Probability

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Lecture 14 - Part I

Sample space

The *sample space*, or outcome space, is the set of all possible outcomes. We denote it as Ω

Suppose we flip a coin once. Then the sample space is

$$\Omega = \{H, T\}$$

If instead we flip a coin twice, then the sample space is

$$\Omega = \{H, T\}^2 = \{HH, HT, TH, TT\}$$

Probability Distribution

First two of Kolmogorov's probability axioms

(1) For any outcome $a \in \Omega$, $P(a) \geq 0$

(2) $P(\Omega) = 1$ (with probability 1, *some* outcome must happen)

A third axiom: finite additivity (weaker than Kolmogorov's third axiom)

(3) For any disjoint subsets $A, B \subseteq \Omega$, $P(A \cup B) = P(A) + P(B)$

Coin-flipping example

Suppose we flip a coin once, so $\Omega = \{H, T\}$

Probability distribution of outcome is specified by the *Bernoulli distribution*.

Let $P(H) = p$. We call p the *success probability*.

A fair coin corresponds to $p = \frac{1}{2}$

Dice example

Suppose we roll a pair of dice; then $\Omega = \{1, 2, \ldots, 6\}^2$

Probability distribution for the outcome (a pair of numbers) is the *uniform distribution*.

The uniform distribution satisfies $P(a) = P(b)$ for all $a, b \in \Omega$

Therefore, we have $P(a) = \frac{1}{|\Omega|}$ for all $a \in \Omega$

In the dice example, $P(i, j) = \frac{1}{36}$ for any $i, j \in \{1, 2, \ldots, 6\}$

Events

An *event* $A \subseteq \Omega$ is a subset of the sample space

Suppose we flip a coin twice. Then $\{HT, TH\}$ is an event

The probability of an event A is $P(A) = \sum_{a \in A} P(a)$

Suppose we roll one die. What is the probability of rolling an even number? We can use shorthand:

$$P(a \text{ is even}) = P(\{a \in \Omega \colon a \text{ is even}\}) = P(\{2, 4, 6\})$$

Random variables

A *random variable* X is a function from the sample space to $V \subseteq \mathbb{R}$

$$X : \Omega \to V$$

Example 1

Suppose we roll a pair of dice and then win an amount of dollars equal to the sum of the rolls.

If the outcome is (a, b), then the amount we win is given by the random variable $X = a + b$.

Random variables

A *random variable* X is a function from the sample space to $V \subseteq \mathbb{R}$

$$X : \Omega \to V$$

Example 2

Suppose that K horses are racing, and we bet money on horse j.
If horse j wins the race, we win \$100; otherwise, we win \$0.

Formally, we have sample space $\Omega = \{1, 2, \ldots, K\}$, where the outcome is i if horse i wins.
The amount we win is given by the random variable:

$$X = 100 \cdot 1[\text{horse } j \text{ wins}] = 100 \cdot 1[a = j]$$

Events based on random variables

We can define events in terms of random variables

Example: For $v \in V$, we can define the event $X = v$

Formally, we have $P(X = v) = P(\{a \in \Omega \colon X(a) = v\})$

Let X be the sum of the numbers when rolling a pair of dice

$$\begin{aligned} P(X = 3) &= P\left(\left\{(a, b) \in \{1, 2, \ldots, 6\}^2 : a + b = 3\right\}\right) \\ &= P(\{(1, 2), (2, 1)\}) \\ &= \frac{2}{36} \end{aligned}$$

Events based on random variables

We can define events in terms of random variables

Example: For $U \subseteq V$, we can define the event $X \in U$

Formally, we have $P(X \in U) = P(\{a \in \Omega \colon X(a) \in U\})$

Let X be the sum of the numbers when rolling a pair of dice

$$
\begin{aligned}
P(X \leq 3) &= P(X = 2) + P(X = 3) \\
&= P(\{(1,1)\}) + P(\{(1,2),(2,1)\}) \\
&= \frac{3}{36}
\end{aligned}
$$

Expected value

For a random variable X, we defined the *expected value* as

$$\mathbb{E}[X] = \sum_{v \in V} v P(X = v)$$

This can be re-expressed as

$$\begin{aligned}
\mathbb{E}[X] = \sum_{v \in V} v P(X = v) &= \sum_{v \in V} v P(\{a \in \Omega : X(a) = v\}) \\
&= \sum_{v \in V} \sum_{a \in \Omega :\, X(a) = v} v P(a) \\
&= \sum_{v \in V} \sum_{a \in \Omega :\, X(a) = v} X(a) P(a) \\
&= \sum_{a \in \Omega} X(a) P(a)
\end{aligned}$$

Expected value - Exercise

Suppose that you are playing the game blackjack, and there are three outcomes: {`blackjack`, `win`, `lose`}.

If the outcome is “`blackjack`”, you win \$150.

If the outcome is “`win`”, you win \$100.

If the outcome is “`lose`”, you win -\$100 (so, you lose \$100).

Let the probability of `blackjack` be 0.02, the probability of `win` be 0.48, and the probability of `lose` be 0.5.

Let the random variable X be the amount of money you win. What is the expected value of X ?

Linearity of expectation

Linearity of expectation:

For random variables X and Y and constants a, b, c, we have:

$\mathbb{E}[aX] = a\mathbb{E}[X]$ and $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

so, in particular, $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$

Independence

Two events A and B are *independent* if $P(A \cap B) = P(A) \cdot P(B)$

Two random variables X and Y are *independent* if,
for any $u, v \in V$, the events $[X = u]$ and $[Y = v]$ are independent

Example: 3 coin flips

We have $P(HHT) = P(H)P(H)P(T)$