Discrete velocity models with general boundary conditions in a slab

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Abstract. We consider a gas between parallel plates, described by a discrete velocity model. At the boundary we impose the most general linear boundary conditions which preserve mass. Using a fixed point theorem we prove the existence of at least a one parameter family of solutions, continuous in x. The velocities are assumed to have a nonzero component in the direction orthogonal to the boundaries.

1. Introduction

The existence theory for the Boltzmann equation has seen remarkable progress thanks to the methods introduced by DiPerna and Lions [DPL] to handle the pure initial value problem. These methods have been extended to the case of mixed initial-boundary value problems [AC][AM][CI].

The case of boundary value problems is harder to handle but some results have been recently obtained [ACI][Ce1][Ce2]. The results have been obtained for the case of a slab, by extending a technique first used by the authors and M. Shinbrot [CIS] in 1987 for a discrete velocity model in a slab. The aim of the present paper is to extend the latter results in another direction.

We recall that a discrete velocity model for a steady solution in a slab reads as follows:

$$\xi_i \partial_x f_i = C_i(f, f) \qquad i = 1, \ldots, n) \tag{1.1}$$

where ξ_i are the x-components of the vectors v_i of the model. f_i gives the nonnegative discretized distribution function. We assume

$$\xi_i \neq 0 \qquad (i = 1, \ldots, n) \tag{1.2}$$

This model has been solved in [CIS] for two kinds of boundary conditions: a) assigned values for $\xi_i > 0$ at $x = 0$ and for $\xi_i < 0$ at $x = d$; b) assigned values for $\xi_i > 0$ at $x = 0$ and given except for a common factor for $\xi_i < 0$ at $x = d$. In case b) the common factor depends on the f_i associated with $\xi_i > 0$ in such a way as to produce a vanishing particle flow at $x = d$. The second result has been extended by Illner and Struckmeier [IS] when this particular case of vanishing particle flow holds at both boundaries.

Here we consider the most general case of vanishing flow at both boundaries. This results in the following boundary conditions at $x = 0$ and $x = d$:

$$\xi_i f_i(0) = \sum_{j\in\Lambda^-} B_{ij}|\xi_j| f_j(0) \quad i \in \Lambda^+$$

$$|\xi_i| f_i(d) = \sum_{j\in\Lambda^+} B_{ij}\xi_j f_j(d) \quad i \in \Lambda^- \tag{1.3}$$

where B_{ij} are given positive coefficients satisfying

$$\sum_{i\in\Lambda^+} B_{ij} = 1 \quad j \in \Lambda^- ; \quad \sum_{i\in\Lambda^-} B_{ij} = 1 \quad j \in \Lambda^+ \tag{1.4}$$

$\Lambda^\pm$ are obviously the index sets of the velocities with $\pm\xi_i > 0$.

We remark that the properties of the B_{ij} imply that if we let $j_+(x) = \sum_{j\in\Lambda^+} \xi_j f_j(x)$ and $j_-(x) = \sum_{j\in\Lambda^-} |\xi_j| f_j(x)$, then $j_+ = j_-$ at $x = 0$ and $x = d$.

Recently Nikkuni and Sakamoto [NS] considered the inhomogeneous version of these boundary conditions when a source term b_i $(i \in \Lambda^\pm)$ is added at $x = 0$ and $x = d$ respectively. If this source does not vanish identically, then their method produces no solution when the coefficients B_{ij} satisfy the above conditions. In fact they had to assume that one replaces $=$ with $<$ at least at one boundary. Their boundary conditions are useful for problems of evaporation and condensation, but the results for the more natural case when the particle number is conserved are not included in [NS]. The purpose of the present paper is to cover this important case.

Concerning the collision term, we just assume that it conserves mass.

2. Basic equations

As in [CIS] we write the model equation as

$$\xi\partial_x f_i + \gamma f_i \rho[f] = C_i^\gamma(f,f) \qquad i = 1,\ldots,n) \tag{2.1}$$

where $\rho[f] = \sum_i f_i$ and γ is a positive constant chosen as to make C_i^γ nonnegative. C_i^γ has the general expression:

$$C_i^\gamma(f,f) = \sum_{k,l} b_{ikl} f_k f_l \tag{2.2}$$

with $b_{ikl} \geq 0$.

We remark that $\rho[f]$ is nonnegative and when it becomes zero C_i^γ/ρ also goes to zero, being bounded by a constant times ρ. Thus there is no harm in dividing the equation by $\rho(x) = \rho[f(x)]$ and by changing the x variable into

$$y = \int_0^x \rho(x')dx' . \tag{2.3}$$

If we let

$$M = \int_0^d \rho(x)dx \tag{2.4}$$

x will vary between 0 and M.

If we let $\tilde{f}_i(y) = f_i(x)$ and then drop the tilda, the system becomes

$$\xi \partial_x f_i + f_i = \sum_{k,l} b_{ikl} \frac{f_k f_l}{\rho} = \tilde{C}_i^\gamma[f] \qquad (i = 1, \dots, n) \tag{2.5}$$

where we let $\gamma = 1$ without any loss of generality. We remark that if $\sum_i C_i(f, f) = 0$ (particle number is conserved in collisions), then

$$\rho = \sum_i \tilde{C}_i^\gamma[f] \tag{2.6}$$

The transformation (2.3)-(2.5) has been first used in [Ce1][Ce2] in discussions of existence for the Boltzmann equation with continuous velocity in a slab.

We shall consider the modified problem that we have just obtained and show that it has a solution (actually a one parameter family of solutions). Finally we shall show that the solutions obtained in this way can be transformed into solutions of the original problem. This will be done in the next section by defining a suitable (nonlinear) operator T such that its fixed points are solutions of our problem and then using Schauder's fixed point theorem. The proof is shorter and simpler than the previous proofs, which applied to simpler boundary conditions.

3. The existence theorem

We can now consider the operator

$$T : g \to f \tag{3.1}$$

defined by

$$\begin{aligned} \xi \partial_x f_i + f_i &= \tilde{C}_i^\gamma[g] \qquad i = 1, \dots, n) \\ \xi_i f_i(0) &= \sum_{j \in \Lambda^-} B_{ij} |\xi_j| g_j(0) \quad i \in \Lambda^+ \\ |\xi_i| f_i(M) &= \sum_{j \in \Lambda^+} B_{ij} \xi_j g_j(M) \quad i \in \Lambda^- \end{aligned} \tag{3.2}$$

where $g_i > 0$, $g_i \in C([0, M]; R_+)$.

The fact that this operator is well defined follows from the same argument as in [CIS]. In addition we have

$$|||Tg||| = |||g||| \tag{3.3}$$

where

$$|||g||| = \int_0^M \rho[g](y)dy + j^-[g](0) + j^+[g](M)\,. \tag{3.4}$$

In fact, if we integrate the equation defining the mapping from 0 to M, sum over i and use the boundary conditions (note that $j^-[f](M) = j^+[g](M)$, $j^+[f](0) = j^-[g](0)$), we obtain:

$$j^-[f](0) - j^-[g](0) + j^+[f](M) - j^+[g](M) + \int_0^M \rho[f](y)dy = \int_0^M \rho[g](y)dy\,. \qquad (3.5)$$

Rearranging, we obtain the desired result.

We consider the set

$$S_R := \{f; |||f||| = R\ (R > 0), \quad f_i \geq 0\} \qquad (3.6)$$

as a subset of $C^n([0,m]; R_+)$, and endowed with the norm $|||\cdot|||$. This set is trivially convex and is mapped on itself by the operator T. Note that the norm is an L^1-norm plus boundary fluxes. In addition TS_R is also compact because it is a bounded and closed subset of the Sobolev space $W^{1,1}$. So T is compact and by Schauder's theorem [Sm] has a fixed point f such that $Tf = f$. Because of the way T was defined this means that

$$\xi\partial_x f_i + f_i = \tilde{C}_i^\gamma[f] \qquad i = 1, \ldots, n)$$

$$\xi_i f_i(0) = \sum_{j\in\Lambda^-} B_{ij}|\xi_j| f_j(0) \quad i \in \Lambda^+$$

$$|\xi_i| f_i(M) = \sum_{j\in\Lambda^+} B_{ij}\xi_j f_j(M) \quad i \in \Lambda^- \qquad (3.7)$$

The solutions $f(y)$ that we have found are parametrized by the parameter R and have a well defined nonnegative density $\rho[f](y)$. In fact, $\rho[f](y) > 0$ for all $y \in [0, M]$, because otherwise it would follow that all f_i would vanish at one point y_0, and by the existence and uniqueness theorem for the initial value problem for ordinary differential equations it would follow that $f_i(y) = 0$ for all y, i, contradicting $|||f||| = R > 0$. We can therefore invert the transformation of the independent variable and return to x to obtain:

$$x = \int_0^x \frac{dy}{\rho[f](y)} dy \qquad (3.8)$$

We have two parameters M and C to play with. M fixes the mass (per unit area if we think in 3 dimensions) and R can be used to obtain any thickness of the slab (if R_0 gives a thickness d_0, we obtain a thickness d by letting $R = R_0 d_0/d$).

We formulate our main result.

Theorem. *If the coefficients B_{ij} are nonnegative and satisfy the condition (1.4), then the boundary value problem (1.1), (1.3) has a one-parameter family of solutions satisfying $\rho[f](x) > 0$ for all $x \in [0, d]$. The solutions are parametrized by their norms $|||f|||$.*

4. Concluding remarks

We have studied the most general boundary value problem with conservation of mass in a slab for discrete velocity models. The proof is not only applicable to more general boundary conditions but is also simpler than any other proof given before for this kind of problems. An extension to the case of continuous velocities appears to be feasible. Although we have not discussed uniqueness, it is easy to see that uniqueness holds if R is sufficiently small.

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