : (v^\infty, \partial_t v^\infty)_{|t=0} \mapsto (u_\pm^\infty, \partial_t u_\pm^\infty)_{|t=0}$$

*are bijective from $\mathcal{E}$ onto itself. The scattering operator $S^\infty$ is then defined by*

$$S^\infty := (\Omega_+^\infty)^{-1} \circ \Omega_-^\infty.$$

*When $A(x_\infty) = Id$, then we simply set $S^\infty = S$ and $\Omega_\pm^\infty = \Omega_\pm$.*

Under the conditions on $\underline{t}$, the two linear concentrating waves $\underline{v}_\pm^\infty$ are then respectively associated to $[\varphi, \psi, \underline{h}, \underline{x}, t_\infty]$ and $[S^\infty(\varphi, \psi), \underline{h}, t_\infty]$.
The remainder part of the proof of the Theorem 1.6 is given into the following three parts.

3.1. **Study Before the Time of Concentration.** By taking $f_n = v_n$ in Lemma 2.1 we will derive the following corollary which is the first assertion of Theorem 1.6.

**Corollary 3.2.** *Assume that the geometric condition $(\mathcal{H}_\mathcal{G})(I, x_\infty)$ is satisfied. Let $K$ be a compact set of $I$. Denoting by*
*$K_{n,\Lambda}^- = \{t \in K$ such that: $t \le t_\infty - \Lambda h_n\}$, then we have*

$$\overline{\lim_n} \||\, u_n - v_n \,\||_{K_{n,\Lambda}^-} \longrightarrow 0 \text{ as } \Lambda \to +\infty.$$

To prove this corollary, we follow the method in [6]. The proof is based on Strichartz's inequalities and the following absorption Lemma.

**Lemma 3.3** ([*bootstrap* Lemma.])**.** *Let $X : [0, T] \to I\!R^+$ be a continuous map such that for all $t \in [0, T]$ we have*

$$X(t) \le a + bX(t)^\gamma.$$

*The constants $a$ and $b$ are positive, the real $\gamma > 1$ satisfying*

$$a < (1 - \frac{1}{\gamma}) \frac{1}{(\gamma b)^{\frac{1}{\gamma - 1}}} \text{ and } X(0) \le \frac{1}{(\gamma b)^{\frac{1}{\gamma - 1}}}.$$

*Then for all real $t \in [0, T]$, we have*

$$X(t) \le \frac{\gamma}{\gamma - 1} a.$$

**Proof.** For the sake of simplicity, we take $K = [0, t_\infty]$ and set $w_n := u_n - v_n$. By applying inequalities (1.4) to the sequence $w_n$, we obtain

$$\|w_n\|_{L^5([0,t_\infty-\Lambda h_n],L^{10}(I\!R^3))} \leq c\Big\{\|v_n\|_{L^5([0,t_\infty-\Lambda h_n],L^{10})}+\|w_n\|_{L^5([0,t_\infty-\Lambda h_n],L^{10})}\Big\}^5$$

By interpolating Strichartz inequalities and using Lemma 2.3 we obtain

$$\overline{\lim_n}\|v_n\|_{L^5([0,t_\infty-\Lambda h_n],L^{10})} \longrightarrow 0 \text{ as } \Lambda \to +\infty.$$

Moreover, there exists a real number $\Lambda_0$ such that for all $\Lambda \geq \Lambda_0$ and for any integer $n \geq n_0(\Lambda)$, the quantity $c\|v_n\|_{L^5([0,t_\infty-\Lambda h_n],L^{10})}$ is sufficiently small. Denoting by

$$X_n(t) := \|w_n\|_{L^5([0,t],L^{10})} \quad \text{and} \quad \delta_n(t) := \|v_n\|_{L^5([0,t],L^{10})},$$

with $t \in [0, t_\infty - \Lambda h_n]$, then we have $X_n(t) \leq c(\delta_n(t) + X_n(t))^5$. Lemma 3.3 enables us to deduce that for any $\Lambda \geq \Lambda_0$ and $n \geq n_0(\Lambda)$, we have

$$X_n(t_\infty - \Lambda h_n) \leq \frac{5}{4} c\delta_n(t_\infty - \Lambda h_n).$$

Then our result follows.

3.2. **Study for Times Close to the Concentration Time.** In what follows, we are studying the nonlinear concentrating wave $\underline{u}$ when the time is “very close ”to $t_\infty$. For that, we begin by the following auxiliary result related to the constant case.
Let $v^\circ$ and $u^\circ_-$ be the solution, respectively of

$$\Box v^\circ = 0, \quad (v^\circ, \partial_s v^\circ)_{|s=0} = (\varphi, \psi),$$

$$\Box u^\circ_- + (u^\circ_-)^5 = 0, \quad \lim_{s\to-\infty} E_0(u^\circ_- - v^\circ, s) = 0.$$

For all real $\Lambda > 0$, let $w_\Lambda$ be the smooth solution of

$$\begin{cases} \Box w_\Lambda + (w_\Lambda)^5 & = & 0 \\ (w_\Lambda, \partial_s w_\Lambda)_{|s=-\Lambda} & = & (v^\circ, \partial_s v^\circ)_{|s=-\Lambda}, \end{cases}$$

belonging to $L^5(I\!R, L^{10}(I\!R^3))$. Then we have the following lemma.

**Lemma 3.4.** *We have*

$$\lim_\Lambda ||| \, u^\circ_- - w_\Lambda \, |||_{I\!R} = 0.$$

**Proof.** We use the technique of deformation of the time described by H. Bahouri and P. Gerard in [1].
Let us denote by $r_\Lambda := w_\Lambda - u^\circ_-$. Then we have $\Box r_\Lambda = (u^\circ_-)^5 - (r_\Lambda + u^\circ_-)^5$.

By applying the inequalities (1.4) on the interval $[-\Lambda,\eta]$, where $\eta \geq -\Lambda$ is arbitrary, we obtain

$$\begin{aligned}\|r_\Lambda\|_{L^5([-\Lambda,\eta],L^{10})} &\leq cE_0^{\frac{1}{2}}(u_-^\circ - v^\circ, s=-\Lambda)\\ &+ c\{\|r_\Lambda\|_{L^5([-\Lambda,\eta],L^{10})}\|u_-^\circ\|^4_{L^5([-\Lambda,\eta],L^{10})}\}\\ &+ c\Big\{\|r_\Lambda\|^2_{L^5([-\Lambda,\eta],L^{10})} + \|r_\Lambda\|^5_{L^5([-\Lambda,\eta],L^{10})}\Big\}.\end{aligned}$$

Using H. Bahouri and J. Shatah's result (see [2]) described in the introduction, we can choose a real number $S$ such that the quantity $c\|u_-^\circ\|_{L^5([-\infty,\eta],L^{10})}$ is sufficiently small. On the other hand, Proposition 3.1 shows that

$$E_0^{\frac{1}{2}}(u_-^\circ - v^\circ, s=-\Lambda) = o(1) \text{ as } \Lambda \to +\infty.$$

which clearly leads to

$$\|r_\Lambda\|_{L^5([-\Lambda,\eta],L^{10})} \leq c\Big\{o(1) + \|r_\Lambda\|^2_{L^5([-\Lambda,\eta],L^{10})} + \|r_\Lambda\|^5_{L^5([-\Lambda,\eta],L^{10})}\Big\}.$$

Using the absorption Lemma 3.3 we obtain $\lim\limits_{\Lambda}\|r_\Lambda\|_{L^5([-\Lambda,\eta],L^{10})} = 0$. The energy estimate applied to $r_\Lambda$ shows that

$$\sup_{s\in[-\Lambda,\eta]} E_0(r_\Lambda,s) \leq c\Big\{E_0(r_\Lambda, s=-\Lambda) + \|(u_-^\circ)^5 - (r_\Lambda + u_-^\circ)^5\|^2_{L^1([-\Lambda,\eta],L^2)}\Big\}$$

and then $\lim\limits_{\Lambda} \sup\limits_{s\in[-\Lambda,\eta]} E_0(r_\Lambda,s) = 0$. Setting

$$\eta_{max} := \sup\big\{\eta \in I\!R, / \quad \lim_{\Lambda}\|r_\Lambda\|_{L^5([-\Lambda,\eta],L^{10})} = 0\big\},$$

and assuming by contradiction that $\eta_{max} < +\infty$.
Let us choose $\eta_1 < \eta_{max}$ such that $c\|u_-^\circ\|_{L^5([\eta_1,\eta_{max}],L^{10})}$ be sufficiently small. By applying the previous arguments, we can first prove that

$$\lim_{\Lambda}|||\ r_\Lambda\ |||_{[-\Lambda,\eta_{max}]} = 0.$$

Then, for $\delta > 0$ small enough, we have $\lim\limits_{\Lambda}|||\ r_\Lambda\ |||_{[\eta_{max},\eta_{max}+\delta]} = 0$, which clearly proves that $\eta_{max} = +\infty$. To achieve the proof on the interval $[-\Lambda,+\infty[$, it suffices to apply, once again, the Strichartz inequalities on the intervals of the type $[\eta,+\infty[$ and conclude the proof as previously done. The proof is completed by time inversion.

**Lemma 3.5.** *Let $\tilde u_n$ be the "rescaled function" associated to $u_n$ that is, $u_n(t,x) = \frac{1}{\sqrt{h_n}}\tilde u_n(\frac{t-t_\infty}{h_n}, \frac{x-x_n}{h_n})$. Then we have*

$$\overline{\lim_{n}}|||\ \tilde u_n - w_\Lambda\ |||_{[-\Lambda,\Lambda]} \to 0 \text{ as } \Lambda \to +\infty. \tag{3.18}$$

**Proof.** We introduce the auxiliary family of function $\tilde u_n^\Lambda$ defined by

$$\begin{cases} \Box_{A(h_n.+x_n)}\tilde u_n^\Lambda + (\tilde u_n^\Lambda)^5 = 0 \\ \\ (\tilde u_n^\Lambda, \partial_s \tilde u_n^\Lambda)_{|s=-\Lambda} = (\tilde v_n, \partial_s \tilde v_n)_{|s=-\Lambda}.\end{cases}$$

To prove Lemma 3.5, it suffices to show that

$$\overline{\lim_n} \||\, w_\Lambda - \tilde{u}_n^\Lambda \,\||_{[-\Lambda,\Lambda]} \to 0 \text{ as } \Lambda \to +\infty, \tag{3.19}$$

and

$$\overline{\lim_n} \||\, \tilde{u}_n - \tilde{u}_n^\Lambda \,\||_{[-\Lambda,\Lambda]} \to 0 \text{ as } \Lambda \to +\infty. \tag{3.20}$$

The proofs of (3.19) and (3.20) are similar to those of Lemma 2.2 and Lemma 3.4. In fact, we set $r_{n,\Lambda} = \tilde{u}_n^\Lambda - w_\Lambda$. Then we get

$$\begin{cases} \Box_{A(h_n.)} r_{n,\Lambda} = (r_{n,\Lambda} + w_\Lambda)^5 + \operatorname{div}_y[(A(h_n. + x_n) - A(0))\nabla_y w_\Lambda] \\ \\ (r_{n,\Lambda}, \partial_s r_{n,\Lambda})_{|s=-\Lambda} = (\tilde{v}_n - \tilde{v}_\Lambda^\circ, \partial_s(\tilde{v}_n - \tilde{v}_\Lambda^\circ)). \end{cases}$$

Lemma 2.2 applied to $w_\Lambda$ shows that the following error term

$$\|\operatorname{div}_y[(A(h_n. + x_n) - A(0))\nabla_y w_\Lambda]\|_{L^1([-\Lambda,\Lambda],L^2(I\!R^3))}$$

tends to 0 with respect to $n$. Meanwhile, for the term $\|r_{n,\Lambda}+w_\Lambda\|^5_{L^5([-\Lambda,\Lambda],L^{10}(I\!R^3))}$, we use the technique of deformation of the time as described above in the proof of Lemma 3.4.
To prove (3.20), we use the result of (3.19) which enables us to replace the sequence $\tilde{u}_n^\Lambda$ by $w_\Lambda$ and then to "absorb" the linear term in the Strichartz inequalities. Finally, we apply the technique of deformation of time to conclude the proof.

3.3. **End of the proof of the Theorem 1.8: Study for times after the concentration time.** Let us denote by $\tilde{v}_{n,+}^\infty$ and $\tilde{v}_\infty^\circ$ the functions respectively defined by

$$\Box_{A(h_n.+x_n)} \tilde{v}_{n,+}^\infty = 0; \quad (\tilde{v}_{n,+}^\infty, \partial_s \tilde{v}_{n,+}^\infty)_{|s=0} = S^\infty(\varphi, \psi),$$

and

$$\Box \tilde{v}_\infty^\circ = 0; \quad (\tilde{v}_\infty^\circ, \partial_s \tilde{v}_\infty^\circ)_{|s=-0} = S^\infty(\varphi, \psi).$$

First, we begin by observing that we can estimate the energy
$E_0(\tilde{u}_n - \tilde{v}_n^\infty, s = \Lambda)$ by

$$E_0(\tilde{u}_n - \tilde{v}_n^\infty, s = \Lambda) \le E_0(\tilde{u}_n - w^\Lambda, \Lambda) + E_0(w^\Lambda - \tilde{v}_\infty^\circ, \Lambda) + E_0(\tilde{v}_\infty^\circ - \tilde{v}_{n,+}^\infty, \Lambda).$$

But according to (3.18), we have $\overline{\lim_n} \||\, \tilde{u}_n - w_\Lambda \,\||_{[-\Lambda,\Lambda]} \to 0$ as $\Lambda \to +\infty$. On the other hand, the analogue of Lemma 2.2 shows that

$$\overline{\lim_n} \sup_{s\in[-\Lambda,\Lambda]} E_0(\tilde{v}_{n,+}^\infty - v_\infty^\circ, s) \to 0 \text{ as } \Lambda \to +\infty.$$

Now, we use the following known result from the scattering theory of the constant case (see Proposition 3.1),

$$\lim_\Lambda \||\, w_\Lambda - v_\infty^\circ \,\||_{[\Lambda,+\infty[} = 0. \tag{3.21}$$

This gives

$$\overline{\lim_n} E_0(\tilde{u}_n - \tilde{v}_n^\infty, s = \Lambda) \to 0 \text{ as } \Lambda \to +\infty$$

To conclude the proof, it suffices to note that the geometric condition $(\mathcal{H}_{\mathcal{G}})(I, x_\infty)$ is made to avoid the lack of compactness of the sequence $(v^\infty_{n,+})_n$ on intervals of types $[t_\infty + \Lambda h_n, T]$ or equivalently

$$\overline{\lim_n} \sup_{t\in[t_\infty+\Lambda h_n,T]} \|v^\infty_{n,+}(t,.)\|_{L^6(I\!R^3)} = \overline{\lim_n} \sup_{s\in[\Lambda,\frac{T-t_\infty}{h_n}]} \|\tilde{v}^\infty_{n,+}(s,.)\|_{L^6(I\!R^3)} \overset{\Lambda\to+\infty}{\longrightarrow} 0.$$

By a similar analysis to that used to prove Corollary 3.2, the proof of Theorem 1.6 is completed.

## Example

Let $A$ be the metric defined by $A(x) = \mathrm{diag}(a_i(x_i))$, where $x = (x_1, x_2, x_3)$ and the coefficients $a_i$ satisfy

$$a_i : \; I\!R \longrightarrow I\!R^*_+, \; \mathcal{C}^\infty \text{ and such that } a_i \equiv 1 \text{ when } \mid x \mid \geq R_0.$$

Let $(x^0, \xi^0) \in I\!R^3 \times I\!R^3\backslash\{0\}$. For $i = 1, 2, 3$, we set $F_i$ the function defined by

$$F_i : I\!R \longrightarrow I\!R, \quad x_i \longmapsto \int_{x_i^0}^{x_i} a_i^{-\frac{1}{2}}(s)ds.$$

Under the above assumptions on the coefficients $a_i$, the functions $F_i$ are bijective from $I\!R$ onto $I\!R$. Denote by $G_i := F_i^{-1}$ and set,

$$x_i(t) := x_i(t, x^0, \xi^0) = G_i(2a_i^{\frac{1}{2}}(x_i^0)\xi_i^0 t),$$

$$\xi_i(t) := \xi_i(t, x^0, \xi^0) = a_i^{-\frac{1}{2}}(x_i(t))a_i^{\frac{1}{2}}(x^0)\xi_i^0.$$

Therefore, $(x(t), \xi(t))$ is a solution of the hamiltonian system associated to the function

$$a_1(x_1)\xi_1^2 + a_2(x_2)\xi_2^2 + a_3(x_3)\xi_3^2,$$

corresponding to the principal symbol of the operator $(-\,\mathrm{div}\, A(x)\nabla.)$. The initial conditions are $(x(0), \xi(0)) = (x^0, \xi^0)$. According to the assumptions on the metric $A$, we can easily observe that for all real $t \in I\!R$ and $y \in I\!R^3$, the map

$$\Pi_1 \circ \Phi_t : I\!R^3\backslash\{0\} \longrightarrow I\!R^3, \; \xi \longmapsto (G_i(2a_i^{\frac{1}{2}}(x_i^0)\xi_i^0 t))_{1\leq i\leq 3}$$

has a maximal rank, which proves that the following set $\{\xi;\ \exp_y(t\xi) = \Pi_1\circ\Phi_t(y, \xi) = x^0\}$ is discrete and therefore $(\mathcal{H}_{\mathcal{G}})(I\!R^*, x^0)$ is globally satisfied.

## 4. Nonlinear Concentrating Wave on the Sphere

Now we study the critical wave equation on the sphere. We start by recalling a few facts.

4.1. **Notations and Preliminary Results.** For $k \in \mathbb{N}$, we denote by $\mathcal{H}_k$ the space of $k$-spherical harmonics of dimension $d+1$. Recall that we have

$$L^2(S^d) = \oplus_{k\in\mathbb{N}} \quad \mathcal{H}_k.$$

Therefore, any function $g \in L^2(S^d)$ uniquely determines a component $g_k \in \mathcal{H}_k$ given by

$$g_k(x) = N(k,d)\int_{S^d} P_k(\tau.x)g(\tau)d\tau, \quad k = 0,1,..., \tag{4.22}$$

such that $g = \sum_{k\in\mathbb{N}} g_k$ in the topology of $L^2$. Here, $P_k$ is denoting the Legendre Polynomial of degree $k$ and of dimension $d+1$ and $N(k,d)$ is the dimension of the space $\mathcal{H}_k$. For more details, see [14]. Taking $(f,g)$ in the energy space $\mathcal{E}$ such that $f = \sum_{k\in\mathbb{N}} f_k$ and $g = \sum_{k\in\mathbb{N}} g_k$ then

$$\|(f,g)\|_{\mathcal{E}}^2 = \sum_{k\in\mathbb{N}} \big(a_k^2\|f_k\|_{L^2}^2 + \|g_k\|_{L^2}^2\big), \tag{4.23}$$

where $a_k = \sqrt{k(k+d-1)}$. Notice that (4.23) is in fact a semi-norm because of constants. In what follows, the notation $o(1)(t)$ means

$$\overline{\lim_n} \sup_{t\in R} \|o(1)(t,.)\|_{\mathcal{E}} = 0.$$

The following lemma gives a quasi-periodicity property of linear waves on the sphere.

**Lemma 4.1.** *Let $\underline{p}$ be the solution of*

$$(\partial_t^2 - \Delta_{S^d})p_n = 0, \quad (p_n, \partial_t p_n)_{|t=0} = (\varphi_n, \psi_n). \tag{4.24}$$

*We suppose that $(\varphi_n, \psi_n)$ converges weakly to (0,0) in the energy space, then we have*

$$p_n(t+\pi, x) = \begin{cases} (-1)^{\frac{d-1}{2}} p_n(t,-x) + o(1)(t), & \text{if } \frac{d-1}{2} \in \mathbb{N}^*, \\ \\ (-1)^{\frac{d}{2}} \tilde{p}_n(t,-x) + o(1)(t), & \text{if } \frac{d}{2} \in \mathbb{N}^*, \end{cases}$$

*where, $\tilde{p}_n$ is the solution of*

$$(\partial_t^2 - \Delta_{S^d})\tilde{p}_n = 0, \quad (\tilde{p}_n, \partial_t\tilde{p}_n)_{|t=0} = \big[-(-\Delta_{S^d})^{-\frac{1}{2}}(\psi_n - \int_{S^d}\psi_n), (-\Delta_{S^d})^{\frac{1}{2}}\varphi_n\big].$$

**Proof.** We note that the solution $p_n$ is explicitly given by expansion into the eigenfunctions of $\Delta_{S^d}$;

$$p_n(t,x) = \sum_{k\in\mathbb{N}} \cos(a_k t)\varphi_{k,n}(x) + \sum_{k\in\mathbb{N}^*} \frac{sin(a_k t)}{a_k}\psi_{k,n}(x) + t\psi_{0,n},$$

where $\varphi_{k,n}$ and $\psi_{k,n}$ are respectively the $k$ spherical-harmonics component of $\varphi_n$ and $\psi_n$. In particular, according to (4.23), we have $\varphi_{k,n} = N(k,n)\int_{S^d} P_k(\tau.x)\varphi_n(\tau)d\tau$ and $\psi_{k,n} = N(k,n)\int_{S^d} P_k(\tau.x)\psi_n(\tau)d\tau$.

(4.25)

First, we consider the case $\frac{d-1}{2} \in \mathbb{N}^*$.
Since spherical harmonics are restrictions to the sphere of homogenous polynomials, we have

$$\begin{aligned}(-1)^{\frac{d-1}{2}} p_n(t,-x) &= \sum_{k\in\mathbb{N}} (-1)^{(k+\frac{d-1}{2})} \cos(a_k t)\varphi_{k,n}(x) \\ &+ \sum_{k\in\mathbb{N}^*} (-1)^{(k+\frac{d-1}{2})} \frac{\sin(a_k t)}{a_k} \psi_{k,n}(x) \\ &+ (-1)^{\frac{d-1}{2}} t\psi_{0,n}.\end{aligned}$$

Therefore we can write

$$p_n(t+\pi,x) - (-1)^{\frac{d-1}{2}} p_n(t,-x) = (i) \quad + \quad (ii)$$

where

$$\begin{aligned}(i) &= \sum_{k\in\mathbb{N}} \big[\cos(a_k t + a_k\pi) - \cos(a_k t + (k+\frac{d-1}{2})\pi)\big]\varphi_{k,n}(x), \\ (ii) &= \sum_{k\in\mathbb{N}^*} \frac{\sin(a_k t + a_k\pi) - \sin(a_k t + (k+\frac{d-1}{2})\pi)}{a_k} \psi_{k,n}(x) \\ &+ (t+\pi-(-1)^{\frac{d-1}{2}} t)\psi_{0,n}.\end{aligned}$$

By estimating $(i)$ and $(ii)$ we get

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|p_n(t+\pi,.) - p_n(t,.)\|^2_{\mathcal{E}} &\leq c\sum_{k\in\mathbb{N}^*} a_k^2 \mid \varphi_{k,n} \mid^2_{L^2(S^d)} (a_k - (k+\frac{d-1}{2}))^2 \\ &+ c\sum_{k\in\mathbb{N}^*} \mid \psi_{k,n} \mid^2_{L^2(S^d)} (a_k - (k+\frac{d-1}{2}))^2 + 2\psi^2_{0,n} \\ &\leq c\sum_{k\in\mathbb{N}} a_k^2 \mid \varphi_{k,n} \mid^2_{L^2(S^d)} \frac{1}{(k+1)^2} \\ &+ c\sum_{k\in\mathbb{N}} \mid \psi_{k,n} \mid^2_{L^2(S^d)} \frac{1}{(k+1)^2}\end{aligned}$$

Using the energy estimate for solutions of (4.24), it is clear that the sequences $(\mid \psi_{k,n} \mid_{L^2(S^d)})_k$ and $(a_k \mid \varphi_{k,n} \mid_{L^2(S^d)})_k$ are uniformly bounded in $n$. Moreover, notice that $(\varphi_n, \psi_n) \rightharpoonup 0$ in the energy space, then by virtue of (4.22), we deduce

$$\lim_n \mid \psi_{k,n} \mid_{L^2(S^d)} = \lim_n \mid \varphi_{k,n} \mid_{L^2(S^d)} = 0.$$

Thus our lemma follows.

From the previous lemma, we will deduce the following result

**Lemma 4.2.** *Let $\underline{p}$ be a linear concentrating wave associated to the concentrating data $[(\varphi,\psi),\underline{h},\underline{x},\underline{t}]$. For all $j\in\mathbb{N}$, we denote by $\underline{p}^{(j)}$, the sequence* *defined by* $p_n^{(j)}(t,x) = p_n(t+j\pi, x)$. *We assume that* $d$ *is odd, then we have*

$$\underline{p}^{(j)} = [\mathcal{A}_d^j(\varphi,\psi), \underline{h}, (-1)^j \underline{x}, \underline{t}] + \ o(1)(t) \tag{4.26}$$

*where the operator* $\mathcal{A}_d$ *is an involution isometry of the energy space defined by*

$$\mathcal{A}_d(\varphi,\psi)(x) = (-1)^{\frac{d-1}{2}}(\varphi,\psi)(-x), \ \textit{and } \mathcal{A}_d^j = \mathcal{A}_d \circ \mathcal{A}_d \circ ... \circ \mathcal{A}_d.$$

**Proof.** Since both $\underline{p}^{(j)}$ and $[\mathcal{A}_d^j(\varphi,\psi), \underline{h}, (-1)^j \underline{x}, \underline{t}]$ satisfy (4.24) then it is sufficient to compute the energy of the difference $\underline{p}^{(j)} - [\mathcal{A}_d^j(\varphi,\psi), \underline{h}, (-1)^j \underline{x}, \underline{t}]$ at time $t_n$. For simplicity, we consider the case $j = 1$.
For all $x$ in the geodesic ball $B'(-x_n, r_0)$, we have

$$[\mathcal{A}_d^1(\varphi,\psi), \underline{h}, (-1)^j \underline{x}, \underline{t}](t_n, x) =$$

$$h_n{}^{1-\frac{d}{2}}\theta(\exp_{-x_n}^{-1}(x))(-1)^{\frac{d-1}{2}}(\varphi, \tfrac{1}{h_n}\psi)(-\tfrac{\exp_{-x_n}^{-1}(x)}{h_n}) + o(1).$$

On the other hand, Lemma 4.1 shows that,

$$p_n^{(1)}(t_n, x) = (-1)^{\frac{d-1}{2}} p_n(t_n, -x) + o(1).$$

when $(-x)$ is in the geodesic ball $B'(x_n, r_0)$, we get

$$p_n^{(1)}(t_n, x) = h_n{}^{1-\frac{d}{2}}\theta(\exp_{x_n}^{-1}(-x))(-1)^{\frac{d-1}{2}}(\varphi, \frac{1}{h_n}\psi)(\frac{\exp_{x_n}^{-1}(-x)}{h_n}) + o(1)$$

Using the property $\exp_{-x_1}^{-1} x_2 = -\exp_{x_1}^{-1}(-x_2)$ on the sphere and the fact that $\theta$ is an even function we derive our result.

The following lemma is analogous to Lemma 2.3. We will need it in the proof of Theorem 1.8.

**Lemma 4.3.** *Let* $\underline{p} = [(\varphi,\psi), \underline{h}, \underline{x}, \underline{t}]$, *be a linear concentrating wave on the sphere. Then* $\underline{p}$ *satisfies the non-concentration property*

$$\overline{\lim_n} \sup_{\Lambda h_n \le |t-t_n| \le \pi - \Lambda h_n} \|p_n(t,.)\|_{L^{p_c+1}(S^d)} \longrightarrow 0; \Lambda \to +\infty.$$

**Proof.** Suppose that $(\varphi,\psi) \in \mathcal{C}_0^\infty(I\!R^d)$ such that
$\mathrm{supp}(\varphi) \bigcup \mathrm{supp}(\psi) \subset B(0,R)$. By finite propagation speed, it follows that

$$\bigcup_{\Lambda h_n \le |t-t_n| \le \pi - \Lambda h_n} \{\mathrm{supp}(p_n(t,.)) \cup \mathrm{supp}(\partial_t p_n(t,.))\} \subset B'(x_n, \pi + (R-\Lambda)h_n),$$

which shows that, for $\Lambda > R$,

$$\bigcup_{\Lambda h_n \le |t-t_n| \le \pi - \Lambda h_n} \{\mathrm{supp}(p_n(t,.)) \cup \mathrm{supp}(\partial_t p_n(t,.))\}$$

is strictly included in $B'(x_n, \pi)$. We introduce the rescaled function $\tilde{p}_n$ defined by

$$\tilde{p}_n(s,y) = h_n^{\frac{d}{2}-1} v_n(h_n s, \exp_{x_n}(h_n y)). \tag{4.27}$$

Hence,

$$\sup_{\Lambda h_n \leq |t-t_n| \leq \pi - \Lambda h_n} \|p_n(t,.)\|_{L^{p_c+1}(S^d)} \leq c \sup_{\Lambda \leq |s| \leq \frac{\pi}{h_n} - \Lambda} \|\tilde{p}_n(s,.)\|_{L^{p_c+1}(B(0,\frac{\pi}{h_n}))}.$$

On the other hand, equation (4.24) for $\tilde{p}_n$ yields

$$\partial_s^2 \tilde{p}_n - \Delta_n \tilde{p}_n = 0 \text{ in } \quad I\!R_s \times B(0, \frac{\pi}{h_n}),$$

with $\Delta_n$ as the rescaled Laplace-Beltrami operator, in particular satisfying

$$| \Delta_n \tilde{p}_n - \Delta \tilde{p}_n | \leq c h_n | \nabla \tilde{p}_n | + c h_n^2 | \nabla^2 \tilde{p}_n | . \tag{4.28}$$

Now let $v^\circ$ be the solution of the free wave equation on $I\!R^d$ with Cauchy data $(\varphi, \psi)$ at $s = 0$. Extending $\tilde{p}_n$ by 0 outside the ball $B(0, \frac{\pi}{h_n})$, and applying the usual energy estimate to $\tilde{p}_n - v^\circ$, we get

$$E_0^{\frac{1}{2}}(\tilde{p}_n - v^\circ, s) \leq c\big[\Big(\int_{|h_n y| \geq \frac{\pi}{2}} (| \nabla \varphi |^2 + | \nabla \psi |^2) dy\Big)^{\frac{1}{2}} + \int_0^s | (\Delta_n - \Delta) \tilde{p}_n(\tau,.) |_{L^2} \, d\tau\big]$$

The Sobolev imbedding $\dot{H}^1 \hookrightarrow L^{p_c+1}$ and the triangle inequality complete the proof.

Now, we are able to prove Theorem 1.8.

4.2. **Proof of Theorem 1.8.** Let $\underline{v} = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$, be a linear concentrating wave on the sphere. For simplicity, we suppose $d = 3$ and for all $n \in N$, $x_n = x_\infty$ and $t_n = t_\infty \in [0, \pi[$. So we have

$$\begin{cases} (v_n, \partial_t v_n)(t_\infty, .) &= h_n^{-\frac{1}{2}} \theta(\exp_{x_\infty}^{-1} .)(\varphi, \frac{1}{h_n}\psi)(\frac{\exp_{x_\infty}^{-1} .}{h_n}) \\ & + \; o(1), \quad \text{in } \mathcal{E}(B(x_\infty, r_0)). \\ (v_n, \partial_t v_n)(t_\infty, .) &= o(1) \quad \text{in } \mathcal{E}(\overline{B}^c(x_\infty, r_0)) \end{cases}$$

Since the proof goes along the same lines as the proof of Theorem 1.6, we omit the details here. The non-concentrating property of linear concentrating waves given by Lemma 4.3, combined with estimate (4.28), obviously imply the first statement of Theorem 1.8.

Beyond the first focus and before the solution focuses again that is $(t_\infty + \pi < t < t_\infty + 2\pi)$, we first recall that the sequence $v_n^{(1)} = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]^{(1)}$ defined by $v_n^{(1)} = v_n(t + \pi, x)$ satisfy, according to Lemma 4.2,

$$\underline{v}^{(1)} = [\mathcal{A}_3 S(\varphi, \psi), \underline{h}, -\underline{x}, \underline{t}]. \tag{4.29}$$

On the other hand, the first statement of Theorem 1.8 shows that

$$\overline{\lim_n} E_0(u_n - v_n^{(1)})(t_\infty - \Lambda h_n) \to 0; \; \Lambda \to +\infty. \tag{4.30}$$

Moreover, the linear concentrating wave $\underline{v}^{(1)}$ satisfies the non-concentration property given by Lemma 4.3. This concludes the proof.

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