GLOBAL CLASSICAL SOLUTIONS OF THE RELATIVISTIC VLASOV-DARWIN SYSTEM WITH SMALL CAUCHY DATA: THE GENERALIZED VARIABLES APPROACH

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Abstract. We show that a smooth, small enough Cauchy datum launches a unique classical solution of the relativistic Vlasov-Darwin (RVD) system globally in time. A similar result is claimed in [15] following the work in [13]. Our proof does not require estimates derived from the conservation of the total energy, nor those previously given on the transverse component of the electric field. These estimates are crucial in the references cited above. Instead, we exploit the formulation of the RVD system in terms of the *generalized* space and momentum variables. By doing so, we produce a simple a-priori estimate on the transverse component of the electric field. We widen the functional space required for the Cauchy datum to extend the solution globally in time, and we improve decay estimates given in [15] on the electromagnetic field and its space derivatives. Our method extends the constructive proof presented in [14] to solve the Cauchy problem for the Vlasov-Poisson system with a small initial datum.

1. Introduction

The relativistic Vlasov-Darwin (RVD) system can be obtained from the Vlasov-Maxwell system by neglecting the transverse component of the displacement current in the Maxwell-Ampère equation. Precisely, consider an ensemble of single species charged particles interacting through the self-induced electromagnetic field. Let $f(t,x,p)$ denote the number of particles per unit volume of the phase-space at a time $t \in]0,\infty[$, where $x \in \mathbb{R}^3$ is position and $p \in \mathbb{R}^3$ denotes momentum. In the regime in which collisions among the particles can be neglected, the time evolution of the distribution function f is given by the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + \left(E + c^{-1} v \times B\right) \cdot \nabla_p f = 0, \quad v = \frac{p}{\sqrt{1 + c^{-2}\,|p|^2}}, \tag{1.1}$$

where v is the relativistic velocity and c the speed of light. Here the mass and charge of the particles have been set to one. $E = E(t,x)$ and $B = B(t,x)$ denote the self-induced electric and magnetic fields, given by the Maxwell equations

$$\nabla \times B - c^{-1}\partial_t E = 4\pi c^{-1} j, \qquad \nabla \cdot B = 0, \tag{1.2}$$

$$\nabla \times E + c^{-1}\partial_t B = 0, \qquad \nabla \cdot E = 4\pi\rho. \tag{1.3}$$

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The Vlasov and Maxwell equations are then coupled via the charge and current densities

$$\rho = \int_{\mathbb{R}^3} f dp \quad \text{and} \quad j = \int_{\mathbb{R}^3} v f dp. \tag{1.4}$$

Equations (1.1)-(1.4) are known as the relativistic Vlasov-Maxwell (RVM) system, which is essential in the study of dilute hot plasmas. Details and an abundant bibliography on this system can be found, for instance, in [6].

We further decompose the electric field into $E = E_L + E_T$, where the longitudinal E_L and transverse E_T components of the electric field satisfy, respectively

$$\nabla \times E_L = 0 \quad \text{and} \quad \nabla \cdot E_T = 0. \tag{1.5}$$

If we now neglect the transverse component of the displacement current $\partial_t E_T$ in the evolution equation (1.2) -the so-called Maxwell-Ampère equation-, then the RVM system reduces to

$$\partial_t f + v \cdot \nabla_x f + \left(E_L + E_T + c^{-1} v \times B\right) \cdot \nabla_p f = 0, \quad v = \frac{p}{\sqrt{1 + c^{-2}|p|^2}}, \tag{1.6}$$

coupled with

$$\begin{aligned} \nabla \times B - c^{-1}\partial_t E_L &= 4\pi c^{-1} j, & \nabla \cdot B &= 0, \qquad (1.7)\\ \nabla \times E_T + c^{-1}\partial_t B &= 0, & \nabla \cdot E_L &= 4\pi\rho, \qquad (1.8) \end{aligned}$$

by means of (1.4). Equations (1.4)-(1.8) are the RVD system. From the physical point of view, the Darwin approximation is valid when the evolution of the electromagnetic field is 'slower' than the speed of light.

In this paper we are concerned with the Cauchy problem for (1.4)-(1.8). Global existence of weak solutions was shown in [2] for small initial data. The smallness assumption was later removed in [13], where the existence and uniqueness of local in time classical solutions was also proved. In [15], it is shown that solutions having the same regularity as the initial data (which is not the case in [13]), can be extended globally in time provided the initial data is small. At the present time, the existence of global in time classical solutions for arbitrary data remains unsolved. Here, we provide a constructive and somewhat simplified proof to the local in time existence and uniqueness result for classical solutions of the RVD system, and we show that the solutions can be extended for all times if the initial data are sufficiently small.

The main difficulty when dealing with the RVD system has been to find an a-priori estimate on the transverse component of the electric field E_T. In contrast to the RVM system, the component E_T does not contribute to the energy of the electromagnetic field, and thus the law for the conservation of the total energy does not provide any control on the L^2-norm of E_T. Indeed, the total energy of the RVD system reads

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} c^2\sqrt{1 + c^{-2}|p|^2} f(t,x,p)dpdx + \frac{1}{8\pi}\int_{\mathbb{R}^3}\left[|E_L(t,x)|^2 + |B(t,x)|^2\right]dx.$$

Hence, by virtue of the underlying elliptic structure of the Darwin equations, duality type arguments and variational methods have previously been used to estimate E_T. Here, we take advantage of the formulation of the RVD system in terms of the generalized variables -defined later on-, and we produce an L^2-bound on $\rho^{1/2}E_T$ instead. This estimate is at the core of our results, and is given in Lemma 8. It is remarkable that by pursuing such an estimate we have obtained, 'almost for free', an L^2-bound on $\partial_x E_T$ as well. In contrast to the results cited above, the law for the conservation of the total energy is not used in our proofs at all.

The structure of the paper is as follows. In Section 2, we present the scalar and vector potentials and introduce the generalized position and momentum variables. We then recast the Vlasov and Darwin equations in terms of the new variables, and treat them as uncoupled linear equations. A representation for the Darwin vector potential is given, and some standard a-priori bounds are obtained as well. Then, in Section 3 we couple both Vlasov and Darwin equations and introduce the RVD system in terms of the potentials. The estimates on the transverse component of the electric field and its space derivative are produced in Subsection 3.1. Finally, we study the Cauchy problem for the RVD system in Section 4. First, we produce the local in time existence result in Subsection 4.1 and then, in Subsection 4.2, we extend local solutions globally in time under the smallness assumption on the Cauchy data. We conclude with an Appendix.

We remark that the RVD is actually an hybrid system, since we are considering relativistic charged particles whose interaction with the electromagnetic field they induce is an order-$(v/c)^2$ approximation [10, 9]. Yet, the RVD system is interesting in its own right, in particular for numerical simulations, since it contains an underlying elliptic feature while preserving a fully coupled magnetic field. This is in contrast to the more involved RVM system, whose hyperbolic structure yields both analytical and numerical challenges. Also, the tools used here are likely to be adapted to the 'proper' physical system, which is (1.4)-(1.8) with $v = p\left(1 - c^{-2}p^2/2\right)$ instead.

The following notations will be used in the paper. As usual, $C^{k,\alpha}(X;Y)$ denotes the space of functions $f : X \to Y$ of class C^k whose k-th derivatives are Hölder continuous with exponent $\alpha \in (0,1)$. $C_0^k(X;Y)$, resp. $C_b^k(X;Y)$, are the spaces of $C^k(X;Y)$-functions with compact support, resp. bounded. $W^{1,\infty}(X;Y)$ stands for the Sobolev space of $L^\infty(X;Y)$-functions whose weak first order partial derivatives belong to $L^\infty(X;Y)$. If I is an interval in $\mathbb{R}$, then by $g \in C^1\left(I, C^k(X);Y\right)$, we mean that $g : I \times X \to Y$, $g = g(t,x)$, and for all $t \in I$, $g(t) \in C^k(X;Y)$ and the function $t \mapsto g(t) \in C^k(X;Y)$ is of class C^1 on I. For such a function, we sometimes write (by abuse of notations) $g \in C^k(X;Y)$ to mean that $g(t) \in C^k(X;Y)$ for all $t \in I$. Similarly, the norm of $g(t)$, say the L^q-norm $\|g(t)\|_{L^q_x}$, will sometimes be denoted by $\|g\|_{L^q_x}$. All other notations in the paper are standard, and the constants may change values from line to line.

2. The Potential Representation

From classical electrodynamics it is known that an electromagnetic field $(E,B) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ that is a smooth solution of the Maxwell equations (1.2)-(1.3) can be represented by a set of potentials $(\Phi, A) : (0,\infty) \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ according to the expressions

$$E(t,x) = -\nabla\Phi(t,x) - c^{-1}\partial_t A(t,x), \tag{2.1}$$

$$B(t,x) = \nabla \times A(t,x). \tag{2.2}$$

These relations can easily be obtained from the two homogeneous Maxwell equations in (1.2)-(1.3). In particular (2.2) follows from the vanishing divergence of the magnetic field, while (2.1) follows after inserting (2.2) into the remaining homogeneous equation. Since for any smooth scalar function Λ we have the identity $\nabla \times \nabla\Lambda \equiv 0$, it is clear that such potentials are not uniquely determined. We may find another pair given by

$$(\Phi', A') = \left(\Phi - c^{-1}\partial_t\Lambda, A + \nabla\Lambda\right) \tag{2.3}$$

which also satisfies (2.1)-(2.2). The two sets of potentials are fully equivalent, in the sense that they produce the same electric and magnetic fields.

This lack of uniqueness allows to impose a condition on the potentials that ultimately determines their dynamical equations. Even after doing so, some arbitrariness remains that can be avoided by imposing an additional restriction on Λ. The resulting restricted class is called a gauge, and all potentials within this class satisfy the same gauge condition. Commonly, the Lorentz gauge condition

$$\nabla \cdot A + c^{-1}\partial_t\Phi = 0, \tag{2.4}$$

or the Coulomb gauge condition

$$\nabla \cdot A = 0 \tag{2.5}$$

is used. The former is relativistically covariant and leads to a class of scalar and vector potentials that satisfy wave equations. This is a natural choice when dealing with the RVM system. It was used in [3] to study the smoothing effect resulting from a coupling of a wave and transport equations. It was also used in [4] to produce an alternative proof of the celebrated result by Glassey and Strauss on the RVM system [7]. On the other hand, the Coulomb gauge condition leads to scalar and vector potentials that satisfy a Poisson and a wave equation, respectively. As we shall see in Subsection 2.2 below, this is the correct choice to introduce the potential representation of the RVD system. In a way, both the Lorentz and Coulomb gauges can be seen as limit cases of a more general class known as the velocity gauge, in which the scalar potential propagates with an arbitrary speed [9].

2.1. **The Vlasov Equation.** We now introduce the generalized variables, which permit to rewrite the Vlasov equation (1.1) in terms of the scalar and vector potentials in a very convenient way. The resulting transport equation is shown to be determined by an incompressible vector field *irrespective of the gauge chosen*. Thus, we can count on the usual a-priori estimates on the distribution function -see Lemma 2 below- no matter which gauge we decide to work in.

To start with, let $I \subset [0,\infty[$ such that $0 \in I$. Assume that the pair $(\Phi, A) \in C^1(I, C^2(\mathbb{R}^3); \mathbb{R}\times\mathbb{R}^3)$ is given, and so in view of (2.1)-(2.2) the electromagnetic field is given as well. Denote $z := (x,p)$ and write $(X,P)(s)$ instead of $(X,P)(s,t,z)$ to ease notation. Then, by virtue of (2.1)-(2.2), the characteristic system associated to the Vlasov equation (1.1) reads

$$\begin{aligned} \dot{X}(s) &= v(P(s)), && (2.6)\\ \dot{P}(s) &= \left[-\nabla\Phi - c^{-1}\partial_s A + c^{-1}v \times (\nabla \times A)\right](s, X(s), P(s)). && (2.7) \end{aligned}$$

Hence, since

$$\dot{A}(s, X(s)) = \left[\partial_s A + (v\cdot\nabla) A\right](s, X(s)),$$

the equation (2.7) can be rewritten as

$$\dot{P}(s) = \left[-c^{-1}\dot{A} - \nabla\Phi + c^{-1}v \times (\nabla\times A) + c^{-1}(v\cdot\nabla) A\right](s, X(s), P(s)). \tag{2.8}$$

The structure of (2.8) suggests that we can define a generalized momentum variable $\pi = p + c^{-1}A$ such that the above equation can be reduced to

$$\dot{\Pi}(s) = \left[-\nabla\Phi + c^{-1}v \times (\nabla\times A) + c^{-1}(v\cdot\nabla) A\right](s, X(s), P(s)).$$

Here we have denoted $\Pi(s) = P(s) + c^{-1}A(s, X(s))$. On the other hand, the relativistic velocity written in terms of the generalized momentum is

$$v_A = \frac{\pi - c^{-1}A}{\sqrt{1 + c^{-2}\left|\pi - c^{-1}A\right|^2}}. \tag{2.9}$$

Therefore, by using the elementary identity

$$v_A \times (\nabla \times A) + (v_A \cdot \nabla)\, A \equiv v_A^i \nabla A^i,$$

we can reformulate the characteristic system (2.6)-(2.7) in terms of the generalized variables $\xi = (x, \pi)$ as

$$\dot{X}(s,t,\xi) \;=\; v_A(s, X(s,t,\xi), \Pi(s,t,\xi)), \tag{2.10}$$

$$\dot{\Pi}(s,t,\xi) \;=\; -\left[\nabla\Phi - c^{-1}v_A^i \nabla A^i\right](s, X(s,t,\xi), \Pi(s,t,\xi)). \tag{2.11}$$

As usual, repeated index means summation. Now, standard results in the theory of first order ordinary differential equations imply that for every fixed $t \in I$ and $\xi \in \mathbb{R}^6$ there exists a unique local solution $\Xi = (X, \Pi)(s,t,\xi)$ of (2.10)-(2.11) satisfying $\Xi(t,t,\xi) = \xi$; see [8, Chapters II and V]. Moreover, $\Xi \in C^1\left(I \times I \times \mathbb{R}^6; \mathbb{R}^6\right)$. In turn, uniqueness implies that

$$Z = (X, \Pi - c^{-1}A)(s,t,x,\pi - c^{-1}A)$$

is the unique solution of (2.6)-(2.7) with initial data $Z(t,t,z) = (x, \pi - c^{-1}A)$, so by having the characteristic curves in the generalized phase space we can recover the characteristic curves in the usual phase space.

As the following lemma shows, the field resulting in the right-hand side of the system of equations (2.10)-(2.11) is an incompressible vector field:

Lemma 1. *For v_A given by (2.9), we have*

$$\nabla_x \cdot v_A + \nabla_\pi \cdot \left(-\nabla\Phi + c^{-1}v_A^i \nabla A^i\right) = 0.$$

Proof. Since trivially $\nabla_\pi \cdot \nabla\Phi = 0$, the result is a consequence of the elementary relation

$$c^{-1}\nabla_\pi \cdot \left(v_A^i \nabla A^i\right) = c^{-1}\frac{\nabla \cdot A - v_A^i\,(v_A \cdot \nabla)\,A^i}{\sqrt{1 + c^{-2}\left|\pi - c^{-1}A\right|}} = -\nabla_x \cdot v_A.$$

□

As a result, solutions of the characteristic system (2.10)-(2.11) satisfy the volume preserving property. Specifically, for any fixed $s, t \in I$, the map $\Xi(s,t,\cdot) : \mathbb{R}^6 \to \mathbb{R}^6$ is a C^1-diffeomorphism with inverse $\Xi^{-1}(s,t,\xi) = \Xi(t,s,\xi)$ and Jacobian determinant; see [8, Corollary V.3.1]

$$\det J_{\Xi_{s,t}}(\xi) = \frac{\partial \Xi(s,t,\xi)}{\partial \xi} = 1.$$

These properties of the characteristic flow lead to the following result:

Lemma 2. *Let $(\Phi, A) \in C(I, C^2(\mathbb{R}^3); \mathbb{R} \times \mathbb{R}^3)$ be given in some gauge and let v_A be given by (2.9). Assume that $\nabla\Phi$ and ∇A^i, $i = 1,2,3$ are bounded on $J \times \mathbb{R}^3$ for every compact subinterval $J \subset I$. Let $f_0 \in C^1\left(\mathbb{R}^6; \mathbb{R}\right)$ and denote by $\Xi = (X, \Pi)$ the characteristic flow solving (2.10)-(2.11). Then, the function $f(t,\xi) = f_0(\Xi(0,t,\xi))$ defined on $I \times \mathbb{R}^6$ is the unique C^1 solution of the Cauchy problem for*

$$\partial_t f + v_A \cdot \nabla_x f - \left[\nabla\Phi - c^{-1}v_A^i \nabla A^i\right] \cdot \nabla_\pi f = 0. \tag{2.12}$$

Moreover, if $f_0 \geq 0$ *then* $f \geq 0$. *Also, for* $t \in I$ *we have that*

$$\boldsymbol{supp} f(t) = \Xi(t, 0, \boldsymbol{supp} f_0),$$

and for each $1 \leq q \leq \infty$, $t \in I$ *we have*

$$\|f(t)\|_{L^q_{x,\pi}} = \|f_0\|_{L^q_{x,\pi}}.$$

Conversely, if f *is a* C^1 *solution of the Cauchy problem for (2.12), then* f *is constant along each solution of the characteristic system (2.10)-(2.11).*

Remark 1. In addition, if $(\Phi, A)(t) \in C^{2,\alpha}(\mathbb{R}^3; \mathbb{R} \times \mathbb{R}^3)$, $0 < \alpha < 1$, $t \in I$, and $f_0 \in C^{1,\alpha}\left(\mathbb{R}^6; \mathbb{R}\right)$, then the unique C^1 solution $f(t, \xi) = f_0(\Xi(0, t, \xi))$ of the Cauchy problem for (2.12) satisfies $f(t) \in C^{1,\alpha}(\mathbb{R}^6; \mathbb{R})$ for every $t \in I$.

Proof of Lemma 2. In view of Lemma 1, the proof follows by the standard Cauchy's method of characteristics; see [8, Chapter VI]. In particular, the properties of f are a direct consequence of the properties of the characteristic flow discussed above. □

We point out that (2.12) is the proper Hamiltonian representation of the Vlasov equation (1.1) in terms of the potentials, since the characteristic equations (2.10)-(2.11) are Hamilton's equations for the Hamiltonian

$$\mathcal{H}(t, x, \pi) = c^2 \sqrt{1 + c^{-2} \left|\pi - c^{-1} A(t, x)\right|^2} + \Phi(t, x) \tag{2.13}$$

of a relativistic charged particle under the influence of an electromagnetic field of potentials (Φ, A). As before, in (2.13) the charge and mass of the particle have been set to one.

2.2. **The Darwin Potentials.** To determine the dynamical equations satisfied by the potentials we shall impose the Coulomb gauge condition, since it leads to the Darwin approximation of the Maxwell equations and ultimately to the RVD system. Throughout this section, unless we specify otherwise, we assume that both the charge and current densities ρ and j are smooth and given, and they satisfy the continuity equation

$$\partial_t \rho + \nabla \cdot j = 0. \tag{2.14}$$

Formally, if we substitute the electric and magnetic fields in (2.1)-(2.2) into the non-homogeneous Maxwell equations in (1.2)-(1.3), we find that Φ and A satisfy

$$\Delta \Phi = -4\pi\rho - c^{-1} \partial_t (\nabla \cdot A), \tag{2.15}$$

$$\Delta A - c^{-2} \partial_t^2 A = -c^{-1} 4\pi j + \nabla \left(\nabla \cdot A + c^{-1} \partial_t \Phi\right). \tag{2.16}$$

Therefore, in the Coulomb gauge (2.5), the potentials satisfy

$$\Delta \Phi = -4\pi\rho, \tag{2.17}$$

$$\Delta A - c^{-2} \partial_t^2 A = -c^{-1} 4\pi j + c^{-1} \nabla \partial_t \Phi. \tag{2.18}$$

On the other hand, any smooth solution (Φ, A) of the above system that satisfies the Coulomb gauge condition initially, will continue to do so for all times, and therefore the induced electromagnetic field will solve (1.2)-(1.3). Indeed, if (Φ, A) is a smooth solution of (2.17)-(2.18) that satisfies $\nabla \cdot A|_{t=0} = 0$ and $\partial_t(\nabla \cdot A)|_{t=0} = 0$, then $g_C = \nabla \cdot A$ is the solution of

$$\Delta g_C - c^{-2} \partial_t^2 g_C = -4\pi c^{-1} \left(\nabla \cdot j + \partial_t \rho\right) = 0,$$
$$g_C|_{t=0} = 0, \quad \partial_t g_C|_{t=0} = 0,$$

and the claim follows. Hence, the system of equations (2.17)-(2.18) complemented with (2.5) is fully equivalent to the set of Maxwell equations (1.2)-(1.3).

We define the Darwin approximation of the Maxwell equations as the quasi-static limit of the system (2.17)-(2.18):

Definition 1. *Let $(\rho, j) : I \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ be given and satisfy the continuity equation (2.14). The set of potentials (Φ, A) is called a classical solution of the Darwin equations if $\Phi \in C^1(I, C^2(\mathbb{R}^3); \mathbb{R})$, $A \in C(I, C^2(\mathbb{R}^3); \mathbb{R}^3)$ and, on $I \times \mathbb{R}^3$,*

$$\Delta\Phi = -4\pi\rho, \tag{2.19}$$

$$\Delta A = -c^{-1}4\pi j + c^{-1}\nabla\partial_t\Phi. \tag{2.20}$$

The system (2.19)-(2.20) has the following explicit solution, as proved below:

Definition 2. *For the charge and current densities $(\rho, j) : I \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ we formally define the set of Darwin potentials $(\Phi_D, A_D) : I \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ by*

$$\Phi_D(t,x) = \int_{\mathbb{R}^3} \rho(t,y)\frac{dy}{|y-x|}, \tag{2.21}$$

$$A_D(t,x) = \frac{1}{2c}\int_{\mathbb{R}^3} [\boldsymbol{id} + \omega \otimes \omega]\, j(t,y)\frac{dy}{|y-x|}, \tag{2.22}$$

where $\omega = (y-x)/|y-x|$ and $\boldsymbol{id}$ denotes the identity matrix.

Lemma 3. *Let $\rho \in C^1(I, C_0^\alpha(\mathbb{R}^3); \mathbb{R})$ and $j \in C(I, C_0^{1,\alpha}(\mathbb{R}^3); \mathbb{R}^3)$, $0 < \alpha < 1$, be given -they do not need to satisfy the continuity equation (2.14)-. Define the field*

$$\mathbb{P}j(t,x) = j(t,x) + \frac{1}{4\pi}\nabla\int_{\mathbb{R}^3} \nabla\cdot j(t,y)\frac{dy}{|y-x|}, \quad t \in I, \quad x \in \mathbb{R}^3. \tag{2.23}$$

Then the following holds:

(a): *The scalar potential Φ_D is the unique solution in $C^1(I, C^{2,\alpha}(\mathbb{R}^3); \mathbb{R})$ of*

$$\Delta\Phi(t,x) = -4\pi\rho(t,x), \quad \lim_{|x|\to\infty}\Phi(t,x) = 0. \tag{2.24}$$

It satisfies

$$\nabla\Phi_D(t,x) = \int_{\mathbb{R}^3}\rho(t,y)\frac{\omega dy}{|y-x|^2}.$$

(b): *$\mathbb{P}j \in C(I, C^{1,\alpha}(\mathbb{R}^3); \mathbb{R}^3)$. It satisfies $\nabla\cdot\mathbb{P}j = 0$ (i.e., $\mathbb{P}j$ is the transverse component of the current density j), and $\mathbb{P}j(x) = O(|x|^{-2})$ for $|x| \to \infty$.*

(c): *The vector potential A_D is the unique solution in $C(I, C^{3,\alpha}(\mathbb{R}^3); \mathbb{R}^3)$ of*

$$\Delta A(t,x) = -4\pi c^{-1}\mathbb{P}j(t,x), \quad \lim_{|x|\to\infty}|A(t,x)| = 0. \tag{2.25}$$

It satisfies

$$\partial_x A_D(t,x) = \frac{1}{2c}\int_{\mathbb{R}^3}\{\omega\otimes j - j\otimes\omega + [3\omega\otimes\omega - \boldsymbol{id}]\,(j\cdot\omega)\}\frac{dy}{|y-x|^2}, \tag{2.26}$$

with $j = j(t,y)$. In particular,

$$\nabla\cdot A_D(t,x) = 0 \quad \textit{and} \quad \nabla\times A_D(t,x) = \frac{1}{c}\int_{\mathbb{R}^3}\omega\times j(t,y)\frac{dy}{|y-x|^2}.$$

Corollary 1. *If ρ and j, as given in Lemma 3, satisfy the continuity equation (2.14), then*

$$\mathbb{P}j(t,x) = j(t,x) - \frac{1}{4\pi}\nabla\partial_t\Phi_D(t,x), \quad t \in I, \quad x \in \mathbb{R}^3,$$

and thus the Darwin potentials (2.21)-(2.22) are the unique classical solution of the Darwin equations (2.19)-(2.20).

Proof of Lemma 3. Without loss of generality we omit the time dependence.

The proof of (a) is a standard result for the Poisson equation. Existence (in a much weaker sense) can be found, for instance, in [11, Theorem 6.21] while the regularity of the solution is given in [11, Theorem 10.3]. Uniqueness is known as Liouville's theorem [12, Theorem 7 Section 4.2].

To prove (b), notice that $\nabla\cdot j \in C_0^\alpha(\mathbb{R}^3;\mathbb{R})$. Hence, as in (a), the integral in the right-hand side of (2.23) is the $C^{2,\alpha}$-solution of the Poisson equation $\Delta u = -4\pi\nabla\cdot j$, $\lim_{|x|\to\infty} u(x) = 0$. That $\nabla u(x) = O(|x|^{-2})$ for $|x| \to \infty$ is well known, which in turn provides the decay for $\mathbb{P}j$, since j has compact support. Moreover,

$$\nabla\cdot\mathbb{P}j(x) = \nabla\cdot j(x) + \frac{1}{4\pi}\Delta\int_{\mathbb{R}^3}\nabla\cdot j(y)\frac{dy}{|y-x|} = 0.$$

As for (c), we first prove the following lemma:

Lemma 4. *The Darwin potential A_D in (2.22) has the equivalent representation*

$$A_D(x) = \frac{1}{c}\int_{\mathbb{R}^3} j(y)\frac{dy}{|y-x|} + \frac{1}{2c}\int_{\mathbb{R}^3}\nabla\cdot j(y)\frac{y-x}{|y-x|}dy. \tag{2.27}$$

Proof. The current density j has compact support, so standard arguments can show that the right-hand side (`RHS`) of the above expression is well defined. The divergence theorem then yields,

$$\begin{aligned}
\texttt{RHS} &= \frac{1}{c}\int_{\mathbb{R}^3} j(y)\frac{dy}{|y-x|} - \frac{1}{2c}\int_{\mathbb{R}^3}(j(y)\cdot\nabla)\,\omega dy \\
&= \frac{1}{c}\int_{\mathbb{R}^3}\left\{j(y) - \frac{1}{2}\left[j(y) - \omega\,(j(y)\cdot\omega)\right]\right\}\frac{dy}{|y-x|} \\
&= \frac{1}{2c}\int_{\mathbb{R}^3}\left[j(y) + \omega\,(j(y)\cdot\omega)\right]\frac{dy}{|y-x|},
\end{aligned}$$

which is precisely the Darwin potential A_D in (2.22). The use of the divergence theorem is justified by the following standard argument: remove a small ball about $x \in \mathbb{R}^3$ in the domain of integration so we can avoid the singularity at $y = x$, then use the divergence theorem and note that the boundary term corresponding to the small ball vanishes as its radii tends to 0. $\square$

We shall now deduce by direct computation from (2.27), the Poisson equation given by (2.25). To this end, we first recall that just as in part (a),

$$\Delta\left\{\frac{1}{c}\int_{\mathbb{R}^3} j(y)\frac{dy}{|y-x|}\right\} = -\frac{4\pi}{c}j(x). \tag{2.28}$$

The integral in curly brackets is in $C^{3,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$, due to the regularity of j. Next, we show that the following equality holds in the sense of distribution,

$$\partial_{x_k}\left\{\int_{\mathbb{R}^3}\nabla\cdot j(y)\omega^i dy\right\} = -\int_{\mathbb{R}^3}\nabla\cdot j(y)\left[\delta_{ik} - \omega^i\omega^k\right]\frac{dy}{|y-x|}. \tag{2.29}$$

Let $r = |y - x| > 0$. First, note that $\partial_{x_k}\omega^i = -r^{-1}\left[\delta_{ik} - \omega^i\omega^k\right]$, and that the integral on the right-hand side of (2.29) is well defined for almost all $x \in \mathbb{R}^3$, since $\nabla\cdot j \in C_0^\alpha(\mathbb{R}^3, \mathbb{R})$ and the kernel is bounded from above by r^{-1}. For $\phi \in C_0^\infty(\mathbb{R}^3; \mathbb{R})$, the function $(x, y) \mapsto \partial_{x_k}\phi(x)\nabla\cdot j(y)\omega^i$ is integrable on $\mathbb{R}^3 \times \mathbb{R}^3$. Hence, we can use Fubini's theorem to find that

$$\begin{aligned}\int_{\mathbb{R}^3} \partial_{x_k}\phi(x)\left\{\int_{\mathbb{R}^3} \nabla\cdot j(y)\omega^i dy\right\} dx &= \int_{\mathbb{R}^3}\left\{\int_{\mathbb{R}^3} \left(\partial_{x_k}\phi(x)\right)\omega^i dx\right\}\nabla\cdot j(y)dy \\ &= -\int_{\mathbb{R}^3}\left\{\int_{\mathbb{R}^3} \phi(x)\partial_{x_k}\omega^i dx\right\}\nabla\cdot j(y)dy,\end{aligned}$$

where the second equality is justified by a standard limiting process and integrations by parts, similar to the argument at the end of the proof of Lemma 4. Then, another use of Fubini's theorem yields (2.29) in the sense of distribution, as claimed.

Actually, the equality in (2.29) holds in the classical sense. By the standard theory of the Poisson equation, the right-hand side of (2.29) is a function in $C^{2,\alpha}(\mathbb{R}^3; \mathbb{R})$; see [11, Theorem 10.3]. Therefore, in view of the theorem for the equivalence of classical and distributional derivatives, the integral in curly brackets on the left-hand side of (2.29) is in $C^{3,\alpha}(\mathbb{R}^3; \mathbb{R})$; see [11, Theorem 6.10].

Now, since $\partial_{x_k} r^{-1} = r^{-2}\omega^k$ and $\omega^k\left[\delta_{ik} - \omega^i\omega^k\right] \equiv 0$, we have

$$\partial_{x_k}\left\{r^{-1}\left[\delta_{ik} - \omega^i\omega^k\right]\right\} = -r^{-1}\omega^i\partial_{x_k}\omega^k = 2r^{-2}\omega^i.$$

Therefore, similar arguments to those used above yield

$$\begin{aligned}&\partial_{x_k}\left\{-\int_{\mathbb{R}^3} \nabla\cdot j(y)\left[\delta_{ik} - \omega^i\omega^k\right]\frac{dy}{|y-x|}\right\} \\ &= -2\int_{\mathbb{R}^3} \nabla\cdot j(y)\frac{\omega^i dy}{|y-x|^2} = -2\partial_{x_i}\int_{\mathbb{R}^3} \nabla\cdot j(y)\frac{dy}{|y-x|}.\end{aligned} \tag{2.30}$$

Hence, since $\Delta \equiv \nabla\cdot\nabla$, we can combine (2.29) and (2.30) to find that

$$\Delta\left\{\frac{1}{2c}\int_{\mathbb{R}^3} \nabla\cdot j(y)\frac{y-x}{|y-x|}dy\right\} = -\frac{1}{c}\nabla\int_{\mathbb{R}^3} \nabla\cdot j(y)\frac{dy}{|y-x|}. \tag{2.31}$$

Then, we add (2.28) and (2.31) to conclude that $\Delta A_D = -4\pi c^{-1}\mathbb{P}j$ holds on $\mathbb{R}^3$, and so A_D is a $C^{3,\alpha}$ solution of (2.25). This solution is unique in view of the Liouville's theorem [12, Theorem 7 Section 4.2].

The representation (2.26) of $\partial_x A$ can be proved as follows. Since j_A is regular enough, we shift the x-variable into the argument of j_A and differentiate (2.22) under the integral. Then, we move the derivative to the kernel of (2.22) helped by the same standard argument at the end of the proof of Lemma 4. In doing so, we notice that for $r > 0$, the imk-th entry of $\partial_x\mathcal{K}$ is

$$\partial_{x_k}\left\{r^{-1}\left[\delta_{im} + \omega^i\omega^m\right]\right\} = r^{-2}\left[\delta_{im}\omega^k - \delta_{km}\omega^i - \delta_{ik}\omega^m + 3\omega^i\omega^k\omega^m\right],$$

which leads to (2.26). Finally, it is not difficult to check that

$$\nabla\cdot A_D = \mathtt{Trace}(\partial_x A_D) = 0 \quad \text{and} \quad (\nabla\times A_D)^i = \frac{1}{2}\left(\partial_x A_D - (\partial_x A_D)^T\right)_{kl},$$

where $(\partial_x A_D)^T$ denotes the transpose of $\partial_x A_D$, and $i, k, l \in \{1, 2, 3\}$ are given according to the cyclic index-permutation. □

It is easy to check that the Darwin equations in Definition 1 are formally equivalent to the equations (1.5) and (1.7)-(1.8) given in the Introduction. To see this let us define

$$E_L = -\nabla\Phi_D, \quad E_T = -c^{-1}\partial_t A_D, \quad B = \nabla\times A_D.$$

We have to show that (E_L, E_T, B) formally solves (1.7)-(1.8) provided that the charge and current densities satisfy the continuity equation. Clearly $\nabla\cdot B = 0$, and since $\nabla\cdot A_D = 0$, we have

$$\nabla\times B = \nabla\left(\nabla\cdot A_D\right) - \Delta A_D = 4\pi c^{-1} j - c^{-1}\partial_t\nabla\Phi_D = 4\pi c^{-1} j + c^{-1}\partial_t E_L,$$

which is (1.7). Easy computations yield (1.5) and (1.8), and the claim follows.

We conclude this section with a-priori estimates on the potentials and their *space* derivatives. For simplicity and without loss of generality, we shall neglect the time dependence.

Lemma 5. *For $1\le m<3$ set $r_0 = 3/(3-m)$ and let $r<r_0<s$. Then there exists a positive constant $C = C(m,r,s)$ such that for any $\Psi\in L^r\cap L^s(\mathbb{R}^3;\mathbb{R})$*

$$\left\|\int_{\mathbb{R}^n}\Psi(y)\frac{dy}{|y-\cdot|^m}\right\|_{L^\infty_x} \le C(m,r,s)\left\|\Psi\right\|^{1-\lambda}_{L^r_x}\left\|\Psi\right\|^{\lambda}_{L^s_x}, \quad \textit{where} \quad \lambda = \frac{1-r/r_0}{1-r/s}.$$

In particular, $C(m,1,\infty) = 3\left(4\pi/m\right)^{m/3}/\left(3-m\right)$.

Proof. cf. [13, Lemma 2.7]. ∎

Lemma 6. *For ρ and j as given in Lemma 3, the Darwin vector potential (2.22) satisfy the estimates:*

$$\left\|A_D\right\|_{L^\infty_x} \le C\left\|j\right\|^{2/3}_{L^1_x}\left\|j\right\|^{1/3}_{L^\infty_x} \quad \textit{and} \quad \left\|\partial_x A_D\right\|_{L^\infty_x} \le C\left\|j\right\|^{1/3}_{L^1_x}\left\|j\right\|^{2/3}_{L^\infty_x}. \tag{2.32}$$

Moreover, for any $0<h\le R$ we have

$$\left\|\partial^2_x A_D\right\|_{L^\infty_x} \le C\left[R^{-3}\left\|j\right\|_{L^1_x} + h\left\|\partial_x j\right\|_{L^\infty_x} + \left(1+\ln\left(R/h\right)\right)\left\|j\right\|_{L^\infty_x}\right],$$

where $C>0$ is independent of h, R, ρ and j. In particular,

$$\left\|\partial^2_x A_D\right\|_{L^\infty_x} \le C\left[\left\|j\right\|_{L^1_x} + \left(1+\left\|j\right\|_{L^\infty_x}\right)\left(1+\ln^+\left\|\partial_x j\right\|_{L^\infty_x}\right)\right]. \tag{2.33}$$

The same estimates hold for the scalar potential Φ_D, with j replaced by ρ.

Proof. The estimates corresponding to Φ_D are well known from the study of the Vlasov-Poisson system. These results can be found, for instance, in [14, Lemma P1] and [1, Propositions 1 and 2]. Here, we shall produce the estimates for the vector potential A_D only. The proof is actually rather similar.

Let $\mathcal{K}(y,x) = |y-x|^{-1}\left[\mathtt{id} + \omega\otimes\omega\right]$. Clearly, $|\mathcal{K}(y,x)|\le C\,|y-x|^{-1}$. Then, the estimates in (2.32) are a straightforward consequence of Lemma 5. To produce the estimates for the second derivatives, consider

$$\begin{aligned}
\partial_l\partial_k A^i_D \quad &\equiv \quad \frac{1}{2c}\left\{\partial_l\int_{\mathbb{R}^3}\left[\delta_{im}\omega^k - \delta_{km}\omega^i - \delta_{ik}\omega^m\right] j^m(y)\frac{dy}{|y-x|^2}\right. \\
&\qquad\quad \left. 3\partial_l\int_{\mathbb{R}^3} j^m\frac{\omega^m\omega^i\omega^k dy}{|y-x|^2}\right\} \\
&= \quad \frac{1}{2c}\left(I_1 + 3I_2\right).
\end{aligned}$$

Here we have introduced the notation $\partial_k = \partial_{x_k}$, $k = 1, 2, 3$; see Lemma 3(c) for the matrix representation of the integrand of $\partial_x A$. Now, the integral I_1 can in turn be split into three integrals, each one essentially the same as the integral corresponding to $\partial_l \partial_k \Phi_D$. Thus, I_1 satisfies the expected estimates, as $\partial_l \partial_k \Phi_D$ does. Therefore, we are led to estimate I_2. To this end, we set $r = |y - x|$, and for $r > 0$ we denote by $\Gamma^m_{ikl}(y - x)$

$$\partial_{y_l}\left[\frac{\omega^m\omega^i\omega^k}{r^2}\right] = \frac{1}{r^3}\left[\delta_{ml}\omega^i\omega^k + \delta_{il}\omega^k\omega^m + \delta_{kl}\omega^i\omega^m - 5\omega^i\omega^k\omega^l\omega^m\right].$$

This kernel is too singular to use Lemma 5. However, since $y^i y^k y^m |y|^{-5}$ is homogeneous of degree -2, for every $0 < R_1 < R_2$ we have

$$\int_{R_1<|y|<R_2} \Gamma^m_{ikl}(y)dy = \int_{|y|=R_2} \frac{y^l}{R_2}\frac{y^i y^k y^m}{|y|^5} dS_y - \int_{|y|=R_1} \frac{y^l}{R_1}\frac{y^i y^k y^m}{|y|^5} dS_y = 0.$$

Thus, for any $h > 0$, we can rewrite I_2 as

$$\begin{aligned} I_2 &= \int_{|y-x|>h} \Gamma^m_{ikl}(y-x) j^m(y)dy + j^m(x)\int_{|\omega|=1} \omega^i\omega^k\omega^l\omega^m d\omega \\ &\quad + \int_{|y-x|\le h} \Gamma^m_{ikl}(y-x)\left[j^m(y) - j^m(x)\right]dy. \end{aligned}$$

The singularity in the last integral at $r = 0$ is now avoided by the difference $j^m(y) - j^m(x)$. Indeed, for $0 \le h \le R$ we produce

$$\begin{aligned} I_2 &\le C\left\{\|j\|_{L^\infty_x}\int_{h<|y-x|\le R}\frac{dy}{|y-x|^3} + \int_{|y-x|>R}|j(y)|\frac{dy}{|y-x|^3}\right. \\ &\qquad \left.\|\partial_x j\|_{L^\infty_x}\int_{|y-x|\le h}\frac{dy}{|y-x|^2} + |j(x)|\right\} \\ &\le C\left[\ln(R/h)\|j\|_{L^\infty_x} + R^{-3}\|j\|_{L^1_x} + h\|\partial_x j\|_{L^\infty_x} + \|j\|_{L^\infty_x}\right]. \end{aligned}$$

This yields the first estimate on $\left\|\partial_x^2 A_D\right\|_{L^\infty_x}$. Then, by setting $R = 1$ and letting $h = \|\partial_x j\|^{-1}_{L^\infty_x}$ if $\|\partial_x j\|^{-1}_{L^\infty_x} \ge 1$, otherwise $h = 1$, the estimate (2.33) follows as well. This completes the proof of the lemma. □

3. The RVD System

If we now combine (2.12) and (2.21)-(2.22) by means of (1.4), then we obtain the following equivalent representation of the RVD system:

$$\partial_t f + v_A\cdot\nabla_x f - \left[\nabla\Phi - c^{-1}v_A^i\nabla A^i\right]\cdot\nabla_p f = 0, \tag{3.1}$$

coupled with

$$\Phi(t,x) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,y,p)\frac{dpdy}{|y-x|}, \tag{3.2}$$

$$A(t,x) = \frac{1}{2c}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left[\mathtt{id} + \omega\otimes\omega\right]v_A f(t,y,p)\frac{dpdy}{|y-x|}, \tag{3.3}$$

where

$$v_A = \frac{p - c^{-1}A}{\sqrt{1 + c^{-2}\left|p - c^{-1}A\right|^2}}. \tag{3.4}$$

For the sake of notation, we have written p instead of π when referring to the generalized momentum variable. We will continue to do so for the rest of the paper. We shall also set $c = 1$ for the speed of light. The goal is to prove that a small enough Cauchy datum launches a unique classical solution of the system (3.1)-(3.4) globally in time. We shall prove this in Subsections 4.1 and 4.2 below, but first we center our attention on (3.3). If f is given, then (3.3) is a nonlinear integral equation of unknown A.

Lemma 7. *Fix $t \in I$ and let $f(t) \in C_0^{1,\alpha}(\mathbb{R}^6;\mathbb{R})$, $0 < \alpha < 1$, be given. Then, there exists an $A(t) \in C_b \cap C^{2,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$ satisfying (3.3)-(3.4).*

Proof. Without loss of generality we shall omit the dependence in time. Let $\bar{C}$ be a constant that may depend on f, to be fixed later on. Define the set

$$\mathcal{D}_{\bar{C}} = \left\{ A \in C_b(\mathbb{R}^3;\mathbb{R}^3) : \|A\|_{L^\infty_x} \leq \bar{C} \right\}.$$

First, we show that there exists an $A_\infty \in \mathcal{D}_{\bar{C}}$ which solves (3.3)-(3.4). To this end, denote the kernel $\mathcal{K}(x,y) = |y-x|^{-1}\left[\mathtt{id} + \omega \otimes \omega\right]$ and let $A \in \mathcal{D}_{\bar{C}}$. Consider the mapping $A \mapsto T[A]$ defined by

$$T[A](x) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathcal{K}(x,y) v_A f(y,p)\,dp\,dy, \quad v_A = \frac{p-A}{\sqrt{1+|p-A|^2}}.$$

We claim that $T[A] \in \mathcal{D}_{\bar{C}}$. Indeed, let $(\mathcal{K})_{ij}(x,y)$ be the ij-entry of $\mathcal{K}(x,y)$. For some u_1, u_2 and u_3 on the line segment between x and z, the mean value theorem implies

$$\begin{aligned}
\left|(\mathcal{K})_{ij}(x,y) - (\mathcal{K})_{ij}(z,y)\right| &\leq \left|\frac{1}{|y-x|} - \frac{1}{|y-z|}\right| + \left|\frac{y^i-x^i}{|y-x|^2} - \frac{y^i-z^i}{|y-z|^2}\right| \\
&\quad + \left|\frac{y^j-x^j}{|y-x|^2} - \frac{y^j-z^j}{|y-z|^2}\right| \\
&\leq C\,|x-z|\left(\frac{1}{|y-u_1|^2} + \frac{1}{|y-u_2|^2} + \frac{1}{|y-u_3|^2}\right).
\end{aligned}$$

Hence, since $|v_A| \leq 1$, a use of Lemma 6 produces

$$\begin{aligned}
|T[A](x) - T[A](z)| &\leq \frac{1}{2}\int_{\mathbb{R}^3} |\mathcal{K}(x,y) - \mathcal{K}(z,y)|\,\rho(y)\,dy \\
&\leq C\,|x-z|\left\|\int_{\mathbb{R}^3}\rho(y)\frac{dy}{|y-\cdot|^2}\right\|_{L^\infty_x} \\
&\leq C(f)\,|x-z|. \qquad (3.5)
\end{aligned}$$

Thus, $T[A]$ is a continuous vector valued function. Also, by Lemma 6

$$\|T[A]\|_{L^\infty_x} \leq 3/2(\pi/2)^{1/3}\,\|\rho\|_{L^1_x}^{2/3}\,\|\rho\|_{L^\infty_x}^{1/3} \equiv \bar{C}. \qquad (3.6)$$

Therefore, $T[A] \in \mathcal{D}_{\bar{C}}$ as claimed.

We now show that T has a fixed point $A_\infty \in \mathcal{D}_{\bar{C}}$. By virtue of the Schauder fixed point theorem [12, Theorem 3 Section 9.1], it suffices to show that T is a continuous mapping and that the closure of the image of T is compact in $\mathcal{D}_{\bar{C}}$. To show the continuity of T, we see that if $A_k \to A$ in $\mathcal{D}_{\bar{C}}$, then by Lemma 6

$$\begin{aligned} |T[A_k](x) - T[A](x)| &\leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v_{A_k} - v_A| f(y,p) \frac{dpdy}{|y-x|} \\ &\leq C(f) \|A_k - A\|_{L^\infty_x} . \end{aligned}$$

To show that $\overline{T\mathcal{D}_{\bar{C}}} \subset \mathcal{D}_{\bar{C}}$ is compact, we first notice that for $A \in \mathcal{D}_{\bar{C}}$

$$|T[A](x)| \leq \|\rho\|_{L^\infty_x} \int_{\mathrm{supp} f} \frac{dy}{|x-y|} \leq C(f) \frac{1}{1+|x|}. \tag{3.7}$$

Now consider the sequence $\{B_n\} \subset T\mathcal{D}_{\bar{C}}$. Let $R > 0$ be fixed. By (3.6) and (3.5), the restriction

$$\{B_n\}|_{\{x\in\mathbb{R}^3:|x|\leq R\}}$$

is clearly bounded and equicontinuous. Then, by Arzelà-Ascoli and a standard diagonal argument we can find a subsequence $\{B_{n_k}\}$ and a continuous, bounded limit vector field B such that $\{B_{n_k}\} \to B$ uniformly on compact sets, and in particular pointwise. Clearly, $\|B\|_{L^\infty_x} \leq \bar{C}$, and since $\{B_{n_k}\}$ satisfies the estimate (3.7), so does B. We only need to show that the convergence $\{B_{n_k}\} \to B$ is uniform. To this end, let $\epsilon > 0$. Choose $R > 0$ such that the right-hand side of (3.7) is less than $\epsilon/2$ for $|x| > R$. Then, for all k we have $|B_{n_k}(x) - B(x)| < \epsilon$ for $|x| > R$, and we can find a $k_0 = k_0(R,\epsilon)$ such that for all $k > k_0$

$$\sup_{|x|\leq R} |B_{n_k}(x) - B(x)| < \epsilon.$$

This proves uniform convergence. Hence, all the hypotheses for the Schauder fixed point theorem are fulfilled, and thus T has a fixed point A_∞ in $\mathcal{D}_{\bar{C}}$.

Next, we have to show that A_∞ has the required regularity. To this end, define v_{A_∞} and then j_{A_∞} according to (3.4) and (1.4), respectively. The vector field A_∞ has the form of a Darwin potential (2.22) with current density $j_{A_\infty} \in C_0(\mathbb{R}^3;\mathbb{R}^3)$. Clearly, the kernel of (2.22) satisfies $|\mathcal{K}(x,y)| \leq C|y-x|^{-1}$ and the derivative estimate $|\partial_x \mathcal{K}(x,y)| \leq C|y-x|^{-2}$. Hence, we can use the standard theory for the Poisson equation to find that $A_\infty \in C^1(\mathbb{R}^3;\mathbb{R}^3)$; see, for instance, [5, Lemma 4.1] or [11, Theorem 10.2 (iii)]. But such a regularity of A_∞ implies that $j_{A_\infty} \in C^1_0(\mathbb{R}^3;\mathbb{R}^3)$. Thus, we also have $j_{A_\infty} \in C^\alpha_0(\mathbb{R}^3;\mathbb{R}^3)$, $0 < \alpha < 1$ and so $A_\infty \in C^{2,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$, as desired. For the latter implication see, for instance, [11, Theorem 10.3]. □

Remark 2. If we consider the time dependence in Lemma 7, and assume that f is C^1 with respect to $t \in I$, then A is also C^1 in $t \in I$ as a consequence of the Implicit Function Theorem in Banach spaces; see [16].

3.1. **Estimates on $\partial_t A$ and its space derivative.** We now turn to the estimates on the time derivative of the vector potential (i.e., the transverse component of the electric field), and its space derivatives.

Throughout this section, we shall assume that the triplet (f,Φ,A) satisfies (3.1)-(3.4) on $I \times \mathbb{R}^3 \times \mathbb{R}^3$ with $f(t)$ having compact support on $\mathbb{R}^3 \times \mathbb{R}^3$. For f as given, define $t \mapsto \bar{Z}(t)$ by

$$\bar{Z}(t) = \sup\{|(x,p)| : \exists 0 \leq s \leq t : f(s,x,p) \neq 0\}. \tag{3.8}$$

The function $\bar{Z}(t)$ is a non-decreasing function of t, which by the compact support of f is bounded on any finite subinterval of I. The following lemma is essential to our results:

Lemma 8. *Let $f \in C^1(I, C_0^{1,\alpha}(\mathbb{R}^6);\mathbb{R})$ and $(\Phi, A) \in C^1(I, C^{2,\alpha}(\mathbb{R}^3);\mathbb{R}^3\times\mathbb{R}^3)$, with $f \geq 0$ and $0 < \alpha < 1$, satisfy (3.1)-(3.4). Define ρ and $\bar{Z}(t)$ according to (1.4) and (3.8), respectively. There exists a positive $C(t) = C(\bar{Z}(t), \|f(t)\|_{L^\infty_{x,p}})$ such that*

$$\|\partial_t \partial_x A(t)\|_{L^2_x} + \left\|\rho^{1/2}(t)\partial_t A(t)\right\|_{L^2_x} \leq C(t), \quad t \in I.$$

Remark 3. For $t \in I$, Lemma 6 and the assumption on the support of f imply $|p - A| \leq C(\bar{Z}(t), \|f(t)\|_{L^\infty_{x,p}}) < \infty$ and so $|v_A| < 1$ strictly on $\mathtt{supp} f(t)$.

Proof of Lemma 8. For v_A as given in (3.4) define the current density

$$j_A(t,x) = \int_{\mathbb{R}^3} v_A f(t,x,p)dp.$$

By Lemma 3(c), the components A^i, $i = 1, 2, 3$ of the vector potential satisfy

$$\Delta A^i(t,x) = -4\pi j_A^i(t,x) - \partial_{x_i} \int_{\mathbb{R}^3} \nabla \cdot j_A(t,y)\frac{dy}{|y-x|}. \tag{3.9}$$

Take the partial time derivative on both sides of the above equation and multiply by $\partial_t A^i$. After integration by parts, dropping the 4π and using the definition of j_A,

$$\begin{aligned}\int_{\mathbb{R}^3} \left|\nabla \partial_t A^i\right|^2(t,x)dx &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t A^i(t,x)\partial_t\left(v_A^i f\right)(t,x,p)dpdx \\ &\quad - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t \partial_{x_i} A^i(t,x)\nabla\cdot\partial_t j_A(t,y)\frac{dxdy}{|y-x|}.\end{aligned} \tag{3.10}$$

Note that the boundary terms vanish. Indeed, since f has a compact support, so does j_A and the boundary term corresponding to the first term on the right-hand side of the above equation is zero. On the other hand, it follows by standard arguments that $\partial_t A^i(x)$ has at least a decay $O(|x|^{-1})$ and $\nabla\partial_t A^i(x) = O(|x|^{-2})$. Moreover, the integral $I(x)$ on the right-hand side of (3.9) has a decay $O(|x|^{-2})$ and so does $\partial_t I(x)$. Therefore, $\partial_t A^i \nabla \partial_t A^i(x) = O(|x|^{-3})$ and $\partial_t A^i \partial_t I(x) = O(|x|^{-3})$, which suffice for the boundary terms to vanish.

Now we add the equations (3.10) for each component of A. We find

$$\begin{aligned}\int_{\mathbb{R}^3} |\partial_x \partial_t A|^2 &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\left(\partial_t A\cdot\partial_t v_A\right) + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(\partial_t A\cdot v_A\right)\partial_t f \\ &\quad -\partial_t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{r}\left(\nabla\cdot A\right)\left(\nabla\cdot\partial_t j_A\right) \\ &= I_1 + I_2 + I_3.\end{aligned} \tag{3.11}$$

But $I_3 \equiv 0$ since the vector potential satisfies the Coulomb gauge condition; see Lemma 3(c). Also, by using the representation of the derivatives of the velocity given in the Appendix, the integral I_1 can be written as

$$I_1 = -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f}{\sqrt{1+g^2}}\left(|\partial_t A|^2 - |v_A\cdot\partial_t A|^2\right),$$

where we have denoted $g = |p - A|$. We shall also denote $K = -\nabla\Phi + v_A^i \nabla A^i$. Hence, after sending I_1 to the left-hand side of (3.11), and by using the Vlasov equation (3.1) in I_2, we find that

$$
\begin{aligned}
\int_{\mathbb{R}^3} |\partial_x \partial_t A|^2 \quad &+ \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f}{\sqrt{1+g^2}} \left(|\partial_t A|^2 - |v_A \cdot \partial_t A|^2 \right) \\
&= \quad -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (v_A \cdot \partial_t A) \left[\nabla_x \cdot (v_A f) + \nabla_p \cdot (K f) \right] \\
&= \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f \partial_t A^i \left(v_A \cdot \nabla_x v_A^i \right) + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f v_A^i \left(v_A \cdot \nabla_x \partial_t A^i \right) \\
&\quad\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f \partial_t A^i \left(K \cdot \nabla_p v_A^i \right). \qquad (3.12)
\end{aligned}
$$

Notice the integration by parts and the use of the product rule in the last equality. We claim that the left-hand side of the above equality has a positive lower bound for every time t. Indeed, we have that

$$
|\partial_t A|^2 - |v_A \cdot \partial_t A|^2 \geq |\partial_t A|^2 - |v_A|^2 |\partial_t A|^2 = \left(1 - |v_A|^2 \right) |\partial_t A|^2 .
$$

Also, by Remark 3 there exists a $g_{\mathtt{max}}(t) = g_{\mathtt{max}}(\bar{Z}(t), \|f(t)\|_{L^\infty_{x,p}}) < \infty$ so that

$$
\frac{1 - |v_A|^2}{\sqrt{1+g^2}} \equiv \frac{1}{(1+g^2)^{3/2}} \geq \frac{1}{(1+g^2_{\mathtt{max}})^{3/2}} \equiv C_{\mathtt{min}} > 0. \qquad (3.13)
$$

Therefore, the left-hand side of (3.12) satisfies

$$
\mathtt{LHS} \geq C_{\mathtt{min}}(t) \left(\|\partial_x \partial_t A(t)\|^2_{L^2_x} + \left\| \rho^{1/2}(t) \partial_t A(t) \right\|^2_{L^2_x} \right). \qquad (3.14)
$$

On the other hand, the known bounds on the derivatives of the potentials given in Lemma 6 lead to

$$
\|\partial_x v_A(t)\|_{L^\infty_{x,p}} + \|\partial_p v_A(t)\|_{L^\infty_{x,p}} + \|K(t)\|_{L^\infty_{x,p}} \leq C(t) \equiv C(\bar{Z}(t), \|f(t)\|_{L^\infty_{x,p}});
$$

see the Appendix for an explicit representation of the derivatives of the velocity. Hence, after a use of the Cauchy-Schwarz inequality and again the use of the compact support of f, the right-hand side of (3.12) can be estimated as

$$
\mathtt{RHS} \leq C(t) \left(\|\partial_x \partial_t A(t)\|_{L^2_x} + \left\| \rho^{1/2}(t) \partial_t A(t) \right\|_{L^2_x} \right). \qquad (3.15)
$$

Finally, since $(a+b)^2 \leq 2\left(a^2 + b^2\right)$, the result follows from (3.14)-(3.15). □

Lemma 9. *Under the assumptions of Lemma 8, we have that*

$$
\begin{aligned}
\|\partial_t A(t)\|_{L^\infty_x} \quad \leq \quad & C \Big[\|\rho(t)\|^{1/3}_{L^1_x} \|\rho(t)\|^{2/3}_{L^\infty_x} \left(1 + \|\rho(t)\|^{2/3}_{L^1_x} \|\rho(t)\|^{1/3}_{L^\infty_x} \right) \\
& + \|\rho(t)\|^{1/6}_{L^1_x} \|\rho(t)\|^{1/3}_{L^\infty_x} \left\| \rho^{1/2}(t) \partial_t A(t) \right\|_{L^2_x} \Big], \quad t \in I.
\end{aligned}
$$

Corollary 2. *Under the assumptions of Lemma 8, we have that*

$$
\|\partial_t A(t)\|_{L^\infty_x} \leq C(t), \quad t \in I,
$$

for some positive $C(t) = C(\bar{Z}(t), \|f(t)\|_{L^\infty_{x,p}})$.

Proof of Lemma 9. Consider the integral representation (3.3) for the vector potential, and take the partial time derivative on both sides of this equation. Denoting the kernel by $\mathcal{K}(x,y)$ and dropping the multiple $1/2$, we have

$$\begin{aligned}\partial_t A(t,x) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathcal{K}(x,y)\left[v_A\partial_t f + \partial_t v_A f\right](t,y,p)dpdy \\ &= I_1 + I_2.\end{aligned}$$

Set $K = -\nabla\Phi + v_A^i \nabla A^i$. A use of the Vlasov equation yields

$$\begin{aligned}I_1 &= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathcal{K}(x,y)v_A\nabla_y\cdot(v_A f) - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{K}(x,y)v_A\nabla_p\cdot(Kf) \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f v_A\cdot\left[\partial_y\mathcal{K}(x,y)v_A + \mathcal{K}(x,y)\partial_y v_A\right] + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} fK\cdot\mathcal{K}(x,y)\partial_p v_A.\end{aligned}$$

Therefore, since $|v_A| \le 1$, also $|\partial_x\mathcal{K}(x,y)| \le C\,|y-x|^2$ (see Lemma 3(c)), and $|\partial_x v_A| \le C\,|\partial_x A|$ and $|\partial_p v_A| \le C$ (see Appendix), we have

$$\begin{aligned}I_1 &\le C\left\{\int_{\mathbb{R}^3}\rho(t,y)\frac{dy}{|y-x|^2} + \left(\|\partial_x\Phi(t)\|_{L_x^\infty} + \|\partial_x A(t)\|_{L_x^\infty}\right)\int_{\mathbb{R}^3}\rho(t,y)\frac{dy}{|y-x|}\right\} \\ &\le C\,\|\rho(t)\|_{L_x^1}^{1/3}\|\rho(t)\|_{L_x^\infty}^{2/3}\left(1+\|\rho(t)\|_{L_x^1}^{2/3}\|\rho(t)\|_{L_x^\infty}^{1/3}\right), \qquad (3.16)\end{aligned}$$

where in the last inequality we used the estimates from Lemmas 5 and 6.

On the other hand, since $|\partial_t v_A| \le C\,|\partial_t A|$ (see Appendix), the integral I_2 can be estimated as

$$\begin{aligned}I_2 &\le C\int_{\mathbb{R}^3}\rho(t,y)\,|\partial_t A(t,y)|\,\frac{dy}{|y-x|} \\ &\le C\,\|\rho(t)\partial_t A(t)\|_{L_x^1}^{1/3}\|\rho(t)\partial_t A(t)\|_{L_x^2}^{2/3},\end{aligned}$$

where the second inequality is a consequence of Lemma 5. Hence, the Cauchy-Schwarz inequality and a direct estimate lead to

$$I_2 \le C\,\|\rho(t)\|_{L_x^1}^{1/6}\|\rho(t)\|_{L_x^\infty}^{1/3}\left\|\rho^{1/2}(t)\partial_t A(t)\right\|_{L_x^2}. \qquad (3.17)$$

The lemma then follows from (3.16) and (3.17). □

Lemma 10. *Under the assumptions of Lemma 8, we also have*

$$\|\partial_t\partial_x A(t)\|_{L_x^\infty} \le C(t)\left(\|\partial_t f(t)\|_{L_{x,p}^\infty} + \|\rho(t)\|_{L_x^1}^{1/3}\|\rho(t)\|_{L_x^\infty}^{2/3}\|\partial_t A(t)\|_{L_x^\infty}\right), t\in I.$$

for some $C(t) = C(\bar{Z}(t))$.

Corollary 3. *Under the assumptions of Lemma 8, we have that*

$$\|\partial_t\partial_x A(t)\|_{L_x^\infty} \le C(t), \quad t \in I,$$

for some positive $C(t) = C(\bar{Z}(t), \|f(t)\|_{L_{x,p}^\infty}, \|\partial_t f(t)\|_{L_{x,p}^\infty})$.

Proof of Lemma 10. Consider

$$\begin{aligned}\partial_t\partial_x A(t,x) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_x\mathcal{K}(x,y)\left[v_A\partial_t f+\partial_t v_A f\right](t,y,p)dpdy \\ &= I_1+I_2,\end{aligned}\tag{3.18}$$

where $|\partial_x\mathcal{K}(x,y)|\le C\,|y-x|^{-2}$. Hence, by Lemma 5, we obtain

$$I_1\le C(\bar{Z}(t))\,\|\partial_t f(t)\|_{L^\infty_{x,p}}\quad\text{ and }\quad I_2\le C\,\|\partial_t A(t)\|_{L^\infty_x}\|\rho(t)\|_{L^1_x}^{1/3}\|\rho(t)\|_{L^\infty_x}^{2/3},$$

where $|\partial_t v_A|\le C\,|\partial_t A|$ has been used. The result readily follows. □

4. The Cauchy Problem for the RVD System

A noticeable advantage of writing the RVD system in terms of the generalized variables and potentials is that it resembles, to some extent, the well-known Vlasov-Poisson (VP) system. Actually, the latter can be formally obtained from (3.1)-(3.4) by letting $c\to\infty$, so that terms involving the vector potential are no longer present. This resemblance allows to adapt previous techniques used for the VP system to the Darwin case. Below, the proofs we present are in the same vein as those given in [14] for the VP system. Obviously, several non-trivial difficulties arise due to the inclusion of the vector potential in the system equations, not present in the Poisson case. Incidentally, we expect that a global in time existence result to the *relativistic* Vlasov-Poisson system for unrestricted Cauchy data, which is still unsolved, will lead to an analogous result for the RVD system.

4.1. Local Solutions.

In this section we shall produce a local in time existence and uniqueness result for classical solutions of the RVD system.

Definition 3. *Let f_0 be given. We call f a classical solution of the RVD system if $f\in C^1(I\times\mathbb{R}^6;\mathbb{R})$; it induces the potentials $(\Phi,A)\in C^1(I,C^2(\mathbb{R}^3);\mathbb{R}\times\mathbb{R}^3)$ via (3.2)-(3.3); for every compact interval $\bar{J}\subset I$ the fields $\nabla\Phi$ and $v_A^i\nabla A^i$ are bounded on $\bar{J}\times\mathbb{R}^3$ and $\bar{J}\times\mathbb{R}^3\times\mathbb{R}^3$ respectively; and the triplet (f,Φ,A) satisfies the system (3.1)-(3.4) on $I\times\mathbb{R}^3\times\mathbb{R}^3$. Moreover, we say that f is a classical solution of the Cauchy problem if $f|_{t=0}=f_0$.*

Theorem 1. *Let $f_0\in C_0^{1,\alpha}(\mathbb{R}^6;\mathbb{R})$, $0<\alpha<1$, $f_0\ge 0$. For some $T>0$, there exists a unique classical solution f on $[0,T[$ of the Cauchy problem for the RVD system. Moreover, for each $0\le t<T$, the function $f(t)$ is in $C^{1,\alpha}(\mathbb{R}^6;\mathbb{R})$, it is non-negative and has compact support. In addition, if $T>0$ is the life span of f, then*

$$\sup\left\{|p| : \exists 0\le t<T, x\in\mathbb{R}^3 : f(t,x,p)\ne 0\right\}<\infty$$

implies that the solution is global in time, i.e., $T=\infty$.

Uniqueness. Consider two solutions (f_1,Φ_1,A_1) and (f_2,Φ_2,A_2) of the RVD system as given by Theorem 1. The Vlasov equation yields

$$\begin{aligned}\partial_t\left(f_1-f_2\right)^2 &+ v_{A_1}\cdot\nabla_x\left(f_1-f_2\right)^2+K_1\cdot\nabla_p\left(f_1-f_2\right)^2 \\ &= 2\left(f_1-f_2\right)\left[\left(v_{A_2}-v_{A_1}\right)\cdot\nabla_x f_2+\left(K_2-K_1\right)\cdot\nabla_p f_2\right]\end{aligned}$$

where $K_1=-\nabla\Phi_1+v_{A_1}^i\nabla A_1^i$ and analogously for K_2. In view of the compact support of the solutions f_1 and f_2, we set $R>0$ such that

$$\mathtt{supp}f_1(t)\cup\mathtt{supp}f_2(t)\subset B_R\times B_R,\quad t\in[0,\bar{T}]\subset[0,T[.$$

Let $Q(t) = \|f_1(t) - f_2(t)\|^2_{L^2_{x,p}}$. Since $f_2 \in C^1([0,T[\times\mathbb{R}^6;\mathbb{R})$ has compact support, $\|\nabla_x f_2(t)\|_{L^\infty_{x,p}} + \|\nabla_p f_2(t)\|_{L^\infty_{x,p}} \leq C_R$. Also, we have $|v_{A_1} - v_{A_2}| \leq C\,|A_1 - A_2|$ and, by Lemma 6, $\|\partial_x(A_1, A_2)\|_{L^\infty_x} \leq C_R$. Then, it is not difficult to check that

$$\begin{aligned} \frac{dQ(t)}{dt} \quad &\leq \quad C_R Q^{1/2}(t)\Big[\ \|\partial_x\Phi_1(t) - \partial_x\Phi_2(t)\|_{L^2_x(B_R)} \\ &\qquad + \|A_1(t) - A_2(t)\|_{L^2_x(B_R)} + \|\partial_x A_1(t) - \partial_x A_2(t)\|_{L^2_x(B_R)}\Big]. \end{aligned} \tag{4.1}$$

From the Poisson equation satisfied by the scalar potentials we deduce

$$\int_{\mathbb{R}^3} |\partial_x\Phi_1(t,x)|^2\,dx = 4\pi\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\rho_1(t,x)\rho_1(t,y)\frac{dxdy}{|y-x|},$$

and analogously for Φ_2. Linearity, the Hardy-Littlewood-Sobolev inequality and Jensen's inequality yield

$$\|\partial_x\Phi_1(t) - \partial_x\Phi_2(t)\|_{L^2_x} \leq C\,\|\rho_1(t) - \rho_2(t)\|_{L^{6/5}_x} \leq C_R Q^{1/2}(t). \tag{4.2}$$

On the other hand, in order to estimate the terms involving the vector potential, we proceed as follows. Define $f_\lambda = \lambda f_1 + (1-\lambda)\,f_2$ for $0 \leq \lambda \leq 1$. Clearly $f_\lambda \geq 0$ has compact support and satisfies $\partial_\lambda f_\lambda = f_1 - f_2$. Let A_λ be the Darwin vector potential induced by f_λ. In view of Lemma 3 we have

$$\Delta A_\lambda(t,x) = -4\pi\int_{B_R} v_{A_\lambda} f_\lambda(t,x,p)dp - \nabla\int_{B_R}\int_{B_R}\nabla\cdot(v_{A_\lambda}f_\lambda)(t,y,p)\frac{dpdy}{|y-x|},$$

and $\nabla\cdot A_\lambda = 0$. Notice that A_1 (resp. A_2) solves the above equation when $\lambda = 1$ (resp. $\lambda = 0$). By virtue of Remark 2 (where t is replaced by λ), we can use the arguments in the proof of Lemma 8 to find

$$\begin{aligned} \int_{\mathbb{R}^3}|\partial_\lambda\partial_x A_\lambda|^2 \quad &+ \quad \int_{B_R}\int_{B_R}\frac{f_\lambda}{\sqrt{1+g_\lambda}}\left(|\partial_\lambda A_\lambda|^2 - |v_{A_\lambda}\cdot\partial_\lambda A_\lambda|^2\right) \\ &= \quad \int_{B_R}\int_{B_R}(\partial_\lambda A_\lambda\cdot v_{A_\lambda})\,\partial_\lambda f_\lambda. \end{aligned} \tag{4.3}$$

The analogous expression in Lemma 8 is the first equality in (3.12) with the right-hand side replaced by the expression of I_2 in (3.11). Note the integration over $B_R\times B_R$ in view of the compact support of $\partial_\lambda f_\lambda$. Hence, since $|v_{A_\lambda}| < 1$ by Remark 3, we can use again the arguments in Lemma 8 and the Cauchy-Schwarz inequality on the right-hand side of (4.3) to obtain

$$\|\partial_\lambda\partial_x A_\lambda(t)\|^2_{L^2_x} + \left\|\rho_\lambda^{1/2}(t)\partial_\lambda A_\lambda(t)\right\|^2_{L^2_x} \leq C_R\,\|\partial_\lambda A_\lambda(t)\|_{L^2_x(B_R)}\,\|\partial_\lambda f_\lambda(t)\|_{L^2_{x,p}},$$

which implies that

$$\|\partial_\lambda\partial_x A_\lambda(t)\|^2_{L^2_x} \leq C_R\,\|\partial_\lambda A_\lambda(t)\|_{L^2_x(B_R)}\,Q^{1/2}(t) \tag{4.4}$$

for some $C_R > 0$ and all $0 \leq \lambda \leq 1$. Poincaré's inequality and (4.4) then yield

$$\|\partial_\lambda A_\lambda(t)\|^2_{L^2_x(B_R)} \leq \tilde{C}_R\,\|\partial_x\partial_\lambda A_\lambda(t)\|^2_{L^2_x} \leq C_R\,\|\partial_\lambda A_\lambda(t)\|_{L^2_x(B_R)}\,Q^{1/2}(t),$$

and thus,

$$\|\partial_\lambda A_\lambda(t)\|_{L^2_x(B_R)} \leq C_R Q^{1/2}(t) \tag{4.5}$$

for all $0 \leq \lambda \leq 1$. Inserting (4.5) into (4.4), we also have for all $0 \leq \lambda \leq 1$

$$\|\partial_\lambda\partial_x A_\lambda(t)\|_{L^2_x} \leq C_R Q^{1/2}(t). \tag{4.6}$$

Now, we observe that by Jensen's inequality,

$$\begin{aligned}\int_{\mathbb{R}^3} |\partial_x A_1(t,x) - \partial_x A_2(t,x)|^2 \, dx &= \int_{\mathbb{R}^3} \left| \int_0^1 \partial_\lambda \partial_x A_\lambda(t,x) d\lambda \right|^2 dx \\ &\leq \int_0^1 \int_{\mathbb{R}^3} |\partial_\lambda \partial_x A_\lambda(t,x)|^2 \, dx d\lambda \\ &\leq \sup_{0\leq\lambda\leq 1} \int_{\mathbb{R}^3} |\partial_\lambda \partial_x A_\lambda(t,x)|^2 \, dx,\end{aligned}$$

and similarly for $\|A_1(t) - A_2(t)\|_{L^2_x}$. Then, we use (4.5) and (4.6) to derive the estimate

$$\|A_1(t) - A_2(t)\|_{L^2_x(B_R)} + \|\partial_x A_1(t) - \partial_x A_2(t)\|_{L^2_x} \leq C_R Q^{1/2}(t). \tag{4.7}$$

Finally, we combine (4.1), (4.2) and (4.7) to conclude that

$$\frac{dQ(t)}{dt} \leq C_R Q(t).$$

Uniqueness then follows as a trivial consequence of Gronwall's lemma. □

Proof of Theorem 1. Let $f_0 \in C_0^{1,\alpha}(\mathbb{R}^6;\mathbb{R})$, $f_0 \geq 0$. Fix $\bar{X}_0 > 0$ and $\bar{P}_0 > 0$ such that $f_0(x,p) = 0$ for $|x| > \bar{X}_0$ or $|p| > \bar{P}_0$. We introduce the following iterative scheme. For $t \in I$ and $z = (x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$, define

$$f^0(t,z) = f_0(z).$$

For $n \in \mathbb{N}$, assume that $f^n : I \times \mathbb{R}^6 \to \mathbb{R}$ is given and define

$$\Phi^n(t,x) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^n(t,y,p) \frac{dpdy}{|y-x|} \tag{4.8}$$

$$A_n(t,x) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[\mathtt{id} + \omega \otimes \omega\right] v_{A_n} f^n(t,y,p) \frac{dpdy}{|y-x|}, \tag{4.9}$$

where

$$v_{A_n} = \frac{p - A_n}{\sqrt{1 + |p - A_n|^2}}.$$

Denote by $Z_n = (X_n, P_n)(s,t,z)$ the solution of the characteristic system

$$\dot{X}_n(s,t,z) = v_{A_n}(s, X_n(s,t,z), P_n(s,t,z)) \tag{4.10}$$

$$\dot{P}_n(s,t,z) = -\left[\nabla\Phi^n - v^i_{A_n} \nabla A^i_n\right](s, X_n(s,t,z), P_n(s,t,z)) \tag{4.11}$$

with $Z_n(t,t,z) = z$. We define the $(n+1)$-th iterate of the distribution function by

$$f^{n+1}(t,z) = f_0(Z_n(0,t,z)).$$

For convenience we shall also define the sequences

$$\rho^n(t,x) = \int_{\mathbb{R}^3} f^n(t,x,p)dp \quad \text{and} \quad j_n(t,x) = \int_{\mathbb{R}^3} v_{A_n} f^n(t,x,p)dp.$$

Step 1: In view of the Lemmas and Remarks in Section 2, and Lemma 7 in Section 3, the sequence $\{(f^n, \Phi^n, A_n)\}$ is well defined. In particular, $f^n \in C^1(I, C^{1,\alpha}(\mathbb{R}^6);\mathbb{R})$, $f^n \geq 0$, and $(\Phi^n, A_n) \in C^1(I, C^{2,\alpha}(\mathbb{R}^3); \mathbb{R}\times\mathbb{R}^3)$. For each n, the regularity in *time* of the potentials is the one of f^n. This is trivial for Φ^n. As for A^n, see Remark 2.

For $t \in I$ set $\bar{P}_0(t) = \bar{P}_0$ and for $n \in \mathbb{N}$ define

$$\begin{aligned} \bar{P}_n(t) &= \big\{|p| : \exists 0 \le s \le t, x \in \mathbb{R}^3 : f^n(s,x,p) \neq 0\big\} \\ &\equiv \{|P_{n-1}(s,0,z)| : 0 \le s \le t, z \in \mathtt{supp} f_0\} \end{aligned}$$

It is clear that $\mathtt{supp} f^n(t) \subseteq \big\{(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| \le \bar{X}_0 + t, |p| \le \bar{P}_n(t)\big\}$. Also,

$$\|f^n(t)\|_{L^q_z} = \|f_0\|_{L^q_z}, \quad 1 \le q \le \infty, \quad t \in I, \quad n \in \mathbb{N}$$

and we have the estimate

$$\|\rho^n(t)\|_{L^\infty_x} \le \frac{4}{3}\pi \|f_0\|_{L^\infty_z} \bar{P}_n^3(t).$$

Since $|j_n| \le |\rho^n|$, the known estimates on the potentials imply that

$$\|\Phi^n(t)\|_{L^\infty_x} + \|A_n(t)\|_{L^\infty_x} \le C(f_0)\bar{P}_n(t),$$

and finally

$$\|\partial_x \Phi^n(t)\|_{L^\infty_x} + \|\partial_x A_n(t)\|_{L^\infty_x} \le C(f_0)\bar{P}_n^2(t). \tag{4.12}$$

Step 2: For some $T > 0$ there is a non-negative, non-decreasing $\mathcal{P} \in C([0,T[;\mathbb{R})$ depending on the Cauchy datum only, such that for all $n \in \mathbb{N} \cup \{0\}$ and $0 \le t < T$

$$\bar{P}_n(t) \le \mathcal{P}(t).$$

Indeed, for $n \in \mathbb{N}$ the characteristic equation (4.11) and the estimate (4.12) imply

$$\begin{aligned} |P_n(s,0,z)| &\le |p| + \int_0^s \Big(\|\partial_x \Phi^n(\tau)\|_{L^\infty_x} + \|\partial_x A_n(\tau)\|_{L^\infty_x}\Big)\, d\tau \\ &\le \bar{P}_0 + C(f_0)\int_0^t \bar{P}_n^2(\tau)d\tau. \end{aligned} \tag{4.13}$$

Let $T > 0$ be the life span of the solution of the integral equation

$$\mathcal{P}(t) = \bar{P}_0 + C(f_0)\int_0^t \mathcal{P}^2(\tau)d\tau. \tag{4.14}$$

Hence, $\bar{P}_0(t) \le \mathcal{P}(t)$. Suppose $\bar{P}_n(t) \le \mathcal{P}(t)$ for some $n \in \mathbb{N}$. Then, in view of (4.13), this estimate also holds for $\bar{P}_{n+1}(t)$, which proves the claim. As a result, all estimates in Step 1 are uniform in n on any subinterval $[0,\bar{T}] \subset [0,T[$. In particular, for all $n \in \mathbb{N}$ and $0 \le t < T$, we have

$$\|\rho^n(t)\|_{L^\infty_x} + \|j_n(t)\|_{L^\infty_x} + \|\partial_x \Phi^n(t)\|_{L^\infty_x} + \|\partial_x A_n(t)\|_{L^\infty_x} \le C^0_{\bar{T}} \equiv C(\bar{T}, f_0). \tag{4.15}$$

For future use, we notice that the maximal solution of (4.14) is given by

$$\mathcal{P}(t) = \bar{P}_0\left(1 - C(f_0)\bar{P}_0 t\right)^{-1}, \quad 0 \le t < T \equiv \left(C(f_0)\bar{P}_0\right)^{-1}, \tag{4.16}$$

with $C(f_0) = 3(2\pi)^{2/3}\|f_0\|^{1/3}_{L^1_{x,p}}\|f_0\|^{2/3}_{L^\infty_{x,p}}$.

Step 3: We claim that for every fixed $0 \le \bar{T} < T$

$$\|\partial_x \rho^n(t)\|_{L^\infty_x} + \|\partial_x j_n(t)\|_{L^\infty_x} + \left\|\partial_x^2 \Phi^n(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A_n(t)\right\|_{L^\infty_x} \le C^0_{\bar{T}}, \tag{4.17}$$

for all $n \in \mathbb{N}$ and $0 \le t \le \bar{T}$.

To start with, we estimate the space derivatives of the characteristic curves. To ease notation, we write $(X_n, P_n)(s) \equiv (X_n, P_n)(s,t,x,p)$. Recall

$$v_{A_n}(s, X_n(s), P_n(s)) \equiv v(P_n(s), A_n(s, X_n(s))),$$

where v is C_b^∞ in its argument. Hence, since (by abuse of notation) we have $(\partial_x X_n(t), \partial_x P_n(t)) = (1,0)$, the uniform bounds in (4.15) lead to

$$\begin{aligned}
|\partial_x X_n(s)| &\leq |\partial_x X_n(t)| + \int_s^t |\partial_x [v(P_n(\tau), A_n(\tau, X_n(\tau)))]| \, d\tau \\
&\leq 1 + C_T^0 \int_0^t (|\partial_x X_n(\tau)| + |\partial_x P_n(\tau)|) \, d\tau.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|\partial_x P_n(s)| &\leq |\partial_x P_n(t)| + \int_s^t \left|\partial_x \left[\nabla\Phi - v^i_{A_n} \nabla A^i_n\right](\tau, X_n(\tau), P_n(\tau))\right| d\tau \\
&\leq C_T^0 \int_0^t \left(1 + \left\|\partial_x^2 \Phi^n(\tau)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_n(\tau)\right\|_{L_x^\infty}\right) \\
&\qquad \times \left(\, |\partial_x X_n(\tau)| + |\partial_x P_n(\tau)| \, \right) d\tau.
\end{aligned}$$

These two estimates and the Gronwall's lemma yield

$$\begin{aligned}
|\partial_x X_n(s)| &+ |\partial_x P_n(s)| \\
&\leq \exp\left\{ C_T^0 \int_0^t \left(1 + \left\|\partial_x^2 \Phi^n(\tau)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_n(\tau)\right\|_{L_x^\infty}\right) d\tau \right\}.
\end{aligned}$$

As a result, we also have

$$\begin{aligned}
\left|\partial_x \rho^{n+1}(t,x)\right| &\leq \int_{|p| \leq \mathcal{P}(t)} |\partial_x [f_0(Z_n(0,t,x,p))]| \, dp \\
&\leq C_{\bar{T}}^0 \exp\left\{ C_T^0 \int_0^t \left(1 + \left\|\partial_x^2 \Phi^n(\tau)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_n(\tau)\right\|_{L_x^\infty}\right) d\tau \right\}.
\end{aligned}$$

Similarly, after using the product rule and the known estimates, there exists a sufficiently large constant $C_{\bar{T}}^0$ such that

$$\begin{aligned}
|\partial_x j_{n+1}(t,x)| &\leq \int_{|p| \leq \mathcal{P}(t)} |\partial_x [v(p, A_{n+1}(t,x)) f_0(Z_n(0,t,x,p))]| \, dp \\
&\leq C_{\bar{T}}^0 \exp\left\{ C_{\bar{T}}^0 \int_0^t \left(1 + \left\|\partial_x^2 \Phi^n(\tau)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_n(\tau)\right\|_{L_x^\infty}\right) d\tau \right\}.
\end{aligned}$$

Hence, in view of Lemma 6, we have for all $0 \leq t \leq \bar{T}$ that

$$\begin{aligned}
\left\|\partial_x^2 \Phi^{n+1}(t)\right\|_{L_x^\infty} &+ \left\|\partial_x^2 A_{n+1}(t)\right\|_{L_x^\infty} \\
&\leq C_{\bar{T}}^0 \left(1 + \ln^+ \left\|\partial_x \rho^{n+1}(t)\right\|_{L_x^\infty} + \ln^+ \|\partial_x j_{n+1}(t)\|_{L_x^\infty}\right) \\
&\leq C_{\bar{T}}^0 + C_{\bar{T}}^0 \int_0^t \left(\left\|\partial_x^2 \Phi^n(\tau)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_n(\tau)\right\|_{L_x^\infty}\right) d\tau.
\end{aligned}$$

Since the right-hand side is bounded for $n = 0$, induction in n yields

$$\left\|\partial_x^2 \Phi^n(t)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_n(t)\right\|_{L_x^\infty} \leq C_{\bar{T}}^0 \exp\left\{C_{\bar{T}}^0 \bar{T}\right\},$$

for all $n \in \mathbb{N}$ and $0 \leq t \leq \bar{T}$. In turn, this provides a uniform bound on the derivatives of the iterates for the current and density functions.

Step 4: We show that $\{f^n\}$ is Cauchy in the uniform norm on $[0,\bar{T}]\times\mathbb{R}^6$. To start with, notice that

$$\begin{aligned} \left|f^{n+1}(t,z)-f^n(t,z)\right| &= |f_0(Z_n(0,t,z))-f_0(Z_{n-1}(0,t,z))| \\ &\leq C\,|Z_n(0,t,z)-Z_{n-1}(0,t,z)|\,. \end{aligned} \tag{4.18}$$

On the other hand, by using the estimates in the previous steps, it is not difficult to check that the characteristics equations lead to

$$\begin{aligned} |X_n(s) &- X_{n-1}(s)| \\ &\leq \int_s^t |v(P_n(\tau),A_n(\tau,X_n(\tau)))-v(P_{n-1}(\tau),A_{n-1}(\tau,X_{n-1}(\tau)))|\,d\tau \\ &\leq C\int_s^t \Big(|X_n(\tau)-X_{n-1}(\tau)|+|P_n(\tau)-P_{n-1}(\tau)| \\ &\qquad +\|A_n(\tau)-A_{n-1}(\tau)\|_{L^\infty_x}\Big)\,d\tau, \end{aligned}$$

and

$$\begin{aligned} |P_n(s) &- P_{n-1}(s)| \\ &\leq \int_s^t \Big(\,\left|\nabla\Phi^n(\tau,X_n(\tau))-\nabla\Phi^{n-1}(\tau,X_{n-1}(\tau))\right| \\ &\qquad +\Big|\,\left(v^i_{A_n}\nabla A^i_n\right)(\tau,X_n(\tau),P_n(\tau)) \\ &\qquad -\left(v^i_{A_{n-1}}\nabla A^i_{n-1}\right)(\tau,X_{n-1}(\tau),P_{n-1}(\tau))\Big|\Big)\,d\tau \\ &\leq C\int_s^t \Big(\,|X_n(\tau)-X_{n-1}(\tau)|+|P_n(\tau)-P_{n-1}(\tau)| \\ &\qquad +\left\|\partial_x\Phi^n(\tau)-\partial_x\Phi^{n-1}(\tau)\right\|_{L^\infty_x} \\ &\qquad +\|A_n(\tau)-A_{n-1}(\tau)\|_{L^\infty_x}+\|\partial_x A_n(\tau)-\partial_x A_{n-1}(\tau)\|_{L^\infty_x}\Big)\,d\tau. \end{aligned}$$

Therefore, after adding the above expressions, Gronwall's inequality yields

$$\begin{aligned} |Z_n(0,t,z) &- Z_{n-1}(0,t,z)| \leq C\int_0^t \Big(\left\|\partial_x\Phi^n(\tau)-\partial_x\Phi^{n-1}(\tau)\right\|_{L^\infty_x} \\ &+ \|A_n(\tau)-A_{n-1}(\tau)\|_{L^\infty_x}+\|\partial_x A_n(\tau)-\partial_x A_{n-1}(\tau)\|_{L^\infty_x}\Big)\,d\tau. \end{aligned} \tag{4.19}$$

Now, to produce a Gronwall's inequality resulting from (4.18) and (4.19), we look for suitable estimates on the right-hand side of (4.19). To start with, let $R=\max\left\{\bar{X}_0+\bar{T},\mathcal{P}(\bar{T})\right\}$. For all $n\in\mathbb{N}$ and $0\leq t\leq\bar{T}$ we have

$$\mathtt{supp} f^n(t)\subset B_R\times B_R.$$

Linearity and Lemma 6 yield

$$\begin{aligned} \left\|\partial_x\Phi^n(\tau)-\partial_x\Phi^{n-1}(\tau)\right\|_{L^\infty_x} &\leq C\left\|\rho^n(\tau)-\rho^{n-1}(\tau)\right\|^{1/3}_{L^1_x}\left\|\rho^n(\tau)-\rho^{n-1}(\tau)\right\|^{2/3}_{L^\infty_x} \\ &\leq C_R\left\|f^n(\tau)-f^{n-1}(\tau)\right\|_{L^\infty_{x,p}}. \end{aligned} \tag{4.20}$$

To estimate the terms involving the vector potential, we proceed as follows. According to the definition of the iterates, it is clear that for each $n\in\mathbb{N}$ they satisfy

$$\partial_t f^{n+1}+v_{A_n}\cdot\nabla_x f^{n+1}-\left[\nabla\Phi^n-v^i_{A_n}\nabla A^i_n\right]\cdot\nabla_p f^{n+1}=0.$$

Hence, by the uniqueness proof now in terms of the iterates, (see (4.7)),

$$\|A_n(\tau) - A_{n-1}(\tau)\|_{L^2_x(B_R)} \le C_R \left\|f^{n+1}(\tau) - f^n(\tau)\right\|_{L^2_{x,p}},$$

which can in turn be estimated in terms of $\left\|f^{n+1}(\tau) - f^n(\tau)\right\|_{L^\infty_{x,p}}$.

Also, if we write respectively $f^n - f^{n-1}$ and $A_n - A_{n-1}$ in Lemma 9 instead of $\partial_t f$ and $\partial_t A$, it is easy to check that

$$\begin{aligned}\|A_n(\tau) \quad &- \quad A_{n-1}(\tau)\|_{L^\infty_x} \\ &\le \quad C_R\left(\left\|f^n(\tau) - f^{n-1}(\tau)\right\|_{L^\infty_{x,p}} + \|A_n(\tau) - A_{n-1}(\tau)\|_{L^2_x(B_R)}\right).\end{aligned}$$

Similarly, after a slight modification of the proof of Lemma 10 (it is actually simpler here since the Vlasov equation is not used), we find

$$\begin{aligned}\|\partial_x A_n(\tau) \quad &- \quad \partial_x A_{n-1}(\tau)\|_{L^\infty_x} \\ &\le \quad C_R\left(\left\|f^n(\tau) - f^{n-1}(\tau)\right\|_{L^\infty_{x,p}} + \|(A_n(\tau) - A_{n-1}(\tau))\|_{L^2_x(B_R)}\right).\end{aligned}$$

Therefore, the previous three estimates yield

$$\begin{aligned}\|A_n(\tau) \quad &- \quad A_{n-1}(\tau)\|_{L^\infty_x} + \|\partial_x A_n(\tau) - \partial_x A_{n-1}(\tau)\|_{L^\infty_x} \\ &\le \quad C_R\left(\left\|f^{n+1}(t) - f^n\right\|_{L^\infty_{x,p}} + \left\|f^n(\tau) - f^{n-1}(\tau)\right\|_{L^\infty_{x,p}}\right).\end{aligned} \tag{4.21}$$

Hence, if we combine (4.18) and (4.19) with (4.20) and (4.21), a use of Gronwall's lemma gives

$$\left\|f^{n+1}(t) - f^n(t)\right\|_{L^\infty_{x,p}} \le C\int_0^t \left\|f^n(\tau) - f^{n-1}(\tau)\right\|_{L^\infty_{x,p}} d\tau,$$

which by induction, readily implies the claim. It follows that $\{f^n\}$ converges uniformly to some $f \in C([0,\bar{T}] \times \mathbb{R}^6; \mathbb{R})$ and for all $0 \le t \le \bar{T}$ we have

$$\mathtt{supp} f(t) \subset B_R \times B_R.$$

Finally, if we respectively define ρ, Φ and A according to (1.4), (3.2) and (3.3), we have that ρ, Φ and A are C_b, and $\rho^n \to \rho$, $\Phi^n \to \Phi$ and $A_n \to A$ hold uniformly on $[0,\bar{T}] \times \mathbb{R}^3$. The latter follows from (4.21). The uniform limits $v_{A_n} \to v_A$ and $v_{A_n} f^n \to v_A f$ can be easily checked, and therefore $j_n \to j_A$ uniformly on $[0,\bar{T}]\times\mathbb{R}^3$, with $j_A \in C_b([0,\bar{T}] \times \mathbb{R}^3; \mathbb{R}^3)$ defined by (1.4).

Step 5: Actually $f \in C^1(I\times\mathbb{R}^6; \mathbb{R})$, as we show next. Indeed, in view of Step 4 and, respectively, (4.20) and (4.21), the sequences $\{\partial_x \Phi^n\}$ and $\{\partial_x A_n\}$ are uniformly Cauchy on $[0,\bar{T}] \times \mathbb{R}^3$. Moreover, by Lemma 6 we have

$$\begin{aligned}\left\|\partial_x^2 A_m(t) \quad \right. &- \quad \left.\partial_x^2 A_n(t)\right\| \le C\left[R^{-3}\|j_m(t) - j_n(t)\|_{L^1_x}\right. \\ &+ \quad \left. h\|\partial_x j_m(t) - \partial_x j_n(t)\|_{L^\infty_x} + \left(1 + \ln(R/h)\|j_m(t) - j_n(t)\|_{L^\infty_x}\right)\right]\end{aligned}$$

and similarly for $\partial_x^2 \Phi^n(t)$. Hence, the known estimates and the fact that we can choose h arbitrary small imply that $\left\{\partial_x^2 \Phi^n\right\}$ and $\left\{\partial_x^2 A_n\right\}$ are uniformly Cauchy on $[0,\bar{T}] \times \mathbb{R}^3$ as well. Therefore, we have (with a slight abuse of notation)

$$(\Phi, A),\ \partial_x(\Phi, A),\ \partial_x^2(\Phi, A)\ \in C([0,\bar{T}] \times \mathbb{R}^3; \mathbb{R} \times \mathbb{R}^3),$$

and so the characteristic flow $Z \in C^1([0,\bar{T}] \times [0,\bar{T}] \times \mathbb{R}^6)$ induced by the limiting field is in turn the limit of the sequence $\{Z_n\}$. As a result, the function

$$f(t,z) = \lim_{n\to\infty} f_0(Z_n(0,t,z)) = f_0(Z(0,t,z))$$

has the claimed regularity and the triplet (f,Φ,A) satisfies (3.1)-(3.4).

Step 6: We show that the potentials Φ and A have the required regularity in time. Define $f_\lambda = \lambda f^m + (1-\lambda) f^n$, $0 \le \lambda \le 1$. This definition is just like the one in the uniqueness proof but in terms of any two elements of the sequence $\{f^n\}$. Let A_λ be the vector potential induced by f_λ. Following the lines in the proof of Lemma 8, it is not difficult to check that (this is analogous to (3.11))

$$\begin{aligned}
\int_{\mathbb{R}^3} |\partial_\lambda \partial_t \partial_x A_\lambda|^2 &= \int_{B_R}\int_{B_R} f_\lambda \partial_\lambda \partial_t A_\lambda \cdot \partial_\lambda \partial_t v_{A_\lambda} \\
&+ \int_{B_R}\int_{B_R} \partial_\lambda \partial_t A \cdot \left[\partial_\lambda v_{A_\lambda} \partial_t f_\lambda + \partial_t v_{A_\lambda} \partial_\lambda f_\lambda + v_{A_\lambda} \partial_\lambda \partial_t f_\lambda\right] \\
&+ \partial_\lambda \partial_t \int_{\mathbb{R}^3}\int_{B_R} \frac{1}{r} \left(\nabla \cdot A_\lambda\right)\left(\nabla \cdot \partial_\lambda \partial_t j_\lambda\right). \qquad (4.22)
\end{aligned}$$

Since $\nabla \cdot A_\lambda = 0$, the third integral in the right-hand side vanishes. On the other hand, by using the notation of the Appendix, we have $\partial_\lambda v_{A_\lambda} = -Dv_{A_\lambda} \partial_\lambda A_\lambda$, also $\partial_t v_{A_\lambda} = -Dv_{A_\lambda} \partial_t A_\lambda$, and

$$\partial_\lambda \partial_t v_{A_\lambda} = -Dv_{A_\lambda} \partial_\lambda \partial_t A - D^2 v_{A_\lambda} \partial_\lambda A_\lambda \partial_t A_\lambda.$$

Therefore, since by Step 5 $|\partial_t f_\lambda| \le C_R$, and by Corollary 2 $|\partial_t A_\lambda| \le C_R$, we obtain from (4.22) that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\partial_\lambda \partial_t \partial_x A_\lambda|^2 + \int_{B_R}\int_{B_R} \frac{f_\lambda}{\sqrt{1+g_\lambda^2}} \left(|\partial_\lambda \partial_t A_\lambda|^2 - |v_{A_\lambda} \cdot \partial_\lambda \partial_t A_\lambda|^2\right) \\
&\le C_R \int_{B_R}\int_{B_R} |\partial_\lambda \partial_t A_\lambda| \left[|\partial_\lambda A_\lambda| + |\partial_\lambda f_\lambda| + |\partial_\lambda \partial_t f_\lambda|\right] \\
&\le C_R \left\|\partial_\lambda \partial_t A_\lambda(t)\right\|_{L^2_x} \left[\left\|\partial_\lambda A_\lambda(t)\right\|_{L^2_x(B_R)} + \left\|\partial_\lambda f_\lambda(t)\right\|_{L^\infty_{x,p}} + \left\|\partial_\lambda \partial_t f_\lambda(t)\right\|_{L^\infty_{x,p}}\right]. \qquad (4.23)
\end{aligned}$$

In the last step we have used the Cauchy-Schwarz inequality. Now, define

$$G_{mn} = \sup_{0\le t\le \bar{T}} \left(\|f^n(t) - f^m(t)\|_{L^\infty_{x,p}} + \|\partial_t f^n(t) - \partial_t f^m(t)\|_{L^\infty_{x,p}}\right),$$

which in view of Steps 4 and 5 converges to zero as $n,m \to \infty$. If we use the estimate (4.5) for the iterates, i.e. $\|\partial_\lambda A_\lambda(t)\|_{L^2_x(B_R)} \le C_R \|\partial_\lambda f_\lambda(t)\|_{L^2_{x,p}}$, we find that the expression in square brackets in the right-hand side of (4.23) can be estimated as

$$\begin{aligned}
&\|\partial_\lambda A_\lambda(t)\|_{L^2_x(B_R)} + \|\partial_\lambda f_\lambda(t)\|_{L^\infty_{x,p}} + \|\partial_\lambda \partial_t f_\lambda(t)\|_{L^\infty_{x,p}} \\
&\le C_R \left(\|\partial_\lambda f_\lambda(t)\|_{L^2_{x,p}} + \|\partial_\lambda f_\lambda(t)\|_{L^\infty_{x,p}} + \|\partial_\lambda \partial_t f_\lambda(t)\|_{L^\infty_{x,p}}\right) \le C_R G_{mn},
\end{aligned}$$

uniformly in λ. On the other hand, since $|v_{A_\lambda}| < 1$ strictly, we can reason as in the proof of Lemma 8 to find a lower bound on the left-hand side of (4.23). This lower bound can then be estimated as

$$\|\partial_\lambda \partial_t \partial_x A_\lambda(t)\|^2_{L^2_x} + \left\|\rho_\lambda^{1/2}(t) \partial_\lambda \partial_t A_\lambda(t)\right\|^2_{L^2_x(B_R)} \le C_R \|\partial_\lambda \partial_t A_\lambda(t)\|_{L^2_x(B_R)} G_{mn}.$$

Consider the first term on the left-hand side. Poincaré's inequality and the above estimate imply that

$$\|\partial_\lambda \partial_t A_\lambda(t)\|^2_{L^2_x(B_R)} \le C_R \|\partial_\lambda \partial_t \partial_x A_\lambda(t)\|^2_{L^2_x} \le C_R \|\partial_\lambda \partial_t A_\lambda(t)\|_{L^2_x(B_R)} G_{mn}.$$

Then, the last two estimates yield

$$\|\partial_\lambda \partial_t A_\lambda(t)\|_{L^2_x(B_R)} + \|\partial_\lambda \partial_t \partial_x A_\lambda(t)\|_{L^2_x} \le C_R G_{mn}. \tag{4.24}$$

On the other hand, by the definition of A_λ, we have

$$\partial_\lambda \partial_t A_\lambda(t,x) = \int_{B_R}\int_{B_R} \mathcal{K}(x,y)\partial_\lambda \partial_t \left[v_{A_\lambda} f_\lambda(t,x,p)\right] dpdy.$$

Therefore, after taking the product rule in the integrand, we may proceed as in Lemma 9 to obtain the estimate

$$\|\partial_\lambda \partial_t A_\lambda(t)\|_{L^\infty_x} \le C_R \left(G_{mn} + \|\partial_\lambda \partial_t A_\lambda(t)\|_{L^2_x(B_R)}\right), \tag{4.25}$$

where again we have used $|\partial_t f_\lambda| \le C_R$ and $|\partial_t A_\lambda| \le C_R$. Similarly, since

$$\partial_\lambda \partial_t \partial_x A_\lambda(t,x) = \int_{B_R}\int_{B_R} \partial_x \mathcal{K}(x,y)\partial_\lambda \partial_t \left[v_{A_\lambda} f_\lambda(t,x,p)\right] dpdy,$$

we can proceed as in Lemma 10 to find

$$\|\partial_\lambda \partial_t \partial_x A_\lambda(t)\|_{L^\infty_x} \le C_R \left(G_{mn} + \|\partial_\lambda \partial_t A_\lambda(t)\|_{L^\infty_x(B_R)}\right). \tag{4.26}$$

Hence, since $|\partial_t A_m - \partial_t A_n| \le \int_0^1 |\partial_\lambda \partial_t A_\lambda|\, d\lambda \le \sup_\lambda |\partial_\lambda \partial_t A_\lambda|$ and similarly for $|\partial_t \partial_x A_m - \partial_t \partial_x A_n|$, we can gather the above estimates to find that

$$\|\partial_t A_m(t) - \partial_t A_n(t)\|_{L^\infty_x} + \|\partial_t \partial_x A_m(t) - \partial_t \partial_x A_n(t)\|_{L^\infty_x} \le C_R G_{mn}.$$

Therefore, the sequences $\{\partial_t A_n\}$ and $\{\partial_t \partial_x A_n\}$ are uniformly Cauchy and we have that $\partial_t A_n \to \partial_t A$ and $\partial_t \partial_x A_n \to \partial_t \partial_x A$ uniformly on $[0,\bar{T}]\times\mathbb{R}^3$. In turn, the former limit and Steps 4 and 5 imply the uniform convergence $\partial_t(v_{A_n} f^n) \to \partial_t(v_A f)$, and so $\partial_t j_n \to \partial_t j_A$. Also, $\partial_t \rho^n \to \partial_t \rho$. Hence, just as in Step 5, the sequences $\left\{\partial_t \partial_x^2 \Phi\right\}$ and $\left\{\partial_t \partial_x^2 A\right\}$ are uniformly Cauchy on $[0,\bar{T}]\times\mathbb{R}^3$. Therefore, since trivially $\partial_t \Phi$ and $\partial_t \partial_x \Phi$ are continuous on $[0,\bar{T}]\times\mathbb{R}^3$, we conclude that

$$\partial_t(\Phi, A),\ \partial_t \partial_x(\Phi, A),\ \partial_t \partial_x^2(\Phi, A) \in C([0,\bar{T}]\times\mathbb{R}^3).$$

Having proved the claim, and since $0 \le \bar{T} < T$ was arbitrary, we conclude that $f \in C^1([0,T[\times\mathbb{R}^6;\mathbb{R})$ is a classical solution of the relativistic Vlasov-Darwin system.

Step 7: Moreover, $f(t) \in C^{1,\alpha}(\mathbb{R}^6;\mathbb{R})$, $0 < \alpha < 1$, for each $0 \le t < T$. In view of Remark 1, this holds if $(\Phi, A)(t) \in C^{2,\alpha}(\mathbb{R}^3;\mathbb{R}\times\mathbb{R}^3)$. But, since we have $(\rho, j_A)(t) \in C^1_0(\mathbb{R}^3;\mathbb{R}\times\mathbb{R}^3) \subset C^\alpha_0(\mathbb{R}^3;\mathbb{R}\times\mathbb{R}^3)$, the regularity needed for the potentials is guaranteed (see the last lines in the proof of Lemma 7).

Step 8: The proof of the continuation criterion is as follows. Let f be the solution of the RVD system previously obtained, which clearly satisfies $f|_{t=0}$. As shown in (4.16), the life span of f is $T \equiv \left(C(f_0)\bar{P}_0\right)^{-1}$ with

$$C(f_0) = 3(2\pi)^{2/3} \|f_0\|^{1/3}_{L^1_{x,p}} \|f_0\|^{2/3}_{L^\infty_{x,p}}.$$

Define $\bar{P}_T = \sup\left\{|p| : \exists 0 \le t < T, x \in \mathbb{R}^3 : f(t,x,p) \ne 0\right\}$ and assume that $\bar{P}_T < \infty$ but $T < \infty$. We claim that this is a contradiction.

Fix $0 < t_0 < T$ and consider $f(t_0)$ as a Cauchy datum of the RVD system, which is guaranteed by Step 7. Known estimates yield

$$\|f(t_0)\|_{L^1_{x,p}} = \|f_0\|_{L^1_{x,p}}, \quad \|f(t_0)\|_{L^\infty_{x,p}} = \|f_0\|_{L^\infty_{x,p}}.$$

Thus, $C(f(t_0)) = C(f_0)$. Define $\epsilon = \left(C(f_0)\bar{P}_T\right)^{-1}$, which does *not* depend on t_0. Steps 1-3 imply that all uniform estimates on the sequence of approximate solutions induced by $f(t_0)$ hold on $[t_0, t_0+\epsilon[$. Then, $f(t_0)$ yields a unique classical solution of the RVD system on that interval.

But we could have fixed t_0 arbitrary close to the life span $T < \infty$ of f and so extend this solution beyond T, which is a contradiction. Hence, we have shown that $\bar{P}_T < \infty$ implies $T = \infty$. This, and the uniqueness result, conclude the proof of Theorem 1. □

4.2. **Global Solutions.** If additional conditions are imposed on the Cauchy datum in Theorem 1, then the local solution found in the previous section can be extended globally in time. We prove this result next. We start by defining the set where the Cauchy datum will be taken from. For $\bar{X}_0 > 0$ and $\bar{P}_0 > 0$ given let

$$\begin{aligned} \mathcal{D} \;=\; & \Big\{ f \in C^{1,\alpha}(\mathbb{R}^6;\mathbb{R}),\ 0<\alpha<1 : \\ & f \geq 0,\ \|f\|_{W^{1,\infty}_{x,p}} \leq 1,\ \mathtt{supp} f \subset B_{\bar{X}_0} \times B_{\bar{P}_0} \Big\}. \end{aligned}$$

Theorem 2. *There exists a $\delta > 0$ such that, if $f_0 \in \mathcal{D}$ with $\|f_0\|_{L^\infty_{x,p}} \leq \delta$, then the classical solution of the RVD system (3.1)-(3.4) with Cauchy datum f_0 is global in time. Moreover, for $t > 0$ this solution satisfies the decay estimates*

$$\|\rho(t)\|_{L^\infty_x} + \|j_A(t)\|_{L^\infty_x} \leq Ct^{-3} \tag{4.27}$$

$$\|\partial_x \Phi(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} \leq Ct^{-2} \tag{4.28}$$

$$\left\|\partial_x^2 \Phi(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} \leq Ct^{-3}\ln(1+t) \tag{4.29}$$

We first introduce some technical results and postpone the actual proof of Theorem 2 to the end of this section. The following lemma shows that a sufficiently small Cauchy datum leads to a classical solution of the RVD system which exists on any given time interval and induces potentials whose derivatives can be made as small as desired.

Lemma 11. *Fix $\epsilon > 0$ and $T > 0$. There exists $\delta = \delta(\epsilon, T) > 0$ such that, if $f_0 \in \mathcal{D}$ with $\|f_0\|_{L^\infty_{x,p}} \leq \delta$, then the classical solution of the RVD system with Cauchy datum f_0 exists on the time interval $[0,T]$ and induces potentials satisfying*

$$\|\partial_t A(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} + \|\partial_x \Phi(t)\|_{L^\infty_x} + \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 \Phi(t)\right\|_{L^\infty_x} < \epsilon \tag{4.30}$$

for all $0 \leq t \leq T$.

Proof. In view of Lemma 6, and since $|j_A| \leq \rho$ and

$$\|\partial_x \rho(t)\|_{L^\infty_x} + \|\partial_x j_A(t)\|_{L^\infty_x} \leq C^0_T, \quad 0 \leq t \leq T$$

hold (the latter proved just as in Step 3 in Theorem 1, with the estimates applied to the solution instead of the iterates), the *space* derivatives of A satisfy the same estimates as the space derivatives of Φ. Hence, the proof is mutatis mutandis the proof of [14, Lemma 4.2] for the Vlasov-Poisson system, as far as the space derivatives of the potentials are concerned. As for $\partial_t A$, the result follows suit in view of the estimates in Lemmas 8 and 9. $\square$

To proceed, we now define the so-called *free streaming condition* for classical solutions of the RVD system.

Definition 4. *Fix $\beta > 0$ and $a > 0$. A classical solution of the RVD system is said to satisfy the free streaming condition of parameter β (FSβ) on the time interval $[0, a]$, if it exists on $[0, a]$ and induces potentials satisfying the estimates*

$$\begin{aligned} \|\partial_t A(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} + \|\partial_x \Phi(t)\|_{L^\infty_x} &\leq \beta(1+t)^{-3/2}, \\ \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 \Phi(t)\right\|_{L^\infty_x} &\leq \beta(1+t)^{-5/2}, \end{aligned}$$

for all $0 \leq t \leq a$.

Lemma 12. *There exists $\delta > 0$, $\beta > 0$ and a positive $C = C(\bar{X}_0, \bar{P}_0)$ such that any classical solution f of the RVD system having a Cauchy datum $f_0 \in \mathcal{D}$ with $\|f_0\|_{L^\infty_{x,p}} \leq \delta$ and satisfying (FSβ) on some interval $[0, a]$, also satisfies the estimates*

$$\|\partial_x \Phi(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} \leq Ct^{-2}, \tag{4.31}$$

$$\left\|\partial_x^2 \Phi(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} \leq Ct^{-3}\ln(1+t), \tag{4.32}$$

for all $0 \leq t \leq a$.

Proof. By virtue of Lemma 6, the following estimates hold

$$\begin{aligned} \|\partial_x \Phi(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} &\leq C\,\|f_0\|^{1/3}_{L^1_{x,p}}\,\|\rho(t)\|^{2/3}_{L^\infty_x}, \\ \left\|\partial_x^2 \Phi(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} &\leq Ct^{-3}\Big[\|f_0\|^{1/3}_{L^1_{x,p}} + \|\partial_x \rho(t)\|_{L^\infty_x} + \|\partial_x j_A(t)\|_{L^\infty_x} \\ &\quad + \left(1 + \ln t^4\right) t^3\,\|\rho(t)\|_{L^\infty_x}\Big], \quad t > 1, \end{aligned}$$

where the latter is a consequence of setting $R = t$ and $h = t^{-3} \leq R$ in the cited lemma. We claim that for some suitable constant $C = C(\bar{X}_0, \bar{P}_0) > 0$ the charge and current densities satisfy

$$\|\rho(t)\|_{L^\infty_x} + \|j_A(t)\|_{L^\infty_x} \leq Ct^{-3}, \tag{4.33}$$

$$\|\partial_x \rho(t)\|_{L^\infty_x} + \|\partial_x j_A(t)\|_{L^\infty_x} \leq C. \tag{4.34}$$

If true, then the lemma follows. To prove the claim, we first introduce some technical results which we present as a sequence of steps.

Step 1: Let $0 \leq s \leq t \leq a$. Denote by $(X, P)(s) = (X, P)(s, t, x, p)$ the solution of the characteristic system

$$\begin{aligned} \dot{X}(s) &= v_A(s, X(s), P(s)) \\ \dot{P}(s) &= -\left[\nabla\Phi + v_A^i \nabla A^i\right](s, X(s), P(s)), \end{aligned}$$

with $(X, P)(t) = (x, p)$. Denote also $Dv_A(s) = Dv_A(P(s), A(s, X(s)))$, where the matrix Dv_A is as given in the Appendix. Consider the system

$$\begin{aligned} \xi(s) &= \partial_p X(s) - (s-t)Dv_A(t) \\ \eta(s) &= Dv_A(s)\partial_p P(s) - Dv_A(t). \end{aligned}$$

Notice that $\xi(t) = \eta(t) = 0$. We show that for some $C = C(\bar{X}_0, \bar{P}_0) > 0$

$$|\xi(s)| \leq \beta C e^{\beta C}(t-s). \tag{4.35}$$

Indeed, on the characteristic curves, we have

$$\begin{aligned}
\dot{\xi}(s) &= \partial_p \dot{X}(s) - Dv_A(t) \\
&= Dv_A(s)\left[\partial_p P(s) - \partial_x A(s, X(s))\partial_p X(s)\right] - Dv_A(t) \\
&= \eta(s) - Dv_A(s)\partial_x A(s, X(s))\left[\xi(s) + (s-t)Dv_A(t)\right].
\end{aligned}$$

Therefore, since $|Dv_A(s)| \leq C$, a use of (FSβ) yields

$$\begin{aligned}
|\xi(s)| &\leq \int_s^t |\eta(\tau)|\, d\tau + \beta C \int_s^t (1+\tau)^{-3/2}\left[|\xi(\tau)| + (t-\tau)\right] d\tau \\
&\leq \beta C e^{\beta C}\left((t-s) + \int_s^t |\eta(\tau)|\, d\tau\right), \qquad (4.36)
\end{aligned}$$

where the Gronwall's inequality has been used in the last step.

On the other hand, we have

$$\dot{\eta}(s) = Dv_A(s)\partial_p \dot{P}(s) + D^2 v_A(s)\left[\dot{P}(s) - \dot{A}(s, X(s))\right]\partial_p P(s).$$

In view of the characteristic system, it is not difficult to check that

$$\begin{aligned}
\left|\partial_p \dot{P}(s)\right| &\leq C\left(\left\|\partial_x^2 \Phi(s)\right\|_{L_x^\infty} + \left\|\partial_x^2 A(s)\right\|_{L_x^\infty} + \left\|\partial_x A(s)\right\|_{L_x^\infty}^2\right)|\partial_p X(s)| \\
&\quad + C\left\|\partial_x A(s)\right\|_{L_x^\infty}|\partial_p P(s)|.
\end{aligned}$$

Hence, since $\dot{A}^i = \partial_s A^i + v_A \cdot \nabla A^i$, $i = 1, 2, 3$ and $\left|D^2 v_A(s)\right| \leq C$, the above inequality and (FSβ) yield

$$\begin{aligned}
|\dot{\eta}(s)| &\leq \left(\left\|\partial_x^2 \Phi(s)\right\|_{L_x^\infty} + \left\|\partial_x^2 A(s)\right\|_{L_x^\infty} + \left\|\partial_x A(s)\right\|_{L_x^\infty}^2\right)|\partial_p X(s)| \\
&\quad + \left(\left\|\partial_x \Phi(s)\right\|_{L_x^\infty} + \left\|\partial_x A(s)\right\|_{L_x^\infty} + \left\|\partial_s A(s)\right\|_{L_x^\infty}\right)|\partial_p P(s)| \\
&\leq 2\beta(1+s)^{-5/2}|\partial_p X(s)| + \beta(1+s)^{-3/2}|\partial_p P(s)|.
\end{aligned}$$

Now, by the definition of $\xi(s)$ and $\eta(s)$, we have $|\partial_p X(s)| \leq |\xi(s)| + C(t-s)$ and $|\partial_p P(s)| \leq C(|\eta(s)| + 1)$, the latter as a result of $\left|Dv_A^{-1}(s)\right| \leq C$, as it can be easily checked. Then, Gronwall's inequality implies

$$\begin{aligned}
|\eta(s)| &\leq \beta C e^{\beta C}\left(\int_s^t (1+\tau)^{-5/2}|\xi(\tau)|\, d\tau\right. \\
&\quad \left. + \int_s^t \left[(1+\tau)^{-5/2}(t-\tau) + (1+\tau)^{-3/2}\right] d\tau\right). \qquad (4.37)
\end{aligned}$$

Both (4.36) and (4.37) then lead to

$$\begin{aligned} |\xi(s)| &\leq \beta C e^{\beta C}\left((t-s)+\int_s^t\int_\tau^t (1+\sigma)^{-5/2}|\xi(\sigma)|\,d\sigma d\tau\right. \\ &\quad \left.+\int_s^t\int_\tau^t \left[(1+\sigma)^{-5/2}(t-\sigma)+(1+\sigma)^{-3/2}\right]d\sigma d\tau\right), \\ &\leq \beta C e^{\beta C}\left((t-s)+\int_s^t\int_s^\sigma (1+\sigma)^{-5/2}|\xi(\sigma)|\,d\tau d\sigma\right. \\ &\quad \left.+\int_s^t\int_s^\sigma \left[(1+\sigma)^{-5/2}(t-\sigma)+(1+\sigma)^{-3/2}\right]d\tau d\sigma\right), \\ &\leq \beta C e^{\beta C}\left((t-s)+\int_s^t (1+\sigma)^{-3/2}|\xi(\sigma)|\,d\sigma\right. \\ &\quad \left.+\int_s^t \left[(1+\sigma)^{-3/2}(t-\sigma)+(1+\sigma)^{-1/2}\right]d\sigma\right). \end{aligned}$$

Finally, since the last integral is less than $3(t-s)$, another use of Gronwall's inequality yields (4.35).

Step 2: For $\beta>0$ small enough, there exists a $C=C(\bar{X}_0,\bar{P}_0)>0$ such that the mapping $X(0,t,x,\cdot):\mathbb{R}^3\to\mathbb{R}^3$ has Jacobian determinant satisfying

$$|\mathtt{det}\partial_p X(0,t,x,p)|\geq Ct^3,\quad 0\leq t\leq a,\quad x\in\mathbb{R}^3,\quad p\in\mathbb{R}^3.$$

For $t=0$ this is obvious. Let $0<t\leq a$. Without loss of generality, we shall assume that $0<\beta\leq 1/2$. Then, by the characteristics and (FSβ) we have

$$|P(t)|\leq \bar{P}_0+\beta\int_0^t (1+\tau)^{-3/2}d\tau\leq \bar{P}_0+1. \tag{4.38}$$

Also, in view of the estimate on the vector potential given in Lemma 6, and recalling that $f_0\in\mathcal{D}$, is not difficult to check that

$$\|A(t)\|_{L^\infty_x}\leq C\bar{X}_0^2\bar{P}_0^2\left(\bar{P}_0+1\right).$$

Denote $g=|p-A|$. Hence $g\leq C(\bar{X}_0,\bar{P}_0)$ and therefore the relativistic velocity satisfies $|v_A|\leq\nu<1$, where ν depends only on $\bar{X}_0$ and $\bar{P}_0$. Now, we have that $Dv_A=\left(1+g^2\right)^{-1/2}[\mathtt{id}-v_A\otimes v_A]$. Then, since by Step 1, $|\xi(0)|\leq\beta Ce^{\beta C}t$ with $\xi(0)=\partial_pX(0)+tDv_A(t)$, we have for some $\beta>0$ small enough that

$$\begin{aligned} \left|\frac{\sqrt{1+g^2}}{t}\partial_pX(0)+\mathtt{id}\right| &\equiv \left|\frac{\sqrt{1+g^2}}{t}\xi(0)+v_A\otimes v_A\right| \\ &\leq \beta Ce^{\beta C}+\nu\ \equiv\ \gamma\ <\ 1. \end{aligned} \tag{4.39}$$

Therefore, a positive constant $C=C(\bar{X}_0,\bar{P}_0)$ exists such that

$$|\mathtt{det}\partial_pX(0)|\equiv\frac{t^3}{(1+g^2)^{3/2}}\left|\mathtt{det}\left[\frac{\sqrt{1+g^2}}{t}\partial_pX(0)+\mathtt{id}-\mathtt{id}\right]\right|\geq Ct^3.$$

Step 3: For every $0<t\leq a$ and $x\in\mathbb{R}^3$, the mapping $X(0,t,x,\cdot):\mathbb{R}^3\to\mathbb{R}^3$ is bijective. Indeed, for $p,q\in\mathbb{R}^3$, let

$$p_\lambda=\lambda p+(1-\lambda)\,q,\quad g_\lambda=g(t,x,p_\lambda)=|p_\lambda-A(t,x)|\,,\quad 0\leq\lambda\leq 1.$$

In view of (4.39) in Step 2, we have

$$\begin{aligned}
|X(0,t,x,p) \quad - \quad X(0,t,x,q)| &= \left|\int_0^1 \partial_p X(0,t,x,p_\lambda)\,(p-q)\,d\lambda\right| \\
&= \left|\int_0^1 \left[-\mathtt{id}+\mathtt{id}+\frac{\sqrt{1+g_\lambda^2}}{t}\partial_p X(0,t,x,p_\lambda)\right]\frac{t\,(p-q)}{\sqrt{1+g_\lambda^2}}d\lambda\right| \\
&\geq t\,|p-q|\int_0^1 \frac{d\lambda}{\sqrt{1+g_\lambda^2}} - \gamma t\,|p-q|\int_0^1 \frac{d\lambda}{\sqrt{1+g_\lambda^2}} \\
&\geq (1-\gamma)\,|p-q|\,t,
\end{aligned}$$

which shows that the mapping is injective. It is also surjective, since the open range $X(0,t,x,\mathbb{R}^3)=\mathbb{R}^3$. If not, there exists a boundary point x_0 so that $X(0,t,x,p_n)\to x_0\notin X(0,t,x,\mathbb{R}^3)$ as $n\to\infty$, for some $p_n\to p_0\in\mathbb{R}^3$. By continuity $X(0,t,x,p_0)=x_0$, which is a contradiction, and the assertion follows.

Step 4: Then, Steps 2 and 3 imply that the mapping $X(0,t,x,\cdot):\mathbb{R}^3\to\mathbb{R}^3$ is a C^1-diffeomorphism. In particular, Step 2 implies that for some constant $C=C(\bar{X}_0,\bar{P}_0)>0$, the inverse mapping $X^{-1}(0,t,x,\cdot):\mathbb{R}^3\to\mathbb{R}^3$ defined by $X\mapsto p(X)$ has Jacobian determinant satisfying

$$\left|\mathtt{det}\partial_p X^{-1}(0,t,x,p(X))\right| \leq Ct^{-3}, \quad 0<t\leq a, \quad x\in\mathbb{R}^3.$$

We can now deduce the estimates (4.33)-(4.34) for the charge and current densities. Indeed, bearing in mind that $f_0\in\mathcal{D}$, we have

$$\begin{aligned}
\rho(t,x) &= \int_{\mathbb{R}^3} f_0(X(0,t,x,p),P(0,t,x,p))dp \\
&= \int_{\mathbb{R}^3} f_0(X,P(0,t,x,p(X)))\left|\mathtt{det}\partial_p X^{-1}(0,t,x,p(X))\right| dX \\
&\leq Ct^{-3},
\end{aligned}$$

where $C=C(\bar{X}_0,\bar{P}_0)>0$. Then, since $|j_A|\leq\rho$, (4.33) indeed holds.

To prove (4.34) we proceed as follows. In view of (FSβ) with $\beta=1/2$, and recalling that $|\partial_x v_A|\leq C\,|\partial_x A|$ and $f_0\in\mathcal{D}$, we have that

$$\|\partial_x\rho(t)\|_{L^\infty_x} \leq C\left(\bar{P}_0+1\right)^3\|\partial_x f(t)\|_{L^\infty_{x,p}} \tag{4.40}$$

$$\begin{aligned}
\|\partial_x j_A(t)\|_{L^\infty_x} &\leq C\left(\bar{P}_0+1\right)^3\left(\|\partial_x A(t)\|_{L^\infty_x}\|f_0\|_{L^\infty_{x,p}}+\|\partial_x f(t)\|_{L^\infty_{x,p}}\right) \\
&\leq C\left(\bar{P}_0+1\right)^3\left(1+\|\partial_x f(t)\|_{L^\infty_{x,p}}\right), \qquad (4.41)
\end{aligned}$$

and

$$|\partial_x f(t,x,p)|\leq|\partial_x P(0,t,x,p)|+|\partial_x X(0,t,x,p)|\,. \tag{4.42}$$

Hence, the proof will be completed if we provide a uniform bound on the space derivatives of the characteristic curves. Similar to the computations in Step 1, it is not difficult to check that for $0\leq s\leq t\leq a$

$$|\partial_x X(s)|\leq 1+C\int_s^t\left(|\partial_x P(\tau)|+\|\partial_x A(\tau)\|_{L^\infty_x}|\partial_x X(\tau)|\right)d\tau,$$

and by (FSβ) and Gronwall's lemma

$$|\partial_x X(s)| \le C\left(1+\int_s^t |\partial_x P(\tau)|\, d\tau\right). \tag{4.43}$$

Also,

$$\begin{aligned}
|\partial_x P(s)| &\le C\int_s^t \left(\left\|\partial_x^2\Phi(s)\right\|_{L_x^\infty} + \left\|\partial_x^2 A(s)\right\|_{L_x^\infty} + \|\partial_x A(s)\|_{L_x^\infty}^2\right)|\partial_x X(\tau)|\, d\tau \\
&\quad + C\int_s^t \|\partial_x A(\tau)\|_{L_x^\infty} |\partial_x P(\tau)|\, d\tau \\
&\le C\int_s^t (1+\tau)^{-5/2}|\partial_x X(\tau)|\, d\tau. \qquad (4.44)
\end{aligned}$$

Therefore, both (4.43) and (4.44) yield

$$\begin{aligned}
|\partial_x X(s)| &\le C + C\int_s^t\int_\tau^t (1+\sigma)^{-5/2}|\partial_x X(\sigma)|\, d\sigma d\tau \\
&\le C + C\int_s^t\int_s^\sigma (1+\sigma)^{-5/2}|\partial_x X(\sigma)|\, d\tau d\sigma \\
&\le C + C\int_s^t (1+\sigma)^{-3/2}|\partial_x X(\sigma)|\, d\sigma.
\end{aligned}$$

Gronwall's lemma then provides a uniform bound on $|\partial_x X(s)|$, which in turn produces a uniform bound on $|\partial_x P(s)|$ via (4.44). As a consequence

$$|\partial_x X(0,t,x,p)| + |\partial_x P(0,t,x,p)| \le C, \qquad 0\le t\le a,\ x\in\mathbb{R}^3,\ p\in\mathbb{R}^3,$$

which implies (4.34) via (4.40)-(4.42). This concludes the proof of the lemma. □

Proof of Theorem 2. By virtue of the previous lemmas, the proof is almost identical to the proof of [14, Theorem 4.1] for the Vlasov-Poisson system.

Indeed, let $\beta,\delta>0$ and $C=C(\bar X_0,\bar P_0)>0$ be suitable for Lemma 12 to hold. Fix $T_0>1$ such that for all $t\ge T_0$

$$Ct^{-2} \le \frac{\beta}{2}(1+t)^{-3/2}, \qquad C(1+\ln t)\, t^{-3} \le \frac{\beta}{2}(1+t)^{-5/2}. \tag{4.45}$$

Now, by letting $\delta>0$ be smaller if necessary, Lemma 11 implies that the Cauchy datum $f_0\in\mathcal{D}$ with $\|f_0\|_{L_{x,p}^\infty}\le\delta$ yields a classical solution f of the RVD system on the maximal existence interval $[0,T[$ with $T>T_0$, and

$$\begin{aligned}
&\|\partial_t A(t)\|_{L_x^\infty} + \|\partial_x A(t)\|_{L_x^\infty} + \|\partial_x \Phi(t)\|_{L_x^\infty} \\
&\qquad + \left\|\partial_x^2 A(t)\right\|_{L_x^\infty} + \left\|\partial_x^2\Phi(t)\right\|_{L_x^\infty} < \frac{\beta}{2}(1+T_0)^{-5/2},
\end{aligned}$$

for all $0\le t\le T_0$. Hence, f satisfies the free streaming condition (FSβ) on $[0,T_0]$. In fact, the continuity of the left-hand side of the above inequality implies that there exists a maximal $T_0<T_1\le T$ such that f satisfies (FSβ) on $[0,T_1[$. Therefore, Lemma 12 and (4.45) imply that for all $T_0\le t<T_1$

$$\begin{aligned}
\|\partial_x\Phi(t)\|_{L_x^\infty} + \|\partial_x A(t)\|_{L_x^\infty} &\le Ct^{-2} \le \frac{\beta}{2}(1+t)^{-3/2}, \\
\left\|\partial_x^2\Phi(t)\right\|_{L_x^\infty} + \left\|\partial_x^2 A(t)\right\|_{L_x^\infty} &\le C(1+\ln t)\, t^{-3} \le \frac{\beta}{2}(1+t)^{-5/2}.
\end{aligned}$$

Then, a continuation argument yields $T_1 = T$, and by (4.38), we deduce

$$\sup \left\{ |p| : \exists 0 \leq t < T, x \in \mathbb{R}^3 : f(t,x,p) \neq 0 \right\} \leq \bar{P}_0 + 1.$$

Therefore the continuation criterion in Theorem 1 implies that $T = \infty$, and thus the solution f is global in time. The proof of Theorem 2 is complete. $\square$

Appendix

$$\text{For} \quad v_A = \frac{p - A}{\sqrt{1 + |p - A|^2}}, \quad \text{set} \quad Dv_A = \frac{\mathtt{id} - v_A \otimes v_A}{\sqrt{1 + |p - A|^2}} = \partial_p v_A.$$

Then $\partial_x v_A = -Dv_A \partial_x A$ and $\partial_t v_A = -Dv_A \partial_t A$. Clearly, $|Dv_A| \leq C$, and so $|\partial_x v_A| \leq C |\partial_x A|$ and $|\partial_t v_A| \leq C |\partial_t A|$. Also, $\left|\partial_p^2 v_A\right| \leq C$; $|\partial_x \partial_p v_A| \leq C |\partial_x A|$; $\left|\partial_x^2 v_A\right| \leq C \left(\left|\partial_x^2 A\right| + |\partial_x A|^2 \right)$; $\left|\partial_t^2 v_A\right| \leq C \left(\left|\partial_t^2 A\right| + |\partial_t A|^2 \right)$;$|\partial_t \partial_p v_A| \leq C |\partial_t A|$ and finally $|\partial_t \partial_x v_A| \leq C \left(|\partial_t \partial_x A| + |\partial_t A| \, |\partial_x A| \right)$.

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