Global well-posedness for semi-linear Wave and Schrödinger equations

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1 Introduction

Nonlinear Wave equation:

$$\begin{cases} (\partial_t^2 - \Delta_x)u = -|u|^{p-1}u, & u : (-T^*, T^*) \times \mathbb{R}^d \longmapsto \mathbb{R} \\ \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1 & (u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d_x) \end{cases} \tag{1}$$

Such equations arise in quantum mechanics.

Nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t v + \Delta v = -|v|^{p-1}v, & v : (-T_*, T^*) \times \mathbb{R}^d \longmapsto \mathbb{C} \\ \\ v(0,x) = v_0(x) & v_0 \in \dot{H}^s(\mathbb{R}^d_x) \end{cases} \tag{2}$$

The Schrödinger equation describes the propagation of an electromagnetic signal through a standard isotropic optical fibre.

We will refer to the initial value problems (1) and (2) with the notation $NLW_p(\mathbb{R}^d)$ and $NLS_p(\mathbb{R}^d)$, respectively.

Interested in the following questions:

- local (in time) well-posedness of the Cauchy problems (1) and (2).
- are the local solutions global ?
- persistence of regularity i.e. does singularity develop?
- Long-time behavior or scattering i.e. does the (global) non-linear solution approach a linear solution when time $t \longrightarrow \pm\infty$?

Facts about equations (1) **and** (2).

- These equations are **Hamiltonian**

$$E(u(t,\cdot)) := \|\partial_t u(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2 + \int_{\mathbb{R}^d} \frac{1}{p+1}|u|^{p+1}(t,x)dx.$$

$$H(v(t,\cdot)) := \|\nabla v(t,\cdot)\|_{L^2}^2 + \int_{\mathbb{R}^d} \frac{1}{p+1}|v|^{p+1}(t,x)dx.$$

- Equations (1) and (2) have a scaling property i.e.
  If u (respectively v) solves (1) (respectively (2)) then, for $\lambda > 0, u^\lambda : (-T_*\lambda^2, T^*\lambda^2) \times \mathbb{R}^d$ defined by

$$u^\lambda(t,x) := \lambda^{2/(1-p)} u(\lambda^{-2}t, \lambda^{-1}x) \tag{3}$$

  also solves (1) (respectively (2)).

- Let $s_c := \frac{d}{2} - \frac{2}{p-1}$. The Banach spaces $\dot{H}^{s_c}(\mathbb{R}^d_x)$ and $L^{p_c}(\mathbb{R}^d_x)$ are relevant in the theory of the initial value problems (1) and (2), since they are invariant under the mapping (3)

Definition 1 *The Cauchy problems* (1) *and* (2) *are said subcritical if $s_c < s$, critical if $s_c = s$, and supercitical if $s_c > s$.*

2 Local well-posedness

Definition 1 is inspired by the following complete trichotomy for the local well-posedness.

Theorem 1 *The Cauchy problems* (1) *and* (2) *are:*

- *locally well-posed if $s_c < s$ with $T_{lwp} = T(\|u_0\|_{H^s})$*
  (Cazenave-Weissler '90)
- *locally well-posed if $s_c = s$ with $T_{lwp} = T(u_0)$,*
  (Ginibre-Velo '85, Cazenave '03)
- *and is "ill-posed" if not.*
  (Christ-Colliander-Tao '04, Lebeau '01)

3 Global well-posedness

To simplify the results, let us restrict our selves to the case when the initial data are in $\dot{H}^1$ i.e. *energy critical* case.

3.1 The case of NLW_P

Theorem 2 *The Cauchy problem* (1) *with initial data* $(u_0, u_1) \in H^1 \times L^2$ *is:*

- *globally well-posed if* $s_c < s = 1$ *(or equivalently* $p < p_c := \frac{d+2}{d-2}$*).*
  (Ginibre-Velo '85)
- *globally well-posed if* $s_c = s = 1$ *(or equivalently* $p = p_c$*).*
  (Grillakis '90, Shatah-Struwe '94)

Moreover, if $p > p_c$ *and the initial data is in* $H^s \times H^{s-1}$ *with* $s < s_c$*, then the Cauchy problem* (1) *is "ill-posed".*

(Christ-Colliander-Tao '04, Lebeau '01-'05)

Remark 1 *Shatah- Struwe result's was extended to the variable coefficients case by (Ibrahim-Majdoub '03) with a "conservative Laplacian":*

$$\Delta_A u := -div(A(\cdot)\nabla u),$$

where A^{-1} *is a Riemannian metric on* $\mathbb{R}^d$ *which is flat outside a fixed compact set.*

3.2 The case of NLS_P

Theorem 3 *The Cauchy problem* (2) *with initial data* $v_0 \in H^1$ *is:*

- *globally well-posed if* $s_c < s = 1$ *(or equivalently* $p < p_c := \frac{d+2}{d-2}$*).*
  (Ginibre-Velo '85, Cazenave '04)
- *globally well-posed if* $s_c = s = 1$ *(or equivalently* $p = p_c$*).*
  (Bourgain '99, Colliander-Keel-Staffilani-Takaoka-Tao '05)

Moreover, if $p > p_c$ *and the initial data is in* H^s *with* $s_c > s$*, then the Cauchy problem* $NLSp$ *is "ill-posed".*

(Christ-Colliander-Tao '04, Burq-Gérard-Tzevtkov '02)

Remark 2 *A refinement and a generalization to the variable coefficients case is now proved by (Burq-Gérard-Ibrahim '06) for both* $NLWp$ *and* $NLSp$ *in any space dimension.*

3.3 Energy criticality in two space dimensions

In dimension two, $p_c = +\infty$ and therefore, the initial value problems $NLWp$ and NLS_p are energy subcritical for all $p > 1$. To identify an "energy critical" nonlinear Wave/Schrödinger initial value problem on $\mathbb{R}^2$, it is thus natural to consider problems with exponential nonlinearities. Consider,

$$\begin{cases} (\partial_t^2 - \Delta_x)u + u = -f(u), & u : (-T_*, T^*) \times \mathbb{R}^2 \longmapsto \mathbb{R} \\ \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1 & (u_0, u_1) \in H^1 \times L^2(\mathbb{R}^2) \end{cases} \tag{4}$$

$$\begin{cases} i\partial_t v + \Delta v = f(v), & u : (-T_*, T^*) \times \mathbb{R}^2 \longmapsto \mathbb{C} \\ v(x,0) = v_0(x) \in H^1(\mathbb{R}^2) \end{cases} \tag{5}$$

where

$$f(z) = z\Big(e^{4\pi|z|^2} - 1\Big). \tag{6}$$

Conserved quantities:

Solutions to the nonlinear wave equation (4) formally satisfy the energy conservation

$$
\begin{aligned}
E(u(t,\cdot)) &:= \|\partial_t u(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2 + \frac{1}{4\pi}\|e^{4\pi|u(t,\cdot)|^2} - 1\|_{L^1(\mathbb{R}^2)} \\
&= E(u(0,\cdot)).
\end{aligned}
$$

and we have conservation of mass and Hamiltonian for Schrödinger equation

$$
\begin{aligned}
M(u(t,\cdot)) &:= \|u(t,\cdot)\|_{L^2}^2 \\
&= M(u(0,\cdot)),
\end{aligned}
$$

$$
\begin{aligned}
H(u(t,\cdot)) &:= \|\nabla u(t,\cdot)\|_{L^2}^2 + \frac{1}{4\pi}\|e^{4\pi|u(t,\cdot)|^2} - 1 - 4\pi|u(t,\cdot)|^2\|_{L^1(\mathbb{R}^2)} \\
&= H(u(0,\cdot)).
\end{aligned}
$$

Definition 2 *The Cauchy problem associated to (4) and with initial data* $(u_0, u_1) \in H^1 \times L^2(\mathbb{R}^2)$ *is said to be*

- *subcritical if* $E(u_0, u_1) < 1$,
- *critical if* $E(u_0, u_1) = 1$ *and,*
- *supercritical if* $E(u_0, u_1) > 1$.

Definition 3 *The Cauchy problem associated to (5) and with initial data* $v_0 \in H^1(\mathbb{R}^2)$ *is said to be*

- *subcritical if* $H(v_0) < 1$,
- *critical if* $H(v_0) = 1$ *and,*
- *supercritical if* $H(v_0) > 1$.

4 Results in two space dimensions

Theorem 4

(Ibrahim-Majdoub-Masmoudi '05)

.

Assume that $E_0 \leq 1$, *then problem* $NLWexp$ *has an unique global solution* u *in the class*

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)) \cap \mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^2)).$$

Moreover, $u \in L^4_{loc}(\mathbb{R},\ \mathcal{C}^{1/4}(\mathbb{R}^2))$ *and satisfies the energy identity.*

Remark 3 *It is important here to note that contrary to problems* $NLWp$ *and* $NLSp$, *we have an "'unconditional uniqueness" results for this type of equations.*

Theorem 5

(Colliander-Ibrahim-Majdoub-Masmoudi '06)
Assume that $H(u_0) \leq 1$*; then problem* $NLSexp$ *has an unique global solution* v *in the class*

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)).$$

Moreover, $u \in L^4_{loc}(\mathbb{R},\, \mathcal{C}^{1/2}(\mathbb{R}^2))$ *and satisfies the conservation of mass and hamiltonian.*

Theorem 6

(Colliander-Ibrahim-Majdoub-Masmoudi '06)
Assume that $E_0 > 1$ *and* $H(v_0) > 1$*; then problems* (4) *and* (5) *are "ill-posed"*

5 Ideas of proofs (the case of $NLSexp$)

- The local-well-posedness idea is:

$$\boxed{NLSexp \quad \sim \quad LSexp}$$

How does the proof of the local well-posedness go ?

Let v_0 be the solution of the free Schrödinger equation

$$i\partial_t v_0 + \Delta v_0 = 0$$
$$v_0(0,x) = u_0.$$

Fix $T > 0$ and define a map

$$i\partial_t \tilde{v} + \Delta \tilde{v} = (v + v_0)\Big(e^{4\pi |v+v_0|^2} - 1\Big), \quad \tilde{v}(0,x) = 0, \tag{7}$$

on a closed neighborhood $X(T)$ around 0 included in the energy space $\mathcal{C}([0,T], H^1)$.

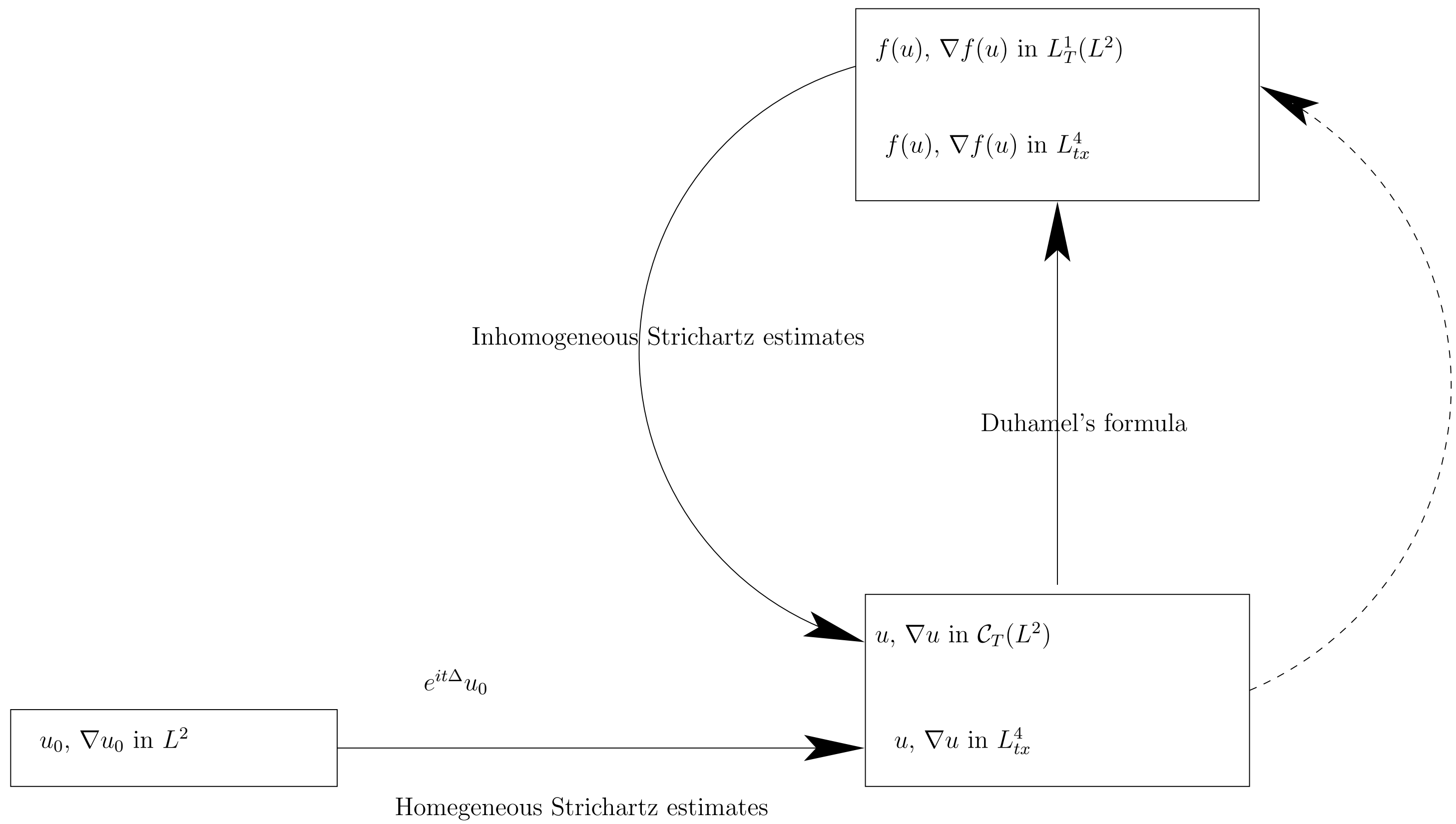

Figure 1: Local well-posedness scheme.

The local well-posedness is obtained by combining the following three fundamental ingredients:

Lemma 1 (Moser-Trudinger Inequality) *Let* $\alpha \in [0, 4\pi)$. *A constant* c_α *exists such that*

$$\| \exp(\alpha |u|^2) - 1 \|_{L^1(\mathbb{R}^2)} \leq c_\alpha \|u\|^2_{L^2(\mathbb{R}^2)} \tag{8}$$

for all u *in* $H^1(\mathbb{R}^2)$ *such that* $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. *Moreover, if* $\alpha \geq 4\pi$, *then (8) is false.*

Lemma 2 (Strichartz estimates) *Let* v_0 *be a function in* $H^1(\mathbb{R}^2)$ *and* $F \in L^1(\mathbb{R}, H^1(\mathbb{R}^2))$. *Denote by* v *the solution of the inhomogeneous linear Schrödinger equation*

$$i\partial_t v + \Delta v = F$$

with initial data $v(0, x) = v_0(x)$.
Then, a constant C *exists such that for any* $T > 0$ *and any admissible couple of Strichartz exponents* (q, r) *i.e*

$0 \leq \frac{2}{q} = 1 - \frac{2}{r} < 1$, we have

$$\|v\|_{L^q([0,T],\mathcal{B}^1_{r,2}(\mathbb{R}^2))} \leq C\left[\|v_0\|_{H^1(\mathbb{R}^2)} + \|F\|_{L^1([0,T],H^1(\mathbb{R}^2))}\right].$$

Lemma 3 (Log Estimate) *(Ibrahim-Majdoub-Masmoudi '05)* . *Let $\beta \in]0,1[$. For any $\lambda > \frac{1}{2\pi\beta}$ and any $0 < \mu \leq 1$, a constant $C_\lambda > 0$ exists such that, for any function $u \in H^1(\mathbb{R}^2) \cap \mathcal{C}^\beta(\mathbb{R}^2)$, we have*

$$\boxed{\|u\|^2_{L^\infty} \leq \lambda \|u\|^2_\mu \log(C_\lambda + \frac{8^\beta \mu^{-\beta} \|u\|_{\mathcal{C}^\beta}}{\|u\|_\mu}),}$$

where we set

$$\|u\|^2_\mu := \|\nabla u\|^2_{L^2} + \mu^2 \|u\|^2_{L^2}.$$

- In the subcritical case, using only the conserved quantities, we can iterate the local-well-posedness result infinitly many times, thus the solution is global.

- In the critical case, it is no longer sufficient to use only the conserved quantities. We prove a result about the distribution of the local mass at different times.

Lemma 4 *Let* u *be a solution of* (5) *on* $[0, T)$ *with* $0 < T \leq +\infty$ *and suppose that* $E := H(u_0) + M(u_0) < \infty$*. For any two positive real numbers* R *and* R' *and for any* $0 < t < T$*, a constant* $C(E)$ *exists such that the following holds:*

$$\int_{B(R+R')} |u(t,x)|^2 dx \geq \int_{B(R)} |u_0(x)|^2 dx - C(E)\frac{t}{R'}. \tag{9}$$

- The instability in the super-critical case is based on the fundamental idea:

$$\boxed{NLSexp \quad \sim \quad ODEexp}$$

Theorem 7 *There exist a sequence of positive real numbers* (t_k)*,* $t_k \longrightarrow 0$ *and tow sequences* (U_k) *and* (V_k) *solutions of* $NLWexp$ *and satisfying the following: for any* $k \in \mathbb{N}$

- $$\|(U_k - V_k)(t=0,\cdot)\|_{H^1}^2 + \|\partial_t(U_k - V_k)(t=0,\cdot)\|_{L^2}^2 = \circ(1), k \to +\infty.$$

- *For any* $\nu > 0$*,*

$$0 < E(U^k, 0) - 1 \leq e^3\nu^2 \text{ and } 0 < E(V^k, 0) - 1 \leq \nu^2,$$

- *and*

$$\liminf_{k \longrightarrow \infty} \|\partial_t(U_k - V_k)(t_k, \cdot)\|_{L^2}^2 \geq \frac{\pi}{4}(e^2 + e^{3-8\pi})\nu^2.$$

How to prove Theorem 7?

1st step: An ODE analysis

Let Φ_k and Ψ_k be the two solutions of $ODEexp$:

$$\frac{d^2}{dt^2}y + ye^{4\pi\ y^2} = 0.$$

with initial data

$$\Phi_k(0) = (1 + \frac{1}{k})\sqrt{\frac{k}{4\pi}}, \quad \frac{d}{dt}\Phi_k(0) = 0,$$

and

$$\Psi_k(0) = \sqrt{\frac{k}{4\pi}}, \quad \frac{d}{dt}\Psi_k(0) = 0.$$

Note that Φ_k is periodic with period $T_k \sim \sqrt{k}\, e^{-(1+\frac{1}{k})^2 k/2}$.

We choose time $t_k \in]0, T_k/4[$ such that

$$\Phi_k(t_k) = (1 + 1/k)\sqrt{\frac{k}{4\pi}} - \left((1 + 1/k)\sqrt{\frac{k}{4\pi}}\right)^{-1}.$$

Then for any $\nu > 0$ and for k large enough, we have

- $$t_k \leq c\frac{\nu}{2}e^{-k/2},$$

- $$|\partial_t \Phi_k(t_k) - \partial_t \psi_k(t_k)|^2 \geq ce^k,$$

  and

- $$\int_{\mathbb{R}^2} |\partial_t(\Phi_k(t_k) - \Psi_k(t_k))|^2 \geq c\nu^2$$

2nd step: PDE-ODE approximation

We construct the following initial data

$$\Big((1+\frac{1}{k})f_k(\frac{x}{\nu}),0\Big) \text{ and } \Big(f_k(\frac{x}{\nu}),0\Big),$$

where f_k is the sequence that violate the sharp moser-Trudinger inequality when the exponent is 4π. The paramete ν is arbitrary.

Using the special form of the sequence f_k, an "enormous gift" is provided by the finite speed of propagation:

$$\boxed{NLWexp \quad = \quad ODEexp}$$

in the backward light cone

$$\{(x,t) : |x| \leq t - \nu e^{-k/2}\}.$$

Remark 4

- *Note that the data are slightly supercritical*
- *For NLSexp, the analogous to Theorem 7 is harder to prove.*
- *Theorem 7 result says no better than the flow map is not uniformly continuous.*

Conclusions

- The novel approach based on the discussion with respect to the size of the initial data in the energy space allows us to obtain a trichotomy almost similar to the power nonlinearity case. We argue that $NLWexp$ and $NLSexp$ are the H^1-critical problems in $\mathbb{R}^2$.
- The long-time behavior of solutions remains unknown. Also in the focusing case, there is no a qualitative study of blow-up.
- The very interesting question of global existence for supercritical problems remains open. Solving such problem may give a good insight to solve the Navier-Stokes system (which is supercritical).

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