Stellar Dynamics and Plasma Physics with Corrected Potentials: Vlasov, Manev, Boltzmann, Smoluchowski

by

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ABSTRACT

We summarize the results from several recent papers in which the effects of a particle interaction via an (attractive or repulsive) potential $\pm 1/r^2$ on the associated Vlasov, Boltzmann, or, most generally, Vlasov-Boltzmann-Smoluchowski equations were discussed. The potential is considered in isolation; it occurs in the literature as a correction ϵ/r^2 to the Coulomb potential, and the associated law is known as Manev potential.

1. Motivation and Background. This paper will summarize the results from Refs. [1,2,5]. We are concerned with particle clouds which interact via a pair potential $u(r) = \pm 1/r^2$. For $u(r) = -1/r^2$, forces are attractive, and the resulting kinetic equations are of stellar dynamic type; for $u(r) = 1/r^2$, the repulsive case, the kinetic equations are Vlasov equations for plasmas (charged particle systems). This kind of potential has been studied since Newton [11], who was the first to describe some of the interesting effects arising in the central force problem. Manev [7-10] studied u as a correction to the Newtonian potential; specifically, he considered

$$-\frac{\gamma}{r} - \frac{\epsilon}{r^2},$$

where γ is assumed to be the universal gravitational constant and $\epsilon = \frac{3\gamma^2}{c^2}$ is thought of as a quasirelativistic correction (for a justification of this interpretation see [5]). This choice of constants predicts the qualitatively correct precession of the perihelion of Mercury. That potentials of this kind would lead to precessional ellipses as orbits was already known to Newton [11].

It is an interesting question, at least from a mathematical point of view, to investigate what happens to kinetic theory if the Manev correction (as we will from now on call it) is incorporated into the kinetic equations. This is exactly was was done in Refs. [1,2,5]. The major milestones, which we will revisit in the current paper, are the following:

1. The initial value problem for a Vlasov equation with self-consistent interparticle potential $1/r^2$ possesses unique local solutions for sufficiently smooth initial data. Nonnegativity and the usual conservation laws (mass, momentum, energy, angular momentum) apply.

2. There are initial values for which global existence does not hold in the attractive case. This is proved indirectly from a relationship linking the moment of inertia with the (invariant) total energy. If the latter is negative, an upper estimate on the maximal existence time is obtained.

3. The Vlasov equation with the self-consistent potential $1/r^2$ possesses an additional fundamental invariant (referred to as "projective invariant") in addition to the usual scalings of time, space, velocity, and Galilei invariance.

4. If particles interact via an (attractive or repulsive) potential $1/r^2$, the addition of Boltzmann collision terms is suggested by dimensional analysis.

5. In the attractive case, inclusion of a Smoluchowski coagulation term is justified by the analysis of pair interactions. Specifically, there are particle encounters which lead to coagulation in the sense that particles will coagulate for small enough collision parameters.

Many of these results are, as indicated, specific for attractive or repulsive forces; they remain valid if the Manev correction is just added to the Newtonian (or Coulomb) potential. As already stated, we will here consider the

Manev correction in isolation; we will also concentrate on the attractive case, which displays the most interesting mathematical features.

2. Equations, Invariants, Non-Global existence. We focus first on the equation in which a Vlasov-type force with an attractive interparticle potential $-\frac{\epsilon}{r^2}$ is responsible for the forces. $f = f(t, x, v)$ denotes the density function at time t in $\mathbb{R}^6$, and $\rho(t, x) = \int f(t, x, v)\, dv$ is the spatial density. The potential is then

$$U[\rho](t, x) = -\epsilon \int \frac{\rho(t, y)}{|x - y|^2}\, dy$$

and the force at location x at time t is

$$\begin{aligned} F(t, x) = -\nabla_x U[\rho](t, x) &= -\epsilon \int \frac{1}{|x - y|^2} \nabla_y \rho(t, y)\, dy \\ &= -2\epsilon \int \frac{x - y}{|x - y|^4} \rho(t, y)\, dy. \end{aligned} \tag{2.1}$$

Note that the last integral on the right must be interpreted as a Cauchy principal value; the integration domain is always all space, so the integral is defined, e.g., if ρ is at least Hölder continuous. The evolution equation for f is

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0, \tag{2.2}$$

with F given as above. We call (2.2) the stellar dynamic equation associated with the potential $-\frac{\epsilon}{r^2}$. It is nonlinear because F depends implicitly on ρ, and ρ is given in terms of f. A major difference with respect to the classical stellar dynamic equation (associated with the potential $-\frac{\gamma}{r}$) is that even the definition of F requires that ρ posesses some smoothness (whereas in the classical case boundedness and integrability are sufficient).

Assuming that we have a sufficiently smooth solution, one readily proves, using the method of characteristics, that for all $1 \leq p \leq \infty$

$$\|f(t, \cdot)\|_{L^p} = \|f_0\|_{L^p} \tag{2.3}$$

(which includes conservation of mass) and that

$$\frac{d}{dt}\int\int vf(t,x,v)\,dv = 0 \tag{2.4}$$

$$\frac{d}{dt}\left[\int\int v^2 f\,dvdx - \epsilon\int\int \frac{\rho(t,x)\rho(t,y)}{|x-y|^2}dxdy\right] = 0 \tag{2.5}$$

(momentum and energy conservation). Conservation of angular momentum also holds.

The proof that the initial value problem is well-posed is actually quite involved and was given in Ref. [5]. One has to work in spaces of functions with Hölder continuous derivatives; in particular, the initial values have to have this degree of regularity. The force field given in (2.1) is undefined at discontinuities of ρ. The model therefore loses its validity if such discontinuities form, or if ρ develops strong enough singularities.

It is remarkably easy to see that this must actually happen. The method is based on a standard calculation used in the theory of N-body problems, the theory of the nonlinear Schrödinger equation, and also in classical stellar dynamics.

Consider the functional $I(t) = \int x^2\rho(t,x)\,dx$ (twice the moment of inertia of the system). A straightforward calculation shows that

$$\frac{d^2}{dt^2}I(t) = E(t) = E(0) \tag{2.6}$$

where $E(t)$ denotes the (time-invariant) energy, given by the expression inside the bracket in (2.5). If $E(0)$ is negative (meaning that there is not a lot of "temperature" in the system), it follows that $I(t)$ must eventually become negative, contradicting the fact that $\rho \geq 0$. Hence the solution must cease to exist at some earlier time. The time when $I(t)$ becomes negative is easily calculated by integrating (2.6) twice. For details, see [1].

3. Projective Invariance. More than 100 years ago Sophus Lie classified the group invariants of the free flow characteristic equations $\frac{d^2}{dt^2}x(t) = 0$ (or

$\dot{x} = v, \quad \dot{v} = 0$). There are 4 of those:

$$\begin{aligned} \text{shift:} \quad & t' = t + \alpha_1, \quad x' = x \\ \text{scale:} \quad & t' = e^{\alpha_2} t, \quad x' = x \\ \text{Galilei:} \quad & t' = t, \quad x' = x + \alpha_3 t \\ \text{Projective:} \quad & t' = \frac{t}{1 - \alpha_4 t}, \quad x' = \frac{x}{1 - \alpha_4 t}. \end{aligned}$$

If x is one-dimensional, there are 4 more by interchanging x and t. The constants $\alpha_1, \ldots, \alpha_4$ are free parameters. We refer to [3] for a detailed discussion of these invariants.

The first three invariants remain valid for the $N-$body problem (i.e., $x \in \mathbb{R}^{3N}$) in the presence of a conservative interparticle potential, like the Newtonian potential. However, the projective invariance is generally lost at this level, with one exception: For interparticle potentials $\pm \frac{\epsilon}{r^2}$, projective invariance remains valid, and remains valid even for the Vlasov equation under consideration here. Specifically, the following theorem holds.

Theorem. *Let f(t,x,v) be a solution of (2.2), where the force is given by (2.1) but can be attractive or repulsive. Suppose that f exists on a time interval $[0, t_0)$. Let $a > 0$ and set*

$$\tau = \frac{t}{1 + at}, \quad y = \frac{x}{1 + at}, \quad w = v(1 + at) - ax.$$

Then $F(t, y, w) := f(t, x, v)$ solves Eqn. (2.2) with respect to τ, y, w on an interval $\tau \in \left[0, \frac{t_0}{1+at_0}\right]$, and for the initial value $F(0, y, w) = f_0(y, w + ay)$.

This projective invariance of the equation has interesting consequences: it allows to transform solutions into other solutions and even to produce some global solutions from local solutions. Note that the transformation $(t, x, v) \to (\tau, y, w)$ is invertible; we can write

$$t = \frac{\tau}{1 - a\tau}, \quad x = \frac{y}{1 - a\tau} \quad v = w(1 - a\tau) + ay,$$

and we see that if $F(\tau, y, w)$ solves the equation on $[0, \tau_0]$, then $f(t, x, v) = F(\tau, y, w)$ assumes the initial value $F_0(x, v - ax)$ and exists on the time interval $\left[0, \frac{\tau_0}{1-a\tau_0}\right]$ if $a < 1/\tau_0$, and on $[0, \infty)$ if $a \geq 1/\tau_0$. It is transparent how the transformation changes the data to new data with different angular momentum.

The proof of the theorem is elementary and uses only the chain rule and the representation of the force field given by (2.1).

Assume that $f = f(x,v)$ is a nonnegative steady solution; the existence of such solutions for the attractive case has recently been shown by Steacy [12]. We can then apply the above transformation and get that $f(\frac{x}{1-at}, v(1-at)+ax)$ is also a solution, assuming the initial value $f(x, v+ax)$ and ceasing to exist at time $t = 1/a$.

4. Why a Boltzmann Collision Term Should be Added. We now discuss a scaling argument first presented in [1] which will suggest that the equation discussed so far is incomplete. For this section, we consider a more general repulsive interparticle potential $u(r) = \frac{\alpha}{r^n}$, where $\alpha > 0$ is a constant and $1 < n \leq 2$. In order to accurately describe close encounters (collisions) between particles, we add a Boltzmann collision term $Q(f,f)$ to the right hand side of the equation. The differential cross-section associated with the potential is

$$\sigma(|v|,\theta) = \left(\frac{\alpha}{m|v|^2}\right)^{2/n} g_n(\cos\theta),$$

where $|v|$ is the relative particle speed, θ is the scattering angle, m is the particle mass and g_n is a function such that $\int_{-1}^{1}(1-x)g_n(x)\,dx < \infty$. The Vlasov-Boltzmann equation is then

$$\partial_t f + v\cdot\nabla_x f + \frac{F}{m}\cdot\nabla_v f = Q(f,f) \tag{4.1}$$

with

$$F = -\nabla_x \int \frac{\alpha}{|x-y|^n}\rho(t,y)\,dy$$

and

$$Q(f,f) = \int\int |v-v_*|\sigma(|v-v_*|,\theta)\left\{f'f'_* - ff_*\right\}\sin\theta d\theta d\varphi dv_*,$$

with the usual notation in the Boltzmann collision term: $f_* = f(t,x,v_*)$, $f'_* = f(t,x,v'_*)$, etc., where v, v_* are pre- and v', v'_* are post-collisional velocities respectively.

The term involving F and the term $Q(f,f)$ in (4.1) represent effects of long-range and short-range forces respectively. To decide which one dominates in the particle system under consideration, we pass to a dimensionless representation: Assume that x_0, v_0, t_0 (with $x_0 = v_0 t_0$) and ρ_0 are typical scales for the system under consideration. We then define new, dimensionless independent and dependent variables via

$$\tilde{x} = \frac{x}{x_0}, \quad \tilde{v} = \frac{v}{v_0}, \quad \tilde{t} = \frac{t}{t_0} \quad \text{and} \quad \tilde{f}(\tilde{t},\tilde{x},\tilde{v}) = \frac{v_0^3}{\rho_0} f(t,x,v).$$

It is tedious but easy to rewrite Eqn. (4.1) in terms of the new variables. After this is done and we drop all the tildas, the equation reads

$$\partial_t f + v \cdot \nabla_x f - C_V \nabla_x \int \frac{\rho(t,y)}{|x-y|^n} dy \cdot \nabla_v f = C_B Q(f,f) \qquad (4.2)$$

The constants C_V and C_B are

$$C_V = \rho_0 \left(\frac{\alpha}{m v_0^2} \right) x_0^{3-n}$$
$$C_B = \rho_0 \left(\frac{\alpha}{m v_0^2} \right)^{2/n} x_0.$$

We define $r_* = \left[\frac{\alpha}{m v_0^2} \right]^{1/n}$. Note that r_* is essentially the square root of the differential cross-section and has therefore the dimension of length. In fact, it can be interpreted as a distance over which typical particles strongly correlate. Furthermore, we define the quotient

$$\lambda = \frac{C_V}{C_B} = \left(\frac{\alpha}{m v_0^2} \right)^{\frac{n-2}{n}} x_0^{2-n} \qquad (4.3)$$

If we interpret $\rho_0^{-1/3}$ as a measure for the typical distance between two particles in the system, the validity of the kinetic description requires that

$r_* << \rho_0^{-1/3} << x_0$. Hence the quotient $\beta := \frac{r_*}{x_0}$ will satisfy $\beta << 1$, and we observe that

$$\lambda = \beta^{n-2} \approx \begin{cases} 0 & \text{if} \quad n > 2 \\ 1 & \text{if} \quad n = 2 \\ \infty & \text{if} \quad n < 2 \end{cases} .$$

Hence if $n > 2$, the Boltzmann collision term dominates over the Vlasov force term (note, however, that this is a purely formal observation as we have made no attempt to define the Vlasov potential for this case— additional regularity assumptions on the density are needed to do so); if $n < 2$, the Vlasov force term dominates. It is exactly for $n = 2$, i.e., the case we are discussing here in detail, that both terms carry similar weight.

5. The scattering problem for attractive potentials. The previous section suggests that Boltzmann collision terms should also be taken seriously when particles interact via an attractive pair potential $-1/r^2$. In fact, two new phenomena arise which require careful attention: One has to address the phenomenon that the relationship between collision parameter and scattering angle is not a one-to-one function for attractive forces, and, for the specific potential $-1/r^2$, that there is the possibility of coagulation. The latter gives rise to yet another term on the right hand side of the equation, namely, a coagulation term of Smoluchowski type.

To start, let us revisit the scattering formulas for the classical repulsive situation. Assume that the interparticle potential $U(r)$ is repulsive and that two particles move at relative speed u and with impact parameter $\rho > 0$, as depicted in Figure 1. r_0 denotes the distance of closest approach between the particles. The angle φ depends on ρ, u and r_0 like

$$\varphi = 2\rho \int_{r_0}^{\infty} \frac{1}{r^2\sqrt{1 - \frac{2U(r)}{mu^2} - \left(\frac{\rho}{r}\right)^2}} dr \qquad (5.1)$$

Here, m is the particle mass. A derivation of (5.1) is a straightforward application of the conservation of energy and momentum and can be found in textbooks on classical mechanics, see, e.g., [6]. r_0 is determined as the distance for which the expression under the square root is zero.

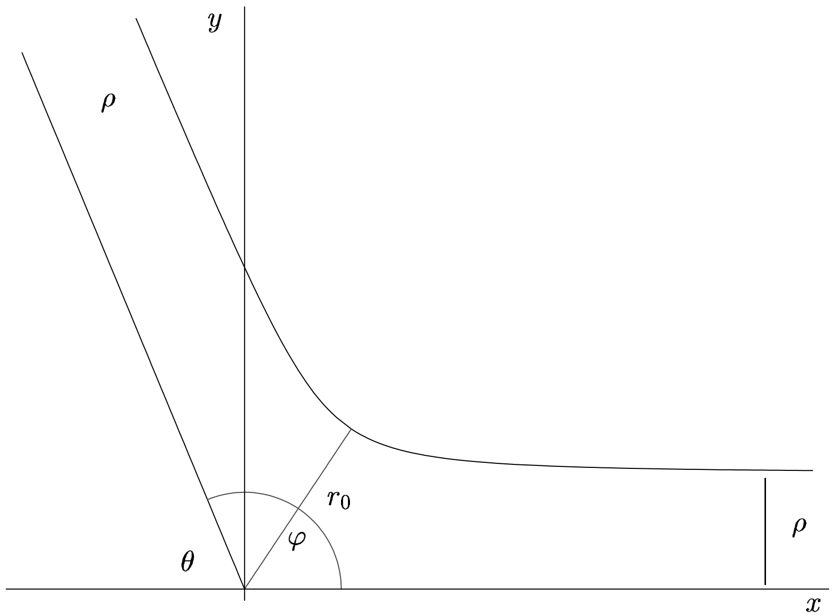

Figure 1. Scattering with a repulsive potential

Note that $\varphi \in [0,\infty]$ with $\varphi(0)=0, \quad \varphi(\infty)=\pi$. The function $\rho \to \varphi$ is 1-1 and invertible. Setting $\theta = \pi - \varphi$, we define $\tilde{\rho}(\theta) = \rho(\pi-\theta)$. The differential cross-section to be used in the Boltzmann collision kernel is

$$\sigma(u,\theta) = \frac{\tilde{\rho}(\theta)}{\sin\theta}\left|\frac{d\tilde{\rho}(\theta)}{d\theta}\right|. \tag{5.2}$$

As is known [4], the function $|u|\sigma(u,\theta)$ is the appropriate kernel for the Boltzmann collision integral associated with the potential $U(r)$ under consideration.

Let us first study a power-potential

$$U(r) = -\frac{\alpha}{r^\gamma}, \quad \alpha > 0, \quad 1 \le \gamma < 2.$$

Formula (5.1) becomes

$$\varphi = 2\rho \int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{1 + \left(\frac{s}{r}\right)^{\gamma} - \left(\frac{\rho}{r}\right)^2}}, \quad \text{with} \quad s^{\gamma} = \frac{2\alpha}{mu^2}, \tag{5.3}$$

or

$$\varphi = \varphi(z) = 2\int_0^{x_0} \frac{dx}{\sqrt{1 + zx^{\gamma} - x^2}},$$

with $z = \frac{2\alpha}{mu^2\rho^{\gamma}}$ and $1 + zx_0^{\gamma} - x_0^2 = 0$.

Consider the special powers $\gamma = 2 - 1/n$, $n = 1, 2, \ldots$, then the substitution $x = (zy)^n$ leads to

$$\varphi(z) = 2n \int_0^{y_0} dy \left[z^{-2n} y^{2-2n} + y(1-y)\right]^{-1/2}$$

with $y_0(1 - y_0) + y^{2-2n} z^{-2n} = 0$.

If $z \to \infty$, i.e., $\rho \to 0$, then

$$\varphi(z) \to \varphi_{max} = 2n \int_0^1 \frac{dy}{\sqrt{y(1-y)}} = 2n\pi. \tag{5.4}$$

Hence simple trajectories (i.e., $\pi < \varphi < 2\pi$) occur only for $n = 1$ (the Newtonian potential), while in the general cases $n = 2, 3, \ldots$ we get spiral trajectories with $n - 1$ loops. See Figure 2 for an example.

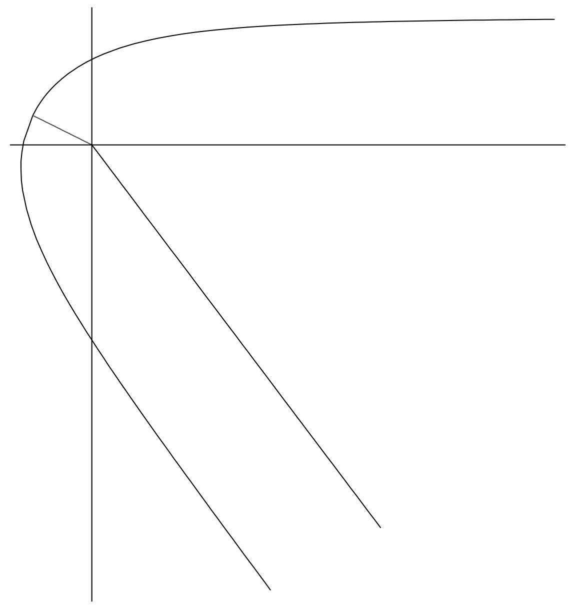

Figure 2: $\pi < \varphi \le 2\pi$, and $\theta = \varphi - \pi$

To compute the scattering angle $\theta \in [0, \pi]$ between the velocities $\vec{u}_-$ and $\vec{u}_+$, we observe the following elementary geometric relationship between $\varphi \in (\pi, \infty)$ and $\theta \in [0, \pi]$ (see Figure 2):

$$\text{If} \quad (2k-1)\pi \le \varphi \le 2k\pi, \quad \text{then} \quad \theta = \varphi - (2k-1)\pi, \tag{5.5a}$$

$$\text{and if} \quad 2k\pi \le \varphi \le (2k+1)\pi, \quad \text{then} \quad \theta = (2k+1)\pi - \varphi. \tag{5.5b}$$

The formulas (5.1) and (5.5) define the scattering angle θ uniquely as a function of the impact parameter $\rho \in (0, \infty)$.

We next consider the scattering problem for the potential

$$U(r) = -\frac{\epsilon}{r^2}, \quad \epsilon > 0. \tag{5.6}$$

For this case, the integral (5.1) gives the simple explicit formula

$$\varphi = \pi \left[1 - \left(\frac{\rho_*}{\rho} \right)^2 \right]^{-1/2}, \quad \text{with} \quad \rho_*^2 = \frac{2\epsilon}{mu^2} \tag{5.7}$$

for $\rho > \rho_*$. Note that $\varphi \to \infty$ as ρ decreases from ∞ to ρ_*.

If $\rho < \rho_*$, then the particle "falls into the center $r = 0$". To analyse this in more detail, recall that the scattering problem appears initially from a reduction of the two body problem. Consider two particles with masses m_i, positions $\vec{x}_i$ and velocities $\vec{v}_i$ $(i = 1, 2)$, interacting via the potential $U(|\vec{x}_1 - \vec{x}_2|)$. A standard transformation to the center-mass frame of reference,

$$\begin{aligned} \vec{X} &= \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}, \quad \vec{x} = \vec{x}_1 - \vec{x}_2, \\ \vec{V} &= \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}, \quad \vec{u} = \vec{v}_1 - \vec{v}_2 \end{aligned} \tag{5.8}$$

reduces the two-body problem to the problem of one body (with mass $m = \frac{m_1 m_2}{m_1 + m_2}$, position $\vec{x}$ and velocity $\vec{u}$) in the central field $U(|\vec{x}|)$. For the potential (5.6) a global (in time) solution of the two-body problem does not exist for sufficiently small ($\rho < \rho_*$) relative angular momentum. For given initial conditions

$$\vec{x}(0) = \vec{x}_0, \quad \vec{u}(0) = \vec{u}_0, \quad \vec{x}_0 \cdot \vec{u}_0 < 0, \quad E = \frac{m\vec{u}_0^2}{2} - \frac{\epsilon}{|\vec{x}_0|^2} < 0,$$

$$M^2 := m^2[x_0^2 u_0^2 - (\vec{x}_0 \cdot \vec{u}_0)^2] < 2\epsilon m,$$

one can prove that there exists a time instant $t_0 \in (0, \infty)$ such that

$$\vec{x}(t) \to 0, \quad |\vec{u}(t)| \to \infty \quad \text{as} \quad t \nearrow t_0.$$

The question arises how to continue the solution of the two-body problem for $t > t_0$. A natural way to do so is to assume that the two particles simply coagulate at $t = t_0$, i.e., they form a larger particle with mass $M = m_1 + m_2$ and velocity $\vec{V}$. This guarantees momentum conservation, while there is in general a loss of energy. This corresponds to the implicit assumption that the new “large” particle possesses internal energy which, however, is irrelevant for our purposes.

Summarizing, we describe collisions between two particles with masses m_i, $i = 1, 2$ and velocities $\vec{v}_i$, $i = 1, 2$ before the collision and interacting with the potential (5.6) as follows. If the impact parameter ρ satisfies $\rho > \rho_*$ (see (5.7)), then an elastic scattering with scattering angle $\theta \in [0, \pi]$ given by (5.1) and (5.5) occurs. If $\rho < \rho_*$ we have coagulation, i.e., the collision results in the formation of a larger particle with mass $M_+ = m_1 + m_2$ and velocity $\vec{V} = (m_1\vec{v}_1 + m_2\vec{v}_2)/M_+$.

This analysis of the pair collision process applies not only to the potential (5.6), but also to other potentials $U(r)$ for which $U(r)r^2 \to -\epsilon$ as $r \to 0$. See [2] for details.

6. Collision integrals for elastic scattering. For a repulsive pair potential defining a scattering cross-section as discussed above, the collision term for the Boltzmann equation is

$$Q(f,f)(v) := \int_{\mathbb{R}^3 \times S^2} dw\, dn\; \sigma(|u|, \phi)|u|\, [f(v')f(w') - f(v)f(w)] \tag{6.1}$$

where $u = v - w$, the arguments x and t have been suppressed, and the post-collisional velocities v' and w' are given by

$$v' = \frac{1}{2}(v + w + |u|n), \quad w' = \frac{1}{2}(v + w - |u|n).$$

Here, $n = (\sin\theta\cos\alpha, \sin\theta\sin\alpha, \cos\theta)$, $dn = \sin\theta d\theta d\alpha$, $0 \le \theta \le \pi$, $0 \le \alpha \le 2\pi$. Compare with (4.1).

In preparation for writing the collision integral for attractive potentials, we rewrite (6.1) in the more compressed form

$$Q(f,f)(v) = \int_{\mathbb{R}^3} dw\, |v - w|\, \Psi(\frac{v + w}{2}, v - w) \tag{6.2}$$

with

$$\Psi(V,U) = \int_{S^2} dn\, \sigma(|U|, \phi)\, \{F(V, |U|n) - F(V,U)\} \tag{6.3}$$

and $F(V,U) = f(V + \frac{U}{2}) f(V - \frac{U}{2})$.

Note that the original form of the Boltzmann collision integral includes an integration over the impact parameter $\rho \in (0, \infty)$, and *not* over the scattering angle θ. With the integral over ρ and (5.2), the integral in (6.3) becomes (we omit the argument V)

$$\Psi(U) = \int_0^\infty d\rho\, \rho \int_0^{2\pi} d\alpha \left\{ F(|U|n) - F(U) \right\}, \tag{6.4}$$

where $n \in S^2$ is still defined as before, but the polar scattering angle θ is now a function of ρ as defined by the scattering problem which we discussed earlier. If the intermolecular potential $U(r)$ is positive and monotonically decreasing with r, we have a one-to-one correspondence between $0 \le \theta \le \pi$ and $0 \le \rho < \infty$, and (5.1) defines the relationship between (6.3) and (6.4).

In the sequel we consider the integral (6.3) for attractive potentials when the dependence between ρ and θ is given by the formulas (5.1) and (5.5). We assume monotone dependence between ρ and φ.

To simplify our notation, we write the integral (6.4) in the compressed form

$$\Psi(U) = \int_0^\infty d\rho\, \rho\, G[\theta(\rho)], \tag{6.5}$$

with $G[\theta(\rho)] = \int_0^{2\pi} d\alpha \left\{ F(|U|n) - F(U) \right\}$, where $n = (\theta(\rho), \alpha)$ is given in spherical coordinates with the polar axis in direction U. Using (5.1), we define ρ_n, $n = 1, 2, \ldots$, by $\rho_1 = \infty$ and $\varphi(\rho_n) = n\pi, \quad n = 2, 3, \ldots$ If there is a φ_{max} such that $\varphi(\rho) \to \varphi_{max}$ as $\rho \to 0$, and if we set $\varphi_{max} = N\pi + \varphi_0$ with $0 \le \varphi_0 \le \pi$, we set $\rho_n \equiv 0$ for $n > N$. Using (5.5), we rewrite (6.4) as

$$\Psi(U) = \sum_{k=1}^{\infty} \left\{ \int_{\rho_{2k}}^{\rho_{2k-1}} d\rho\, \rho G[\varphi(\rho) - (2k-1)\pi] + \int_{\rho_{2k+1}}^{\rho_{2k}} d\rho\, \rho G[(2k+1)\pi - \varphi(\rho)] \right\} \tag{6.6}$$

$\varphi(\rho)$ being defined in (5.1).

Let

$$\begin{aligned} \tilde{\rho}^{(2k)}(\theta) &= \rho[(2k-1)\pi + \theta], \quad k = 1, 2, \ldots \\ \tilde{\rho}^{(2k+1)}(\theta) &= \rho[(2k+1)\pi - \theta], \quad k = 1, 2, \ldots \end{aligned} \tag{6.7}$$

$0 \leq \theta \leq \pi$. We remark that the inverse function $\rho(\varphi)$ exists in view of our assumption that $\varphi(\rho)$ is strictly monotone.

It is now natural to introduce a set of "partial" differential cross section (see (5.2) for comparison)

$$\sigma_n(\theta) = \frac{\tilde{\rho}^{(n)}(\theta)}{\sin\theta} \left| \frac{d}{d\theta} \tilde{\rho}^{(n)}(\theta) \right|, \quad n = 2, 3, \ldots \tag{6.8}$$

and to set $\rho = \tilde{\rho}^{(2k)}(\theta)$ or $\rho = \tilde{\rho}^{(2k+1)}(\theta)$ in each of the integrals in (6.6). Returning to the integration variable $\theta \in [0, \pi]$, the result is

$$\Psi(U) = \int_0^\pi d\theta \, \sin\theta \, F(\theta) \, \hat{\sigma}(|U|, \theta), \tag{6.9}$$

with

$$\hat{\sigma}(|U|, \theta) = \sum_{n=2}^{\infty} \sigma_n(|U|, \theta). \tag{6.10}$$

If $\varphi_{max} < \infty$, the sum in (6.10) is actually a finite sum. For example, $\sigma_n(|U|, \theta) = 0$ for $n = 3, 4, \ldots$ for the Newtonian potential with $\gamma = 1$, while $\sigma_2(|U|, \theta)$ is merely the classical Rutherford cross-section in this case (the integral (6.5) is then actually divergent, a fact we ignore for the current formal discussion).

The partial differential cross-sections $\sigma_n(\theta)$ in (6.8) allow the following physical interpretation: Each function $\sigma_n(\theta)$ is associated with the relative contribution of those trajectories which have exactly $n-1$ intersections with a real axis directed along u.

To summarize, we obtain for an attractive potential $U(r)$ the usual Boltzmann collision integral (6.1), provided that $|U(r)|r^2 \to 0$ as $r \to 0$. The only difference is that we have to use a generalized differential cross-section $\hat{\sigma}(|U|, \theta)$ as given by (6.10) to replace the usual cross-section $\sigma(|U|, \theta)$.

We now discuss the case $U(r) = -\epsilon/r^2$. Elastic scattering is again described by the Boltzmann collision integral in the form (6.2)-(6.4), provided that we integrate over $d\rho$ in (6.4) *not* from 0 to ∞, but from $\rho = \rho_{\min} = \rho_*$

(as given by (5.7)) to ∞. All the above arguments carry over. Inverting the explicit formula (5.7), we find

$$\rho^2 = \rho_*^2 \left[1 - (\pi/\varphi)^2\right]^{-1}, \tag{6.11}$$

hence

$$\begin{aligned} \rho \frac{d\rho}{d\varphi} &= -\frac{\rho_*^2 \pi^2}{\varphi^3} \left[1 - (\pi/\varphi)^2\right]^{-2} \\ &= -\frac{(\pi\rho_*)^2 \varphi}{[\varphi^2 - \pi^2]^2}. \end{aligned} \tag{6.12}$$

From (6.7), (6.8)

$$\begin{aligned} \sin\theta\ \sigma_{2k}(\theta) &= (\pi\rho_*)^2 \frac{(2k-1)\pi + \theta}{\{[(2k-1)\pi + \theta]^2 - \pi^2\}^2}, \\ \sin\theta\ \sigma_{2k+1}(\theta) &= (\pi\rho_*)^2 \frac{(2k+1)\pi - \theta}{\{[(2k+1)\pi - \theta]^2 - \pi^2\}^2}, \end{aligned} \tag{6.13}$$

$k = 1, 2, \ldots$ The formulas (6.10) and (6.13) (note that the convergence in (6.10) is straightforward) define a generalized elastic cross-section $\hat{\sigma}(|U|, \theta)$ for the potential $U(r) = -\epsilon/r^2$, $\epsilon > 0$.

7. Boltzmann-Smoluchowski collision integrals. We focus on the potential

$$U(r) = -\frac{\epsilon}{r^2}, \quad \epsilon > 0. \tag{7.1}$$

As discussed earlier, a classical solution of the scattering problem for this potential exists only for sufficiently large impact parameters $\rho^2 > \rho_*^2 = \frac{2\epsilon}{\mu u^2}$ and reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$. If $\rho < \rho_*$, we have to consider a generalized solution of the scattering problem which incorporates a coagulation process. A consequence is that we have to consider particles with different masses in any case, because particle masses change in the collision (coagulation) process.

1. We replace the distribution function $f(x, v, t)$ of a simple gas by a new distribution function $f(m, x, v, t)$, where $m > 0$ denotes a particle mass. The total number of particles is now given by the formula

$$N = \int\int dx dv \int_0^\infty dm\, f(m, x, v, t). \tag{7.2}$$

2. We assume that a pair collision with an impact parameter $\rho < \rho_*$ results in coagulation of particles, i.e., in the "reaction"

$$(m_1, v_1) + (m_2, v_2) \longrightarrow (M, V) \tag{7.3}$$

with $M = m_1 + m_2$, $V = \frac{m_1 v_1 + m_2 v_2}{M}$. Note that if $\rho < \rho_*$ the postcollisional velocity V does not depend on the impact parameter ρ. The coagulation process is therefore completely determined by a total coagulation cross-section

$$\sigma_c(u; m_1, m_2) = \pi \rho_*^2. \tag{7.4}$$

An elastic differential cross-section can now be calculated in the same way as described in Section 6. We set

$$\rho_*^2 = \frac{2\epsilon(m_1 + m_2)}{m_1 m_2 u^2} \tag{7.5}$$

in (6.13) and calculate a differential cross-section $\hat{\sigma}(u, \theta; m_1, m_2)$ by the formulas (6.10-13).

We are now ready to write the Boltzmann-Smoluchowski equation for a distribution density $f(m, x, v, t)$:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q_B + I_{Sm} \tag{7.6}$$

where Q_B and I_{Sm} denote the Boltzmann and Smoluchowski collision integrals, respectively. As discussed in section 4, a Vlasov term should also be included; we omit this term in the present section for the sake of simplicity.

First calculate cross-sections $\hat{\sigma}(u, \theta; m_1, m_2)$ and $\sigma_c(u; m_1, m_2)$ as described earlier. Then

$$\begin{aligned} Q_B = \int_0^\infty dm_1 \int_{\mathbb{R}^3 \times S^2} & dw\, dn |U| \hat{\sigma}(|U|, \theta; m, m_1) \\ & \times \{f(m, v') f(m_1, w') - f(m, v) f(m_1, w)\} \end{aligned} \tag{7.7}$$

with $U = v - w$, $U \cdot n = |U| \cos\theta$, $v' = V + \frac{\mu}{m} U \cdot n$, $w = V - \frac{\mu}{m} U \cdot n$, $V = \frac{mv + m_1 w}{m + m_1}$, $\mu = \frac{m_1 m}{(m_1 + m)}$. This is the obvious generalization of the usual Boltzmann collision integral for particles with different masses.

The Smoluchowski collision integral is

$$
\begin{aligned}
I_{Sm} = \frac{1}{2}\int_0^\infty dm_1 \int_0^\infty dm_2 \int_{\mathbb{R}^3\times\mathbb{R}^3} dv_1 dv_2 |U|\sigma_c(|U|; m_1, m_2) \\
f(m_1, v_1) f(m_2, v_2)\delta(m_1+m_2-m)\delta\left(\frac{m_1v_1+m_2v_2}{m_1+m_2} - v\right) \qquad (7.8) \\
- f(m, v)\int_0^\infty dm_1 \int dw\, f(m_1, w)|v-w|\,\sigma_c(|v-w|; m_1, m_2)
\end{aligned}
$$

with $U = v_1 - v_2$. The factor 1/2 in front of the first integral gives the correct number of inelastic collisions. We have omitted the arguments x and t throughout.

The cross sections $\hat{\sigma}(|U|, \theta; m_1, m_2)$ is a symmetric function of the masses m_1 and m_2. The presented form of the coagulation term is convenient for the calculation of inner products

$$
(\psi, I_{Sm}) = \int_0^\infty dm \int_{\mathbb{R}^3} dv\, \psi(m, v) I_{Sm}(m, v)
$$

with test functions ψ. After some elementary transformations, one finds

$$
\begin{aligned}
(\psi, I_{Sm}) = \frac{1}{2}\int dm_1 \int dm_2 \int_{\mathbb{R}^3\times\mathbb{R}^3} dv_1 dv_2\, f(m_1, v_1) f(m_2, v_2) \\
\times|U|\sigma_c(|U|; m_1, m_2)\left[\psi(m_1+m_2, \frac{m_1v_1+m_2v_2}{m_1+m_2}) - \psi(m_1, v_1) - \psi(m_2, v_2)\right],
\end{aligned}
$$

$U = v_1 - v_2$, from which conservation of mass and momentum readily follow:

$$
(m, I_{Sm}) = 0 = (mv, I_{Sm}).
$$

We conclude by presenting explicit formulas for (7.1). Let (see (6.10-13))

$$
\begin{aligned}
g(\theta; m_1, m_2) &= 2\frac{\pi^2\epsilon(m_1+m_2)}{m_1m_2} g(\theta), \\
\sin\theta\ g(\theta) &= \sum_{k=1}^{\infty}\left\{\frac{(2k-1)\pi+\theta}{\{[(2k-1)\pi+\theta]^2-\pi\}^2} + \frac{(2k+1)\pi-\theta}{\{[(2k+1)\pi-\theta]^2-\pi\}^2}\right\}, \qquad (7.9) \\
p(m_1, m_2) &= 2\frac{\pi\epsilon(m_1+m_2)}{m_1m_2}.
\end{aligned}
$$

The collision terms in (7.6) then become

$$Q_B = \int_0^\infty dm_1 \int_{\mathbb{R}^3\times\mathbb{R}^3} dw\, dn \frac{g(\theta; m_1, m_2)}{|v-w|} \times \{f(m,v')f(m_1,w') - f(m,v)f(m_1,w)\} \tag{7.10}$$

and

$$I_{Sm} = \frac{1}{2}\int_0^m ds \int_{\mathbb{R}^3} du \frac{p(s,m-s)}{|U|} f(s, v+\frac{s}{m}U) f(m-s, v-\frac{m-s}{m}U) - f(m,v)\int_0^\infty ds \int dw \frac{p(m,s)}{|v-w|} f(s,w), \tag{7.11}$$

where v' and w' are defined after (7.7).

For generalizations to other potentials, the two-dimensional case and additional details the reader is referred to [2].

Concluding remarks. We have presented a survey on kinetic equations for particle systems in which the particles interact via pair potentials $\pm\epsilon/r^n$, with $\epsilon > 0$ and $1 < n \leq 2$. The emphasis of the discussion was on the case $-\epsilon/r^2$. i.e., the attractive (stellar-dynamic) potential known as the Manev correction. We showed that for this potential, the kinetic equation depicting the time evolution of a particle cloud should include Vlasov interaction, Boltzmann collision and Smoluchowski coagulation integrals, and we derived the corresponding cross-sections.

If one considers the full Manev potential $-\gamma/r - \epsilon/r^2$, in which the term $-\epsilon/r^2$ may be interpreted as a quasirelativistic correction to the Newtonian potential, it follows that the corresponding kinetic equations should include corrective Boltzmann collision and Smoluchowski coagulation terms. A possible application of such equations would be to the dynamics of globular clusters, where close encounters between stars are sufficiently frequent to justify the inclusion of diffusive collision terms in the kinetic equation (in astrophysical applications, the Fokker-Planck collision term [4] is usually employed in this context).

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REFERENCES

[1] A.V. Bobylev, P. Dukes, R. Illner and H.D. Victory, On Vlasov-Manev Equations, I: Foundations, Properties and Non-Global Existence, Journal Stat. Phys. 88, 885-912 (1997)

[2] A. V. Bobylev and R. Illner, Collision Integrals for Attractive Potentials, Journal Stat. Phys. 95 (3/4), 633-650 (1999)

[3] A. V. Bobylev and G. Vilasi, Projective Invariance for Classical and Quantum Systems, J. Gp. Th. Phys. 3 (1), 1995

[4] C. Cercignani, Theory and Application of the Boltzmann Equation. Springer-Verlag, New York 1988

[5] R. Illner, H. D. Victory, P. Dukes and A. V. Bobylev, On Vlasov-Manev Equations, II: Local Existence and Uniqueness, Journal Stat. Phys. 91 (3/4), 625-654 (1998)

[6] L. D. Landau, E. M. Lifshitz, Mechanics. Pergamon Press, Oxford-London-Paris 1960

[7] G. Manev, La gravitation et le principe de l'égalité de l'action et de la réaction, C.R.A.S. 178, 2159-2161 (1924)

[8] G. Manev, Die Gravitation und das Prinzip von Wirkung und Gegenwirkung, Zeitschrift für Physik 31, 786-802 (1925)

[9] G. Manev, Le principe de la moindre action et la gravitation, C.R.A.S. 190, 963-965 (1930)

[10] G. Manev, La gravitation et l'énergie au zéro, C.R.A.S. 190, 1374-1377 (1930)

11] I. Newton, in Principia, Book I, Article IX, Theorem IV, Corollary 2

[12] R. Steacy, private communication, manuscript in preparation