Remarks on a class of kinetic models of granular media: asymptotics and entropy bounds

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Abstract

We obtain new a priori estimates for spatially inhomogeneous solutions of a kinetic equation for granular media, as first proposed in [3] and, more recently, studied in [1]. In particular, we show that a family of convex functionals on the phase space is non-increasing along the flow of such equations, and we deduce consequences on the asymptotic behaviour of solutions. Furthermore, using an additional assumption on the interaction kernel and a "potential for interaction", we prove a global entropy estimate in the one-dimensional case.

Keywords: Kinetic granular media, global in time estimates, asymptotic behavior, entropy bounds.

Mathematics subject classification: 82C22, 82C40.

1 Introduction

We are concerned with kinetic models of granular media as derived in [3, 4, 1]. More precisely, let $d \geq 1$ be an integer, and consider a system of N identical particles (e.g., grains) moving in $\mathbb{R}^d$. Assume that the particles move freely up to an instant when two of them occupy the same position; then they collide (inelastically) at this position according to an interaction rule to be defined later. After collision, they acquire new velocities, and then continue to move freely until another collision occurs. Let $x_i(t) \in \mathbb{R}^d$ and $v_i(t) \in \mathbb{R}^d$ denote the respective position and velocity of particle $i \in \{1, 2, \cdots, N\}$ at time $t \in [0, \infty)$, and let (x_i^0, v_i^0) be its initial position and velocity. Then (very formally) the motion of the N particles is described by the system of ODE's considered in [3]

$$\left\{ \begin{array}{rcl} \dot{x}_i(t) & = & v_i(t) \\ \dot{v}_i(t) & = & \alpha \sum_{j=1}^N \delta(x_i - x_j) \nabla W(v_j - v_i) \\ x_i(0) & = & x_i^0; \quad v_i(0) = v_i^0 \end{array} \right. \tag{1.1}$$

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where W is an interaction potential describing the interaction rule between particles, and $\alpha > 0$ is a constant measuring the degree of the inelasticity of the collisions between particles. Here δ denotes the Dirac measure centered at the origin. The second equation in (1.1), having a measure in the right-hand side, does not really make sense. A reasonable way to correct this conceptual error from [3] is to replace the Dirac measure δ by a $C_c^\infty(\mathbb{R}^d)$ approximation ξ_ε, where eventually $\varepsilon \to 0$ (see [4]). The mollified equation of (1.1) then becomes

$$\begin{cases} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= \alpha \sum_{j=1}^N \xi_\varepsilon(x_i - x_j)\nabla W(v_j - v_i) \\ x_i(0) &= x_i^0; \quad v_i(0) = v_i^0. \end{cases} \tag{1.2}$$

The mollified system (1.2) expresses the fact that collisions between particles occur when they are within a distance $\varepsilon > 0$ to each other, as opposed to (1.1) where collisions are only allowed when the particles are exactly at the same position.

Since the number of particles is assumed to be very large, $N \to \infty$, it is reasonable to describe the system with a kinetic equation. In this case, following the arguments in [3, 4, 1], one can show that when $N \to \infty$ and $\alpha \to 0$ with the scaling limit assumption $N\alpha \to \lambda$, where $\lambda > 0$ is a parameter, the kinetic equation corresponding to the system (1.2) is

$$\partial_t f + v \cdot \nabla_x f = \lambda \,\mathrm{div}_v \left[(G_\varepsilon \star f) f \right], \qquad f|_{t=0} = f_0 \tag{1.3}$$

where

$$G_\varepsilon(x, v) = \xi_\varepsilon(x)\nabla W(v)$$

and the convolution $G_\varepsilon \star f$ is with respect to both variables x and v, i.e.,

$$\begin{aligned} [G_\varepsilon \star f](t, x, v) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} G_\varepsilon(x - y, v - u) f(t, y, u) \,\mathrm{d}y \,\mathrm{d}u \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi_\varepsilon(x - y)\nabla W(v - u) f(t, y, u) \,\mathrm{d}y \,\mathrm{d}u. \end{aligned} \tag{1.4}$$

Here, $f(t, x, v)$ denotes the one-particle distribution function, that is, the probability density of particles which at time $t > 0$ occupy a position $x \in \mathbb{R}^d$ and move with a velocity $v \in \mathbb{R}^d$, and $f_0(x, v)$ is the corresponding initial probability density. In fact, $f(t, x, v)$ (resp. $f_0(x, v)$) can be viewed as the limit of the discrete probability measure $\mu_t = \frac{1}{N}\sum_{j=1}^N \delta_{(x_j(t), v_j(t))}$ (resp. $\mu_0 = \frac{1}{N}\sum_{j=1}^N \delta_{(x_j^0, v_j^0)}$) as the number of particles $N \to \infty$, where $(x_j(t), v_j(t))$ solves (1.2) for $j = 1, 2, \cdots, N$, (see [15]).

Finally, sending $\varepsilon \to 0$ in (1.4), we formally have,

$$\lim_{\varepsilon \to 0} [G_\varepsilon \star f](t, x, v) = \int_{\mathbb{R}^d} \nabla W(v - u) f(t, x, u) \,\mathrm{d}u = (\nabla W *_v f)(t, x, v)$$

so that the kinetic equation associated with the discrete system (1.1) is the limiting equation of (1.3) as $\varepsilon \to 0$, which reads

$$\partial_t f + v \cdot \nabla_x f = \lambda \,\mathrm{div}_v \Big((\nabla W *_v f) f \Big), \; f|_{t=0} = f_0, \tag{1.5}$$

where the convolution $\nabla W *_v f$ is with respect to the velocity variable only i.e. $(\nabla W *_v f)(x, v) = \int_{\mathbb{R}^d} \nabla W(v - u) f(x, u) \mathrm{d}u$. Throughout the paper, we will assume that

- the Cauchy datum f_0 is a bounded probability density on the phase space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, $f_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $f_0 \geq 0$, $\int_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x, v) \mathrm{d}x\mathrm{d}v = 1$, it is compactly supported i.e. there exist $R_1 > 0$ and $R_2 > 0$ such that

$$\operatorname{supp}(f_0) \subset B_{R_1} \times B_{R_2} \subset B_R \times B_R \text{ with } R := \max(R_1, R_2), \tag{1.6}$$

which in particular implies:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (|x|^2 + |v|^2) f_0(x, v) \mathrm{d}x\mathrm{d}v < +\infty. \tag{1.7}$$

- the interaction potential $W : \mathbb{R}^d \to [0, \infty)$ is strictly convex, C^2, and radially symmetric, i.e.,

$$W(z) = w(|z|), \tag{1.8}$$

where $w : [0, \infty) \to [0, \infty)$ is a strictly convex, non-decreasing C^2 function with $w'(0) = 0$.

Typical examples of such interaction potentials are $W(v) = |v|^p/p$ where $p \geq 2$, see [3, 4, 16]. We are interested in global estimates for solutions to the kinetic equation (1.5). Let us remark that the spatially homogeneous case (i.e. f depending on t and v only) associated with (1.5) has been very much studied (see [3, 8, 9, 14, 6, 10] and the references therein), and existence, uniqueness and long-time behavior are well understood in this case. In fact, the spatially homogeneous version of (1.5) can be seen as the Wasserstein gradient flow of the interaction energy associated to λW, and then well-posedness results can be viewed as a consequence of the powerful theory of Wasserstein gradient flows (see [2]).

In contrast, for the spatially inhomogeneous kinetic equation (1.5), very few existence results are available in the literature. Understanding under which conditions one can hope for global existence or on the contrary expect explosion in finite time is an open question. Regarding the question of existence of solutions to the kinetic equation (1.5), local existence and uniqueness of a classical solution was proved in one dimension in [3] for the potential $W(v) = |v|^3/3$ when the initial datum f_0 is a non-negative integrable function satisfying $f_0 \in C^1 \cap W^{1,\infty}(\mathbb{R} \times \mathbb{R})$ with compact support in the velocity space. As for global existence of solutions to (1.5), it was also proved in [3] again in one dimension and for the cubic potential, for a compactly (in position and velocity) suppported f_0 and under an additional smallness assumption on the parameter λ, i.e. $\lambda < \lambda_0$ for some $\lambda_0 = \lambda_0 (f_0)$ depending on the support and L^∞ norm of the initial datum f_0. The global existence proof of [3] uses the method of characteristics, a fixed point argument and an a priori L^∞ bound. We will show in section 2.1 that this L^∞ a priori bound naturally extends to any dimension $d \geq 1$, and to any interaction potential of the form $W(v) = |v|^p/p$, provided $p > 3 - d$.

In [1], the first author has extended the local existence result of [3] to more general interaction potentials W and to any dimension, $d \geq 1$. More precisely, he proved that when W satisfies the assumptions imposed above, and $0 \leq f_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with compact support in the velocity space, then (1.5) has a weak solution in some time interval $[0, T_0)$, where $T_0 = \frac{1}{C\|f_0\|_{L^\infty}}$ and C is a constant that depends on the dimension d and the velocity support of f_0. The proof given in [1] is based on a splitting of the kinetic equation (1.5) into a free transport equation in x, and a collision equation in v that is interpreted as the gradient flow of a convex interaction energy with respect to the quadratic Wasserstein distance. The splitting scheme of [1] just requires an L^∞ bound on f_0, so as soon as one has an L^∞ bound up to some time $T \geq T_0$, one can extend the solution after time T. One can therefore define maximal solutions on some interval $[0, T^*)$ with $T^* \in [T_0, +\infty]$ and in case where such solutions are not global i.e. $T^* < +\infty$, $\|f_t\|_{L^\infty}$ necessarily tends to $+\infty$ as $t \to T^*$. As for the long-time behavior of solutions when $T^* = +\infty$, to our knowledge, there were no results in the literature.

Still, the general global existence/non existence question is mainly open: *what happens for other values of the parameter $\lambda > 0$? Do solutions still exist globally in time, or do they concentrate in finite time (i.e., is there a formation of a Dirac in finite time)*? Answering this question in full generality is clearly very difficult, and we cannot provide an answer in this note. However, we are able to provide some a priori estimates which shed some light on these issues. As we shall see later, our a priori estimates seem to suggest that the global in time existence or eventually finite-time blow-up of solutions to (1.5) depends on the nature of the interaction potential W.

Section 2 is devoted to preliminary results. In section 3, we observe that integrals of convex functions of $(x - tv, v)$ are nonincreasing along the flow of (1.5) and deduce various consequences from this observation, in particular the asymptotic behavior of solutions to (1.5). In section 4, we obtain, in dimension one, a global entropy bound under the assumption that W'' is subquadratic near zero and show, considering the quadratic kernel, that this bound cannot be true in general.

2 Preliminaries

The following notations will be used in the paper. For a Borel set $B \subset \mathbb{R}^d$, $|B|$ will denote the Lebesgue measure of B, and $\mathbf{1}_B$ will be the characteristic function of B. The support of a function f will be denoted by $\mathrm{supp}(f)$, and B_R (resp. $B_R(x)$) will stand for the closed ball in $\mathbb{R}^d$ centered at the origin (resp. at x) with radius R. Throughout the paper C will denote a positive constant that may change values from one line to another. In what follows f will denote a solution of (1.5) defined on a maximal time interval $[0, T^*)$ with $T^* \in (0, +\infty]$. We shall sometimes denote $f(t, x, v)$ as $f_t(x, v)$ and $f(t, ., .)$ as f_t, and for convenience, we sometimes omit the volume elements in the integrals.

We start by recalling some properties of the kinetic equation (1.5); we refer to [1] for the proofs.

- **Mass conservation:** the total mass, $\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t, x, v)\,\mathrm{d}x\mathrm{d}v$, is conserved along (1.5):
$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t, x, v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x, v)\,\mathrm{d}x\mathrm{d}v \quad \forall\, t \in [0, T^*), \tag{2.1}$$
so f_t are *probability* densities since we have assumed that f_0 has total mass 1.

- **Momentum conservation:** the momentum, $\int_{\mathbb{R}^d\times\mathbb{R}^d} vf(t,x,v)\,\mathrm{d}x\mathrm{d}v$, is conserved along (1.5):

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} vf(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^d\times\mathbb{R}^d} vf_0(x,v)\,\mathrm{d}x\mathrm{d}v \quad \forall\, t\in[0,T^*). \tag{2.2}$$

- **Decrease of moments of order $p\geq 2$:** all the p-moments in v for $p\geq 2$, $\int_{\mathbb{R}^d\times\mathbb{R}^d}|v|^p f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v$, decrease along (1.5). This implies (letting $p\to\infty$) that the velocity support of a solution $f(t,x,v)$ to (1.5) stays compactly supported for all times [1],

$$\mathrm{supp}\,(f(t,x,.))\subset B_{R_2} \quad \forall\, x\in\mathbb{R}^d,\ t\in[0,T^*). \tag{2.3}$$

And since the equation satisfied by x in the characteristic system associated with (1.5) is

$$\dot{x}(t)=v(t)\in B_{R_2},\quad x(0)=x_0\in B_{R_1},$$

it easily follows that

$$\mathrm{supp}\,(f_t)\subset B_{R_1+tR_2}\times B_{R_2}. \tag{2.4}$$

In fact, we shall actually see in remark 3.2 in section 3 a slightly more precise result:

$$\mathrm{supp}\,(f_t)\subset Q(t):=\{(x,v)\ :\ (x-tv,v)\in B_{R_1}\times B_{R_2}\}, \tag{2.5}$$

which in particular implies that

$$\mathrm{supp}\,(f_t(x,.))\subset S(x,t):=B_{\frac{R_1}{t}}\left(\frac{x}{t}\right)\cap B_{R_2}, \tag{2.6}$$

and then also

$$|\,\mathrm{supp}\,(f_t(x,.))\,|\leq \omega_d\Big(\min(\frac{R_1}{t},R_2)\Big)^d,\ \mathrm{diam}(\mathrm{supp}\,(f_t(x,.)))\leq 2\min(\frac{R_1}{t},R_2), \tag{2.7}$$

where $\omega_d:=|B_1|$.

2.1 L^∞ a priori bound in $\mathbb{R}^d$ for potentials $W(v)=|v|^p/p$

Assume here that $W(v)=|v|^p/p$ with $p>3-d$. Following [3, section 3], we also assume that (1.5) has a classical solution $f\in C^1\left([0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d\right)$. We have the following L^∞ a priori bound on f_t, $t\in[0,\infty)$.

Lemma 2.1. *If $\lambda<\lambda_0:=\frac{1}{4C\gamma\|f_0\|_{L^\infty}}$, where $C>0$ is a constant depending on d, p and $R=\max(R_1,R_2)$, and $\gamma=\int_0^\infty h(t)^{p+d-2}\,dt<\infty$ with $h(t):=\min(R,R/t)$, then*

$$\sup_{t\in[0,\infty)}\|f_t\|_{L^\infty}\leq 2\|f_0\|_{L^\infty}. \tag{2.8}$$

Proof. Denoting $\mathcal{F} = -\nabla W *_v f$, the characteristic system associated with (1.5) is

$$\begin{aligned} \dot{X}(t,x,v) &= V(t,x,v) \qquad & \dot{V}(t,x,v) &= \lambda \mathcal{F}\left(t, X(t,x,v), V(t,x,v)\right) \\ X(0,x,v) &= x, & V(0,x,v) &= v. \end{aligned}$$

Then rewriting (1.5) as

$$\partial_t f + v \cdot \nabla_x f + \lambda \mathcal{F} \cdot \nabla_v f = \lambda f(\Delta W *_v f), \qquad f|_{t=0} = f_0,$$

we have along the characteristics, that f solves

$$\frac{d}{dt}[f\left(t, X(t), V(t)\right)] = \lambda \left[f(\Delta W *_v f)\right](t, X(t), V(t))\,.$$

Integration over $[0,t]$ yields

$$f\left(t, X(t), V(t)\right) = f_0(x,v) + \lambda \int_0^t \left[f(\Delta W *_v f)\right](s, X(s), V(s))\ \mathrm{d}s. \tag{2.9}$$

Now, we estimate $f(t,x,v)\,(\Delta W *_v f)\,(t,x,v)$. First recall that from (2.6), $\operatorname{supp}\left(f(t,\cdot,x)\right) \subset S(x,t) = B_{\frac{R_1}{t}}(\frac{x}{t}) \cap B_{R_2}$. Since $S(x,t)$ has diameter less than $2h(t)$ and measure less than $\omega_d h(t)^d$, and using $\Delta W(u) = (p+d-2)|u|^{p-2}$ we get:

$$\begin{aligned} f(t,x,v)\,(\Delta W *_v f)\,(t,x,v) &\le \|f_t\|_{L^\infty} \mathbf{1}_{S(x,t)}(v) \int_{S(x,t)} \Delta W(v-u) f_t(x,u) \mathrm{d}u \\ &\le C\|f_t\|^2_{L^\infty} |S(x,t)| \operatorname{diam}(S(x,t))^{p-2} \le C\|f_t\|^2_{L^\infty} h(t)^{p+d-2}. \end{aligned}$$

Combining the previous inequality with (2.9), integrating over time and setting $\delta := \sup_{t\in[0,\infty)} \|f_t\|_{L^\infty}$, we thus have

$$\delta \le \|f_0\|_{L^\infty} + \lambda C \gamma \delta^2. \tag{2.10}$$

Then using the continuity of $t \mapsto f_t$, we conclude (2.8) provided $\lambda < \lambda_0 := \frac{1}{4C\gamma\|f_0\|_{L^\infty}}$. ∎

Remark 2.2. As explained in the proof of [3, Theorem 3.2], the above L^∞ a priori-bound is the main step to obtain the global existence of a classical solution to (1.5), provided the parameter λ is small enough, $\lambda < \lambda_0$, as defined in Lemma 2.1. One could of course rephrase the previous result in terms of smallness of the initial datum instead of the parameter λ: the previous L^∞ bound similarly holds if $\lambda = 1$ and $\|f_0\|_{L^\infty} < \frac{1}{4C\gamma}$.

For simplicity, we assume from now on that $\lambda = 1$, otherwise we just replace the interaction potential W by λW. Then the kinetic equation (1.5) becomes

$$\partial_t f + v \cdot \nabla_x f = \operatorname{div}_v\Big((\nabla W *_v f) f\Big), \quad f|_{t=0} = f_0. \tag{2.11}$$

2.2 A reverse H-theorem

The fact that solutions of (2.11) obey a reverse H-theorem was first observed in [3]; more precisely, the following was established in [1].

Lemma 2.3. *If $U : [0,\infty) \to \mathbb{R}$ is $C^1(0,\infty)$, convex and satisfies $U(0) = 0$, then the functional $\mathcal{U}(f)(t) := \int_{\mathbb{R}^d\times\mathbb{R}^d} U\left(f(t,x,v)\right) dx\,dv$ is nondecreasing along (2.11):*

$$\frac{d\mathcal{U}(f)}{dt} = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Delta W(v-u)\left[P_U(f)\right](t,x,v) f(t,x,u)\, dx\,du\,dv \geq 0, \tag{2.12}$$

where $P_U(r) = rU'(r) - U(r)$ denotes the pressure associated with U.

In particular if $U(r) = r\ln r$, then the entropy satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f_t \ln f_t\, dx\,dv = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Delta W(v-u) f(t,x,v) f(t,x,u)\, dx\,du\,dv \geq 0. \tag{2.13}$$

Proof. Since U is convex and $U(0) = 0$, then $P_U(r) \geq 0$ for all $r > 0$. Also since $U(0) = 0$, we have that $U\left(f(t,x,v)\right) = 0$ on the subset of $(x,v) \in \mathbb{R}^d\times\mathbb{R}^d$ where $f(t,x,v)$ vanishes; so the internal energy can be written as $\mathcal{U}(f)(t) := \int_{[f>0]} U\left(f(t,x,v)\right) \mathrm{d}x\,\mathrm{d}v$. Therefore, we can assume w.l.o.g. that $f > 0$ in $\mathcal{U}(f)(t)$. Integrating by parts, we have

$$\begin{aligned}
\frac{d\mathcal{U}(f)}{dt} &= \int_{\mathbb{R}^d\times\mathbb{R}^d} U'(f)\,\mathrm{div}_v\left(f(\nabla W *_v f)\right) \mathrm{d}x\,\mathrm{d}v - \int_{\mathbb{R}^d\times\mathbb{R}^d} U'(f)\,\mathrm{div}_x(vf)\,\mathrm{d}x\,\mathrm{d}v \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} fU'(f)\,\mathrm{div}_v\left(\nabla W *_v f\right) \mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{R}^d\times\mathbb{R}^d} U'(f)\nabla_v f\cdot(\nabla W *_v f)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} f\nabla_x\left(U'(f)\right)\cdot v\,\mathrm{d}x\,\mathrm{d}v \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} fU'(f)\,\mathrm{div}_v\left(\nabla W *_v f\right) \mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\left(U(f)\right)\cdot(\nabla W *_v f)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_x\left(P_U(f)\right)\cdot v\,\mathrm{d}x\,\mathrm{d}v \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} P_U(f)\,\mathrm{div}_v\left(\nabla W *_v f\right) \mathrm{d}x\,\mathrm{d}v,
\end{aligned}$$

that is (2.12). If $U(r) = r\ln r$, then $P_U(r) = r$ and (2.13) follows. ◻

Note in particular that all L^p norms of f_t are nondecreasing in t, by choosing $U(r) = r^p$ in Lemma 2.3.

3 Asymptotics

Let us define the density g_t (solution evaluated on the free flow) by

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y,v) g_t(y,v) dy\mathrm{d}v = \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x-tv,v) f_t(x,v)\mathrm{d}x\mathrm{d}v,\ \forall\varphi\in C_c(\mathbb{R}^d\times\mathbb{R}^d)$$

so that $g_0 = f_0$ and $g_t(y,v) = f_t(y+tv,v)$. Denoting by $C_b(\mathbb{R}^d \times \mathbb{R}^d)$ the space of continuous and bounded functions on $\mathbb{R}^d \times \mathbb{R}^d$, we then have the following result; the key step in the proof is an adaptation of an argument of Illner and Rein [13]:

Theorem 3.1. *Let f_t be a solution of (2.11) globally defined on the time interval $[0,T^*)$, and g_t be the density defined as above. Then we have*

1. *for every φ convex on $\mathbb{R}^d \times \mathbb{R}^d$, the map*

$$t \in [0,T^*) \mapsto \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y,v) g_t(y,v) dy\, dv = \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x-tv,v) f_t(x,v) dx\, dv$$

is nonincreasing,

2. *there exists a probability measure g^* on $\mathbb{R}^d \times \mathbb{R}^d$ such that g_t converges weakly to g^* as $t \to T^*$ i.e.*

$$\lim_{t\to T^*} \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y,v) g_t(y,v) dy\, dv = \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y,v) dg^*(y,v) \tag{3.1}$$

for every $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. 1. Let $\varphi = \varphi(y,v)$ be some (smooth, say) convex function. Following [13], we have,

$$\begin{aligned}\frac{d}{dt}\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x-tv,v) f_t(x,v) \mathrm{d}x\mathrm{d}v\Big) = &-\int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_y \varphi(x-tv,v)\cdot v f_t(x,v)\mathrm{d}x\mathrm{d}v \\ &+ \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x-tv,v)\partial_t f_t(x,v)\mathrm{d}x\mathrm{d}v.\end{aligned}$$

To compute the second term, we set $\psi_t(x,v) := \varphi(x-tv,v)$, use (2.11) and perform integrations by parts (recall that f_t is compactly supported thanks to (2.4)), to get:

$$\begin{aligned}\frac{d}{dt}\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x-tv,v) f_t(x,v)\mathrm{d}x\mathrm{d}v\Big) &= -\int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_y \varphi(x-tv,v)\cdot v f_t(x,v)\mathrm{d}x\mathrm{d}v \\ &\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_y \varphi(x-tv,v)\cdot v f_t(x,v)\mathrm{d}x\mathrm{d}v \\ &\quad - \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v \psi_t(x,v)(\nabla W *_v f_t)(x,v) f_t(x,v)\mathrm{d}x\mathrm{d}v \\ &= -\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \nabla W(v-u)\cdot\nabla_v\psi_t(x,v) f_t(x,u) f_t(x,v)\mathrm{d}x\mathrm{d}u\mathrm{d}v \\ &= -\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \nabla W(u-v)\cdot\nabla_v\psi_t(x,u) f_t(x,v) f_t(x,u)\mathrm{d}x\mathrm{d}v\mathrm{d}u,\end{aligned}$$

and then using the fact that ∇W is odd, we get

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi g_t\,\mathrm{d}x\mathrm{d}v = -\frac{1}{2}\int_{\mathbb{R}^{3d}} \nabla W(u-v)\cdot(\nabla_u\psi_t(x,u) - \nabla_v\psi_t(x,v)) f_t(x,v) f_t(x,u)\mathrm{d}x\mathrm{d}v\mathrm{d}u. \tag{3.2}$$

We finally use the radial symmetry of W:

$$\nabla W(u-v) = w'(|u-v|)\frac{u-v}{|u-v|}$$

and the convexity of $\psi_t(x,.)$ to deduce that the right hand side of (3.2) is nonpositive. The case of a general not necessarily smooth convex φ follows by standard approximation arguments.

2. Applying 1. to $\varphi(y,v) := |y|^2 + |v|^2$ we see that

$$\sup_{t\in[0,T^*)} \int_{\mathbb{R}^{2d}} (|y|^2 + |v|^2) g_t(y,v) dy \mathrm{d}v \leq \int_{\mathbb{R}^{2d}} (|x|^2 + |v|^2) f_0(x,v) \mathrm{d}x\mathrm{d}v. \tag{3.3}$$

In particular the family of probability measures $(g_t)_{t\in[0,T^*)}$ is tight. Thanks to Prokhorov's theorem, this implies that there exists a probability measure g^* on $\mathbb{R}^d \times \mathbb{R}^d$ (with finite second moment) and a sequence t_n converging to T^* such

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y,v) g_{t_n}(y,v) dy \mathrm{d}v \to \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y,v) dg^*(y,v),\ \forall \varphi \in C_b(\mathbb{R}^d\times\mathbb{R}^d). \tag{3.4}$$

Now we shall use assertion 1. to prove that the whole family g_t converges weakly to g^* as $t\to T^*$. Let us first take a convex function φ such that for some $C\geq 0$, one has

$$-C \leq \varphi(y,v) \leq C(1+|y|+|v|),\ \forall (y,v)\in \mathbb{R}^d\times\mathbb{R}^d. \tag{3.5}$$

We know from assertion 1. that $\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi g_t \,\mathrm{d}x\mathrm{d}v$ is nonincreasing. Since it is also bounded from below, it converges as $t\to T^*$. We shall now prove that it necessarily converges to $\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi dg^*(x,v)$. For $R>0$, let χ_R be some smooth cutoff function: $0\leq \chi_R \leq 1$ with $\chi_R = 1$ on $B_R\times B_R$ and $\chi_R = 0$ outside of $B_{R+1}\times B_{R+1}$. Then on the one hand, thanks to (3.3), we have for some constant M and every $t\in[0,T^*)$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |\varphi|(1-\chi_R) d(g_t+g^*)(x,v) \leq \frac{M}{1+R}.$$

On the other hand, thanks to (3.4), for any $R>0$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_R\varphi g_{t_n} \mathrm{d}x\mathrm{d}v = \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_R \varphi dg^*(x,v).$$

Since we have

$$\begin{aligned}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi g_{t_n} \mathrm{d}x\mathrm{d}v - \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi dg^*(x,v)\right| &\leq \left|\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_R\varphi g_{t_n} - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_R\varphi dg^*\right| \\ &\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} |\varphi|(1-\chi_R) d(g_{t_n}+g^*) \\ &\leq \left|\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_R\varphi g_{t_n} - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_R\varphi dg^*\right| + \frac{M}{1+R},\end{aligned}$$

we deduce that $\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi g_{t_n}\mathrm{d}x\mathrm{d}v$ converges to $\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi dg^*(x,v)$. Using the monotonicity of $t\mapsto\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi g_t\mathrm{d}x\mathrm{d}v$, we deduce that $\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi g_t\mathrm{d}x\mathrm{d}v$ converges to $\int_{\mathbb{R}^d}\varphi dg^*(x,v)$ as $t\to T^*$.

Let us now take $\varphi\in C^2_c(\mathbb{R}^d\times\mathbb{R}^d)$ supported on $B_R\times B_R$, and let Φ be a convex nonnegative function on $\mathbb{R}^d\times\mathbb{R}^d$ which satisfies (3.5) for some C and coincides with $|y|^2+|v|^2$ on $B_{R+1}\times B_{R+1}$. Then for M such that $M\geq\|D^2\varphi\|_{L^\infty}$, $M\Phi-\varphi$ is convex (and obeys a sublinear estimate of type (3.5)). Since φ is the difference of the two convex functions (with at most linear growth) $M\Phi$ and $M\Phi-\varphi$, the previous step implies that (3.1) holds for any $\varphi\in C^2_c(\mathbb{R}^d\times\mathbb{R}^d)$. Passing from $C^2_c(\mathbb{R}^d\times\mathbb{R}^d)$ to $C_b(\mathbb{R}^d\times\mathbb{R}^d)$ in (3.1) then follows from (3.3) and classical truncation/mollification arguments.

□

The previous result has a certain number of straightforward but useful consequences.

Remark 3.2. If $f_0=g_0$ has a support included in a compact and convex set K then taking $\mathrm{dist}(.,K)$ as convex test function in assertion 1, we deduce that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{dist}((y,v),K)g_t(y,v)dy\mathrm{d}v\leq\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{dist}((x,v),K)f_0(x,v)\mathrm{d}x\mathrm{d}v=0$$

so that $\mathrm{supp}(g_t)\subset K$ for every t hence $\mathrm{supp}(f_t)\subset\{(x,v)\ :\ (x-tv,v)\in K\}$. In particular taking $K=B_{R_1}\times B_{R_2}$ we exactly obtain (2.5).

Remark 3.3. Taking $\varphi(y,v)=|y|^2$, we immediately deduce from assertion 1. that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-tv|^2f_t(x,v)\mathrm{d}x\mathrm{d}v\leq\int_{\mathbb{R}^d\times\mathbb{R}^d}|x|^2f_0(x,v)\mathrm{d}x\mathrm{d}v$$

and then

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}|\frac{x}{t}-v|^2f_t(x,v)\mathrm{d}x\mathrm{d}v\leq\frac{1}{t^2}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x|^2f_0(x,v)\mathrm{d}x\mathrm{d}v.\tag{3.6}$$

Remark 3.4. The marginals of g_t converge weakly as $t\to T^*$ to the corresponding marginals of g^*; in particular the v-marginal of f_t weakly converges to that of g^* as $t\to T^*$ (and not only up to a subsequence) which we shall denote η^* i.e.

$$\lim_{t\to T^*}\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(v)f_t(x,v)\mathrm{d}x\mathrm{d}v=\int_{\mathbb{R}^d}\varphi(v)d\eta^*(v),\ \forall\varphi\in C_b(\mathbb{R}^d).\tag{3.7}$$

Remark 3.5. Identity (3.2) actually tells us more than just the fact that $\int_{\mathbb{R}^{2d}}\varphi g_t\mathrm{d}y\mathrm{d}v$ is nonincreasing in t since it also implies that the right-hand side of (3.2) is integrable with respect to t. Taking for instance $\varphi(y,v)=|y|^2$, we actually get

$$\int_0^{T^*}t^2\int_{\mathbb{R}^{3d}}\nabla W(v-u)\cdot(v-u)f_t(x,v)f_t(x,u)\mathrm{d}x\mathrm{d}v\mathrm{d}u\mathrm{d}t<+\infty\tag{3.8}$$

which as soon as W is strictly convex will also imply that for every $\delta > 0$ one has

$$\int_0^{T^*} t^2 \int_{\{(x,u,v)\in\mathbb{R}^{3d}\,:\,|u-v|\geq\delta\}} f_t(x,v) f_t(x,u) \mathrm{d}x\mathrm{d}v\mathrm{d}u\mathrm{d}t < +\infty. \tag{3.9}$$

In the special case (as considered in [3]) where $W(z) = \frac{1}{3}|z|^3$, the previous estimate (3.8) becomes

$$\int_0^{T^*} t^2 \Big(\int_{\mathbb{R}^{3d}} |u-v|^3 f_t(x,u) f_t(x,v) \mathrm{d}x\mathrm{d}u\mathrm{d}v \Big) \mathrm{d}t < +\infty. \tag{3.10}$$

When $W(z) = \frac{1}{3}|z|^3$, taking as convex test function $\varphi(y,v) = |y|^q$ with $q > 2$, we similarly obtain

$$t \int_{\mathbb{R}^{3d}} |u-v|(u-v)\cdot(|x-tu|^{q-2}(tu-x) - |x-tv|^{q-2}(tv-x)) f_t(x,u) f_t(x,v) \mathrm{d}x\mathrm{d}u\mathrm{d}v$$

is integrable with respect to t. First, using homogeneity, and setting $a := u - x/t$, $b = v - x/t$, we can rewrite

$$t(u-v)\cdot(|x-tu|^{q-2}(tu-x) - |x-tv|^{q-2}(tv-x)) = t^q (a-b)\cdot(|a|^{q-2}a - |b|^{q-2}b).$$

Then we use the well-known inequality (see for instance Lemma 4.4 in [12]):

$$(a-b)\cdot(|a|^{q-2}a - |b|^{q-2}b) \geq \mu |a-b|^q$$

which holds for any $(a,b) \in \mathbb{R}^d \times \mathbb{R}^d$ and for a positive constant μ depending on $q > 2$ and d, to deduce that

$$\int_0^{T^*} t^q \Big(\int_{\mathbb{R}^{3d}} |u-v|^{q+1} f_t(x,u) f_t(x,v) \mathrm{d}x\mathrm{d}u\mathrm{d}v \Big) \mathrm{d}t < +\infty.$$

This implies that for every $\delta > 0$

$$\int_0^{T^*} t^q \Big(\int_{\{(x,u,v)\in\mathbb{R}^{3d}\,:\,|u-v|\geq\delta\}} f_t(x,u) f_t(x,v) \mathrm{d}x\mathrm{d}u\mathrm{d}v \Big) \mathrm{d}t < +\infty. \tag{3.11}$$

Inequalities like (3.6)-(3.10)-(3.11) indicate that in some sense, conditionally on the position, the velocity distribution concentrates on a single velocity. To give a meaning to this, we shall rescale the position by dividing it by t. More precisely, under the assumption that global in time solutions exist ($T^* = \infty$), we have the following asymptotic result:

Proposition 3.6. *Assume that there is global existence i.e. $T^* = +\infty$, and let g^* be as in Theorem 3.1 and η^* be the v-marginal of g^*. Then*

$$\lim_{t\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi\Big(\frac{x}{t}, v\Big) f_t(x,v)\, dx\, dv = \int_{\mathbb{R}^d} \varphi(v,v)\, d\eta^*(v)$$

for every $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. Clearly (setting $\widetilde{\varphi}(y,v)=\varphi(y+v,v)$), the desired result amounts to proving that

$$\lim_{t\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d}\widetilde{\varphi}\Big(\frac{y}{t},v\Big)g_t(y,v)\mathrm{d}y\mathrm{d}v=\int_{\mathbb{R}^d\times\mathbb{R}^d}\widetilde{\varphi}(0,v)\mathrm{d}g^*(y,v)=\int_{\mathbb{R}^d}\widetilde{\varphi}(0,v)\mathrm{d}\eta^*(v).$$

Introducing the cutoff function χ_R as in the proof of Theorem 3.1, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\widetilde{\varphi}\Big(\frac{y}{t},v\Big)g_t(y,v)\mathrm{d}y\mathrm{d}v-\int_{\mathbb{R}^d\times\mathbb{R}^d}\widetilde{\varphi}(0,v)\mathrm{d}g^*(y,v)\Big|\\
&\le\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi_R\Big|\widetilde{\varphi}\Big(\frac{y}{t},v\Big)-\widetilde{\varphi}(0,v)\Big|g_t(y,v)\mathrm{d}y\mathrm{d}v\\
&+2\sup|\widetilde{\varphi}|\int_{\mathbb{R}^d\times\mathbb{R}^d}(1-\chi_R)g_t\\
&+\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\widetilde{\varphi}(0,v)g_t(y,v)\mathrm{d}y\mathrm{d}v-\int_{\mathbb{R}^d\times\mathbb{R}^d}\widetilde{\varphi}(0,v)\mathrm{d}g^*(y,v)\Big|.
\end{aligned}$$

Thanks to the moment bound (3.3), we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(1-\chi_R)g_t\le\int_{\{(x,v)\in\mathbb{R}^{2d}:\,|x|^2+|v|^2\ge R^2\}}g_t\le\frac{1}{R^2}\int_{\mathbb{R}^d\times\mathbb{R}^d}(|x|^2+|v|^2)f_0(x,v)\mathrm{d}x\mathrm{d}v.$$

Let $\varepsilon>0$ and choose $R>0$ such that the right-hand side of the inequality above is less than $\varepsilon/3$. Using Theorem 3.1, we know that for t large enough,

$$\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\widetilde{\varphi}(0,v)g_t(y,v)\mathrm{d}y\mathrm{d}v-\int_{\mathbb{R}^d\times\mathbb{R}^d}\widetilde{\varphi}(0,v)\mathrm{d}g^*(y,v)\Big|\le\frac{\varepsilon}{3}.$$

Since $\widetilde{\varphi}$ is uniformly continuous on compact sets, for t large enough, we also have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi_R\Big|\widetilde{\varphi}\Big(\frac{y}{t},v\Big)-\widetilde{\varphi}(0,v)\Big|g_t(y,v)\mathrm{d}y\mathrm{d}v\le\sup_{|v|\le R,\,|z|\le R/t}|\widetilde{\varphi}(z,v)-\widetilde{\varphi}(0,v)|\le\frac{\varepsilon}{3}.$$

All this proves the desired result. □

4 Entropy bounds in dimension one

For this section, we further assume that $d=1$. In this one-dimensional geometry, the "potential for interaction" first used by Bony (see ([7, 11, 5]) provides additional control.

Lemma 4.1. *The following estimate holds*

$$\int_0^{T^*}\int_{\mathbb{R}^3}(u-v)^2f_t(x,u)f_t(x,v)\,dx\,du\,dv\,dt<+\infty \tag{4.1}$$

Proof. For $t \in [0, T^*)$, define

$$I(t) := \int_{\mathbb{R}^4} (u-v)\mathbf{1}_{\{x<y\}} f_t(x,u) f_t(y,v) \mathrm{d}x\mathrm{d}u\mathrm{d}y\mathrm{d}v$$

(where $\mathbf{1}_{\{x<y\}} = 1$ if $x < y$ and 0 otherwise). By our bound on the velocity support, I is bounded. To compute the time derivative of I, it is convenient to observe that (2.11) can be rewritten as

$$\partial_t f + v\partial_x f = F \tag{4.2}$$

where $F = \partial_v(f(W' *_v f))$ satisfies

$$\int_{\mathbb{R}} F(x,v)\mathrm{d}v = 0, \; \int_{\mathbb{R}} vF(x,v)\mathrm{d}v = 0 \tag{4.3}$$

(the first equality comes from the fact that $f(x,.)(W' *_v f(x,.))$ has compact support, the second one is obtained by an integration by parts and using the fact that W' is odd). We then have

$$\begin{aligned}
\frac{dI}{dt}(t) &= \int_{\mathbb{R}^4} (u-v)\mathbf{1}_{\{x<y\}} (\partial_t f_t(x,u) f_t(y,v) + f_t(x,u)\partial_t f_t(y,v)) \mathrm{d}x\mathrm{d}u\mathrm{d}y\mathrm{d}v \\
&= \int_{\mathbb{R}^4} (u-v)\mathbf{1}_{\{x<y\}} (-u\partial_x f_t(x,u) + F_t(x,u)) f_t(y,v)) \mathrm{d}x\mathrm{d}u\mathrm{d}y\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^4} (u-v)\mathbf{1}_{\{x<y\}} (-v\partial_x f_t(y,v) + F_t(y,v)) f_t(x,u)) \mathrm{d}x\mathrm{d}u\mathrm{d}y\mathrm{d}v.
\end{aligned}$$

Thanks to (4.3), the integrals containing F are zero, and using

$$\int_{\mathbb{R}} \partial_x f(x,u)\mathbf{1}_{\{x<y\}} \mathrm{d}x = f(y,u) \text{ and } \int_{\mathbb{R}} \partial_x f(y,u)\mathbf{1}_{\{x<y\}} dy = -f(x,u),$$

we are left with

$$\frac{dI}{dt}(t) = -\int_{\mathbb{R}^3} (u-v)^2 f_t(x,u) f_t(x,v) \mathrm{d}x\mathrm{d}u\mathrm{d}v.$$

The bound (4.1) is then obtained by integration and using the fact that I is bounded from below. ☐

We showed in Lemma 2.3 that the entropy is nondecreasing along the flow of (2.11). However, the estimate of the previous lemma turns out to be useful to deduce an entropy bound if the laplacian of the interaction kernel has subquadratic behavior near zero:

Proposition 4.2. *Assume that there exist $\delta > 0$ and $M \geq 0$ such that the interaction kernel W satisfies:*

$$W''(\xi) \leq M\xi^2, \forall \xi \in [-\delta, \delta] \tag{4.4}$$

and that $\int_{\mathbb{R}^2} f_0 \ln(f_0) < +\infty$. Then there exists C such that for a.e. $t \in [0, T^*)$, one has

$$\int_{\mathbb{R}^2} f_t(x,v) \ln(f_t(x,v))\, dx\, dv \leq C. \tag{4.5}$$

Proof. The computation of the time-derivative of the entropy follows from Lemma 2.3:

$$\frac{d}{dt}\int_{\mathbb{R}^2} f_t(x,v)\ln(f_t(x,v))\mathrm{d}x\mathrm{d}v = \int_{\mathbb{R}^3} W''(u-v)f_t(x,u)f_t(x,v)\mathrm{d}x\mathrm{d}u\mathrm{d}v.$$

Then we split the last integral in the right hand side into two parts, one on $|u-v|\leq\delta$ for which we use (4.4) and (4.1) to get

$$\int_0^{T^*}\int_{\mathbb{R}^3} \mathbf{1}_{\{|u-v|\leq\delta\}}W''(v-u)f_t(x,u)f_t(x,v)\mathrm{d}x\mathrm{d}u\mathrm{d}v\mathrm{d}t \leq C,$$

and for the other part where $|u-v|>\delta$, recalling that $f(x,.)$ has a compact support (say included in $[-R,R]$) uniformly in x and t, we bound $W''(v-u)$ by its supremum on $[-2R,2R]$ and use (3.9) to obtain

$$\int_0^{T^*}\int_{\mathbb{R}^3} \mathbf{1}_{\{|u-v|>\delta\}}W''(v-u)f_t(x,u)f_t(x,v)\mathrm{d}x\mathrm{d}u\mathrm{d}v\mathrm{d}t \leq C.$$

Those two estimates give the desired entropy bound (4.5). □

Typical examples of interaction potentials satisfying (4.4) are $W(v)=|v|^p/p$ where $p\geq 4$.

We have seen in Theorem 3.1 that g_t (defined as $g_t(y,v)=f_t(y+tv,v)$) converges weakly to some limit g^* as $t\to T^*$ and also that as soon as f_0 is compactly supported, so is g_t uniformly in t. Since obviously g_t and f_t have the same entropy, we deduce that if W'' satisfies the subquadratic assumption (4.4) and f_0 is compactly supported, then g_t is uniformly integrable and, thanks to the lower semicontinuity of the entropy, $g^*\in L^1$ and $\int_{\mathbb{R}^2} g^*(y,v)\ln(g^*(y,v))dy\mathrm{d}v$ is finite. Denoting by η^* the v-marginal of g^* (which is also the weak limit of the v-marginal of f_t as $t\to T^*$), writing $g^*(y,v)=\eta^*(v)g^*(y|v)$ and denoting by $[-R,R]$ a segment supporting $g^*(.,v)$ for every v, we then have

$$\begin{aligned}\int_{\mathbb{R}^2} g^*(y,v)\ln(g^*(y,v))dy\mathrm{d}v &= \int_{\mathbb{R}} \eta^*(v)\ln(\eta^*(v))\mathrm{d}v \\ &\quad + \int_{\mathbb{R}}\Big(\int_{[-R,R]} g^*(y|v)\ln(g^*(y|v))\mathrm{d}y\Big)\eta^*(v)\mathrm{d}v \\ &\geq \int_{\mathbb{R}} \eta^*(v)\ln(\eta^*(v))\mathrm{d}v - \frac{2R}{e}\end{aligned}$$

(where in the last line we have used $\inf_{g>0} g\ln(g) = -\frac{1}{e}$ and the fact that η^* is a probability measure) so that η^* also has a finite entropy.

The next result concerning the quadratic kernel (which does *not* satisfy (4.4)) shows that additional assumptions on the kernel are necessary to derive global entropy bounds. Applying Lemma 2.3 to the quadratic interaction potential $W(v)=v^2/2$ in one-dimension, $v\in\mathbb{R}$, we have:

Lemma 4.3. *If $W(v)=v^2/2$, $v\in\mathbb{R}$, then $\forall\, t\in[0,T^*)$,*

$$\int_{\mathbb{R}\times\mathbb{R}} f_t\ln f_t\, dx\,dv \geq \int_{\mathbb{R}\times\mathbb{R}} f_0\ln f_0\, dx\,dv + \frac{1}{2R}\ln(1+t). \tag{4.6}$$

Proof. Using $W(v) = v^2/2$ and $d = 1$ in (2.13), we have $\Delta W(v) = W''(v) = 1$, so that

$$\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}} f_t \ln f_t \mathrm{d}x\mathrm{d}v = \int_{\mathbb{R}} \rho(t,x)^2 \,\mathrm{d}x, \quad \rho(t,x) := \int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v.$$

Then thanks to (2.5), we can rewrite the above expression as

$$\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}} f_t \ln f_t \,\mathrm{d}x\mathrm{d}v = 2(1+t)R\int_{\mathbb{R}} \rho(t,x)^2\, d\mu(x), \text{ where } \quad d\mu(x) = \frac{1}{2(1+t)R}\mathbf{1}_{B_{(1+t)R}}(x)\,\mathrm{d}x,$$

which gives (by Jensen's inequality)

$$\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}} f_t \ln f_t \,\mathrm{d}x\mathrm{d}v \geq \frac{1}{2(1+t)R}\left(\int_{B_{(1+t)R}} \rho(t,x)\,\mathrm{d}x\right)^2 = \frac{1}{2(1+t)R}.$$

Integration over $[0,t]$ yields (4.6). □

In case $T^* = +\infty$, letting $t \to \infty$ in (4.6), we have $\int_{\mathbb{R}\times\mathbb{R}} f_t \ln f_t \,\mathrm{d}x\mathrm{d}v \to \infty$, which shows that there can be no global entropy bound. Also note that the quadratic kernel in dimension one does not satisfy the integrability requirement of Lemma 2.1.

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