CONSTRAINED OPTIMIZATION:
THEORY AND ECONOMIC EXAMPLES

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These notes provide a brief review of methods for constrained optimization. They cover equality-constrained problems only. Part 1 outlines the basic theory. Part 2 provides a number of economic examples to illustrate the methods.

PART 1: THEORY

1.1 THE CONSTRAINED OPTIMIZATION PROBLEM

We begin with a constrained optimization problem of the type

$$\max_x \quad f(x_1,...,x_n) \quad \text{subject to} \quad g(x_1,...,x_n) = b$$

The function $f(x_1,...,x_n)$ is called the **objective function** or **maximand**; the equation $g(x_1,...,x_n) = b$ is called the **constraint**.

Remarks

1. We are restricting attention here to equality-constrained problems. An inequality-constrained problem would arise where the constraint is $g(x_1,...,x_n) \le b$. The techniques we develop here can be extended easily to that case.
2. A minimization problem with objective function $f(x)$ can be set up as a maximization problem with objective function $-f(x)$.

An Example

Utility maximization subject to a budget constraint.

(1.1) $$\max_x \; u(x_1,...,x_n) \quad \text{subject to} \quad \sum_{i=1}^{n} p_i x_i = m$$

In section 1.5 below we will consider a specific case where $n = 2$ and $u(x_1, x_2) = x_1^a x_2^b$ (Cobb-Douglas utility).

1.2 CHARACTERISTICS OF THE OPTIMUM

At the maximum of the objective function subject to the constraint, infinitesimal changes in the variables $x_1, x_2,..., x_n$ which satisfy the constraint must have no effect on the value of the objective function. Otherwise, we could not be at a maximum.

Thus, a necessary condition for the maximum is that $df = 0$ whenever $dg = 0$. That is, we require

(1.2) $$\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_n}dx_n = 0$$

for all dx_1, dx_2, …, dx_n satisfying

(1.3) $$\frac{\partial g}{\partial x_1}dx_1 + \frac{\partial g}{\partial x_2}dx_2 + ... + \frac{\partial g}{\partial x_n}dx_n = 0$$

where $\partial f / \partial x_i$ is the partial derivative of *f* with respect to x_i, and $\partial g / \partial x_i$ is the partial derivative of *g* with respect to x_i.

These conditions tells us that at the optimum there must be no way in which we can change the x_i's such that we can change the value of the function (and in particular, increase the value of the function) and still satisfy the constraint.

1.3 THE UNCONSTRAINED OPTIMUM

Note that in the absence of the constraint we would be seeking conditions under which (1.2) holds for all dx_1, dx_2, …, dx_n, rather than only those changes in *x* that satisfy (1.3). That is, we would require that the dx_i have zero coefficients in (1.2):

(1.4) $$\frac{\partial f}{\partial x_i} = 0 \quad \forall i$$

Note that this is a set of n equations in *n* unknowns.

An Example

Profit maximization for a "competitive" firm with Cobb-Douglas technology, given by

(1.5) $$h(x) = x_1^a x_2^b$$

The profit maximization problem is

(1.6) $$\max_x \quad p x_1^a x_2^b - w_1 x_1 - w_2 x_2$$

with first-order conditions

(1.7) $$apx_1^{a-1}x_2^b = w_1$$

and

(1.8) $$bpx_1^a x_2^{b-1} = w_2$$

We can solve these equations by first taking the ratio of (1.7) and (1.8) to obtain

(1.9) $$\frac{ax_2}{bx_1} = \frac{w_1}{w_2}$$

Now rearrange (1.9) to obtain:

(1.10) $$x_2 = \frac{bw_1x_1}{aw_2}$$

Substitute (1.10) into (1.7) or (1.8) and solve for x_1:

(1.11) $$x_1(p,w) = \left(\frac{bp}{w_2}\right)^{\frac{1}{1-a-b}}\left(\frac{aw_2}{bw_1}\right)^{\frac{1-b}{1-a-b}}$$

Then substitute (1.11) into (1.10) to obtain $x_2(p,w)$. These are the **factor demands** or **input demands**. We can then construct the **supply function** by substituting these factor demands into the production function:

(1.12) $$y(p,w) = x_1(p,w)^a x_2(p,w)^b$$

1.4 THE CONSTRAINED OPTIMUM: SOLUTION BY SUBSTITUTION

Rewrite (1.3) to isolate dx_1:

(1.13) $$dx_1 = -\frac{\sum_{i\neq 1}(\partial g/\partial x_i)dx_i}{\partial g/\partial x_1}$$

Now substitute this expression for dx_1 into (1.2) to obtain

(1.14) $$-\frac{\partial f}{\partial x_1}\left(\frac{\sum_{i\neq 1}(\partial g/\partial x_i)dx_i}{\partial g/\partial x_1}\right) + \sum_{i\neq 1}\frac{\partial f}{\partial x_i}dx_i = 0$$

This can be written as

(1.15) $$\sum_{i\neq 1}\left(-\frac{(\partial f/\partial x_1)(\partial g/\partial x_i)dx_i}{\partial g/\partial x_1}\right)+\sum_{i\neq 1}\frac{\partial f}{\partial x_i}dx_i=0$$

Then by collecting terms under the summation operator, we have

(1.16) $$\sum_{i\neq 1}\left(\frac{\partial f}{\partial x_i}-\frac{(\partial f/\partial x_1)(\partial g/\partial x_i)}{\partial g/\partial x_1}\right)dx_i=0$$

The only solution to this equation is to set all of the coefficients on the dx_i 's equal to zero since the equation must hold for <u>all</u> possible values of the dx_i 's. That is,

(1.17) $$\frac{\partial f}{\partial x_i}=\frac{(\partial f/\partial x_1)(\partial g/\partial x_i)}{\partial g/\partial x_1}\quad\forall i\neq 1$$

This can in turn be written as

(1.18) $$\frac{\partial f/\partial x_i}{\partial g/\partial x_i}=\frac{\partial f/\partial x_1}{\partial g/\partial x_1}\quad\forall i\neq 1$$

Note that (1.18) comprises $n-1$ equations. Together with the constraint itself we therefore have *n* equations which can be solved for the *n* unknowns (the x_i 's).

1.5 EXAMPLE: UTILITY MAXIMIZATION

Consider the case where $n=2$. In that case, the set of equations in (1.18) become a single equation:

(1.19) $$\frac{\partial f/\partial x_2}{\partial g/\partial x_2}=\frac{\partial f/\partial x_1}{\partial g/\partial x_1}$$

This in turn can be rearranged as

(1.20) $$\frac{\partial f/\partial x_1}{\partial f/\partial x_2}=\frac{\partial g/\partial x_1}{\partial g/\partial x_2}$$

In the utility maximization problem we have $\partial f/\partial x_i\equiv\partial u/\partial x_i$ and $\partial g/\partial x_i\equiv p_i$. Thus, (1.20) becomes

(1.21) $$\frac{\partial u/\partial x_1}{\partial u/\partial x_2}=\frac{p_1}{p_2}$$

This is the familiar tangency condition, stating that the slope of the indifference curve (the marginal rate of substitution) is equal to the slope of the budget constraint at the optimum.

In the specific case of Cobb-Douglas (CD) utility, (1.21) becomes

(1.22) $$\frac{ax_1^{a-1}x_2^b}{bx_1^a x_2^{b-1}} \equiv \frac{ax_2}{bx_1} = \frac{p_1}{p_2}$$

Thus, in the case of CD utility, the consumption ratio is inversely proportional to the price ratio.

Note that (1.22) is one equation in two unknowns. It tells us the relationship between x_1 and x_2 at the optimum but cannot be solved for unique values of x_1 and x_2. In the geometric interpretation, it tells us that we must have a tangency but it does not tells us <u>where</u> that tangency must be. For that we need additional information: the <u>position</u> of the budget constraint (as opposed to its slope). That information is contained in the budget constraint itself, which in the $n = 2$ case is

(1.23) $$p_1 x_1 + p_2 x_2 = m$$

Combining equations (1.22) and (1.23), we have two equations in two unknowns, which can be solved by simple substitution. In particular, express (1.22) as

(1.24) $$p_1 x_1 = \frac{ap_2 x_2}{b}$$

and substitute into the budget constraint to obtain

(1.25) $$\frac{ap_2 x_2}{b} + p_2 x_2 = m$$

Solving for x_2 yields

(1.26) $$x_2(p,m) = \frac{bm}{(a+b)p_2}$$

Substituting this solution for x_2 into (1.24) then yields the solution for x_1:

(1.27) $$x_1(p,m) = \frac{am}{(a+b)p_1}$$

Equations (1.26) and (1.27) are the **Marshallian demands**; they relate the demand for each good to the prices, and to income.

1.6 THE LAGRANGE MULTIPLIER APPROACH

The Lagrange multiplier approach to the constrained maximization problem is a useful mathematical algorithm that allows us to reconstruct the constrained problem as an unconstrained problem which yields (1.18) as its solution.

Consider the problem

(1.28) $$\max_x \quad f(x) \quad \text{subject to} \quad g(x,b)=0$$

where $x=\{x_1,...,x_n\}$ and b is a parameter in the constraint, identified explicitly here because we have some particular interest in it.

To solve this problem, we define the **Lagrangean** by introducing a new variable λ called the **Lagrange multiplier** (LM):

(1.29) $$L(x,\lambda)=f(x)+\lambda g(x)$$

We then solve the unconstrained maximization problem

(1.30) $$\max_{x,\lambda} \; L(x,\lambda)$$

The necessary (first-order) conditions for a maximum are

(1.31) $$\frac{\partial L}{\partial x_i}=0 \quad \forall i$$

and

(1.32) $$\frac{\partial L}{\partial \lambda}=0$$

Note that (1.31) comprises n equations, so together with (1.32) we have $n+1$ equations in $n+1$ unknowns (including λ). Take these derivatives of $L(x,\lambda)$ to yield

(1.33) $$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i} = 0 \quad \forall i$$

and

(1.34) $$\frac{\partial L}{\partial \lambda} = g(x,b) = 0$$

Now take the ratio of any two equations from (1.33), say for $i = 1$ and $i = j$:

(1.35) $$\frac{\partial f / \partial x_j}{\partial f / \partial x_1} = \frac{\partial g / \partial x_j}{\partial g / \partial x_1} \quad \forall j \neq 1$$

Note that the LM has now been eliminated. This expression can be rearranged to yield

(1.36) $$\frac{\partial f / \partial x_j}{\partial g / \partial x_j} = \frac{\partial f / \partial x_1}{\partial g / \partial x_1} \quad \forall j \neq 1$$

These $n-1$ equations are the same as conditions (1.18). Thus, the LM method yields the solution to the constrained optimization problem. The additional information we need to complete the solution is the constraint itself, and this is given by (1.34). Thus, conditions (1.34) and (1.36) describe a complete solution to the constrained problem.

<u>Example 1: Utility Maximization Revisited</u>

Recall the constrained optimization problem for CD utility:

(1.37) $$\max_x \; x_1^a x_2^b \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 = m$$

Construct the Lagrangean:

(1.38) $$L = x_1^a x_2^b + \lambda(m - p_1 x_1 - p_2 x_2)$$

Derive the first-order conditions:

(1.39) $$a x_1^{a-1} x_2^b - \lambda p_1 = 0$$

(1.40) $$b x_1^a x_2^{b-1} - \lambda p_2 = 0$$

(1.41) $$m - p_1 x_1 - p_2 x_2 = 0$$

Take the ratio of (1.39) and (1.40) to obtain

(1.42) $$\frac{ax_2}{bx_1} = \frac{p_1}{p_2}$$

and rearrange this to yield

(1.43) $$x_2 = \frac{bp_1 x_1}{ap_2}$$

Now substitute (1.43) into the budget constraint (1.41):

(1.44) $$m = p_1 x_1 + p_2 \left(\frac{bp_1 x_1}{ap_2} \right)$$

and solve for x_1:

(1.45) $$x_1(p,m) = \frac{am}{(a+b)p_1}$$

Then substitute (1.45) into (1.43) to yield

(1.46) $$x_2(p,m) = \frac{bm}{(a+b)p_2}$$

<u>Example 2: Generalized Log-Linear Utility</u>

(1.47) $$\max_x \sum_{i=1}^{n} a_i \log x_i \quad \text{subject to} \quad \sum_{i=1}^{n} p_1 x_i = m$$

Construct the Lagrangean:

(1.48) $$L = \sum_{i=1}^{n} a_i \log x_i + \lambda(m - \sum_{i=1}^{n} p_i x_i)$$

Derive the first-order conditions:

(1.49) $$\frac{a_i}{x_i} - \lambda p_i = 0 \quad \forall i$$

(1.50) $$m-\sum_{i=1}^{n} p_i x_i = 0$$

In this example, taking the ratio of any pair of equations from (1.49) will yield the usual tangency condition, but it is not the most efficient way to solve the problem. Instead we will use an alternative solution method that usually performs better when we have $n > 2$ variables.

Rearrange (1.49) to yield

(1.51) $$a_i = \lambda p_i x_i \quad \forall i$$

Now take the <u>sum</u> over *i* on both sides of (1.51) to yield

(1.52) $$\sum_{i=1}^{n} a_i = \lambda \sum_{i=1}^{n} p_i x_i$$

Substitute *m* for expenditure in the RHS term to yield:

(1.53) $$\sum_{i=1}^{n} a_i = \lambda m$$

and rearrange to solve for λ:

(1.54) $$\lambda = \frac{\sum_{i=1}^{n} a_i}{m}$$

Now substitute (1.54) for λ in (1.51) and solve for x_i:

(1.55) $$x_i(p,m) = \frac{a_i m}{p_i \sum_{i=1}^{n} a_i}$$

Note that in the special case where $n = 2$, these solutions become

(1.56) $$x_1(p,m) = \frac{a_1 m}{(a_1 + a_2) p_1}$$

(1.57) $$x_2(p,m) = \frac{a_2 m}{(a_1 + a_2) p_2}$$

Compare these with the CD Marshallian demands from (1.45) and (1.46). They are the same solutions (with $a_1 = a$ and $a_2 = b$). Why? The CD utility function and the log-linear utility function represent exactly the same preferences; one function is a monotonic transform of the other. A monotonic transform does not change the underlying preferences because preferences have no cardinal interpretation; they are an ordinal notion only.

1.7 THE VALUE FUNCTION AND THE ENVELOPE THEOREM

Consider again the generalized utility-maximization problem from Section 1.6. Substitute the Marshallian demands back into the utility function to obtain utility as a function of prices and income:

(1.58) $$v(p,m) = u(x(p,m)) = \sum_{i=1}^{n} a_i \log\left(\frac{a_i m}{p_i A}\right)$$

where $A = \sum_{i=1}^{n} a_i$. This function tells us the maximized value of utility at any given prices and income. It is called the **indirect utility function**, and is a special case of a **value function**. In general, the value function associated with a constrained optimization problem tells us the maximized value of the objective function as a function of the constraint parameters.

The value function has a special relationship to the Lagrange multiplier. To see this, expand the expression from (1.58) above to obtain

(1.59) $$v(p,m) = A\log(m) + \sum_{i=1}^{n} a_i \log\left(\frac{a_i}{p_i A}\right)$$

and take the derivate with respect to *m* to obtain

(1.60) $$\frac{\partial v(p,m)}{\partial m} = \frac{A}{m}$$

This derivate tells us the amount by which utility rises with a marginal increase in income. The derivate is called the **marginal utility of income**. Note from (A1.54) above that this derivative is equal to the value of the Lagrange multiplier at the optimum. This link between the Lagrange multiplier and the value function is an implication of the following important theorem.

The Envelope Theorem

Consider a slightly generalized form of our optimization problem from section A1.1:

(1.61) $$\max_x \; f(x,b) \text{ subject to } g(x,b)=0$$

This generalizes our earlier problem by allowing the constraint parameter to enter the objective function itself.

Now define the associated Lagrangean,

(1.62) $$L = f(x,b) + \lambda g(x,b)$$

and let $x^*(b)$ denote the solution. Furthermore, let $v(b) \equiv f(x^*(b),b)$ denote the associated value function. Then the envelope theorem states that

(1.63) $$v'(b) = \frac{\partial f}{\partial b} + \lambda \frac{\partial g}{\partial b}$$

Proof. By differentiation of $v(b)$:

(1.64) $$v'(b) = \sum_{i=1}^{n} \left(\frac{\partial f(x^*(b),b)}{\partial x_i} \right) \left(\frac{\partial x_i^*}{\partial b} \right) + \frac{\partial f(x^*(b),b)}{\partial b}$$

But since $x^*(b)$ is optimal for *b*, it must satisfy the FOCs and the constraint. Thus,

(1.65) $$\frac{\partial f(x^*(b),b)}{\partial x_i} = -\lambda \frac{\partial g(x^*(b),b)}{\partial x_i} \quad \forall i$$

and

(1.66) $$g(x^*(b),b) = 0$$

Differentiating equation (1.66) yields

(1.67) $$\sum_{i=1}^{n} \left(\frac{\partial g(x^*(b),b)}{\partial x_i} \right) \left(\frac{\partial x_i^*}{\partial b} \right) + \frac{\partial g(x^*(b),b)}{\partial b} = 0$$

Substituting (1.65) and (1.67) into (1.64) yields

(1.68) $$v'(b) = \frac{\partial f}{\partial b} + \lambda \frac{\partial g}{\partial b}$$

which proves the result.♣

Interpretation. A change in the parameter does cause a change in the value of $x^*(b)$ but at the margin that change in x^* has no effect on *L* since x^* is a turning point of *L*.

In our utility-maximization problem, the constraint is $g(x,m) = m - \sum_{i=1}^{n} p_i x_i$, where m takes the role of *b*. This is linear in m , so $\partial g / \partial b = 1$ for our problem. Moreover, m does not enter the utility function directly in our problem, so $\partial g / \partial b = 0$ for our problem. Thus, for our problem, the envelope theorem tells us that $v'(b) = \lambda$; the marginal utility of income is equal to the Lagrange multiplier.

PART 2: OTHER ECONOMIC EXAMPLES

This section presents a number of examples from microeconomics, primarily related to consumer theory and industrial organization. They are presented in the form of a question and a solution.

2.1 UTILITY MAXIMIZATION WITH ADDITIVELY SEPARABLE UTILITY

A consumer has the following utility function:

$$u(x) = x_1^{1/2} + x_2^{1/2}$$

Consider the associated utility maximization problem:

$$\max_x \quad x_1^{1/2} + x_2^{1/2} \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 = m$$

Find the solution to this problem.

Solution

This is a constrained optimization problem. Set up the Lagrangean:

$$L(x,\lambda) = x_1^{1/2} + x_2^{1/2} + \lambda[m - (p_1 x_1 + p_2 x_2)]$$

The first-order conditions are

(1) $$\frac{\partial L}{\partial x_1} = \frac{x_1^{-1/2}}{2} - \lambda p_1 = 0$$

(2) $$\frac{\partial L}{\partial x_2} = \frac{x_2^{-1/2}}{2} - \lambda p_2 = 0$$

(3) $$\frac{\partial L}{\partial \lambda} = m - (p_1 x_1 + p_2 x_2) = 0$$

Take the ratio of (1) and (2) so as to eliminate the LM, and square both sides:

(4) $$\frac{x_2}{x_1} = \left(\frac{p_1}{p_2}\right)^2$$

Rearrange (4) to express x_2 as the subject:

(5) $$x_2 = x_1\left(\frac{p_1}{p_2}\right)^2$$

Substitute (5) into (3) – the constraint – to yield

(6) $$p_1 x_1 + p_2 x_1\left(\frac{p_1}{p_2}\right)^2 = m$$

Collect terms in x_1 and rearrange to yield

(7) $$x_1(p,m) = \frac{mp_2}{p_1^2 + p_1 p_2}$$

Substitute (7) into (5) to yield

(8) $$x_2(p,m) = \frac{mp_1}{p_2^2 + p_1 p_2}$$

Economic Interpretation

Equations (7) and (8) are **Marshallian demands**. Some properties of these particular Marshallian demands:

(9) $$\frac{\partial x_1(p,m)}{\partial m} = \frac{p_2}{p_1^2 + p_1 p_2} > 0$$

(10) $$\frac{\partial x_1(p,m)}{\partial p_1} = -\frac{mp_2(2p_1 + p_2)}{p_1^2(p_1 + p_2)^2} < 0$$

(11) $$\frac{\partial x_1(p,m)}{\partial p_2} = \frac{m}{(p_1 + p_2)^2} > 0$$

2.2 EXPENDITURE MINIMIZATION AND THE HICKSIAN DEMAND CURVES

A consumer has the following utility function:

$$u(x) = x_1^{1/2} + x_2^{1/2}$$

Consider the associated expenditure minimization problem:

$$\min_x \; p_1 x_1 + p_2 x_2 \;\; \text{subject to} \;\; x_1^{1/2} + x_2^{1/2} = \bar{u}$$

Find the solution to this problem.

Solution

This is a constrained optimization problem. Set up the Lagrangean:

$$L(x,\lambda) = p_1 x_1 + p_2 x_2 + \lambda[\bar{u} - (x_1^{1/2} + x_2^{1/2})]$$

The first-order conditions are

(1) $$\frac{\partial L}{\partial x_1} = p_1 - \lambda \frac{x_1^{-1/2}}{2} = 0$$

(2) $$\frac{\partial L}{\partial x_2} = p_2 - \lambda \frac{x_2^{-1/2}}{2} = 0$$

(3) $$\frac{\partial L}{\partial \lambda} = \bar{u} - (x_1^{1/2} + x_2^{1/2}) = 0$$

Take the ratio of (1) and (2) so as to eliminate the LM, and square both sides:

(4) $$\frac{x_2}{x_1} = \left(\frac{p_1}{p_2}\right)^2$$

Rearrange (4) to express x_2 as the subject:

(5) $$x_2 = x_1\left(\frac{p_1}{p_2}\right)^2$$

Substitute (5) into (3) – the constraint – to yield

(6) $$\bar{u} = x_1^{1/2} + \left(x_1\left(\frac{p_1}{p_2}\right)^2\right)^{1/2}$$

Collect terms in x_1 and rearrange to yield

(7)

Substitute (7) into (5) to yield

(8) $$x_2(p,\bar{u}) = \bar{u}^2\left(\frac{p_1}{p_1 + p_2}\right)^2$$

Economic Interpretation

Equations (7) and (8) are **Hicksian demands** (or **compensated demands**). They measure only the substitution effect associated with a price change (as opposed to the Marshallian demands, which measure both the substitution effect and the income effect).

2.3 PROFIT MAXIMIZATION WITH ADDITIVELY SEPARABLE PRODUCTION

A firm has the following production function

$$y = (x_1^{1/2} + x_2^{1/2})$$

It faces a product price p, and input prices w_1 and w_2 for inputs x_1 and x_2 respectively. Solve its profit maximization problem:

$$\max_{x_1, x_2} \; p(x_1^{1/2} + x_2^{1/2}) - (w_1 x_1 + w_2 x_2)$$

Solution

This is an unconstrained optimization problem. The first-order conditions with respect to x_1 and x_2 are

(1) $$\frac{p x_1^{-1/2}}{2} - w_1 = 0$$

(2) $$\frac{p x_2^{-1/2}}{2} - w_2 = 0$$

These two equations solve independently of each other. Simply rearrange (1) to obtain:

(3) $$x_1(p, w) = \left(\frac{p}{2w_1}\right)^2$$

and rearrange (2) to obtain

(4) $$x_2(p, w) = \left(\frac{p}{2w_2}\right)^2$$

Economic Interpretation

Equations (3) and (4) and the **input demands**. The fact that $x_1(p,w)$ is independent of p_2, and $x_2(p,w)$ is independent of p_1 is a special property reflective of the additively separable production function in this example.

Substituting these input demands back into the production function yields the **supply function**:

(5) $$y(p,w) = x_1(p,w)^{1/2} + x_2(p,w)^{1/2}$$

That is,

(6) $$y(p,w) = \left(\left(\frac{p}{2w_1}\right)^2\right)^{1/2} + \left(\left(\frac{p}{2w_2}\right)^2\right)^{1/2} = \frac{p(w_1 + w_2)}{2w_1 w_2}$$

2.4 COST MINIMIZATION WITH ADDITIVELY SEPARABLE PRODUCTION

A firm has the following production function

$$y = (x_1^{1/2} + x_2^{1/2})$$

It faces input prices w_1 and w_2 for inputs x_1 and x_2 respectively. Solve its cost minimization problem:

$$\min_{x_1, x_2} \; w_1 x_1 + w_2 x_2 \;\; \text{subject to} \;\; \bar{y} = (x_1^{1/2} + x_2^{1/2})$$

Solution

This is a constrained optimization problem. Set up the Lagrangean:

$$L(x,\lambda) = w_1 x_1 + w_2 x_2 + \lambda[\bar{y} - (x_1^{1/2} + x_2^{1/2})]$$

The first-order conditions are

(1) $$\frac{\partial L}{\partial x_1} = w_1 - \frac{\lambda x_1^{-1/2}}{2} = 0$$

(2) $$\frac{\partial L}{\partial x_2} = w_2 - \frac{\lambda x_2^{-1/2}}{2} = 0$$

(3) $$\frac{\partial L}{\partial \lambda} = \bar{y} - (x_1^{1/2} + x_2^{1/2}) = 0$$

Take the ratio of (1) and (2) to obtain

(4) $$\frac{w_1}{w_2} = \left[\frac{x_2}{x_1}\right]^{1/2}$$

Express x_2 in terms of x_1, substitute into (3) – the constraint – and solve for x_1:

(5) $$x_1(w, \bar{y}) = \bar{y}^2 \left[\frac{w_2}{w_1 + w_2}\right]^2$$

Substitute (5) into (4) and solve for x_2:

(6) $$x_2(w, \bar{y}) = \bar{y}^2 \left[\frac{w_1}{w_1 + w_2}\right]^2$$

Economic Interpretation

Equations (5) and (6) are the **conditional input demands**. They tell us how much of each input the firm will demand in order to produce a given level of output $\bar{y}$. Equation (4) is a tangency condition: it tells us that the slope of the isocost line (the LHS) is equal to the slope of the isoquant (the RHS), also called the **marginal rate of technical substitution**.

We can construct the **cost function** from the conditional input demands. In particular, substitute the conditional input demands into the expression for cost to obtain

(7) $$c(w, \bar{y}) = w_1 x_1(w, \bar{y}) + w_2 x_2(w, \bar{y})$$

The cost function tells us the minimum cost of producing a given level of output $\bar{y}$ for any given input prices w_1 and w_2. In our example, substituting for $x_1(w, \bar{y})$ and $x_2(w, \bar{y})$ yields

(8) $$c(w, \bar{y}) = \bar{y}^2 \left[\frac{w_1 w_2^2 + w_2 w_1^2}{(w_1 + w_2)^2}\right] = y^2 \left[\frac{w_1 w_2 (w_1 + w_2)}{(w_1 + w_2)^2}\right]$$

Simplifying yields

(9) $$c(w,\bar{y}) = \bar{y}^2\left[\frac{w_1 w_2}{w_1 + w_2}\right]$$

Note that this function is strictly convex in $\bar{y}$. That is, marginal cost is upward-sloping. This reflects the fact that the production function exhibits decreasing returns to scale (ie., it is homogeneous of degree less than one).

2.5 PROFIT MAXIMIZATION USING THE COST FUNCTION

Reconsider the firm from Example 4. We know from Example 5 that it's cost function is given by

(1) $$c(w,y) = y^2\left[\frac{w_1 w_2}{w_1 + w_2}\right]$$

for any given level of output y. Find the profit-maximizing level of output for this firm, using the cost function:

$$\max_y \quad py - c(w,y)$$

Solution

This is an unconstrained optimization problem in a single variable, y. The problem is

$$\max_y \quad py - y^2\left(\frac{w_1 w_2}{w_1 + w_2}\right)$$

This first-condition is

(2) $$p - 2y\left(\frac{w_1 w_2}{w_1 + w_2}\right) = 0$$

Solving (2) for y yields

(3) $$y(p,w) = \frac{p(w_1 + w_2)}{2w_1 w_2}$$

Economic Interpretation

Note that (3) is exactly the same supply function we obtained in Example 2.3; see equation (6) from that example. Examples 2.4 and 2.5 correspond to the two stages of a two-stage approach to the profit-maximization problem for a competitive firm that must yield the same result obtained from the single-stage direct profit maximization problem in Example 2.3.

2.6 MONOPOLY PROFIT MAXIMIZATION

Suppose a monopolist has the production function from Example 2.5 above, and faces input prices w_1 and w_2 for inputs x_1 and x_2 respectively. Then its cost function is given by (9) from Example 2.5. For simplicity of notation, define

$$c \equiv \left[\frac{w_1 w_2}{w_1 + w_2} \right]$$

Then the cost function is given by

(1) $$c(y) = cy^2$$

for any given level of output *y*. Now suppose this monopolist faces an inverse market demand curve given by

(2) $$p(y) = a - by$$

Solve the profit-maximization problem for this monopolist:

$$\max_y \quad p(y)y - c(y)$$

and verify that your does indeed yield a maximum.

Solution

This is an unconstrained maximization problem in a single variable. The problem is

$$\max_y \quad (a - by)y - cy^2$$

The first-order condition is

(3) $$a - 2by - 2cy = 0$$

Solving (3) for y yields

(4) $$y^M = \frac{a}{2(b+c)}$$

To verify that we have found a maximum, we need to check second-order conditions. We know that the first-order conditions are necessary and sufficient if the objective function is strictly concave. Take the second derivative of profit with respect to *y* to yield

(5) $$-2(b+c) < 0 \quad \text{for} \quad b > 0 \text{ and } c > 0$$

That is, the objective function is strictly concave in *y* if the inverse demand curve is down-ward sloping and the cost function is upward-sloping.

Economic Interpretation

The general form of the first-order condition is

(5) $$p'(y)y + p(y) - c'(y) = 0$$

which we usually write as $MR = MC$:

(6) $$p'(y)y + p(y) = c'(y)$$

In the case of the specific functional forms we have used, $MR = a - 2by$ and $MC = 2cy$.

Note that solution in (4) is not a supply function; in particular, it does not specify a level of output as a profit-maximizing response to a particular market price. The monopolist is not a price-taker, and hence, it does not have a supply curve. It chooses price and quantity to maximize profit.

2.7 MONOPOLY PROFIT MAXIMIZATION WITH TWO MARKETS

Suppose the monopolist from Example 2.6 sells its output in two distinct markets. Inverse demand in market 1 is given by

(1) $$p_1(y_1) = a_1 - by_1$$

and inverse demand in market 2 is given by

(2) $$p_2(y_2) = a_2 - by_2$$

Assume that consumers cannot trade with other each across these markets. Thus, the firm can set different prices in these two markets. The firm produces all of its output in a single plant – with the cost function specified in (1) from Example 2.6; that is, $c(y) = cy^2$. Assume $c = \frac{1}{2}$. How much will the monopolist sell in each market?

Solution

This is a constrained optimization problem in three variables:

$$\max_{y_1, y_2, y} \; p_1(y_1)y_1 + p_2(y_2)y_2 - c(y) \quad \text{subject to} \quad y_1 + y_2 = y$$

This can be solved easily as an unconstrained problem by substituting $y_2 = y - y_1$, but for illustrative purposes we will use the LM method instead.

The Lagrangean is

$$L(y_1, y_2, y, \lambda) = (a_1 - b_1 y_1)y_1 + (a_2 - b_2 y_2)y_2 - \frac{y^2}{2} + \lambda[y - (y_1 + y_2)]$$

The first-order conditions are

(3) $$\frac{\partial L}{\partial y_1} = a_1 - 2by_1 - \lambda = 0$$

(4) $$\frac{\partial L}{\partial y_2} = a_2 - 2by_2 - \lambda = 0$$

(5) $$\frac{\partial L}{\partial y} = -y + \lambda = 0$$

(6) $$\frac{\partial L}{\partial \lambda} = y - (y_1 + y_2) = 0$$

Use (5) to substitute for λ in (3) and (4) to obtain, respectively,

(7) $$a_1 - 2by_1 = y$$

(8) $$a_2 - 2by_2 = y$$

Rearrange these expressions to obtain

(9) $$y_1 = \frac{a_1 - y}{2b}$$

(10) $$y_2 = \frac{a_2 - y}{2b}$$

respectively. Then substitute (9) and (10) into (6) – the constraint – to obtain

(11) $$y = \frac{a_1 - y}{2b} + \frac{a_2 - y}{2b}$$

Solve (11) for *y*:

(12) $$y = \frac{a_1 + a_2}{2(b+1)}$$

Substitute (12) back into (9) and (10) to solve for y_1 and y_2 respectively:

(13) $$y_1 = \frac{a_1(2b+1) - a_2}{4b(b+1)}$$

(14) $$y_2 = \frac{a_2(2b+1) - a_1}{4b(b+1)}$$

Economic Interpretation

The essential relationships are equations (7) and (8). These state that $MR_1 = MC$ and $MR_2 = MC$, respectively. The logic is as follows. First, marginal revenue must be equated in the two markets. If not, then total revenue could be increased by reallocating output from one market to the other. Second, marginal revenue (in both markets) must be equated to marginal cost, or else profit could be increased by raising or reducing total output.

2.8 MONOPOLY PROFIT MAXIMIZATION WITH TWO PRODUCTION PLANTS

Consider a monopolist that sells to a single market but draws output from two different plants. The cost function for plant number 1 is

(1) $$c_1(y_1) = c_1 y_1^2$$

and the cost function for plant number 2 is

(2) $$c_2(y_2) = c_2 y_2^2$$

where c_1 and c_2 are positive constants. Inverse demand is given by $p(y) = a - by$. How much will the monopolist produce in each plant?

Solution

This is a constrained optimization problem in three variables:

$$\max_{y_1, y_2, y} \; p(y)y - [c_1(y_1) + c_2(y_2)] \;\; \text{subject to} \;\; y_1 + y_2 = y$$

This can be solved easily as an unconstrained problem by substituting $y_2 = y - y_1$, but for illustrative purposes we will use the LM method instead.

The Lagrangean is

$$L(y_1, y_2, y, \lambda) = (a - by)y - c_1 y_1^2 - c_2 y_2^2 + \lambda[y - (y_1 + y_2)]$$

The first-order conditions are

(3) $$\frac{\partial L}{\partial y_1} = -2c_1 y_1 - \lambda = 0$$

(4) $$\frac{\partial L}{\partial y_2} = -2c_2 y_2 - \lambda = 0$$

(5) $$\frac{\partial L}{\partial y} = a - 2by + \lambda = 0$$

(6) $$\frac{\partial L}{\partial \lambda} = y - (y_1 + y_2) = 0$$

Use (5) to substitute for λ in (3) and (4) to obtain, respectively,

(7) $$2c_1 y_1 = a - 2by$$

(8) $$2c_2 y_2 = a - 2by$$

Rearrange these expressions to obtain

(9) $$y_1 = \frac{a - 2by}{2c_1}$$

(10) $$y_2 = \frac{a - 2by}{2c_2}$$

respectively. Then substitute (9) and (10) into (6) – the constraint – to obtain

(11) $$y = \frac{a - 2by}{2c_1} + \frac{a - 2by}{2c_2}$$

Solve (11) for *y*:

(12) $$y^M = \frac{a(c_1 + c_2)}{2[c_1 c_2 + b(c_1 + c_2)]}$$

Substitute (12) back into (9) and (10) to solve for y_1 and y_2 respectively:

(13) $$y_1^M = \frac{ac_2}{2[c_1 c_2 + b(c_1 + c_2)]}$$

(14) $$y_2^M = \frac{ac_1}{2[c_1 c_2 + b(c_1 + c_2)]}$$

Economic Interpretation

The essential relationships are equations (7) and (8). These state that $MC_1 = MR$ and $MC_2 = MR$, respectively. The logic is as follows. First, marginal cost must be equated in the two plants. If not, then total cost could be reduced by reallocating production from one plant to the other. Second, marginal cost (in both plants) must be equated to marginal revenue, or else profit could be increased by raising or reducing total output.

It is interesting to consider a special case where $c_1 = c_2 = c$; that is, the two plants are identical. Making this substitution into (12), (13) and (14) yields

(15) $$y^M = \frac{a}{c + 2b}$$

(16) $$y_1^M = \frac{a}{2(c + 2b)} = \frac{y}{2}$$

(17) $$y_2^M = \frac{a}{2(c+2b)} = \frac{y}{2}$$

That is, total production is split equally between the two plants. Note also that total production is greater than if the firm has only one of the plants; compare (15) with (4) from Example 2.6, where we assumed the same inverse demand function and the same cost function. Why? Overall costs are lower if production is split between two plants – due to the decreasing returns to scale – and so is profitable to produce more.

2.9 A SIMPLE DUOPOLY MODEL

Consider a setting with two identical firms selling into a single market. Firm 1 has a cost function given by

(1) $$c(y_1) = cy_1^2$$

and firm 2 has a cost function given by

(2) $$c(y_2) = cy_2^2$$

where c is a positive constant. Inverse demand in the market is given by

(3) $$p(y) = a - by$$

where y is the sum of output from the two firms; that is, $y = y_1 + y_2$.

The firms choose their output at the same time (simultaneous moves). Thus, each firm chooses its own output, taking as given the output produced by the other firm. Let $\tilde{y}_1$ be expected output by firm 1 from the perspective of firm 2, and let $\tilde{y}_2$ be expected output by firm 2 from the perspective of firm 1.

Solve the profit maximization problem for firm 1. Verify that your solution is indeed a maximum.

Solution

This is solved most easily as an unconstrained optimization problem. The problem for firm 1 is

$$\max_{y_1} \; [a - b(y_1 + \tilde{y}_2)]y_1 - cy_1^2$$

The first-order condition is

(4) $$a - b(y_1 + \tilde{y}_2) - by_1 - 2cy_1 = 0$$

Note that firm 1 treats $\tilde{y}_2$ as a constant because it has no power to choose this. Taking the second derivative of the objective function with respect to y_1 yields

(5) $$-2b - 2c < 0$$

Thus, the objective function is strictly concave in y_1 and so the first-order condition is necessary and sufficient for a maximum.

Solving (4) for y_1 yields

(6) $$y_1(\tilde{y}_2) = \frac{a - b\tilde{y}_2}{2(b+c)}$$

Economic Interpretation

Equation (6) is a **best-response function**. It specifies the optimal output for firm 1 as a response to its expectation of what firm 2 will produce. Note that it is <u>not</u> a response to what firm 2 actually produces, since they both produce at the same time; there is no sequentiality to the actions of these players.

Firm 2 solves an equivalent problem, and its best response function is

(7) $$y_2(\tilde{y}_1) = \frac{a - b\tilde{y}_1}{2(b+c)}$$

In a **Nash equilibrium** each firm expects the other firm to act in a profit-maximizing way, and their expectations are correct. Thus, the Nash equilibrium values of $\tilde{y}_2$ and $\tilde{y}_1$ are

(8) $$\tilde{y}_2 = \frac{a - b\tilde{y}_1}{2(b+c)}$$

(9) $$\tilde{y}_1 = \frac{a - b\tilde{y}_2}{2(b+c)}$$

respectively. Solving (8) and (9) – substitute (8) for $\tilde{y}_2$ in (9) and solve – yields the Nash equilibrium outputs:

(10) $$\hat{y}_1 = \frac{a}{3b+2c}$$

(11) $$\hat{y}_2 = \frac{a}{3b+2c}$$

Note that both firms chose the same output because they have the same cost function.

It is interesting to compare the total output in this equilibrium with the multi-plant monopoly case from Example 2.8. In particular, total output in the duopoly is

(12) $$\hat{y} = \hat{y}_1 + \hat{y}_2 = \frac{2a}{3b+2c}$$

In comparison, if a monopoly firm operates both plants (rather than operation by two competing firms) then we have the solution from Example 2.8 with $c_1 = c_2 = c$, given by (15) in that example:

(13) $$y^M = \frac{a}{c+2b}$$

Note that

(14) $$\hat{y} - y^M = \frac{ab}{(2b+c)(3b+2c)} > 0$$

That is, $\hat{y} > y^M$; more is produced under duopoly than would be produced by a monopolist operating both plants. Why? The duopoly firms are in competition with each other and this drives price down – and total sales up – relative to the monopoly outcome.

2.10 PUBLIC GOODS AND FREE-RIDING

Consider an economy in which *n* identical agents each have the following utility function

(1) $$u(y, G) = y + \log G$$

where *y* is a private good and *G* is a continuous public good. Each agent has income *m* (in terms of the private good) which she divides between consumption of the private good and a contribution *g* to the provision of the public good, such that

(2) $$G = \sum_{i=1}^{n} g_i$$

<u>Part 1</u>: Find the Nash Equilibrium in voluntary contributions.

Solution to Part 1

Agent *i* solves

$$\max_{g_i} \; (m - g_i) + \log(g_i + G_{-i})$$

where G_{-i} is the total contribution from agents other than agent *i*. The first-order condition is

(3) $$-1 + \frac{1}{g_i + G_{-i}} = 0$$

Simplifying yields

(4) $$g_i = 1 - G_{-i}$$

This represents the best response function for agent *i*. Note that it is downward-sloping; the more agent *i* expects others to provide, the less she will provide herself. This reflects the free-rider problem.

In a symmetric Nash equilibrium, $g_i = g \;\; \forall i$ and so $G_{-i} = (n-1)g$. Thus, in equilibrium

(5) $$\hat{g} = \frac{1}{n}$$

The aggregate contribution is

(6) $$\hat{G} = n\hat{g} = 1$$

Part 2: Compare the Nash equilibrium with the efficient solution.

Solution to Part 2

There is a continuum of efficient solutions, each one corresponding to a different distribution of utility. We will focus on a symmetric solution in which each agent derives the same utility.

The most straightforward way to solve for a symmetric efficient solution when agents are identical is to maximize the utility of a representative agent:

(7) $$\max_{G,y} \; y + \log G \quad \text{subject to} \quad ny + G = nm$$

where *nm* is the total amount of the private good available in the economy for allocation between direct consumption and transformation into the public good; thus, the constraint is the **resource constraint** for this economy. To solve the problem it is easiest to substitute the resource constraint directly into the objective function and solve the unconstrained problem for *G*:

(8) $$\max_{G} \; \frac{nm - G}{n} + \log G$$

The first-order condition is

(9) $$-\frac{1}{n} + \frac{1}{G} = 0$$

which solves for

(10) $$G^* = n$$

Comparing this with $\hat{G}$ reveals that the Nash equilibrium level of *G* is inefficiently low. Note too that in this example,

(11) $$\frac{\partial \left(G^* / \hat{G} \right)}{\partial n} > 0$$

The inefficiency associated with free-riding is worse for larger populations.

2.11 A COURNOT OLIGOPOLY MODEL WITH IDENTICAL FIRMS

Recall the duopoly model from Example 2.9. Here we extend that model to *n* firms. Suppose there are *n* identical firms each with marginal cost *c*, and suppose inverse demand is given by $p(Y) = a - bY$. Find the Nash equilibrium outputs.

Solution

The problem for representative firm *i* is

(1) $$\max_{y_i} \; [a - bY]y_i - cy_i \quad \text{subject to } Y = y_i + \tilde{Y}_{-i}$$

where

(2) $$\tilde{Y}_{-i} = \sum_{j \neq i}^{n} \tilde{y}_j$$

is the expected total output from all firms other than firm *i*. Substitute the constraint directly for *Y* in the revenue function and derive the first-order condition (the best response function):

(3) $$y_i = \frac{a - c - b\tilde{Y}_{-i}}{2b} \quad \forall i$$

The Nash Equilibrium

Each firm rationally expects all other firms to choose their outputs based on a best-response function like (3). Since firms are identical, it is natural to look for a *symmetric* Nash equilibrium in which each firm chooses the same equilibrium output. Let $\hat{y}$ denote that equilibrium output. Then in the symmetric equilibrium,

(4) $$\hat{y}_i = \hat{y} \quad \forall i \quad \text{and} \quad \tilde{Y}_{-i} = (n-1)\hat{y} \quad \forall i$$

Making these substitutions into (3) and solving yields

(5) $$\hat{y} = \frac{a - c}{b(n+1)}$$

Note that setting $n = 2$ in (4) yields the duopoly result from Example 2.9.

The Nash equilibrium price is

(6) $$\hat{p} = a - bn\hat{y} = \frac{a+nc}{n+1}$$

Special Cases

1. Perfect competition. Take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \hat{p} = c$$

2. Monopoly. Set $n = 1$:

$$\hat{p}\Big|_{n=1} = \frac{a+c}{2}$$

2.12 A COURNOT OLIGOPOLY MODEL WITH HETEROGENEOUS FIRMS

Consider a generalization of Example 2.11 in which we allow all firms to be different with respect to their marginal cost. In particular, the cost function for firm is *i* is $c_i(y_i) = c_i y_i$.

Solution

The problem for firm *i* is

(1) $$\max_{y_i} \ [a - bY]y_i - c_i y_i \ \text{ subject to } Y = y_i + \tilde{Y}_{-i}$$

where

(2) $$\tilde{Y}_{-i} = \sum_{j\neq i}^{n} \tilde{y}_j$$

is the expected total output from all firms other than firm *i*. Substitute the constraint directly for *Y* in the revenue function and derive the first-order condition (the best response function):

(3) $$y_i = \frac{a - c_i - b\tilde{Y}_{-i}}{2b} \quad \forall i$$

So far this is a simple generalization Example 2.9 (with c replaced by c_i in the best-response function). Things become more complicated when we solve for the Nash equilibrium.

The Nash Equilibrium

Each firm rationally expects all other firms to choose their outputs based on a best-response function like (3) but we can no longer impose a symmetry condition, since firms are not identical. To find the NE, first let $\hat{y}_i$ denote the equilibrium output from firm i and let $\hat{Y}$ denote the total equilibrium output. Then the rational expectation for firm i with respect to Y_{-i} is

(4) $$\tilde{Y}_{-i} = \hat{Y} - \hat{y}_i$$

Making this substitution in (3) yields

(5) $$\hat{y}_i = \frac{a - c_i - b(\hat{Y} - \hat{y}_i)}{2b} \quad \forall i$$

Solve (5) for $\hat{y}_i$:

(6) $$\hat{y}_i = \frac{a - c_i}{b} - \hat{Y} \quad \forall i$$

Now sum both sides across i to obtain

(7) $$\sum_{i=1}^{n} \hat{y}_i = \frac{na - \sum_{i=1}^{n} c_i}{b} - n\hat{Y}$$

But $\sum_{i=1}^{n} \hat{y}_i = \hat{Y}$, so (7) can be rewritten as

(8) $$\hat{Y} = \frac{na - \sum_{i=1}^{n} c_i}{b} - n\hat{Y}$$

We can now solve (8) for $\hat{Y}$:

(9) $$\hat{Y} = \frac{na - \sum_{i=1}^{n} c_i}{b(n+1)}$$

Equilibrium outputs for each firm can then be found by substituting (9) into (6):

(10) $$\hat{y}_i = \frac{a - nc_i + C_{-i}}{b(n+1)}$$

where

$$C_{-i} = \sum_{j \neq i}^{n} c_j \tag{11}$$

We can recover the identical firm case from Example 2.11 by setting $c_i = c \quad \forall i$. In that case, $C_{-i} = (n-1)c$. Making this substitution in (10) yields

$$\hat{y}_i = \frac{a-c}{b(n+1)} \quad \forall i \tag{12}$$