Some mathematical remarks concerning the localization of planetary waves in a stochastic background flow

Peter Imkeller, Adam Hugh Monahan, and Lionel Pandolfo

Abstract. In this article we develop some mathematically rigorous ideas to explain the phenomena of localization of planetary waves in a stochastic background flow as presented in the physical companion paper. For this purpose the linearized equation for the zonally averaged vorticity subject to a zonal background wind and driven by a local source is transformed into Sturm-Liouville problems with random potential function. There is a first background effect of localization symmetrical with respect to the equator which is due to the nodes of the potential function (*critical lines*). A second more subtle effect causes localization around the source, by superposition of the source term with the Green's kernel expressed in terms of the eigenfunctions of the spectral resolution of the random Sturm-Liouville operators involved. On an average, this effect is moderate for zero damping, and stronger for small non-zero damping.

1. Introduction

In this second part we shall attempt some more rigorous mathematical approaches of the questions of localization of waves which came up in the physical first part [12]. As we shall rapidly see, the problem of analyzing a small perturbation of the zonally averaged vorticity equation by a random background zonal wind leads into the outskirts of a well developed and understood area of mathematics known as *random Schrödinger operators* (see for example Carmona, Lacroix [2] or Stollmann [17]). In fact, simple algebraic operations allow to transfer the key equation (9) of [12] into an equation of the Sturm-Liouville type with a random potential term. The main term underlying the random forcing being a stationary random process, one could suspect that asymptotic properties such as the ones we are interested in, namely decay rates of waves propagating through a random medium, could be tackled using the powerful tools of ergodic theory. They would then correspond to Lyapunov exponents, objects by now well understood (see Arnold [1]). Yet, the situation is somewhat unfortunate. The parameter space of our random process is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, physically the interval between south and north pole. This interval is simply too short to make the main argument of ergodic theory work, according to which averages over the parameter space on the long run can be replaced by phase space averages. Due to this annoying fact, we take resort to a pathwise analysis of decay of planetary waves. As we shall point out precisely below, the hypotheses made in [12] on the random perturbation R of the potential function U lead to very smooth paths. We show that the stationary process has in fact $C^\infty-$paths. Since the wave propagation and attenuation effects we are interested in are stable when passing from a C^∞ path to a still smoother one, we start with considering pathwise analytic potential functions. Once this is accepted, we use the tools of the classical theory of singular differential equations and special functions, accompanied by the Liouville-Green approximations of solutions, commonly known as WKB approximations (see Olver [13]). Assuming *typical* path properties qualitatively derived from Rice's formula and other tools of the theory of stationary processes, we can then explain some of the reasons causing the localization patterns described in [12]. We organize the presentation of the main results in the following way.

In section 2 we explain some smoothness and oscillation properties of stationary stochastic processes such as the random component R of U with properties chosen in [12].

Section 3 is devoted to transforming the key equation (9) of [12] into Sturm-Liouville equations with random potential function. A clear distinction is made between the cases of *zero damping* $\mu = 0$ and *non-zero damping.* The former leads to random differential operators of the second order roughly of the form $Lu = (V^2\, u')' + pVu$ with a random potential function V and a slowly varying nonrandom positive function p. In the latter the part of L is taken by a more complicated operator K of the form $Kv = (V^2\, w')' + p\beta V$ with a new random quantity β which fluctuates rather fast and changes sign relatively often on an average, even inside *excursion intervals* of the potential V, i.e. intervals bounded by consecutive zeroes.

In the subsequent sections we account for different rather subtle effects of localization.

There is, first of all, a background localization effect due to the presence of *critical lines* (lines in the sense of parallels on the earth's surface), i. e. lines at which the waves become extinct. The critical lines are given by the zeroes of the random potential function $U(\phi) = U_{eq}\cos\phi + \eta\, U_{max} R(\phi)$, which, on an average, are symmetrical around the equator. In section 4, critical lines are discussed for zero damping, in section 5 for small non-zero damping.

To describe the more subtle effects of localization around the source, we have to study the random spectra of the operators L (for zero damping) and K (for non-zero damping), respectively restricted to excursion intervals of the potential V. In section 6, we show that the continuous part of the spectrum of L is given by $[0,\infty[$, while the more interesting pure point part lies in the interval $]-\infty, 0[$. In case $V < 0$, the pure point part is empty, while it contains at most finitely many negative eigenvalues in case $V > 0$. Eigenfunctions of negative eigenvalues decay exponentially near the nodes of V. Since propagating waves are described by superposing the Green's kernel composed of the eigenfunctions with the source term, this leads, for typical sample functions of V, to a moderate localization effect around the source (Theorem 6.1). In section 7, finally, this localization effect is studied for non-zero damping. By the appearance of the strongly fluctuating factor β in K, which is due to its dependence on the second derivative of R, there is, first of all, no distinction between intervals of positive resp. negative sign of V, and, secondly, negative ground states lie deeper in narrower potential wells, and thus cause, by superposition with the source term, a more pronounced effect of localization around the source (Theorem 7.1).

2. Some remarks concerning path properties of R

We start with a few remarks concerning the consequences of the hypotheses made on the covariance function of the stationary process R on the behaviour of the paths. The basic hypothesis is

$$r(\phi_0) = E(R(\phi)\, R(\phi + \phi_0)) = \exp(-\frac{|\phi_0|^2}{2\tau^2})\, \cos(2\alpha\, \phi_0),$$

for $\phi, \phi_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The following calculation involving Hermite polynomials will clarify the structure of r and its bearing on path regularity. We may write

$$r(\phi) = Re[\exp(-\frac{\phi^2}{2\tau^2} + 2i\,\alpha\,\phi)],$$

and

$$\exp(-\frac{\phi^2}{2\tau^2} + 2i\,\alpha\,\phi) = \exp(-2\,\alpha^2\,\tau^2)\; \exp(-\frac{1}{2}[\frac{\phi}{\tau} - 2i\alpha\,\tau]^2).$$

If we change variables according to $\gamma = \frac{\phi}{\tau} - 2i\alpha\tau$, and define

$$\rho(\gamma) = R(\phi), \quad s(\gamma) = r(\phi) = \exp(-\frac{1}{2}\gamma^2)\, \exp(-2\alpha^2\tau^2)$$

we obtain $R' = \frac{1}{\tau}\,\rho'$, and therefore for $k \geq 0$, via the well known relationship

$$s^{(2k)}(\gamma) = \exp(-2\alpha^2\tau^2)\,(\frac{d}{d\gamma})^{2k}\; \exp(-\frac{1}{2}\gamma^2) = \exp(-2\alpha^2\tau^2)\, H_{2k}(\gamma)\, \exp(-\frac{1}{2}\gamma^2)$$

the equation

$$\begin{aligned} r^{(2k)}(0) &= \frac{1}{\tau^{2k}}\; s^{(2k)}(-2i\alpha\tau) \\ &= \frac{1}{\tau^{2k}}\; \exp(-2\alpha^2\tau^2)\, H_{2k}(-2i\alpha\tau)\; \exp(-\frac{1}{2}(-2i\alpha\tau)^2) = \frac{1}{\tau^{2k}}\; H_{2k}(-2i\alpha\tau). \end{aligned}$$

Here H_l denotes the Hermite polynomial of degree l defined implicitly in the preceding equation. Using well known formulas for Hermite polynomials we finally arrive at

$$r^{(2k)}(0) = (-1)^k \frac{1}{\tau^{2k}} \sum_{l=0}^{k} \frac{(2k)!}{2^l\, l!\, (2(k-l))!} (4\alpha^2\tau^2)^{k-l},$$

while the derivatives of odd order at 0 vanish.

It is well known (see Cramer, Leadbetter [4]) that R' exists as a real valued process under (2), is itself stationary and possesses the covariance functional $-r''$. Consequently be iteration $R^{(k)}$ exists as a real valued process with continuous sample paths, is stationary, and its covariance function is given by $(-1)^k r^{(2k)}$. This in particular means that by (2) R is a stochastic process with C^∞ trajectories. Due to the exponential decay of the second moments of the Fourier coefficients, and the assumption that they be independent, the polynomials used in the simulations converge uniformly on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to the trajectories of R. Since this uniform convergence pertains to the derivative processes, we also know that at least non degenerate level crossing points and extrema converge. Therefore the subsequent pathwise analysis of localization, in which we take R to be analytic (as the polynomials used in the simulation) will pertain when generalized to the stationary process R satisfying (2) as chosen in Monahan, Pandolfo, Imkeller [12].

This pathwise analysis will largely depend on the *oscillation behaviour* of processes closely related with R, and therefore directly with the oscillation behaviour of R itself. Let us therefore add a few remarks concerning the expected number of zero level crossings, extrema etc. which emerge from the famous *Rice formula*. According to the classical formula, the expected number of zero level crossings N_0 of R over the whole parameter range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is given by

$$E(N_0) = [-\frac{r''(0)}{r(0)}]^{\frac{1}{2}} = [\frac{1}{\tau^2}(1 + 4\alpha^2\tau^2)]^{\frac{1}{2}},$$

the expected number of zero level crossings N_0' of R' correspondingly by

$$E(N_0') = [-\frac{r^{(4)}(0)}{r''(0)}]^{\frac{1}{2}} = [\frac{1}{\tau^2}\,\frac{3 + 24\alpha^2\tau^2 + 16\alpha^4\tau^4}{1 + 4\alpha^2\tau^2}]^{\frac{1}{2}}.$$

A similar formula exists for R'' which is relevant below. There are explicit formulas for the law of the height of a maximum (depth of a well) conditional on its position, if the law of the stationary process is Gaussian (see Cramer, Leadbetter [4], p. 247) and similar quantities specifying the oscillation of the trajectories.

To explain the localization phenomena observed in the simulations we took resort to a pathwise analysis for the following reason. There is no obvious way to use ergodic theory, despite the stationarity of R: the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ on which it lives is "too short". This situation is somewhat unfortunate, considering the fact that the powerful tools of the well developed theory of random Schrödinger operators are therefore not available. They are exhibited for example in Carmona, Lacroix [2] which employ multiplicative ergodic theory, and are thus able to express localization in terms of the *Lyapunov exponents* of the system. If the use of stationary and *self similar* fields like *fractional Brownian motion* known for example from Mandelbrot's works, instead of R, would physically make sense, this approach would come into reach.

3. Transformation into Sturm-Liouville problems

We consider the equation for the spectral component of wave number 1 of a small perturbation of the zonally averaged vorticity ψ subject to a background zonal wind U, which is given by

$$\begin{aligned}(U - i\,\mu\,a\,\cos\phi)\ [\cos\phi\,\frac{d}{d\phi}(\cos\phi\,\frac{d}{d\phi}\,\psi) - \psi] \qquad\qquad (1)\\ -\cos^2\phi\,\frac{d}{d\phi}[\frac{1}{\cos\phi}\ \frac{d}{d\phi}(U\ \cos\phi)]\,\psi + 2\Omega\,a\,\cos^3\phi\ \psi = S(\phi),\end{aligned}$$

equation (9) in Monahan, Pandolfo, Imkeller [12]. Here U is a random process for example given by

$$U(\phi) = \tilde{U}(\phi) + \eta\,U_{max}\,R(\phi),$$

where $\tilde{U}$ describes the mean background wind, which in the important case of the superrotation flow is given by

$$\tilde{U}(\phi) = U_{eq}\ \cos\phi,$$

and R a stationary centered stochastic process of unit variance, indexed by the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We shall concentrate on the case of a superrotation flow for most of the following notes. We fix a sample path of R and denote this function by the same symbol. Since in the numerical simulation R is taken to be a trigonometric polynomial, we shall assume for our analysis that the path of U we fix is real analytic. In fact, as has been explained in the previous section, the sample path properties of our stationary process R under (2) are such that, besides explaining the simulation result, our qualitative results will pertain as we pass to the limit process.

Our first aim is to transscribe (1) into an extended Sturm-Liouville problem on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ or on $\mathbf{R}$ by introducing renormalizations of the functions involved, and eventually new natural coordinates. Abbreviating $\psi' = \frac{d}{d\phi}\psi, \psi'' = \frac{d^2}{d\phi^2}\psi$, and noting

$$\cos\phi\,\frac{d}{d\phi}(\cos\phi\,\frac{d}{d\phi}\,\psi)(\phi) = -\sin\phi\ \cos\phi\ \psi'(\phi) + \cos^2\phi\ \psi''(\phi),$$

$$\cos^2\phi\,\frac{d}{d\phi}[\frac{1}{\cos\phi}\ \frac{d}{d\phi}(U\ \cos\phi)] = -U(\phi) - \sin\phi\,\cos\phi\ U'(\phi) + \cos^2\phi\ U''(\phi).$$

Using these identities, and dividing by $\cos^2\phi$, we obtain the equation

$$\begin{aligned}[U\,\psi'' - \psi\,U''](\phi) - \tan\phi\ [U\,\psi' - \psi\,U'](\phi) \qquad\qquad (2)\\ -i\mu a[-\frac{1}{\cos\phi}\,\psi(\phi) - \sin\phi\,\psi'(\phi) + \cos\phi\,\psi''(\phi)] + 2\Omega a\ \cos\phi\,\psi(\phi) = \frac{S(\phi)}{\cos^2\phi}.\end{aligned}$$

We now normalize the functions to account for geometric factors, as is also done in the plots. For $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we let

$$\chi(\phi) = \psi(\phi)\,\sqrt{\cos\phi}, \quad V(\phi) = U(\phi)\,\sqrt{\cos\phi}.$$

One easily verifies the equations

$$\psi'(\phi) = \frac{1}{\sqrt{\cos\phi}}\,[\chi'(\phi) + \frac{1}{2}\,\tan\phi\,\chi(\phi)],$$

$$\psi''(\phi) = \frac{1}{\sqrt{\cos\phi}}[\chi''(\phi) + \tan\phi\,\chi'(\phi) + \frac{3}{4}\,\tan^2\phi\,\chi(\phi) + \frac{1}{2}\,\chi(\phi)],$$

and analogous ones for V. Exploiting (3) and (3), one can see that (2) simplifies to the following equation

$$[V\chi'' - \chi V''](\phi) + i\mu a\frac{1}{\sqrt{\cos\phi}}[(\frac{1}{2} + \frac{1}{4}\,\sin^2\phi)\,\chi(\phi) - \cos^2\phi\,\chi''(\phi)] + 2\Omega a\cos^{\frac{3}{2}}\phi\,\chi(\phi) = \frac{S(\phi)}{\cos\phi}. \tag{3}$$

We shall study the propagation and attenuation of ψ, and investigate mathematically the different factors made responsible for its localization in the physical interpretation. Recalling that the set of zeroes of U is finite, we fix therefore any interval $[\phi_1, \phi_2]$ bounded by *critical lines* on the interior of which U does not vanish, i.e. such that $U(\phi_i) = 0, i = 1, 2, U|_{]\phi_1,\phi_2[} \neq 0$. These intervals are the same for the renormalized V. On the interval $]\phi_1, \phi_2[$, we may finally introduce another normalization of χ which turns out to be quite practical for investigating the decay of ψ near the critical lines. For $\phi \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, we let

$$u(\phi) = \frac{\chi(\phi)}{V(\phi)} = \frac{\psi(\phi)}{U(\phi)}.$$

Then the obvious equation

$$V^2\,u' = V\,\chi' - \chi\,V'$$

implies that on the interval $]\phi_1, \phi_2[$ we obtain the differential equation

$$(V^2\,u')'(\phi) + \frac{1}{\sqrt{\cos\phi}}[i\mu a\,\frac{\cos^2\phi}{V(\phi)}\,(V^2\,u')'(\phi) + i\mu a[(\frac{1}{2} + \frac{1}{4}\sin^2\phi)V(\phi) + \cos^2\phi\,V''(\phi)]\,u(\phi) + 2\Omega a\,\cos^2\phi\,V u(\phi)] = \frac{S(\phi)}{\cos\phi)}. \tag{4}$$

To abbreviate, define the positive geometric functions

$$g(\phi) = \frac{\frac{1}{2} + \frac{1}{4}\,\sin^2\phi}{\sqrt{\cos\phi}}, \quad h(\phi) = \cos^{\frac{3}{2}}\phi, \quad \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}],$$

and the functions

$$\gamma = 2\Omega a h^2 - V''\,h + g\,V, \quad T = \frac{S}{\cos}.$$

We next define the differential operator of the second order on $]\phi_1, \phi_2[$

$$L\,u = (V^2\,u')' + 2\Omega ah\,Vu.$$

Then (4) can be written as

$$(1 - i\mu a\,\frac{h}{V})\,Lu + i\mu a\gamma\,u = T.$$

In case the friction μ vanishes, (3) takes the simpler form of the Sturm-Liouville type

$$L\,u = \frac{S}{\cos}.$$

In case of non-vanishing friction, we may put $u = v + iw$ and derive equations for the real and imaginary parts, giving

$$L\,v + \mu a\frac{h}{V}\,L\,w - \mu a\,\gamma\,w = T,$$

$$L\,w - \mu a\frac{h}{V}\,L\,v + \mu a\,\gamma\,v = 0.$$

By substituting one into the other, (3) and (3) may finally be transformed into the equations

$$L\,v - \frac{(\mu a)^2\,\frac{h}{V}\,\gamma}{1+(\mu a\frac{h}{V})^2}\,v - \frac{\mu a\,\gamma}{1+(\mu a\frac{h}{V})^2}\,w = \frac{T}{1+(\mu a\frac{h}{V})^2},$$

$$L\,w - \frac{(\mu a)^2\,\frac{h}{V}\,\gamma}{1+(\mu a\frac{h}{V})^2}\,w + \frac{\mu a\,\gamma}{1+(\mu a\frac{h}{V})^2}\,v = \frac{\mu a\frac{h}{V}\,T}{1+(\mu a\frac{h}{V})^2}.$$

Now note that if we finally set

$$\beta = \frac{V^2 + \mu^2\,\frac{a}{2\Omega}(V''\,h - V\,g)}{V^2 + (\mu a h)^2},$$

and define a second order differential operator K by

$$K\,v = (V^2\,v')' + 2\Omega a h\,\beta\,V v,$$

a little algebra shows

$$L\,v - \frac{(\mu a)^2\,\frac{h}{V}\,\gamma}{1+(\mu a\frac{h}{V})^2}\,v = K\,v,$$

and therefore (3) and (3) are given by

$$K\,v = \frac{V^2}{V^2+(\mu a h)^2}\,(T + \mu a\gamma w),$$

$$K\,w = \frac{V^2\,\mu a}{V^2+(\mu a h)^2}\,(\frac{h}{V}T - \gamma v).$$

To finally compare the operators K and L, note that

$$\frac{K\,v - (V^2\,v')'}{L\,v - (V^2 v')'} = \beta.$$

Due to the presence of V'', β fluctuates fast, whereas the geometric function h determining L is slowly fluctuating.

4. Critical lines for $\mu = 0$

Here we shall briefly discuss the behaviour of the wave function near the critical lines, in particular its decay there, in the simple case of null friction. We shall deal with (3) and show that ψ decays as fast as U, or equivalently χ as fast as V. For this purpose we shall study the homogeneous part of (3) first.

We shall assume that $\phi_1 > -\frac{\pi}{2}, \phi_2 < \frac{\pi}{2}$, and that $V'(\phi_1), V'(\phi_2) \neq 0$. This is the generic case. To save some writing, let $\phi_1 = 0, \phi_2 = 1$. Let us consider the singularity 0 of the differential equation first. Due to our assumptions, for x in a neighborhood of 0, we have the equation

$$u''(x) + \frac{1}{x}\, h_1(x)\, u'(x) + \frac{1}{x}\, h_2(x)\, u(x) = 0,$$

with h_1, h_2 analytic in a neighborhood of 0, and $h_1(0) = 2, h_2(0) = \frac{2\Omega a\, h(0)}{V'(0)}$. To describe the singularities of (4), we follow Walter [19] or Coddington, Levinson [3]. One may consider the singular system of the first order

$$y' = \frac{1}{x} A(x)\, y,$$

where

$$A(x) = \begin{bmatrix} 0 & 1 \\ -x h_2(x) & 1 - h_1(x) \end{bmatrix}.$$

The zeroes of the characteristic polynomial $\lambda \mapsto (\lambda I - A(0))$ are then given by $\lambda_1 = -1, \lambda_2 = 0$. In this case it is well known that a fundamental system of solutions is given by

$$\hat{u}_1(x) = p(x),$$

$$\hat{u}_2(x) = \frac{1}{x}[p_1(x) + \log x\, p_2(x)],$$

with p, p_1, p_2 analytic in a neighborhood of 0. Let us next discuss the singularity 1. We have the expansion

$$u''(x) + \frac{1}{x-1}\, k_1(x)\, u'(x) + \frac{1}{x-1}\, k_2(x)\, u(x) = 0,$$

where k_1, k_2 are analytic in a neighborhood of 1, and $k_1(1) = 2, k_2(1) = \frac{2\Omega a h(1)}{V'(1)}$. The situation is thus the same as above, modulo the affine transformation $x \mapsto x + 1$, and we obtain the fundamental system of solutions with analytic functions q, q_1, q_2 in a neighborhood of 1

$$\hat{v}_1(x) = q(x),$$

$$\hat{v}_2(x) = \frac{1}{x-1}\, [q_1(x) + \log(1-x)\, q_2(x)].$$

Recalling local uniqueness in the interior of $]0, 1[$, by matching values of the functions and their derivatives, we can now combine (4) and (4) as well as (4) and (4) to obtain a fundamental system of solutions of the homogeneous part of (3). We return to the original coordinates and write the solutions as follows:

$$u_1(\phi) = \frac{a_1}{\phi - \phi_1} + \log(\phi - \phi_1) + v_1(\phi),$$

$$u_2(\phi) = \frac{a_2}{\phi_2 - \phi} + \log(\phi_2 - \phi) + v_2(\phi),$$

$\phi \in]\phi_1, \phi_2[$, where a_1, a_2 are constants, v_1, v_2 are bounded and continuous on $[\phi_1, \phi_2]$. In fact, the functions are analytic except in a finite number of points. It is well known that in this context the Green's kernel is given by

$$\Gamma(x, y) = u_1(x)\, u_2(y)\, 1_{[\phi_1, x]}(y) + u_2(x)\, u_1(y)\, 1_{[x, \phi_2]}(y),$$

$x, y \in [\phi_1, \phi_2]$. Thus a solution of the inhomogeneous problem (3) is well defined and given by the formula

$$u(\phi) = \int_{\phi_1}^{\phi_2} \Gamma(\phi, y)\, T(y)\, dy.$$

By our assumptions $T = \frac{S}{\cos}$ is bounded and continuous on $[\phi_1, \phi_2]$, so that by our choice of fundamental system we have

$$u(\phi) = \int_{\phi_1}^{\phi} T(y)\, u_2(y)\, dy \cdot u_1(\phi) + \int_{\phi}^{\phi_2} T(y)\, u_1(y)\, dy \cdot u_2(\phi),$$

$\phi \in]\phi_1, \phi_2[$. But the properties of the solutions u_1, u_2 imply immediately that u can be continuously extended to the whole of $[\phi_1, \phi_2]$. Hence

$$\psi = U\, u$$

is a solution of (1) for $\mu = 0$ on the interval $[\phi_1, \phi_2]$, which satisfies $\psi(\phi_i) = 0, i = 1, 2$, and $|\psi(\phi)| \leq c|U(\phi)|$ for all $\phi \in [\phi_1, \phi_2]$. Hence the solution of (1) decays like U near the critical lines. This in particular means that ψ cannot survive the critical lines next to the source S, and decays like U there. We formulate this fact more precisely. Let $C \subset]-\frac{\pi}{2}, \frac{\pi}{2}[$ be the compact support of S. Recall our hypothesis that U possesses roots, and let

$$\begin{aligned} \underline{\phi} &= \sup\{\phi : U(\phi) = 0, [-\frac{\pi}{2}, \phi] \cap C = \emptyset\}, \\ \overline{\phi} &= \inf\{\phi : U(\phi) = 0, [\phi, \frac{\pi}{2}] \cap C = \emptyset\}. \end{aligned}$$

We can then apply the solution algorithm discussed above on all intervals in $[\underline{\phi}, \overline{\phi}]$ bounded by consecutive roots of U. We obviously obtain the following statement.

Theorem 4.1. *Suppose $\mu = 0$, and that U has at least two roots in $]-\frac{\pi}{2}, \frac{\pi}{2}[$. Then there is a solution ψ of (1) which satisfies*

$$|\psi(\phi)| \leq c\, |U(\phi)|$$

for some constant c, and

$$\psi|_{[-\frac{\pi}{2}, \underline{\phi}]} = \psi|_{[\overline{\phi}, \frac{\pi}{2}]} = 0.$$

Any solution of (1) satisfies (4.1).

5. Critical lines for general μ

The most important object in the deduction of properties of the solutions in the preceding section was the Green's kernel Γ of the operator L. Instead of (3) we now have to solve the pair of equations (3) and (3). This means we need information about the Green's kernel of the operator K instead. We fix again an interval between two consecutive zeroes ϕ_1, ϕ_2 of U or V. It is now essential to remark that the singularities of the fundamental system of solutions of the homogeneous part of (3) were completely determined by $h_1(x) = \frac{(x-\phi_1)\,V'(x)}{V(x)}$, resp. $k_1(x) = \frac{(x-\phi_2)\,V'(x)}{V(x)}$. So the essential difference between L and K, expressed in (3) through the fast fluctuating β, only enters through the non-singular parts of the solutions. Hence we may again choose a pair of fundamental solutions of the equation $K\,v = 0$ given by

$$u_1(\phi) = \frac{b_1}{\phi - \phi_1} + \log(\phi - \phi_1) + w_1(\phi),$$

$$u_2(\phi) = \frac{b_2}{\phi_2 - \phi} + \log(\phi_2 - \phi) + w_2(\phi),$$

$\phi \in]\phi_1, \phi_2[$, where b_1, b_2 are constants, w_1, w_2 are bounded and continuous on $[\phi_1, \phi_2]$. The Green's kernel of K therefore has the same structure as in the preceding section

$$\Lambda(x,y) = u_1(x)\,u_2(y)\,1_{[\phi_1,x]}(y) + u_2(x)\,u_1(y)\,1_{[x,\phi_2]}(y),$$

$x, y \in [\phi_1, \phi_2]$. Via convolution

$$\Lambda p(x) = \int_{\phi_1}^{\phi_2} \Lambda(x,y)\,p(y)\,dy, \quad x \in [\phi_1, \phi_2],$$

Λ well defines a linear operator on the space of continuous functions $C([\phi_1, \phi_2])$. It is not hard to show that it is bounded with respect to the *sup*−norm $||.||_\infty$. Indeed, we have

$$\begin{aligned} ||\Lambda\,p||_\infty &= \sup_{x\in[\phi_1,\phi_2]} |u_1(x) \int_{\phi_1}^{x} u_2(y)\,p(y)\,dy + u_2(x) \int_{x}^{\phi_2} u_1(y)\,p(y)\,dy| \\ &\leq [\sup_{x\in[\phi_1,\phi_2]} |u_1(x)| \int_{\phi_1}^{x} |u_2(y)|\,dy + |u_2(x)| \int_{x}^{\phi_2} |u_1(y)|\,dy]\ ||p||_\infty, \end{aligned}$$

and (5) and (5) imply that the constant

$$\sup_{x\in[\phi_1,\phi_2]} |u_1(x)| \int_{\phi_1}^{x} |u_2(y)|\,dy + |u_2(x)| \int_{x}^{\phi_2} |u_1(y)|\,dy|$$

is finite. With this in mind, a few comments on the solvability of (3) and (3) are now in order.

Abbreviate $\alpha = \frac{V^2}{V^2+(\mu a h)^2}$, define

$$\begin{aligned} w_0 &= 0, \\ v_1 &= \Lambda(\alpha T), \\ w_1 &= \Lambda[\alpha(\frac{h}{V}T - \gamma v_1)], \end{aligned}$$

and recursively

$$\begin{aligned} v_{n+1} &= \Lambda[\alpha(T + \mu a \,\gamma\, w_n)], \\ w_{n+1} &= \mu a \;\Lambda[\alpha\,(\frac{h}{V}T - \gamma\, v_{n+1})]. \end{aligned}$$

Since Λ is bounded, a standard contraction argument provides the convergence of the sequences $(v_n)_{n\in\mathbf{N}}$ and $(w_n)_{n\in\mathbf{N}}$ in the space $C([\phi_1,\phi_2])$ with the norm $||.||_\infty$. In fact, the limits are even smoother, provide solutions of (3) and (3), and, still for $\mu < \frac{1}{a\,||\Lambda\,\alpha\,\gamma||}$ can be expressed in the Neumann series

$$v = \sum_{n=0}^{\infty}(-1)^n\,(\mu a)^{2n}\,(\Lambda\,\alpha\gamma)^{2n}\,\Lambda\,\alpha T + \sum_{n=0}^{\infty}(-1)^n\,(\mu a)^{2n+2}\,(\Lambda\,\alpha\gamma)^{2n+1}\,\Lambda\,\alpha\,\frac{h}{V}T,$$

$$w = \sum_{n=0}^{\infty}(-1)^n\,(\mu a)^{2n+1}\,(\Lambda\,\alpha\gamma)^{2n}\,\Lambda\,\alpha\,\frac{h}{V}T - \sum_{n=0}^{\infty}(-1)^n\,(\mu a)^{2n+1}\,(\Lambda\,\alpha\gamma)^{2n+1}\,\Lambda\,\alpha\,T.$$

By the continuity of v and w we can now finish our argument just as in the preceding section. Recalling $u = v + iw$, we have

$$\psi = U\,u$$

is a solution of (1) on the interval $[\phi_1,\phi_2]$, which satisfies $\psi(\phi_i) = 0, i = 1,2$, and $|\psi(\phi)| \leq c|U(\phi)|$ for all $\phi \in [\phi_1,\phi_2]$. Hence the solution of (1) decays like U near the critical lines. This again means that ψ cannot survive the critical lines next to the source S, and decays like U there. We formulate these facts in the following Theorem.

Theorem 5.1. *Suppose that U has at least two roots in $]-\frac{\pi}{2},\frac{\pi}{2}[$. Then for $\mu < \frac{1}{a\,||\Lambda\,\alpha\,\gamma||}$ there is a solution ψ of (1) which satisfies*

$$|\psi(\phi)| \leq c\,|U(\phi)|$$

for some constant c, and

$$\psi|_{[-\frac{\pi}{2},\underline{\phi}]} = \psi|_{[\overline{\phi},\frac{\pi}{2}]} = 0.$$

Any solution of (1) satisfies (5.1).

Due to the fact that we express solutions in terms of Neumann series, in Theorem 5.1 we are forced to make statements only for μ inside the interval of convergence of these series. We omit the mathematical treatment of the case of non-small μ at this place, not without remarking that this case is clearly more relevant to atmospheric flow.

So far, the effects on wave decay caused by the intervals of negative or positive U or the fast fluctuations of the function β remain unexplained. To find a mathematical explanation, we need the spectral decomposition of the Green's kernel Λ.

6. The spectrum of L

We shall first consider the problem without damping, i. e. investigate the operator L. Here we have to distinguish two completely different cases with essentially different localization patterns. Of course, as before we shall consider the situation on an interval $[\phi_1, \phi_2]$ bounded by critical lines. In case $U|_{]\phi_1,\phi_2[}$ is negative, we shall see that the spectrum of L is purely continuous and coincides with the one of the Laplace operator. As a consequence, the localization of waves related to L is not essentially stronger than already given by the impact of the critical lines. In case $U|_{]\phi_1,\phi_2[}$ is positive, besides the continuous spectrum $[0, \infty[$ of the Laplace operator there is a finite discrete set of negative eigenvalues, whose eigenfunctions are exponentially strongly localized, eventually near the negative absolute minimum of the effective potential function. We describe essential features, in particular the decay pattern, of these eigenfunctions by looking at their Liouville-Green approximation.

This one not being symmetric around the equator, convolutions of the spectral resolution of the associated Green's kernels Γ on the union of the finite number of excursion intervals of U with the source term will favor components which belong to negative *ground states* and at the same time are strongly localized and have a strong overlap with the source function. This seems to be responsible for the localization *around the source* as opposed to localization *by critical lines*, which emerges for larger values of η as poited out above. Though we shall not explicitly derive statements about the random *typical* location of the absolute minimum of the potential function from the knowledge of the covariance function of R, it shall become qualitatively clear from the following discussion that this explanation is mathematically reasonable.

We therefore consider the eigenvalue problem

$$Lu + \lambda u = 0.$$

To be able to refer to known results, let us transform the Sturm-Liouville form (6) into a more familiar Schrödinger type equation by using the classical variable change

$$x = T(\phi) = \int_{\phi_0}^{\phi} \frac{1}{V^2(\theta)}\, d\theta, \quad \phi_1 \le \phi \le \phi_2,$$

with some $\phi_0 \in]\phi_1, \phi_2[$, and set, for $x \in \mathbf{R}$

$$\begin{aligned} w(x) &= u(T^{-1}(x)), \\ W(x) &= V(T^{-1}(x)), \\ p(x) &= 2\Omega a h(T^{-1}(x)). \end{aligned}$$

Then (6) turns into

$$w'' + p\,W\,w + \lambda\,w = 0.$$

Of course, (6) and (6) have the same spectra. Now note that $\lim_{|x|\to\infty} |p\,W(x)| = 0$. So if we can show that $p\,W$ is locally in the *Kato class* K_1, a well known criterion based on Weyl's characterization (see Carmona, Lacroix [2], p. 76) will imply that the essential spectrum of L, i. e. the set of $\lambda \in \mathbf{R}$ for which the rank of the spectral projection in any vicinity of λ is infinite, is identical to the purely continuous spectrum $[0,\infty[$ of the Laplace operator. K_1 is just the set of measurable functions, say f on $\mathbf{R}$, for which the condition

$$\sup_{x\in\mathbf{R}} \int_{\{|x-y|\leq 1\}} |f(y)|dy < \infty$$

is valid. Due to boundedness of p it is enough to show this for W. If K is a compact subset of $\mathbf{R}$, $T^{-1}(K)$ is a compact subset of $]\phi_1,\phi_2[$. Therefore,

$$\int_K |W(x)|\,dx = \int_{T^{-1}(K)} |V(\phi)|\,\frac{1}{V^2(\phi)}\,d\phi < \infty.$$

To comment on the type of localization observed in (6), let us first consider the eigenfunctions corresponding to the continuous part of the spectrum. For this purpose we briefly look again at the solutions in the vicinity of the singular points ϕ_1, ϕ_2, and concentrate on ϕ_1. In a neighborhood of ϕ_1 the differential equation is described by

$$u''(\phi) + \frac{1}{\phi-\phi_1}\,h_1(\phi)\,u'(\phi) + \frac{1}{(\phi-\phi_1)^2}\,h_2(\phi)\,u(\phi) = 0,$$

with h_1, h_2 analytic, and $h_1(\phi_1) = 2, h_2(\phi_1) = \frac{\lambda}{V'(\phi_1)^2}$. Arguing as above, the singularities of the solutions are determined by the roots of the characteristic polynomial

$$\mu(\mu+1) + \frac{\lambda}{V'(\phi_1)^2} = 0,$$

i. e. by the exponents

$$\mu_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\lambda}{V'(\phi_1)^2}}.$$

Keep in mind in the following discussion that, to return to eigensolutions of the eigenvalue problem corresponding to the original equation (1), we have to multiply the solutions discussed by V on the interval $]\phi_1,\phi_2[$. This way e.g. sublinear growth will have to be interpreted as a type of *polynomial decay* for (1). Let us first discuss the case $\lambda > \frac{1}{4}\,V'(\phi_1)^2$. In this case the solutions are of oscillatory nature. A fundamental system is given by

$$u_1^\lambda(\phi) = (\phi-\phi_1)^{-\frac{1}{2}}[\cos(\sqrt{\frac{\lambda}{V'(\phi_1)^2}-\frac{1}{4}}\ \log(\phi-\phi_1)) \qquad (5)$$
$$+i\ \sin(\sqrt{\frac{\lambda}{V'(\phi_1)^2}-\frac{1}{4}}\ \log(\phi-\phi_1))]\ p_1(\phi),$$

$$u_2^\lambda(\phi) = (\phi-\phi_1)^{-\frac{1}{2}}[\cos(\sqrt{\frac{\lambda}{V'(\phi_1)^2}-\frac{1}{4}}\ \log(\phi-\phi_1)) \qquad (6)$$
$$-i\ \sin(\sqrt{\frac{\lambda}{V'(\phi_1)^2}-\frac{1}{4}}\ \log(\phi-\phi_1))]\ p_2(\phi),$$

with p_1, p_2 analytic, as long as $2\sqrt{\frac{\lambda}{V'(\phi_1)^2}-\frac{1}{4}} \notin i\,\mathbf{Z}$, and with an additional term containing
$(\phi-\phi_1)^{-\frac{1}{2}}\ \log(\phi-\phi_1)$ instead of just $(\phi-\phi_1)^{-\frac{1}{2}}$ if it is an integer on the imaginary axis.

Let us now consider the case $0 \leq \lambda \leq \frac{\lambda}{V'(\phi_1)^2}$. Here we have a fundamental system of functions with sublinear growth given by

$$u_1^\lambda(\phi) = (\phi-\phi_1)^{-\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\lambda}{V'(\phi_1)^2}}}\ p_1(\phi),$$

$$u_2^\lambda(\phi) = (\phi-\phi_1)^{-\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{\lambda}{V'(\phi_1)^2}}}\ p_2(\phi),$$

with p_1, p_2 analytic, plus an eventual logarithmic term.

To come to the part of the spectrum located on the negative half axis, we now have to make the crucial distinction. Let us first argue for the simpler case $U|_{]\phi_1,\phi_2[} < 0$, i. e. $W < 0$. A typical sample function of the potential function pW may then look like this.

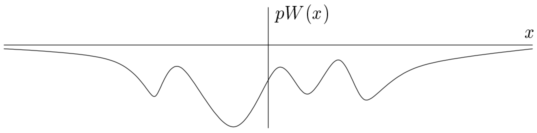

Figure 1

To explain the spectrum, we make use of the well known relationship between spectral theory and oscillation theory of eigenfunctions, as for example presented in Dunford, Schwartz [5]. First of all, if $\lambda < 0$, a pair of independent solutions is still given by formulas (6) and (6). But now (6) does no more respect the critical lines, since we have

$$\lim_{\phi\to\phi_1} V(\phi)\, u_2^\lambda(\phi) \in \{\pm\infty\}.$$

Hence there is only one solution satisfying the boundary condition

$$\lim_{\phi\to\phi_1} V(\phi)\, u^\lambda(\phi) = 0$$

at ϕ_1, namely u_1^λ. But since the same situation is met near ϕ_2, an argument involving analyticity and growth at the singular points shows that u_1^λ has to vanish. Hence the point spectrum below the infimum 0 of the essential spectrum is void.

Let us now consider the case $U|_{]\phi_1,\phi_2[} > 0$, i. e. $W > 0$. A typical sample of the potential function pW then has the shape depicted in Figure 2.

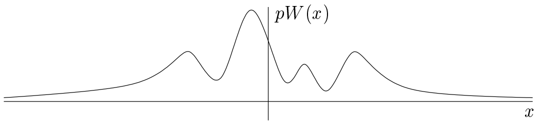

Figure 2

The operator $-L$ is bounded below by the constant $-\sup_{x\in\mathbf{R}} pW(x)$. Moreover, (6) and (6) and their analogues near ϕ_2 and regularity of solutions entail that there is at least one solution for $\lambda = 0$ which has only finitely many zeroes. Hence the hypotheses of Theorem 55 in Dunford, Schwartz [5], p. 1480, are satisfied and oscillation theory implies that the spectrum between $-\infty$ and 0 is pure point and contains a finite number of eigenvalues. We shall now give a qualitative description of a typical solution using the Liouville-Green approximation, usually called WKB approximation. A nice presentation of this approximation which also keeps track of the errors made can be found in Olver [13]. Fix a negative eigenvalue λ. Abbreviate

$$f(x) = -pW(x) - \lambda.$$

Since $\lim_{|x|\to\infty} pW(x) = 0$, $\mathbf{R}$ can be decomposed into finitely many intervals $]-\infty,\kappa_o],[\kappa_0,\kappa_1],\cdots,[\kappa_{n-1},\kappa_n],[\kappa_n,\infty[$ such that in the interior of each of the compact ones f has exactly one up- or downcrossing of level 0, and such that on the unbounded ones f is positive and bounded below by $\frac{|\lambda|}{2}$. To describe the behaviour of the eigensolution on one of the compact intervals, let us consider $[\kappa_1,\kappa_2]$ on which f has an upcrossing, say, at x_0. We follow Olver [13], pp. 397-400, define

$$\begin{aligned}
r(x) &= [\frac{f(x)}{x-x_0}]^{\frac{1}{2}},\\
s(x) &= \frac{1}{(x-x_0)^{\frac{3}{2}}}\int_{x_0}^{x}(t-x_0)^{\frac{1}{2}}\, r(t)\, dt,\\
\xi(x) &= (x-x_0)\,[\frac{3}{2}s(x)]^{\frac{2}{3}},\\
\hat{f}(x) &= r^2(x)\,[\frac{3}{2}\, s(x)]^{\frac{2}{3}},
\end{aligned}$$

$x \in \mathbf{R}$. Then according to Olver [13], p. 399, our equation on $[\kappa_1, \kappa_2]$ has the independent solutions

$$w_1^\lambda(x) = \hat{f}^{-\frac{1}{4}}(x)[Bi(\xi(x)) + \epsilon_1(x)],$$

$$w_2^\lambda(x) = \hat{f}^{-\frac{1}{4}}(x)[Ai(\xi(x)) + \epsilon_2(x)],$$

with uniformly bounded errors ϵ_1, ϵ_2, and *Airy's integrals*

$$\begin{aligned} Ai(x) &= \frac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}t^3 + x\,t)\,dt, \\ Bi(x) &= \frac{1}{\pi}\int_0^\infty [\exp(-\frac{1}{3}t^3 + x\,t) + \sin(\frac{1}{3}t^3 + x\,t)]\,dt, \end{aligned}$$

$x \in \mathbf{R}$. Let us now come to the essential point of localization, and describe the behaviour of the solutions on one of the unbounded intervals, say $[\kappa_n, \infty[$. Here we may, still following Olver [13], p. 193, find the solution

$$w^\lambda(x) = f(x)^{-\frac{1}{4}} \exp(-\int_{\kappa_n}^x \sqrt{f(y)}\,dy)\,(1 + \epsilon(x)),$$

with an error function ϵ which is bounded on $[\kappa_n, \infty[$. Most importantly, due to the fact that f is bounded below by $\frac{|\lambda|}{2}$, the solution decays exponentially as $x \to \infty$, where the exponential decay rate is at least as big as the (random) *spectral gap* between 0 and the closest negative eigenvalue. Of course, by matching boundary values, solutions on the finite set of intervals can be reasonably glued together to yield the eigenfunction for λ. To make the main finding still clearer, let us restate it in the framework of the old coordinates. Given an eigenvalue $\lambda < 0$, there is an interval $[\kappa_1, \kappa_2] \subset]\phi_1, \phi_2[$, depending crucially on the geometry of the potential function $2\Omega a h V$, and typically not symmetric with respect to the equator 0, such that the eigenfunction u^λ decays exponentially outside $[\kappa_1, \kappa_2]$ with rate at least half the spectral gap.

We may summarize our findings in the following Theorem.

Theorem 6.1. *Suppose $\mu = 0$. Let $\phi_1, \phi_2 \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ be such that $U(\phi_i) = 0, U'(\phi_i) \neq 0, i = 1, 2, U|_{]\phi_1,\phi_2[} \neq 0$. Then the essential part of the spectrum of L is given by $[0, \infty[$. The part of the spectrum contained in $]-\infty, 0[$ is pure point, and consists of at most finitely many simple eigenvalues.*

Eigenfunctions for $\lambda > \frac{1}{4}V'(\phi_1)^2$ near ϕ_1 are of oscillatory nature with a square root singularity, and given by (5), (6). Eigenfunctions for $0 \leq \lambda \leq \frac{1}{4}V'(\phi_1)^2$ near ϕ_1 have a singularity of sublinear growth, and are given by (6), (6). Similar results hold near ϕ_2.

In case $U|_{]\phi_1,\phi_2[} < 0$, the point spectrum is empty.

In case $U|_{]\phi_1,\phi_2[} > 0$, the point spectrum is generically non-empty.

For eigenvalues $\lambda < 0$ there exist intervals $[\kappa_1, \kappa_2] \subset]\phi_1, \phi_2[$ generically non-symmetric w.r.t. 0, such that the corresponding eigenfunctions decay exponentially *in* $(\phi-\phi_1)^{-\frac{1}{2}}$ *resp.* $(\phi_2-\phi)^{-\frac{1}{2}}$ *near* ϕ_1 *resp.* ϕ_2*, with a rate at least half the spectral gap.*

Corresponding eigenfunctions for the spectral decomposition related to (1) on the interval $[\phi_1,\phi_2]$ *for* $\lambda > \frac{1}{4}V'(\phi_1)^2$ *and near* ϕ_1 *are ocillatory and have a square root decay, for* $0 \le \lambda \le \frac{1}{4}V'(\phi_1)^2$ *near* ϕ_1 *have a decay of a sublinear power. Similar results hold near* ϕ_2*. For the finitely many negative eigenvalues outside of compact subintervals generically non-symmetric w.r.t. 0 eigenfunctions decay exponentially in* $(\phi-\phi_1)^{-\frac{1}{2}}$ *resp.* $(\phi_2-\phi)^{-\frac{1}{2}}$ *near* ϕ_1 *resp.* ϕ_2*, with a rate at least half the spectral gap.*

The results of Theorem 6.1 carry obvious consequences for the localization of solutions of (1) with a strongly local source S. One may write out a spectral resolution of the Green's kernel Λ using the information of the Theorem, and then take the convolution of the kernel with the source function. According to the statements of the Theorem, in intervals of negative U bounded by critical lines, the decay of eigenfunctions is at most polynomial near the critical lines. Hence the localization effect caused by the mere presence of critical lines will not be topped by much. In particular the geometry of the random potential U will not play a big role in these intervals.

However, since we consider the example of a superrotation flow, and the source is concentrated near 0, intervals of positive U are much more important. In fact, generically there is a big interval of positive U containing the equator, bounded by critical lines. If for some negative eigenvalue position and size of the characteristic interval outside of which exponential localization takes place, strongly overlaps with the support of the source, the contribution of its eigenfunction to the solution of (1) will be big, and the localization consequently centered around the source, and non-symmetric w.r.t. the equator. So if the fluctuation of U governed by the parameter η is sufficiently big, there will be a localization effect around the source which overrides the effect of the critical lines described above.

Due to the stronger fluctuations of U'' as compared to U the localization effect just described will still be enhanced when passing to the case of non-zero friction, which we shall treat in the following section.

7. The spectrum of K

Let us fix again an interval $[\phi_1,\phi_2]$ bounded by critical lines, on which U does not vanish. As was pointed out when discussing the critical lines for general μ, localization of (1) is closely related to the spectral decomposition of the Green's kernel Λ of K. We shall now point out that this one is again different from the one of L. Due to the appearance of the function β in the potential function of K which strongly depends on the second derivative of R, the system undergoes stronger fluctuations than without friction, and therefore the localizing effect of the negative part of the spectrum will be enhanced. So generically we have the same picture as in the intervals of positive U in the preceding section, just with a stronger role of the point spectrum.

Gluing together the intervals bounded by critical lines as usual, and using spectral resolutions of the Green's kernels on each of the intervals, we see that this leads to overall the same localization picture as in the preceding section, just with a stronger role of localization around the source.

We have to consider the eigenvalue problem

$$Ku + \lambda u = 0$$

on the fixed excursion interval $[\phi_1, \phi_2]$ of V from 0. Recall from (3) that the essential difference between L and K is the more heavily fluctuating function β. Let us start our analysis with a few, still superficial, remarks about the oscillation behaviour of β, recall that we are discussing the superrotation flow, and set, for simplicity, $U_{max} = 1$. So we have

$$U(\phi) = \cos\phi + \eta\, R(\phi).$$

Using the notation of previous sections and the definition of V, a little algebra gives us the equations

$$\begin{aligned}[V''\, h - V\, g](\phi) &= -2\,\cos^3\phi + \eta[R''(\phi)\cos^2\phi - R'(\phi)\,\sin\phi\,\cos\phi\, - R(\phi)] \\ &= -2\,\cos^3\phi + \eta[\cos\phi\,\,(R'\cos\phi)' - R(\phi)],\end{aligned}$$

and thus, abbreviating $\epsilon = \frac{\mu^2 a}{2\Omega}$,

$$\begin{aligned}\beta\,[V^2 + (\mu a h)^2](\phi) &= [V^2 + \epsilon\,(V''\,h - V\,g)](\phi) \\ &= \cos^2\phi\,(1 - 2\epsilon\,\cos\phi) + \eta^2\,R^2(\phi) \\ &\quad +\eta[(2\cos\phi - \epsilon)\,R(\phi) + \epsilon\,\cos\phi\,(R'\,\cos\phi)'(\phi)].\end{aligned} \tag{7}$$

(7) describes the fluctuation of β around zero level. If we recall from the remarks made in the beginning about the moments and regularity of R that

$$\begin{aligned}E(R(0)^2) &= 1, \\ E(R'(0)^2) &= \frac{1}{\tau^2}[1 + 4\alpha^2\,\tau^2], \\ E(R''(0)^2) &= \frac{1}{\tau^4}[3 + 24\alpha^2\,\tau^2 + 16\alpha^4\,\tau^4],\end{aligned}$$

it becomes plausible that the term in (7) which contributes most to the fluctuation of β around 0 is given by $\cos\phi(R'\,\cos\phi)'$. By the presence of the two cos functions in this term its fluctuations will in addition concentrate more around the equator 0. To stick to our pathwise analysis, we choose a typical trajectory of R for which β in the interval $[\phi_1, \phi_2]$ will have several zero crossings and local minima below 0 of varying depth. Let us, as in the preceding section,transform the Sturm-Liouville form (6) into a more familiar Schrödinger type equation by the change of variables

$$x = T(\phi) = \int_{\phi_0}^{\phi} \frac{1}{V^2(\theta)}\, d\theta, \quad \phi_1 \le \phi \le \phi_2,$$

with some $\phi_0 \in]\phi_1, \phi_2[$. Then, writing

$$\begin{aligned} w(x) &= u(T^{-1}(x)), \\ W(x) &= V(T^{-1}(x)), \\ p(x) &= 2\Omega a h(T^{-1}(x)), \\ \gamma(x) &= \beta(T^{-1}(x)), \end{aligned}$$

we obtain the eigenvalue problem

$$w'' + p\,W\,\gamma\,w + \lambda\,w = 0.$$

The typical trajectory of $p\,W\,\gamma$ looks like Figure 3.

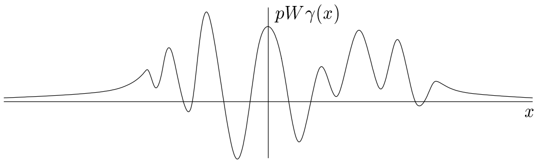

Figure 3

With the same arguments as in the preceding section we see that the potential function is in the Kato class K_1, and obviously we have

$$\lim_{|x|\to\infty} pW\gamma(x) = 0.$$

Therefore according to Carmona, Lacroix [2] the essential spectrum of the operator K is the same as the continuous spectrum of the Laplace operator $[0,\infty[$. Again $-K$ is bounded below by the constant $-\sup_{x\in\mathbf{R}} pW\gamma(x)$. Hence we may apply oscillation theory as before to obtain the main result.

Theorem 7.1. *Suppose $\mu \neq 0$. Let $\phi_1, \phi_2 \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ be such that $U(\phi_i) = 0, U'(\phi_i) \neq 0, i = 1, 2, U|_{]\phi_1,\phi_2[} \neq 0$. Then the essential part of the spectrum of K is given by $[0,\infty[$. The part of the spectrum contained in $]-\infty, 0[$ is pure point, and consists of at most finitely many simple eigenvalues.*

Eigenfunctions for $\lambda > \frac{1}{4}V'(\phi_1)^2$ near ϕ_1 are of oscillatory nature with a square root singularity, and given by (5), (6). Eigenfunctions for $0 \leq \lambda \leq \frac{1}{4}V'(\phi_1)^2$ near ϕ_1 have a singularity of sublinear growth, and are given by (6), (6). Similar results hold near ϕ_2.

For eigenvalues $\lambda < 0$ there exist intervals $[\kappa_1, \kappa_2] \subset]\phi_1, \phi_2[$ generically non-symmetric w.r.t. 0, such that the corresponding eigenfunctions decay exponentially in $(\phi - \phi_1)^{-\frac{1}{2}}$ resp. $(\phi_2 - \phi)^{-\frac{1}{2}}$ near ϕ_1 resp. ϕ_2, with a rate at least half the spectral gap.

Corresponding eigenfunctions for the spectral decomposition related to (1) on the interval $[\phi_1, \phi_2]$ for $\lambda > \frac{1}{4}V'(\phi_1)^2$ and near ϕ_1 are ocillatory and have a square root decay, for $0 \leq \lambda \leq \frac{1}{4}V'(\phi_1)^2$ near ϕ_1 have a decay of a sublinear power. Similar results hold near ϕ_2. For the finitely many negative eigenvalues outside of compact subintervals generically non-symmetric w.r.t. 0 eigenfunctions decay *exponentially in* $(\phi - \phi_1)^{-\frac{1}{2}}$ *resp.* $(\phi_2 - \phi)^{-\frac{1}{2}}$ *near* ϕ_1 *resp.* ϕ_2*, with a rate at least half the spectral gap.*

Although on the qualitative level we are working on we cannot precisely state this, here are a few remarks concerning the difference of localization described above with the one in the preceding section. First of all, since the moments of $R''(0)$ are essentially higher than the ones of $R(0)$, an asymptotic estimate for the number of negative eigenvalues which in the given setting is another challenge (considering that we cannot employ ergodic theory in an obvious manner), should reveal that the proportion of the point spectrum to the entire spectrum is much higher for K than for L. Also, the eigenvalues should be located deeper on an average, and therefore the localization effect outside small characteristic intervals defined by the deepest potential wells should be stronger generically. So we would expect localization around the source to overrun the basic localization by the critical lines at lower values of the parameter η. To make these statements precise, one should perhaps make assumptions on the stationary process R which allow to bring ergodic theory into the game, especially for applying multiplicative ergodic theory, and to be able argue with Lyapunov exponents for describing localization.

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Institut für Mathematik,
Humboldt-Universität zu Berlin,
Unter den Linden 6,
10099 Berlin,
Germany
E-mail address: `imkeller@mathematik.hu-berlin.de`

Earth and Ocean Sciences and Crisis Point Group,
Peter Wall Institute for Advanced Studies,
University of British Columbia,
Vancouver, B. C.,
Canada
E-mail address: `monahan@ocgy.ubc.ca, lionel@ocgy.ubc.ca`